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GROUND STATE SOLUTIONS FOR ASYMPTOTICALLY PERIODIC SCHRÖDINGER-POISSON SYSTEMS IN $\mathbb{R}^2$

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Abstract. This article concerns the planar Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + V(x)u + \phi u = f(x,u), \quad x \in \mathbb{R}^2, \\ \Delta\phi = u^2, \quad x \in \mathbb{R}^2, \end{gathered}$$

where $V(x)$ and $f(x,u)$ are periodic or asymptotically periodic in x. By combining the variational approach, the non-Nehari manifold approach and new analytic techniques, we establish the existence of ground state solutions for the above problem in the periodic and asymptotically periodic cases. In particular, in our study, f is not required to satisfy the Ambrosetti-Rabinowitz type condition or the Nehari-type monotonic condition.

1. Introduction

In this article, we consider the planar Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + V(x)u + \lambda\phi u = f(x,u), \quad x \in \mathbb{R}^2, \\ \Delta\phi = u^2, \quad x \in \mathbb{R}^2, \end{gathered} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, V and f satisfy the following assumptions:

- (A1) $V \in L^\infty(\mathbb{R}^2,\mathbb{R})$ and $\inf_{x\in\mathbb{R}^2} V(x) > 0$;
- (A2) $f \in \mathcal{C}(\mathbb{R}^2\times\mathbb{R},\mathbb{R})$, and there exist constants $C_0 > 0$ and $p \in (2,\infty)$ such that
$$|f(x,t)| \le C_0\left(1+|t|^{p-1}\right), \quad \forall (x,t) \in \mathbb{R}^2\times\mathbb{R};$$
- (A3) $f(x,t) = o(|t|)$ as $t \to 0$, uniformly in $x \in \mathbb{R}^2$.

System (1.1) is a special form of the Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + V(x)u + \lambda\phi u = f(x,u), \quad x \in \mathbb{R}^N, \\ \Delta\phi = u^2, \quad x \in \mathbb{R}^N, \end{gathered} \tag{1.2}$$

where $\lambda \in \mathbb{R}$, $V \in \mathcal{C}(\mathbb{R}^N,(0,\infty))$ and $f \in \mathcal{C}(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$. It is well known that the solutions of (1.2) are related to the solitary wave solutions to the Schrödinger-Poisson system

$$\begin{gathered} -i\psi_t - \Delta\psi + E(x)\psi + \lambda\phi\psi = f(x,\psi), \\ \Delta\phi = |\psi|^2, \end{gathered} \tag{1.3}$$

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in $\mathbb{R}^N \times \mathbb{R}$, where $\psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ is the wave function, E is a real external potential, $\lambda \in \mathbb{R}$ is a parameter, ϕ represents an internal potential for a nonlocal self-interaction of the wave function and the nonlinear term f describes the interaction effect among many particles. It has a profound physical meaning because it appears in quantum mechanics models (see e.g. [5, 6, 20]) and in semiconductor theory [4, 23, 25]. For more details in the physical applications, we refer the readers to [3, 4].

From a mathematical point of view, the second equation in (1.2) determines $\phi : \mathbb{R}^N \to \mathbb{R}$ only is up to harmonic functions. It is natural to choose ϕ as the negative Newton potential of u^2, i.e., the convolution of u^2 with the fundamental solution Γ_N of the Laplacian, which is given by

$$\Gamma_N(x) = \begin{cases} \frac{1}{2\pi} \ln |x|, & N = 2, \\ \frac{1}{N(2-N)\omega_N} |x|^{2-N}, & N \neq 2, \end{cases}$$

here ω_N is the volume of the unit N-ball. With this formal inversion of the second equation in (1.2), we obtain the integro-differential equation

$$-\Delta u + V(x)u + \lambda(\Gamma_N * u^2)u = f(x, u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

Let $\phi_{N,u}(x) = (\Gamma_N * u^2)(x)$. At least formally, the energy functional associated to (1.2) becomes

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x + \frac{\lambda}{4} \int_{\mathbb{R}^N} \phi_{N,u} u^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(x, u)\mathrm{d}x,$$

where, and in the sequel, $F(x, t) := \int_0^t f(x, s)\mathrm{d}s$. If u is a critical point of J, then the pair $(u, \phi_{N,u})$ is a weak solution of (1.2). For the sake of simplicity, in many cases we just say u, instead of $(u, \phi_{N,u})$, is a weak solution of (1.2).

In recent years, there has been increasing attention on the existence of positive solutions ground state solutions and multiple solutions for to systems of the form (1.2). The greatest part of the literature focuses on the study of (1.2) with $N = 3$ and $\lambda < 0$. In this case, by Hardy-Littlewood-Sobolev inequality (see [21] or [22, page 98]), J is a well-defined of class $\mathcal{C}^1$ functional on space

$$H_V = \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x < +\infty\right\}.$$

Moreover, the competing nonlocal term

$$\lambda \int_{\mathbb{R}^3} \phi_{3,u} u^2 \mathrm{d}x = -\frac{\lambda}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x - y|} \mathrm{d}x\mathrm{d}y$$

is positive and homogeneous of degree 4, the mountain pass geometry can be easily verified provided $f(x, t)$ is superlinear at $t = 0$ and super-cubic at $t = \infty$. In this situation, the existence or multiplicity of solutions have been obtained under various assumptions on V and f, see e.g. [1, 2, 7, 8, 11, 12, 17, 15, 16, 19, 27, 28, 31, 37, 39, 40].

As described above, there are many results for (1.2) with $N = 3$. In contrast, the literature is scantier for the planar case. Unlike the three dimensional case, the logarithmic integral kernel

$$\phi_{2,u}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x - y| u^2(y)\mathrm{d}y$$

is sign-changing and neither bounded from above nor from below, which may behave like $\frac{1}{2\pi}\|u\|_2^2 \ln|x|$ at infinity. Moreover, J is not well defined on $H^1(\mathbb{R}^2)$ even if $V \in L^\infty(\mathbb{R}^2)$ and $\inf_{\mathbb{R}^2} V > 0$. Hence, variational methods for (1.2) with $N = 3$ can not be directly applied to (1.1). This is one of the reasons why much less is known in the planar case.

In this work we focus on (1.1) in the case $N = 2$ and $\lambda > 0$, and by rescaling we may assume $\lambda = 1$. More precisely, we are dealing with System (1.1), the associated scalar equation

$$-\Delta u + V(x)u + (\Gamma_2 * u^2)u = f(x,u), \quad x \in \mathbb{R}^2. \tag{1.5}$$

Inspired by Stubbe [30], Cingolani and Weth [10] developed a variational framework for the above equation with a smaller Hilbert space

$$E := \big\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} [V(x) + \ln(1+|x|)]\, u^2 \mathrm{d}x < \infty\big\} \tag{1.6}$$

equipped with the norm

$$\|u\|_E = \Big(\int_{\mathbb{R}^2} \big[|\nabla u|^2 + V(x)u^2 + \ln(1+|x|)u^2(x)\big]\,\mathrm{d}x\Big)^{1/2}.$$

It is easy to see that the corresponding energy functional associated with (1.5)

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^2} \big(|\nabla u|^2 + V(x)u^2\big)\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2} \phi_{2,u}(x)u^2 \mathrm{d}x - \int_{\mathbb{R}^2} F(x,u)\mathrm{d}x, \tag{1.7}$$

for $u \in E$, is a well-defined of class $\mathcal{C}^1$ functional on E under assumptions (A1)–(A3), see also in Section 2. When V satisfies the assumption

(A4) $V \in \mathcal{C}(\mathbb{R}^2, (0,\infty))$ and $V(x)$ is 1-periodic in x_1 and x_2,

and $f(x,t) = b|t|^{p-2}t$ with $b \geq 0$ and $p \geq 4$, by a strong compactness condition (modulo translation) for Cerami sequences at arbitrary positive energy levels, Cingolani and Weth [10] proved that (1.1) admits high energy solutions, and every minimizer u of Φ on the Nehari manifold

$$\mathcal{N} := \{u \in E : u \neq 0,\ \langle \Phi'(u), u\rangle = 0\}$$

is a solution which obeys the minimax characterization

$$\Phi(u) = \inf_{\mathcal{N}} \Phi = \inf_{u \in E\setminus\{0\}} \sup_{t \geq 0} \Phi(tu) > 0. \tag{1.8}$$

When $V \equiv 1$ and $f(x,t) = b|t|^{p-2}t$ with $b > 0$ and $p > 2$, based on the strong compactness condition introduced by Cingolani and Weth [10], and a scaling technique developed by Jeanjean [18], Du and Weth [13] constructed a Cerami sequence with a key additional property related to the Pohozaev identity, and proved the boundedness of this Cerami sequence when $2 < p < 4$, which is the main obstacle in [13]. Hence, they can relax the restriction $p \geq 4$ to $p > 2$. Very recently, Chen and Tang [9] established the existence of nontrivial solutions and ground state solutions in the axially symmetric functions space.

It is worth pointing out that the approach used in [10, 13] heavily rely on the fact that V is a positive constant or $\mathbb{Z}^2$-translation invariance and $f(x,t) = b|t|^{p-2}t$. They can not directly applied to (1.1) with variable potential and nonlinearity, even if $V(x)$ and $f(x,t)$ are asymptotically periodic in x.

Motivated by [10, 13], in the present paper, by combining the approach developed in [10] with some new tricks, we shall establish the existence of ground state solutions for (1.1) in the periodic and asymptotically periodic cases. In particular, in our set of hypotheses, f is not required to satisfy the Ambrosetti-Rabinowitz type condition

(AR) $0 < 4F(x,t) \le f(x,t)t$ for all $x \in \mathbb{R}^2$ and $t \in \mathbb{R} \setminus \{0\}$,

which would readily imply the boundedness of Palais-Smale sequences; nor does the Nehari-type monotonic condition:

(MT) the function $t \mapsto f(x,t)/|t|^3$ is nondecreasing on $\mathbb{R} \setminus \{0\}$,

which prevents us from using Nehari manifold and fibering methods as e.g. in [26, 32, 33].

Here, we point out some difficulties involving this subject. (1) The norm of E is not translation invariant even if the functional Φ is translation invariant; (2) The quadratic part of Φ is not coercive on E; (3) The Nehari manifold approach is not applicable without the monotonicity on $f(x,t)/|t|^3$; (4) Φ loses the $\mathbb{Z}^2$-translation invariance in the asymptotically periodic case.

Difficulties (1) and (2) have been overcome in [10]. To overcome difficulty (3), we shall use the non-Nehari manifold approach developed by Tang [36], i.e., finding a minimizing Cerami sequence for Φ outside $\mathcal{N}$ by the diagonal method, see Lemma 2.9. Difficulty (4) can be overcome by showing that the minimizer of Φ on $\mathcal{N}$ is a critical point (because f is only assumed to be continuous, $\mathcal{N}$ may not be a $\mathcal{C}^1$-manifold of E), see Lemma 4.2 below.

Before presenting our theorems, we fix notation. Let

$$\mathcal{B} = \{u \in L^\infty(\mathbb{R}^2, \mathbb{R}) : \operatorname{meas}\{x \in \mathbb{R}^2 : |u(x)| \ge \epsilon\} < \infty,\ \forall \epsilon > 0\}.$$

In addition to (A2)–(A4), we introduce the following assumptions:

(A5) $V(x) = V_0(x) + V_1(x)$, $\inf_{\mathbb{R}^2} V > 0$, $V_0 \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$, $V_0(x)$ is 1-periodic in x_1 and x_2, and $V_1 \in \mathcal{C}(\mathbb{R}^2, (-\infty, 0]) \cap \mathcal{B}$;

(A6) $f(x,t)$ is 1-periodic in x_1 and x_2;

(A6') $f(x,t) = f_0(x,t) + f_1(x,t)$, $f_0 \in \mathcal{C}(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$, $f_0(x,t)$ is 1-periodic in x_1 and x_2, and $f_1 \in \mathcal{C}(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$ satisfies that

$$\begin{gathered} f_1(x,t)t \ge 0, \quad \frac{1}{4}V_1(x)t^2 + \frac{1}{4}f_1(x,t)t - F_1(x,t) \le 0, \\ |f_1(x,t)| \le a(x)\left(|t| + |t|^{p_0-1}\right) \quad \text{with } a \in \mathcal{B}, \end{gathered} \tag{1.9}$$

where $F_1(x,t) = \int_0^t f_1(x,s)\mathrm{d}s$ and $p_0 \in (2,\infty)$;

(A7) $\inf_{x\in\mathbb{R}^2, t\in\mathbb{R}\setminus\{0\}} \frac{F(x,t)}{|t|^2} > -\infty$;

(A8) there exists $\theta \in (0,1)$ such that

$$\frac{1}{4}f(x,t)t - F(x,t) + \frac{\theta}{4}V(x)t^2 \ge 0, \quad \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R};$$

(A8') there exists $\theta \in (0,1)$ such that

$$\Big[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,t\tau)}{(t\tau)^3}\Big] \operatorname{sign}(1-t) + \theta V(x)\frac{|1-t^2|}{(t\tau)^2} \ge 0, \tag{1.10}$$

for all $x \in \mathbb{R}^2$, $t > 0$, $\tau \ne 0$.

Now, we state our results of this paper.

Theorem 1.1. *Assume that* (A2)–(A4), (A6)–(A8) *hold. Then* (1.5) *or* (1.1) *with* $\lambda = 1$ *has a nontrivial solution of mountain pass type* $u_0 \in E$ *such that* $\Phi(u_0) > 0$.

Theorem 1.2. *Assume that* (A2)–(A4), (A6), (A7) (A8') *hold. Then* (1.5) *or* (1.1) *with* $\lambda = 1$ *has a ground state solution* $u_0 \in E$ *such that* $\Phi(u_0) = \inf_{\mathcal{N}} \Phi > 0$.

Theorem 1.3. *Assume that* (A2), (A3), (A5), (A6'), (A7), (A8') *hold. Then* (1.5) *or* (1.1) *with* $\lambda = 1$ *has a ground state solution* $u_0 \in E$ *such that* $\Phi(u_0) = \inf_{\mathcal{N}} \Phi > 0$.

Remark 1.4. By (A8'), we have

$$\begin{aligned} &\frac{1-t^4}{4}\tau f(x,\tau) + F(x,t\tau) - F(x,\tau) + \frac{\theta V(x)}{4}\left(1-t^2\right)^2\tau^2 \\ &= \int_t^1 \Big[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,s\tau)}{(s\tau)^3} + \theta V(x)\frac{(1-s^2)}{(s\tau)^2}\Big] s^3\tau^4 \mathrm{d}s \\ &\geq 0, \quad \forall x \in \mathbb{R}^2,\ t \geq 0,\ \tau \neq 0. \end{aligned} \tag{1.11}$$

Let $t = 0$ in (1.11), one can deduce (A8). This shows that (A8') implies (A8).

Theorem 1.3 is new even if $f \equiv 0$. For the asymptotically periodic case, in contrast to the case $N = 3$, it is removed that f_0 satisfies the Nehari-type monotonic condition or the condition similar to (1.10), see [8, Theorem 1.2].

Besides $f(x,t) = b|t|^{p-2}t$ with $b \geq 0$ and $p \geq 4$ considered in [10], there are many functions satisfying (A6), (A7) and (A8'), for example:

$$f(x,t) = K(x)|t|^{p-2}t - V(x)|t|^{3/2}t + V(x)|t|t, \quad \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R},$$

where $p \geq 4$, $K \in \mathcal{C}(\mathbb{R}^2, (0,+\infty))$ and $K(x)$ is 1-periodic in x_1 and x_2, and V satisfies (A4). Moreover, it is easy to see that the above function does not satisfy the usual Nehari-type monotonic condition (MT).

Under (A2), (A3) and (A7), it is difficult to find a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$ for every $u \in E \setminus \{0\}$. Hence, one can not obtain the minimax characterization (1.8) as in [10]. In the present paper, we introduce a new set

$$\begin{aligned} \Lambda := \Big\{u \in E : &\int_{\mathbb{R}^2} \left[V(x)u^2 - f(x,u)u\right] \mathrm{d}x \\ &+ \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u^2(x)u^2(y)\mathrm{d}y\mathrm{d}x < 0\Big\} \end{aligned}$$

and construct a similar minimax characterization

$$\inf_{u\in\mathcal{N}} \Phi(u) := m = \inf_{u\in\Lambda}\max_{t\geq 0} \Phi(tu) > 0,$$

see Lemmas 2.6–2.8 below.

This article is organized as follows. In Section 2, we give the variational setting and preliminaries. We complete the proofs of Theorems 1.1–1.3 in Sections 3 and 4.

Throughout this article, we let $u_t(x) := u(tx)$ for $t > 0$, and denote the norm of $L^s(\mathbb{R}^2)$ by $\|u\|_s = \left(\int_{\mathbb{R}^2} |u|^s \mathrm{d}x\right)^{1/s}$ for $s \in [2,\infty)$, $B_r(x) = \{y \in \mathbb{R}^2 : |y-x| < r\}$, and positive constants possibly different in different places, by $C_1, C_2, \dots$.

2. Variational setting and preliminaries

Under assumption (A1), we endow $H^1(\mathbb{R}^2)$ with the scalar product and norm

$$(u,v) = \int_{\mathbb{R}^2} \left(\nabla u \cdot \nabla v + V(x)uv\right) \mathrm{d}x, \quad \|u\| = \Big(\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x\Big)^{1/2}.$$

Define the symmetric bilinear forms

$$(u,v)\mapsto A_1(u,v)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\big(1+|x-y|\big)u(x)v(y)\mathrm{d}x\mathrm{d}y, \tag{2.1}$$

$$(u,v)\mapsto A_2(u,v)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\Big(1+\frac{1}{|x-y|}\Big)u(x)v(y)\mathrm{d}x\mathrm{d}y, \tag{2.2}$$

$$(u,v)\mapsto A_0(u,v)=A_1(u,v)-A_2(u,v)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u(x)v(y)\mathrm{d}x\mathrm{d}y, \tag{2.3}$$

where the definition is restricted, in each case, to measurable functions $u,v:\mathbb{R}^2\to\mathbb{R}$ such that the corresponding double integral is well defined in Lebesgue sense. Note that $0\le\ln(1+r)\le r$ for $r\ge 0$, it follows from the Hardy-Littlewood-Sobolev inequality (see [21] or [22, page 98]) that

$$|A_2(u,v)|\le\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{|x-y|}|u(x)v(y)|\mathrm{d}x\mathrm{d}y\le\mathcal{C}_1\|u\|_{4/3}\|v\|_{4/3} \tag{2.4}$$

with a constant $\mathcal{C}_1>0$. Using (2.1), (2.2) and (2.3), we define the functionals:

$$\begin{gathered}
I_1:H^1(\mathbb{R}^2)\to[0,\infty],\quad I_2:L^{8/3}(\mathbb{R}^2)\to[0,\infty),\quad I_0:H^1(\mathbb{R}^2)\to\mathbb{R}\cup\{\infty\},\\
I_1(u)=A_1(u^2,u^2)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\left(1+|x-y|\right)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y,\\
I_2(u)=A_2(u^2,u^2)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln\big(1+\frac{1}{|x-y|}\big)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y,\\
I_0(u)=A_0(u^2,u^2)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ln|x-y|u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y.
\end{gathered}$$

Here I_2 takes only finite values on $L^{8/3}(\mathbb{R}^2)$. Indeed, (2.4) implies

$$|I_2(u)|\le\mathcal{C}_1\|u\|_{8/3}^4,\quad\forall u\in L^{8/3}(\mathbb{R}^2). \tag{2.5}$$

As in [10], we define, for any measurable function $u:\mathbb{R}^2\to\mathbb{R}$

$$\|u\|_*^2=\int_{\mathbb{R}^2}\ln(1+|x|)u^2(x)\mathrm{d}x\in[0,\infty].$$

Then the set

$$E=\big\{u\in H^1(\mathbb{R}^2):\|u\|_*<+\infty\big\}$$

is a Hilbert space equipped with the norm

$$\|u\|_E=\big(\|u\|^2+\|u\|_*^2\big)^{1/2}.$$

It is easy to see that E is compactly embedded in $L^s(\mathbb{R}^2)$ for all $s\in[2,\infty)$. Moreover, since

$$\ln(1+|x-y|)\le\ln(1+|x|+|y|)\le\ln(1+|x|)+\ln(1+|y|),\quad\forall x,y\in\mathbb{R}^2, \tag{2.6}$$

we have

$$\begin{aligned}
&0\le A_1(uv,wz)\\
&\le\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}[\ln(1+|x|)+\ln(1+|y|)]\,|u(x)v(x)||w(y)z(y)|\mathrm{d}x\mathrm{d}y\\
&\le\|u\|_*\|v\|_*\|w\|_2\|z\|_2+\|u\|_2\|v\|_2\|w\|_*\|z\|_*,\quad\forall u,v,w,z\in E.
\end{aligned} \tag{2.7}$$

According to [10, Lemma 2.2], we have I_0, I_1 and I_2 are of class $\mathcal{C}^1$ on E, and

$$\langle I_i'(u), v\rangle = 4A_i(u^2, v), \quad \forall u, v \in E,\ i = 0, 1, 2. \tag{2.8}$$

Then, (A1)–(A3) and (2.8) imply that Φ is a well-defined of class $\mathcal{C}^1$ functional on E (see [10]), and that

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}[I_1(u) - I_2(u)] - \int_{\mathbb{R}^2} F(x, u)\mathrm{d}x, \tag{2.9}$$

$$\begin{aligned}\langle \Phi'(u), v\rangle = &\int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + V(x)uv)\,\mathrm{d}x + A_1(u^2, uv) - A_2(u^2, uv)\\ &- \int_{\mathbb{R}^2} f(x, u)v\mathrm{d}x.\end{aligned} \tag{2.10}$$

Hence, the solutions of (1.1) with $\lambda = 1$ are the critical points of the reduced functional (2.9).

To prove the existence of nontrivial solutions for (1.1) with $\lambda = 1$, we use the following version of the Mountain Pass Theorem, see [14, 29].

Lemma 2.1. *Let X be a real Banach space and let $\Psi \in \mathcal{C}^1(X, \mathbb{R})$. Let S be a closed subset of X which disconnects (archwise) X in distinct connected components X_1 and X_2. Suppose further that $\Psi(0) = 0$ and*

1. *$0 \in X_1$ and there is $\rho_0 > 0$ such that $\Psi|_S \geq \rho_0 > 0$,*
2. *there is $e \in X_2$ such that $\Psi(e) \leq 0$.*

Then there exists a sequence $\{u_n\} \subset X$ satisfying

$$\Psi(u_n) \to c \geq \rho_0 > 0, \quad \|\Psi'(u_n)\|(1 + \|u_n\|) \to 0,$$

where $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Psi(\gamma(t))$ and

$$\Gamma = \{\gamma \in \mathcal{C}([0, 1], X) : \gamma(0) = 0,\ \gamma(1) \in X_2,\ \Psi(\gamma(1)) < 0\}.$$

Now, we apply Lemma 2.1 to obtain a Cerami sequence of Φ.

Lemma 2.2. *Assume that* (A1)–(A3), (A7) *hold. Then there exist a constant $c > 0$ and a sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c > 0, \quad \|\Phi'(u_n)\|_{E^*}(1 + \|u_n\|_E) \to 0. \tag{2.11}$$

Proof. By (A2) and (A3), for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$f(x, t)t \leq \varepsilon t^2 + C_\varepsilon|t|^p, \quad F(x, t) \leq \varepsilon t^2 + C_\varepsilon|t|^p, \quad \forall (x, t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{2.12}$$

By (2.9) and (2.12), there exist $\delta_0 > 0$ and $\rho_0 > 0$ such that

$$\Phi(u) \geq 0,\ \forall \|u\| \leq \delta_0, \quad textand \quad \Phi(u) \geq \rho_0,\ \forall \|u\| = \delta_0. \tag{2.13}$$

Note that for each fixed $u \in E$ with $u \neq 0$,

$$\begin{aligned} I_0(t^2u_t) &= \frac{t^4}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x - y|u^2(tx)u^2(ty)\mathrm{d}(tx)\mathrm{d}(ty)\\ &= \frac{t^4}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (\ln|tx - ty| - \ln t)\, u^2(tx)u^2(ty)\mathrm{d}(tx)\mathrm{d}(ty)\\ &= \frac{t^4}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (\ln|x - y| - \ln t)\, u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y\\ &= t^4 I_0(u) - \frac{t^4\ln t}{2\pi}\|u\|_2^4, \quad \forall t > 0. \end{aligned} \tag{2.14}$$

Moreover, by (A2), (A3), (A7), there exists a constant $\kappa_1 \geq 0$ such that

$$F(t^{-1}x, t^2u) \geq -\kappa_1 |t^2 u|^2, \quad \forall x \in \mathbb{R}^2,\ t > 0. \tag{2.15}$$

Then, it follows from (A1), (1.7), (2.14) and (2.15) that

$$\begin{aligned}
\Phi(t^2u_t) &= \frac{1}{2}\int_{\mathbb{R}^2} \big[t^4|\nabla u|^2 + t^2 V(t^{-1}x)u^2\big]\,\mathrm{d}x + \frac{t^4}{4} I_0(u) - \frac{t^4 \ln t}{8\pi}\|u\|_2^4 \\
&\quad - \int_{\mathbb{R}^2} \frac{F(t^{-1}x, t^2u)}{t^2}\mathrm{d}x \\
&\leq \frac{t^4}{2}\|\nabla u\|_2^2 + (\|V\|_\infty + \kappa_1)\, t^2 \|u\|_2^2 + \frac{t^4}{4} I_0(u) - \frac{t^4 \ln t}{8\pi}\|u\|_2^4,
\end{aligned} \tag{2.16}$$

which implies

$$\Phi(t^2u_t) \to -\infty \quad \text{as } t \to +\infty. \tag{2.17}$$

Taking $e = T^2 u_T$ for $T > 0$ large, we have $\Phi(e) < 0 = \Phi(0)$. Applying Lemma 2.1, there exists a sequence $\{u_n\} \subset E$ satisfying (2.11). $\square$

Lemma 2.3 ([10, Lemma 2.1]). *Let $\{u_n\}$ be a sequence in $L^2(\mathbb{R}^2)$ such that $u_n \to u \in L^2(\mathbb{R}^2) \setminus \{0\}$ a.e. on $\mathbb{R}^2$. If $\{v_n\}$ be a bounded sequence in $L^2(\mathbb{R}^2)$ such that*

$$\sup_{n\in\mathbb{N}} A_1(u_n^2, v_n^2) < \infty,$$

then $\{\|v_n\|_*\}$ is bounded. If, moreover,

$$A_1(u_n^2, v_n^2) \to 0 \quad \textit{and} \quad \|v_n\|_2 \to 0 \quad \textit{as } n \to \infty,$$

then $\|v_n\|_* \to 0$ as $n \to \infty$.

To find ground state solutions for (1.1) with $\lambda = 1$, we give the following lemmas.

Lemma 2.4. *Assume that (A1)–(A3) (A8') hold. Then*

$$\Phi(u) \geq \Phi(tu) + \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-\theta)\left(1-t^2\right)^2}{4}\|u\|^2, \quad \forall u \in E,\ t \geq 0. \tag{2.18}$$

Proof. By (1.11), (2.9) and (2.10), one has

$$\begin{aligned}
\Phi(u) - \Phi(tu) &= \frac{1-t^2}{2}\|u\|^2 + \frac{1-t^4}{4} I_0(u) + \int_{\mathbb{R}^2} [F(x,tu) - F(x,u)]\mathrm{d}x \\
&= \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{\left(1-t^2\right)^2}{4}\|u\|^2 \\
&\quad + \int_{\mathbb{R}^2} \Big[\frac{1-t^4}{4} f(x,u)u + F(x,tu) - F(x,u)\Big]\mathrm{d}x \\
&\geq \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-\theta)\left(1-t^2\right)^2}{4}\|u\|^2 \\
&\quad + \int_{\mathbb{R}^2} \Big[\frac{1-t^4}{4} f(x,u)u + F(x,tu) - F(x,u) + \frac{\theta V(x)}{4}(1-t^2)^2 u^2\Big]\mathrm{d}x \\
&\geq \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-\theta)\left(1-t^2\right)^2}{4}\|u\|^2, \quad t \geq 0.
\end{aligned}$$

This shows that (2.18) holds. $\square$

Corollary 2.5. *Assume that* (A1)–(A3), (A8') *hold. Then for* $u \in \mathcal{N}$,

$$\Phi(u) = \max_{t \geq 0} \Phi(tu). \tag{2.19}$$

To obtain the minimax characterization of m, we define the set

$$\Lambda = \Big\{u \in E : \int_{\mathbb{R}^2} \big[V(x)u^2 - f(x,u)u\big]\,\mathrm{d}x + I_0(u) < 0\Big\}.$$

Lemma 2.6. *Assume that* (A1)–(A3), (A7), (A8) *hold. Then* $\Lambda \neq \emptyset$ *and* $\mathcal{N} \subset \Lambda$.

Proof. For each fixed $u \in E \setminus \{0\}$, in the similar way as in (2.14), we have

$$I_0(tu_t) = I_0(u) - \frac{\ln t}{2\pi}\|u\|_2^4, \quad \forall t > 0. \tag{2.20}$$

By (A2), (A3) and (A7), there exists a constant $\kappa_2 \geq 0$ such that

$$F(t^{-1}x, tu) \geq -\kappa_2|tu|^2, \quad \forall x \in \mathbb{R}^2,\ t > 0, \tag{2.21}$$

which, together with (A8), yields

$$\int_{\mathbb{R}^2} \frac{f(t^{-1}x, tu)tu}{t^2}\mathrm{d}x \geq -\int_{\mathbb{R}^2} \left[4\kappa_2 + \theta V(x)\right] u^2 \mathrm{d}x. \tag{2.22}$$

Then, it follows from (A1), (2.20) and (2.22) that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \big[V(x)(tu_t)^2 - f(x, tu_t)tu_t\big]\,\mathrm{d}x + I_0(tu_t) \\
&= \int_{\mathbb{R}^2} V(t^{-1}x)u^2\mathrm{d}x - \int_{\mathbb{R}^2} \frac{f(t^{-1}x, tu)tu}{t^2}\mathrm{d}x + I_0(u) - \frac{\ln t}{2\pi}\|u\|_2^4 \\
&\leq \left(4\kappa_2 + 2\|V\|_\infty\right)\|u\|_2^2 + I_0(u) - \frac{\ln t}{2\pi}\|u\|_2^4,
\end{aligned} \tag{2.23}$$

which implies

$$\int_{\mathbb{R}^2} \big[V(x)(tu_t)^2 - f(x, tu_t)tu_t\big]\,\mathrm{d}x + I_0(tu_t) \to -\infty \quad \text{as } t \to +\infty. \tag{2.24}$$

Taking $v = Tu_T$ for T large, we have $v \in \Lambda$. Hence, $\Lambda \neq \emptyset$. Using (2.10), it is easy to see that $\mathcal{N} \subset \Lambda$. $\square$

Lemma 2.7. *Assume that* (A1)–(A3), (A7), (A8') *hold. Then, for any* $u \in \Lambda$, *there exists a unique* $t(u) > 0$ *such that* $t(u)u \in \mathcal{N}$.

Proof. Since (A8') implies (A8), we have $\Lambda \neq \emptyset$ by Lemma 2.6. For any fixed $u \in \Lambda$, we define a function $g(t) := \langle \Phi'(tu), tu\rangle$ on $[0, \infty)$. By (A8'), one has

$$f(x, t\tau)t\tau \geq f(x,\tau)\tau t^4 - \theta V(x)(t^2-1)(t\tau)^2, \quad \forall x \in \mathbb{R}^2,\ t \geq 1,\ \tau \in \mathbb{R}, \tag{2.25}$$

which yields

$$\int_{\mathbb{R}^2} \big[\theta V(x)(t\tau)^2 - f(x, t\tau)t\tau\big]\,\mathrm{d}x \leq t^4 \int_{\mathbb{R}^2} \big[\theta V(x)\tau^2 - f(x,\tau)\tau\big]\,\mathrm{d}x, \tag{2.26}$$

for all $t \geq 1$, $\tau \in \mathbb{R}$.

From (2.10) and (2.26) it follows that

$$g(t) \leq t^2\|u\|^2 + t^4 \int_{\mathbb{R}^2} \big[V(x)u^2 - f(x,u)u\big]\,\mathrm{d}x + t^4 I_0(u) - \theta t^2 \int_{\mathbb{R}^2} V(x)u^2\mathrm{d}x, \tag{2.27}$$

for all $t \geq 1$. Using (2.10), (2.12) and (2.27), it is easy to verify that $g(0) = 0$, $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for t large due to $u \in \Lambda$. Therefore, there exists a $\hat{t} = t(u) > 0$ so that $g(\hat{t}) = 0$ and $t(u)u \in \mathcal{N}$. Arguing as in [8] or [35], we prove that $t(u)$ is unique for any $u \in \Lambda$. In fact, for any given $u \in \Lambda$, let $t_1, t_2 > 0$ such that $g(t_1) = g(t_2) = 0$. Jointly with (2.18), we have

$$\begin{aligned} \Phi(t_1 u) &\geq \Phi(t_2 u) + \frac{t_1^4 - t_2^4}{4t_1^4}\langle \Phi'(t_1 u), t_1 u\rangle + \frac{(1-\theta)(t_1^2 - t_2^2)^2}{4t_1^2}\|u\|^2 \\ &= \Phi(t_2 u) + \frac{(1-\theta)(t_1^2 - t_2^2)^2}{4t_1^2}\|u\|^2 \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} \Phi(t_2 u) &\geq \Phi(t_1 u) + \frac{t_2^4 - t_1^4}{4t_2^4}\langle \Phi'(t_2 u), t_2 u\rangle + \frac{(1-\theta)(t_2^2 - t_1^2)^2}{4t_2^2}\|u\|^2 \\ &= \Phi(t_1 u) + \frac{(1-\theta)(t_2^2 - t_1^2)^2}{4t_2^2}\|u\|^2. \end{aligned} \tag{2.29}$$

Then, (2.28) and (2.29) imply $t_1 = t_2$. Hence, $t(u) > 0$ is unique for any $u \in \Lambda$. $\square$

Lemma 2.8. *Assume that* (A1)–(A3), (A7), (A8') *hold. Then*

$$\inf_{u\in\mathcal{N}} \Phi(u) := m = \inf_{u\in\Lambda}\max_{t\geq 0}\Phi(tu) > 0.$$

Proof. Corollary 2.5 and Lemma 2.7 imply that $m = \inf_{u\in\Lambda}\max_{t\geq 0}\Phi(tu)$. Using (A2) and (A3), it is easy to see that there exist $C_1 > 0$ and $q > 4$ such that

$$|f(x,t)t| \leq \frac{\gamma_2}{2}t^2 + C_1|t|^q, \quad \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \tag{2.30}$$

where $\gamma_s := \inf_{u\in H^1(\mathbb{R}^2), \|u\|_s = 1}\|u\|^2$ for $s \geq 2$. By (2.5), (2.10), (2.30) and the Sobolev embedding theorem, we have

$$\begin{aligned} \|u\|^2 &\leq \|u\|^2 + I_1(u) = I_2(u) + \int_{\mathbb{R}^2} f(x,u)u\mathrm{d}x \\ &\leq C_2\|u\|^4 + \frac{1}{2}\|u\|^2 + C_3\|u\|^q, \quad \forall u \in \mathcal{N}, \end{aligned}$$

which implies

$$\|u\| \geq \min\left\{2^{-1/2}(C_2 + C_3)^{-1/2}, 1\right\} := \sigma_0, \quad \forall u \in \mathcal{N}. \tag{2.31}$$

Thus, it follows from (2.18) with $t = 0$ and (2.31) that $m \geq (1-\theta)\sigma_0^2/4 > 0$. $\square$

Next we find a minimizing Cerami sequence for Φ outside $\mathcal{N}$ by the diagonal method, this idea goes back to [36], which is a key in the proof of Theorems 1.2 and 1.3.

Lemma 2.9. *Assume that* (A1)–(A3), (A7), (A8') *hold. Then there exist a constant* $c_* \in (0, m]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c_*, \quad \|\Phi'(u_n)\|_{E^*}(1 + \|u_n\|_E) \to 0. \tag{2.32}$$

Proof. In view of Lemmas 2.7 and 2.8, we choose $v_k \in \mathcal{N} \subset \Lambda$ such that

$$m \leq \Phi(v_k) < m + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{2.33}$$

Since $\langle \Phi'(v_k), v_k\rangle = 0$, then (2.18) implies that $\Phi(tv_k) < 0$ for large $t > \delta_0/\|v_k\|$. Moreover, (2.13) implies that $\Phi(tv_k) \geq \rho_0 > 0 = \Phi(0)$ for $t\|v_k\| = \delta_0$. Applying Lemma 2.1, there exists a sequence $\{u_{k,n}\}_{n\in\mathbb{N}} \subset E$ satisfying

$$\Phi(u_{k,n}) \to c_k, \quad \|\Phi'(u_{k,n})\|_{E^*}(1 + \|u_{k,n}\|_E) \to 0, \quad k \in \mathbb{N}, \tag{2.34}$$

where $c_k \in [\rho_0, \sup_{t\geq 0} \Phi(tv_k)]$. By Corollary 2.5, one has $\Phi(v_k) = \sup_{t\geq 0} \Phi(tv_k)$. Hence, by (2.33) and (2.34), one has

$$\Phi(u_{k,n}) \to c_k \in [\rho_0, m + \frac{1}{k}), \quad \|\Phi'(u_{k,n})\|_{E^*}(1 + \|u_{k,n}\|_E) \to 0, \quad k \in \mathbb{N}. \tag{2.35}$$

Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\Phi(u_{k,n_k}) \in [\rho_0, m + \frac{1}{k}), \quad \|\Phi'(u_{k,n_k})\|_{E^*}(1 + \|u_{k,n_k}\|_E) < \frac{1}{k}, \quad k \in \mathbb{N}. \tag{2.36}$$

Let $u_k = u_{k,n_k}, k \in \mathbb{N}$. Then, going if necessary to a subsequence, we have

$$\Phi(u_n) \to c_* \in [\rho_0, m], \quad \|\Phi'(u_n)\|_{E^*}(1 + \|u_n\|_E) \to 0.$$

□

3. The periodic case

In this section, we give the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. In view of Lemma 2.2, there exists a sequence $\{u_n\} \subset E$ satisfying (2.11), then

$$\Phi(u_n) \to c > 0, \quad \langle \Phi'(u_n), u_n\rangle \to 0. \tag{3.1}$$

By (A8), (2.9), (2.10) and (3.1), one has

$$\begin{aligned} &c + o(1)\\ &= \Phi(u_n) - \frac{1}{4}\langle \Phi'(u_n), u_n\rangle\\ &= \frac{\theta}{4}\|\nabla u_n\|_2^2 + \frac{1-\theta}{4}\|u_n\|^2 + \int_{\mathbb{R}^2} \big[\frac{1}{4}f(x,u_n)u_n - F(x,u_n) + \frac{\theta V(x)}{4}u_n^2\big]\\ &\geq \frac{1-\theta}{4}\|u_n\|^2. \end{aligned} \tag{3.2}$$

This shows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_2(y)} |u_n|^2 \mathrm{d}x = 0,$$

then by Lion's concentration compactness principle [24] or [38, Lemma 1.21], $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $s > 2$. Then, (2.5) implies that $I_2(u_n) \to 0$. Note that $\|u_n\|_2 \leq M_1$ with some constant $M_1 > 0$. By (2.12), for $\varepsilon = c/2M_1$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} \limsup_{n\to\infty} \int_{\mathbb{R}^2} \big|\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\big|\mathrm{d}x &\leq \frac{3}{2}\varepsilon \sup_{n\in\mathbb{N}} \|u_n\|_2^2 + C_\varepsilon \lim_{n\to\infty} \|u_n\|_p^p\\ &\leq \frac{3c}{4}. \end{aligned} \tag{3.3}$$

Thus, it follows from (2.9), (2.10), (3.1) and (3.3) that

$$\begin{aligned} c &= \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle + o(1)\\ &= -\frac{1}{4}I_1(u_n) + \frac{1}{4}I_2(u_n) + \int_{\mathbb{R}^3} \big[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\big]\mathrm{d}x + o(1)\\ &\leq \frac{3c}{4} + o(1). \end{aligned} \tag{3.4}$$

This contradiction shows that $\delta > 0$.

Going to a subsequence, if necessary, we assume the existence of $k_n \in \mathbb{Z}^2$ such that

$$\int_{B_2(k_n)} |u_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{3.5}$$

Let $\tilde{u}_n(x) = u_n(x + k_n)$. Then

$$\int_{B_2(0)} |\tilde{u}_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{3.6}$$

Note that

$$\|\tilde{u}_n\|_*^2 = \int_{\mathbb{R}^2} \ln(1 + |x - k_n|) u_n^2 \mathrm{d}x \le \|u_n\|_*^2 + \ln(1 + |k_n|)\|u_n\|_2^2, \quad \forall n \in \mathbb{N}, \tag{3.7}$$

then $\tilde{u}_n \in E$ for every $n \in \mathbb{N}$. Since $V(x)$ and $f(x, u)$ are periodic in x, and $I_i(\tilde{u}_n) = I_i(u_n)$ for $i = 0, 1, 2$, then (3.1) implies

$$\Phi(\tilde{u}_n) \to c > 0, \quad \langle \Phi'(\tilde{u}_n), \tilde{u}_n \rangle \to 0. \tag{3.8}$$

Passing to a subsequence, we have $\tilde{u}_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^2)$, $\tilde{u}_n \to u_0$ in $L^s_{\rm loc}(\mathbb{R}^2)$, $s \in [2, \infty)$ and $\tilde{u}_n(x) \to u_0(x)$ a.e. on $\mathbb{R}^2$. Thus, (3.6) implies that $u_0 \neq 0$. By (2.5), (2.12), (3.8) and the Sobolev embedding theorem that

$$\begin{aligned} \|\tilde{u}_n\|^2 + I_1(\tilde{u}_n) &= I_2(\tilde{u}_n) + \int_{\mathbb{R}^2} f(x, \tilde{u}_n)\tilde{u}_n \mathrm{d}x \\ &\le \mathcal{C}_1 \|\tilde{u}_n\|_{8/3}^4 + \|\tilde{u}_n\|_2^2 + C_1 \|\tilde{u}_n\|_p^p \\ &\le C_2 \|\tilde{u}_n\|^4 + C_3 \|\tilde{u}_n\|^2 + C_4 \|\tilde{u}_n\|^p, \end{aligned} \tag{3.9}$$

which implies that $\sup_{n\in\mathbb{N}} I_1(\tilde{u}_n) = \sup_{n\in\mathbb{N}} A_1(\tilde{u}_n^2, \tilde{u}_n^2) < \infty$. Applying Lemma 2.3, we have $\{\|\tilde{u}_n\|_*\}$ is bounded. Hence, $\{\tilde{u}_n\}$ is bounded in E. We may thus assume, passing to a subsequence again if necessary, that

$$\tilde{u}_n \rightharpoonup u_0 \text{ in } E, \quad \tilde{u}_n \to u_0 \text{ in } L^s(\mathbb{R}^2), s \in [2, \infty), \quad \tilde{u}_n(x) \to u_0(x) \text{ a.e. on } \mathbb{R}^2. \tag{3.10}$$

Now, we prove that $\Phi'(u_0) = 0$. To this end, we claim that

$$\langle \Phi'(u_0), w \rangle = \lim_{n\to\infty} \langle \Phi'(\tilde{u}_n), w \rangle = \lim_{n\to\infty} \langle \Phi'(u_n), w(\cdot - k_n) \rangle = 0, \quad \forall w \in E. \tag{3.11}$$

In fact, it is easy to see that

$$\|w(\cdot - k_n)\|_E^2 = \|w\|^2 + \int_{\mathbb{R}^2} \ln(1 + |x + k_n|) w^2 \mathrm{d}x \le \|w\|_E^2 + \ln(1 + |k_n|)\|w\|_2^2, \tag{3.12}$$

for all $w \in E$. Moreover, by (3.6), we have

$$\begin{aligned} \|u_n\|_*^2 &= \int_{\mathbb{R}^2} \ln(1 + |x - k_n|) \tilde{u}_n^2 \mathrm{d}x \\ &\ge \int_{B_2(0)} \ln(1 + |x - k_n|) \tilde{u}_n^2 \mathrm{d}x \\ &\ge \frac{\delta \ln(|k_n| - 1)}{2} \ge \frac{\delta \ln(1 + |k_n|)}{4}, \quad \forall k_n \ge 3. \end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we conclude that

$$\|w(\cdot - k_n)\|_E^2 \le \|w\|_E^2 + \Big(\frac{4\|u_n\|_*^2}{\delta} + \ln 4\Big)\|w\|_2^2, \quad \forall n \in \mathbb{N}. \tag{3.14}$$

Thus, it follows from (2.10), (2.32) and (3.14) that

$$
\begin{aligned}
\langle \Phi'(\tilde{u}_n), w\rangle &= \langle \Phi'(u_n), w(\cdot - k_n)\rangle \\
&\le \|\Phi'(u_n)\|_{E^*}\big[\|w\|_E^2 + \big(\frac{4\|u_n\|_*^2}{\delta} + \ln 4\big)\|w\|_2^2\big]^{1/2} \\
&= o(1), \quad \forall w \in E.
\end{aligned}
\tag{3.15}
$$

Then, (3.15) implies

$$
\begin{aligned}
\langle \Phi'(\tilde{u}_n), u_0\rangle &= \langle \Phi'(u_n), u_0(\cdot - k_n)\rangle \\
&\le \|\Phi'(u_n)\|_{E^*}\big[\|u_0\|_E^2 + \big(\frac{4\|u_n\|_*^2}{\delta} + \ln 4\big)\|u_0\|_2^2\big]^{1/2} = o(1).
\end{aligned}
\tag{3.16}
$$

According to [10, Lemma 2.6], we have

$$
A_1\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)w\right) = o(1), \quad \forall w \in E. \tag{3.17}
$$

Thus, it follows from (3.8), (3.10), (3.16), (3.17) and Lebesgue's dominated convergence theorem that

$$
\begin{aligned}
0 &= \langle \Phi'(\tilde{u}_n), \tilde{u}_n - u_0\rangle + o(1) \\
&= \|\tilde{u}_n\|^2 - \|u_0\|^2 + A_1\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)^2\right) + A_1\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)u_0\right) \\
&\quad - A_2\left(\tilde{u}_n^2, \tilde{u}_n(\tilde{u}_n - u_0)\right) - \int_{\mathbb{R}^2} f(x, \tilde{u}_n)(\tilde{u}_n - u_0)\mathrm{d}x + o(1) \\
&= \|\tilde{u}_n\|^2 - \|u_0\|^2 + A_1\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)^2\right) + o(1),
\end{aligned}
\tag{3.18}
$$

which, together with $\tilde{u}_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^2)$, yields

$$
\|\tilde{u}_n - u_0\| \to 0, \quad A_1\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)^2\right) \to 0. \tag{3.19}
$$

Applying Lemma 2.3, we have $\|\tilde{u}_n - u_0\|_* \to 0$. Hence, $\|\tilde{u}_n - u_0\|_E \to 0$. By (2.7), we have

$$
A_1(\tilde{u}_n^2 - u_0^2, u_0 w) \le \|\tilde{u}_n - u_0\|_*\|\tilde{u}_n + u_0\|_*\|u_0\|_2\|w\|_2 = o(1). \tag{3.20}
$$

By (2.10), (3.10), (3.17), (3.20) and Lebesgue's dominated convergence theorem, we have

$$
\begin{aligned}
&\langle \Phi'(\tilde{u}_n) - \Phi'(u_0), w\rangle \\
&= (\tilde{u}_n - u_0, w) + A_1\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)w\right) + A_1\left(\tilde{u}_n^2 - u_0^2, u_0 w\right) \\
&\quad - A_2\left(\tilde{u}_n^2, (\tilde{u}_n - u_0)w\right) - A_2\left(\tilde{u}_n^2 - u_0^2, u_0 w\right) \\
&\quad - \int_{\mathbb{R}^2}\left[f(x, \tilde{u}_n) - f(x, u_0)\right] w\mathrm{d}x = o(1).
\end{aligned}
\tag{3.21}
$$

Therefore, (3.11) follows from (3.15) and (3.21). This shows that $u_0 \in E$ is a nontrivial solution of (1.5), and $\Phi(u_0) = c > 0$. □

Proof of Theorem 1.2. In view of Lemma 2.9, there exists a sequence $\{u_n\} \subset E$ satisfying (2.32). Then

$$
\Phi(u_n) \to c^* \in (0, m], \quad \langle \Phi'(u_n), u_n\rangle \to 0. \tag{3.22}
$$

By the same argument as in the last part of the proof of Theorem 1.1, we conclude that there exists $u_0 \in E \setminus \{0\}$ such that $\Phi'(u_0) = 0$ and $\Phi(u_0) = c^* \in (0, m]$. Moreover, since $u_0 \in \mathcal{N}$, we have $\Phi(u_0) \ge m$. This shows that $u_0 \in E$ is a ground state solution for (1.5) with $\Phi(u_0) = m = \inf_{\mathcal{N}} \Phi > 0$. □

4. The asymptotically periodic case

In this section, we have $V(x) = V_0(x) + V_1(x)$ and $f(x,u) = f_0(x,u) + f_1(x,u)$. We define the functional

$$\Phi_0(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V_0(x)u^2\right) \mathrm{d}x + \frac{1}{4}\left[I_1(u) - I_2(u)\right] - \int_{\mathbb{R}^2} F_0(x,u)\mathrm{d}x, \quad (4.1)$$

where $F_0(x,u) := \int_0^u f_0(x,s)\mathrm{d}s$. Then (A2), (A3), (A5) and (A6') imply that $\Phi_0 \in \mathcal{C}^1(E,\mathbb{R})$ and

$$\begin{aligned}\langle \Phi_0'(u), v\rangle = \int_{\mathbb{R}^2} \left(\nabla u\cdot\nabla v + V_0(x)uv\right) \mathrm{d}x + A_1(u^2, uv) - A_2(u^2, uv)\\ - \int_{\mathbb{R}^2} f_0(x,u)v\mathrm{d}x.\end{aligned} \quad (4.2)$$

By a standard argument, we can obtain the following lemma.

Lemma 4.1. *Assume that* (A2), (A3), (A5), (A6') *hold. If* $u_n \rightharpoonup 0$ *in* $H^1(\mathbb{R}^2)$, *then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} V_1(x)u_n^2\mathrm{d}x = 0, \quad \lim_{n\to\infty}\int_{\mathbb{R}^2} V_1(x)u_n v\mathrm{d}x = 0, \quad \forall v\in H^1(\mathbb{R}^2), \quad (4.3)$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} F_1(x,u_n)\mathrm{d}x = 0, \quad \lim_{n\to\infty}\int_{\mathbb{R}^2} f_1(x,u_n)v\mathrm{d}x = 0, \quad \forall v\in H^1(\mathbb{R}^2). \quad (4.4)$$

In the asymptotically periodic case, we prove that the minimizer of Φ on $\mathcal{N}$ is a critical point.

Lemma 4.2. *Assume that* (A1)–(A3), (A7), (A8') *hold. If* $u_0 \in \mathcal{N}$ *and* $\Phi(u_0) = m$, *then* u_0 *is a critical point of* Φ.

Proof. Assume that $u_0 \in \mathcal{N}$, $\Phi(u_0) = m$ and $\Phi'(u_0) \neq 0$. Then there exist $\delta > 0$ and $\varrho > 0$ such that

$$\|u - u_0\|_E \le 3\delta \Rightarrow \|\Phi'(u)\| \ge \varrho. \quad (4.5)$$

In view of Lemma 2.4, one has

$$\begin{aligned}\Phi(tu_0) &\le \Phi(u_0) - \frac{(1-\theta)(1-t^2)^2}{4}\|u_0\|^2\\ &= m - \frac{(1-\theta)(1-t^2)^2}{4}\|u_0\|^2, \quad \forall t\ge 0.\end{aligned} \quad (4.6)$$

For $\varepsilon := \min\{3(1-\theta)\|u_0\|^2/64, 1, \varrho\delta/8\}$, $S := B(u_0,\delta)$, [38, Lemma 2.3] yields a deformation $\eta \in \mathcal{C}([0,1]\times E, E)$ such that

(i) $\eta(1,u) = u$ if $\Phi(u) < m - 2\varepsilon$ or $\Phi(u) > m + 2\varepsilon$;
(ii) $\eta\left(1, \Phi^{m+\varepsilon}\cap B(u_0,\delta)\right) \subset \Phi^{m-\varepsilon}$;
(iii) $\Phi(\eta(1,u)) \le \Phi(u)$, for all $u\in E$;
(iv) $\eta(1,u)$ is a homeomorphism of E.

By Corollary 2.5, $\Phi(tu_0) \le \Phi(u_0) = m$ for $t\ge 0$, then it follows from (ii) that

$$\Phi(\eta(1,tu_0)) \le m - \varepsilon, \quad \forall t\ge 0,\ |t-1| < \delta/\|u_0\|. \quad (4.7)$$

On the other hand, by (iii) and (4.6), one has

$$\begin{aligned}\Phi(\eta(1,tu_0)) &\le \Phi(tu_0)\\ &\le m-\frac{(1-\theta)(1-t^2)^2}{4}\|u_0\|^2\\ &\le m-\frac{(1-\theta)\delta^2}{4},\quad \forall t\ge 0,\ |t-1|\ge \delta/\|u_0\|.\end{aligned}\tag{4.8}$$

Combining (4.7) with (4.8), we have

$$\max_{t\in[1/2,\sqrt{7}/2]}\Phi(\eta(1,tu_0))<m.\tag{4.9}$$

We prove that $\eta(1,tu_0)\cap\mathcal{N}\ne\emptyset$ for some $t\in[1/2,\sqrt{7}/2]$, contradicting to the definition of m. Define

$$\Psi_0(t):=\langle\Phi'(tu_0),tu_0\rangle,\quad \Psi_1(t):=\langle\Phi'(\eta(1,tu_0)),\eta(1,tu_0)\rangle,\quad \forall t\ge 0.$$

Since $u_0\in\Lambda$, by Lemma 2.7 and degree theory, $\deg(\Psi_0,(1/2,\sqrt{7}/2),0)=1$. Using (4.6) and i), it is easy to verify that $\eta(1,tu_0)=tu_0$ for $t=1/2$ and $t=\sqrt{7}/2$. Thus, $\deg(\Psi_1,(1/2,\sqrt{7}/2),0)=\deg(\Psi_0,(1/2,\sqrt{7}/2),0)=1$. Since $u_0\ne 0$, it follows from (iv) that $\eta(1,tu_0)\ne 0$ for all $t>0$. Hence, $\Psi_1(t_0)=0$ for some $t_0\in(1/2,\sqrt{7}/2)$, that is $\eta(1,t_0u_0)\in\mathcal{N}$, which is a contradiction. □

Proof of Theorem 1.3. Lemma 2.9 implies the existence of a sequence $\{u_n\}\subset E$ satisfying (2.32). Similar to the proof of (3.2), we can deduce that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. Passing to a subsequence, we may assume that $u_n\rightharpoonup\bar{u}$ in $H^1(\mathbb{R}^2)$, $u_n\to\bar{u}$ in $L^s_{\rm loc}(\mathbb{R}^2)$, $s\in[2,\infty)$ and $u_n(x)\to\bar{u}(x)$ a.e. on $\mathbb{R}^2$. There are two possible cases: $\bar{u}=0$ and $\bar{u}\ne 0$.

Case (i): $\bar{u}=0$. Then $u_n\rightharpoonup 0$ in $H^1(\mathbb{R}^2)$, and so $u_n\to 0$ in $L^s_{\rm loc}(\mathbb{R}^2)$, $s\in[2,\infty)$ and $u_n(x)\to 0$ a.e. on $\mathbb{R}^2$. Note that

$$\|u\|^2=\int_{\mathbb{R}^2}\left(|\nabla u|^2+V_0(x)u^2\right)\mathrm{d}x+\int_{\mathbb{R}^2}V_1(x)u^2\mathrm{d}x,\quad u\in H^1(\mathbb{R}^2),\tag{4.10}$$

$$\Phi_0(u)=\Phi(u)-\frac{1}{2}\int_{\mathbb{R}^2}V_1(x)u^2\mathrm{d}x+\int_{\mathbb{R}^2}F_1(x,u)\mathrm{d}x,\tag{4.11}$$

$$\langle\Phi_0'(u),v\rangle=\langle\Phi'(u),v\rangle-\int_{\mathbb{R}^2}V_1(x)uv\mathrm{d}x+\int_{\mathbb{R}^2}f_1(x,u)v\mathrm{d}x.\tag{4.12}$$

By (2.32), (4.3), (4.4), (4.11) and (4.12), one has

$$\Phi_0(u_n)\to c^*\in(0,m],\quad \langle\Phi_0'(u_n),u_n\rangle\to 0,\quad \Phi_0'(u_n)\to 0.\tag{4.13}$$

Analogous to the proof of (3.5), there exists $k_n\in\mathbb{Z}^2$, going to a subsequence, if necessary, such that

$$\int_{B_2(k_n)}|u_n|^2\mathrm{d}x>\frac{\delta}{2}>0.$$

Let us define $v_n(x)=u_n(x+k_n)$ so that

$$\int_{B_2(0)}|v_n|^2\mathrm{d}x>\frac{\delta}{2}.\tag{4.14}$$

Since $V_0(x)$ and $f_0(x,u)$ are periodic in x, it follows from (4.13) that

$$\Phi_0(v_n)\to c^*\in(0,m],\quad \langle\Phi_0'(v_n),v_n\rangle\to 0.\tag{4.15}$$

Passing to a subsequence, we have $v_n \rightharpoonup \bar{v}$ in $H^1(\mathbb{R}^2)$, $v_n \to \bar{v}$ in $L^s_{\rm loc}(\mathbb{R}^2)$, $s \in [2,\infty)$ and $v_n(x) \to \bar{v}(x)$ a.e. on $\mathbb{R}^2$. Thus, (4.14) implies that $\bar{v} \neq 0$. Arguing as in the proof of Theorem 1.1, we conclude that $\{v_n\}$ is bounded in E. We may thus assume, passing to a subsequence again if necessary, that

$$v_n \rightharpoonup \bar{v} \text{ in } E, \quad v_n \to \bar{v} \text{ in } L^s(\mathbb{R}^2),\ s \in [2,\infty), \quad v_n(x) \to \bar{v}(x)\text{a.e. on } \mathbb{R}^2. \tag{4.16}$$

By (1.9), (4.2), (4.12), (4.15) and (4.16), one has

$$\langle \Phi'(\bar{v}), \bar{v}\rangle \le \langle \Phi_0'(\bar{v}), \bar{v}\rangle \le \liminf_{n\to\infty} \langle \Phi_0'(v_n), v_n\rangle = 0.$$

In view of Lemma 2.7, there exists a unique $t_0 = t(\bar{v}) \in (0,1]$ such that $t_0\bar{v} \in \mathcal{N}$, and so $\Phi(t_0\bar{v}) \ge m$. Now, we prove that $\Phi(t_0\bar{v}) = m$. By (A8'), we have

$$f(x,t\tau)t\tau \le f(x,\tau)\tau t^4 + \theta V(x)(1-t^2)(t\tau)^2, \quad \forall x \in \mathbb{R}^2,\ 0 \le t \le 1,\ \tau \in \mathbb{R}. \tag{4.17}$$

Note that (1.11) implies

$$F(x,t\tau) \ge \frac{t^4-1}{4} f(x,\tau)\tau + F(x,\tau) - \frac{1-2t^2+t^4}{4}\theta V(x)\tau^2, \tag{4.18}$$

for all $x \in \mathbb{R}^2$, $t \ge 0$, $\tau \in \mathbb{R}$.

Then, (4.17) and (4.18) imply

$$\frac{1}{4} f(x,t\tau)t\tau - F(x,t\tau) + \frac{\theta V(x)}{4}(t\tau)^2 \le \frac{1}{4} f(x,\tau)\tau - F(x,\tau) + \frac{\theta V(x)}{4}\tau^2, \tag{4.19}$$

for all $x \in \mathbb{R}^2$, $0 \le t \le 1$, $\tau \in \mathbb{R}$.

Thus, it follows from (1.9), (2.9), (2.10), (4.1), eqrefPhd0, (4.15), (4.16), (4.19) and Lebesgue's dominated convergence theorem that

$$\begin{aligned}
m &\le \Phi(t_0\bar{v}) = \Phi(t_0\bar{v}) - \frac{1}{4}\langle \Phi'(t_0\bar{v}), t_0\bar{v}\rangle \\
&= \frac{t_0^2}{4}\int_{\mathbb{R}^2} \big[|\nabla \bar{v}|^2 + (1-\theta)V(x)|\bar{v}|^2\big]\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2} \big[\frac{1}{4} f(x,t_0\bar{v})t_0\bar{v} - F(x,t_0\bar{v}) + \frac{\theta V(x)}{4}(t_0\bar{v})^2\big]\mathrm{d}x \\
&\le \frac{1}{4}\int_{\mathbb{R}^2} \big[|\nabla \bar{v}|^2 + (1-\theta)V(x)|\bar{v}|^2\big]\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2} \big[\frac{1}{4} f(x,\bar{v})\bar{v} - F(x,\bar{v}) + \frac{\theta V(x)}{4}(\bar{v})^2\big]\mathrm{d}x \\
&= \Phi_0(\bar{v}) - \frac{1}{4}\langle \Phi_0'(\bar{v}), \bar{v}\rangle + \int_{\mathbb{R}^2} \big[\frac{1}{4} f_1(x,\bar{v})\bar{v} - F_1(x,\bar{v}) + \frac{V_1(x)}{4}(\bar{v})^2\big]\mathrm{d}x \\
&\le \Phi_0(\bar{v}) - \frac{1}{4}\langle \Phi_0'(\bar{v}), \bar{v}\rangle \\
&= \frac{1}{4}\int_{\mathbb{R}^2} \big[|\nabla \bar{v}|^2 + V_0(x)|\bar{v}|^2\big]\,\mathrm{d}x + \int_{\mathbb{R}^2} \big[\frac{1}{4} f(x,\bar{v})\bar{v} - F(x,\bar{v})\big]\mathrm{d}x \\
&\le \liminf_{n\to\infty} \Big\{\frac{1}{4}\int_{\mathbb{R}^2} \big[|\nabla v_n|^2 + V_0(x)v_n^2\big]\,\mathrm{d}x + \int_{\mathbb{R}^2} \big[\frac{1}{4} f(x,v_n)v_n - F(x,v_n)\big]\mathrm{d}x\Big\} \\
&= \lim_{n\to\infty} \big[\Phi_0(v_n) - \frac{1}{4}\langle \Phi_0'(v_n), v_n\rangle\big] \\
&= c^* \le m.
\end{aligned}$$

This shows that $\Phi(t_0\bar{v}) = m$. Let $u_0 = t_0\bar{v}$. Then $u_0 \in \mathcal{N}$ and $\Phi(u_0) = m$. In view of Lemma 4.2, we obtain $\Phi'(u_0) = 0$. This shows that $u_0 \in E$ is a ground state solution for (1.5) with $\Phi(u_0) = m = \inf_{\mathcal{N}} \Phi > 0$.

Case ii: $\bar{u} \neq 0$**.** Similar to the proof of (3.9), we can deduce that $\sup_{n\in\mathbb{N}} I_1(u_n) = \sup_{n\in\mathbb{N}} A_1(u_n^2, u_n^2) < \infty$. By Lemma 2.3, we have $\{\|u_n\|_*\}$ is bounded, and so $\{u_n\}$ is bounded in E. We may assume, passing to a subsequence, that $u_n \rightharpoonup \bar{u}$ in E, $u_n \to \bar{u}$ in $L^s(\mathbb{R}^2)$, $s \in [2,\infty)$ and $u_n(x) \to \bar{u}(x)$ a.e. on $\mathbb{R}^2$. By the same fashion as the last part of the proof of Theorem 1.1, we can obtain that $\|u_n - \bar{u}\|_E \to 0$ and $\Phi'(\bar{u}) = 0$, and so $\Phi(\bar{u}) = c^* \in (0, m]$. Since $\bar{u} \in \mathcal{N}$, we have $\Phi(\bar{u}) \geq m$. This shows that $\bar{u} \in E$ is a ground state solution for (1.5) with $\Phi(\bar{u}) = m = \inf_{\mathcal{N}} \Phi > 0$. $\square$

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