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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF POSITIVE LEAST ENERGY SOLUTIONS FOR COUPLED NONLINEAR CHOQUARD EQUATIONS

SONG YOU, PEIHAO ZHAO, QINGXUAN WANG

Abstract. In this article, we study the coupled nonlinear Schrödinger equations with Choquard type nonlinearities

$$
\begin{gathered}
-\Delta u+\nu_1 u=\mu_1(\frac{1}{|x|^\alpha}*u^2)u+\beta(\frac{1}{|x|^\alpha}*v^2)u \quad \text{in } \mathbb{R}^N,\\
-\Delta v+\nu_2 v=\mu_2(\frac{1}{|x|^\alpha}*v^2)v+\beta(\frac{1}{|x|^\alpha}*u^2)v \quad \text{in } \mathbb{R}^N,\\
u,v\geq 0 \quad \text{in } \mathbb{R}^N, \quad u,v\in H^1(\mathbb{R}^N),
\end{gathered}
$$

where ν_1,ν_2,μ_1,μ_2 are positive constants, $\beta>0$ is a coupling constant, $N\geq 3$, $\alpha\in(0,N)\cap(0,4)$, and "$*$" is the convolution operator. We show that the nonlocal elliptic system has a positive least energy solution for positive small β and positive large β via variational methods. For the case in which $\nu_1=\nu_2$, $\mu_1\neq\mu_2$, $N=3,4,5$ and $\alpha=N-2$, we prove the uniqueness of positive least energy solutions. Moreover, the asymptotic behaviors of the positive least energy solutions as $\beta\to 0^+$ are studied.

1. Introduction

We consider the time-dependent coupled nonlinear Schrödinger equations with Choquard type nonlinearities in the following form (see [12, 36]):

$$
\begin{gathered}
-i\frac{\partial}{\partial t}\Phi_1=\Delta\Phi_1+\mu_1(V(x)*|\Phi_1|^2)\Phi_1+\beta(V(x)*|\Phi_2|^2)\Phi_1 \quad \text{in } \mathbb{R}^N,\\
-i\frac{\partial}{\partial t}\Phi_2=\Delta\Phi_2+\mu_2(V(x)*|\Phi_2|^2)\Phi_2+\beta(V(x)*|\Phi_1|^2)\Phi_2 \quad \text{in } \mathbb{R}^N,\\
\Phi_j=\Phi_j(x,t)\in\mathbb{C}, \quad j=1,2,\\
\Phi_j(x,t)=0, \quad x\in\mathbb{R}^N,\ t>0,\ j=1,2,
\end{gathered} \tag{1.1}
$$

where i is the imaginary unit, and "$*$" is the convolution operator. System (1.1) appears in many physical problem, especially in nonlinear optics. Physically, the solution Φ_j denotes the j-th component of the beam in Kerr-like photorefractive media (see [22, 23]). The positive constant μ_j indicate the self-focusing in the j-th components of the beam. $V(x)$ is the response function which possesses information on the mutual interaction. The coupling constant β is the interaction between the

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two components of the beam. The problem (1.1) also arises in the basic quantum chemistry model of small number of electrons interacting with static nucleii which can be approximated by Hartree or Hartree-Fock minimization problems (see [13, 14, 17]).

To obtain solitary wave solutions of system (1.1), we set $\Phi_1(x,t) = e^{i\nu_1 t}u(x)$ and $\Phi_2(x,t) = e^{i\nu_2 t}v(x)$. Then system (1.1) turns into the elliptic system

$$\begin{aligned} -\Delta u + \nu_1 u &= \mu_1 (V(x) * u^2)u + \beta(V(x) * v^2)u \quad x \in \mathbb{R}^N, \\ -\Delta v + \nu_2 v &= \mu_2 (V(x) * v^2)v + \beta(V(x) * u^2)v \quad x \in \mathbb{R}^N. \end{aligned} \tag{1.2}$$

If the response function is a Dirac-delta function, *i.e.* $V(x) = \delta(x)$, then (1.2) turns to be the following semilinear elliptic system with local nonlinearities:

$$\begin{aligned} -\Delta u + \nu_1 u &= \mu_1 u^3 + \beta u v^2 \quad x \in \mathbb{R}^N, \\ -\Delta v + \nu_2 v &= \mu_2 v^3 + \beta v u^2 \quad x \in \mathbb{R}^N. \end{aligned} \tag{1.3}$$

Here, $\mu_1, \mu_2 > 0$ and $\beta \neq 0$ is a coupling constant. The existence and multiplicity of solutions to (1.3) have been the subject of extensive mathematical studies in recent years, see [3, 4, 5, 6, 7, 8, 19, 20, 26, 27, 28, 29, 34] and references therein.

In this paper we consider system (1.2) with a response function of Riesz potential, i.e. $V(x) = |x|^{-\alpha}$, then (1.2) is reduced to the nonlocal elliptic system

$$\begin{aligned} -\Delta u + \nu_1 u &= \mu_1 (\frac{1}{|x|^\alpha} * u^2)u + \beta(\frac{1}{|x|^\alpha} * v^2)u \quad x \in \mathbb{R}^N, \\ -\Delta v + \nu_2 v &= \mu_2 (\frac{1}{|x|^\alpha} * v^2)v + \beta(\frac{1}{|x|^\alpha} * u^2)v \quad x \in \mathbb{R}^N. \end{aligned} \tag{1.4}$$

Here, $\alpha \in (0,N) \cap (0,4), \nu_1, \nu_2 > 0, \mu_1, \mu_2 > 0$ and $\beta \neq 0$ is a coupling constant.

Before proceeding to state our resutls, we introduce the following classical Hardy-Littlewood-Sobolev inequality (see [16]).

Lemma 1.1. *Let $p, r > 1$ and $0 < \alpha < N$ with $\frac{1}{p} + \frac{\alpha}{N} + \frac{1}{r} = 2$, $f \in L^p(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(N, \alpha, p)$ such that*

$$\Big| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\alpha} \, dx \, dy \Big| \leq C(N, \alpha, p) |f|_p |h|_r, \tag{1.5}$$

where $|\cdot|_q$ is the $L^q(\mathbb{R}^N)$-norm with $q \in [1, \infty]$.

Assume that $f, g \in L^1_{loc}(\mathbb{R}^N)$ and $\alpha \in (0,N)$, as [16] we define

$$D(f,g) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\alpha} \, dx \, dy.$$

The following lemma is important for considering (1.4), and its proof is given in [16, Theorem 9.8].

Lemma 1.2. *If $D(|f|, |f|) < \infty$, then*

$$D(f,f) \geq 0,$$

and there is equality if and only if $f \equiv 0$. Moreover, if $D(|g|, |g|) < \infty$, then

$$|D(f,g)|^2 \leq D(f,f) D(g,g). \tag{1.6}$$

Suppose that $u, v \in H^1(\mathbb{R}^N)$ and $\alpha \in (0,N)\cap(0,4)$, then by Lemma 1.1 we have

$$D(u^2,u^2) \le C|u^2|^2_{\frac{2N}{2N-\alpha}} = C|u|^4_{\frac{4N}{2N-\alpha}}. \tag{1.7}$$

It is well known that the solutions of (1.4) correspond to the critical points of the C^1 functional $E: H \to \mathbb{R}$ given by

$$\begin{aligned} E(u,v) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \nu_1 u^2 + |\nabla v|^2 + \nu_2 v^2) \\ &\quad - \frac{1}{4}\int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * u^2)u^2 + 2\beta(\frac{1}{|x|^\alpha} * u^2)v^2 + \mu_2(\frac{1}{|x|^\alpha} * v^2)v^2, \end{aligned} \tag{1.8}$$

where $H := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. From (1.6) and (1.7), it is easy to check that E is well defined in H. This allows to consider *positive least energy solution*, which is defined as solution (u,v) of (1.4) with positive components and achieving the level

$$\inf\{E(u,v) : E'(u,v) = 0, (u,v) \in H,\ u > 0 \text{ and } v > 0\}.$$

We call a solution (u,v) *semi-trivial* if $u = 0$ or $v = 0$. A solution (u,v) *nontrivial* if both $u \not\equiv 0$ and $v \not\equiv 0$. A nontrivial solution (u,v) *positive* if both $u > 0$ and $v > 0$.

Note that system (1.4) admits a trivial solution $(0,0)$ and a pair of semi-trivial solutions $(\omega_1, 0)$ or $(0, \omega_2)$, where ω_i is the positive least energy solution of (see [24])

$$-\Delta u + \lambda u = \mu(\frac{1}{|x|^\alpha} * u^2)u \quad u \in H^1(\mathbb{R}^N), \tag{1.9}$$

with $(\lambda,\mu) = (\nu_1,\mu_1)$ for ω_1, and $(\lambda,\mu) = (\nu_2,\mu_2)$ for ω_2 respectively. The existence of solutions to (1.9) has received great interest recently, see [1, 2, 9, 10, 11, 15, 18, 21, 24, 25, 33] and references therein. Next, we will pay close attention to the existence of nontrivial solutions to (1.4).

Recently, Wang and Shi [30] studied the existence and various qualitative properties of positive least energy solutions to system (1.4) with $N = 3, \alpha = 1$. In [31] the authors acquired the existence and multiplicity of nontrivial solutions of (1.4) with perturbations. In [32] the authors studied the existence and nonexistence of $L^2(\mathbb{R}^N)$-normalized solutions of (1.4) with trapping potentials.

To the best of our knowledge, there are no papers considering system (1.4) with $\alpha \in (0,N)\cap(0,4)$. In present paper, we will focus on providing conditions on the coupling constant β that insures the existence of positive least energy solutions. Moreover, we will investigate the asymptotic behaviors of those solutions.

We define

$$\begin{aligned} \mathcal{N} = \Big\{ u \not\equiv 0, v \not\equiv 0, \int_{\mathbb{R}^N} |\nabla u|^2 + \nu_1 u^2 = \int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * u^2)u^2 + \beta(\frac{1}{|x|^\alpha} * u^2)v^2, \\ \int_{\mathbb{R}^N} |\nabla v|^2 + \nu_2 v^2 = \int_{\mathbb{R}^N} \mu_2(\frac{1}{|x|^\alpha} * v^2)v^2 + \beta(\frac{1}{|x|^\alpha} * u^2)v^2 \Big\}. \end{aligned}$$

Then any nontrivial solution of (1.4) belongs to $\mathcal{N}$. Let

$$A := \inf_{(u,v)\in\mathcal{N}} E(u,v) = \inf_{(u,v)\in\mathcal{N}} \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u|^2 + \nu_1 u^2 + |\nabla v|^2 + \nu_2 v^2. \tag{1.10}$$

Now, we list our main results. First, we consider the case $\nu_1 = \nu_2 = \nu$. Let φ be any a positive least energy solution of (1.9) with $\lambda = \nu$ and $\mu = 1$. Then we have the following two Theorems.

Theorem 1.3. *Assume that $N \geq 3, \alpha \in (0,N) \cap (0,4)$ and $\nu_1 = \nu_2 = \nu > 0$.*

(I) *If $0 < \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, then A is attained by $(\sqrt{k}\varphi, \sqrt{l}\varphi)$, where $k, l > 0$ satisfy*

$$\begin{aligned} &\mu_1 k + \beta l = 1, \\ &\beta k + \mu_2 l = 1. \end{aligned} \tag{1.11}$$

Therefore, $(\sqrt{k}\varphi, \sqrt{l}\varphi)$ is a positive least energy solution of (1.4).

(II) *If $\beta \in [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$ and $\mu_1 \neq \mu_2$, then (1.4) does not have a nontrivial nonnegative solution.*

Theorem 1.4. *Assume that $\nu_1 = \nu_2 = \nu > 0$, and let $0 < \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$. Let (u, v) be any a least energy nontrivial solution of (1.4), then $(u, v) = (\sqrt{k}\varphi, \sqrt{l}\varphi)$, where (k, l) satisfies (1.11). In particular, when $N = 3, 4, 5$ and $\alpha = N - 2, (\sqrt{k}\varphi, \sqrt{l}\varphi)$ is a unique positive least energy solution of (1.4) up to a translation.*

For the general case in which $\nu_1 \neq \nu_2$, we have the following theorem.

Theorem 1.5. *Assume that $N \geq 3$ and $\alpha \in (0,N) \cap (0,4)$.*

(1) *There exists $\beta_1 > 0$ such that for any $\beta \in (0, \beta_1)$, (1.4) has a positive least energy solution (u, v), which is radially symmetric.*
(2) *There exists $\beta_2 > 0$ such that for any $\beta \in (\beta_2, +\infty)$, (1.4) has a positive least energy solution (u, v), which is radially symmetric.*
(3) *Assume that $0 < \nu_1 \leq \nu_2$ and $\mu_2 < \mu_1$. If $\mu_2 \leq \beta \leq \mu_1$, then (1.4) does not have a nontrivial nonnegative solution.*

In fact, we can give an accurate definition of β_1 in Lemma 4.1 and β_2 in Lemma 4.5, but do not give it here to avoid introducing heavy notation at this stage.

Remark 1.6. System (1.4) is critical when $\alpha = 4$ in the sense of the Hardy-Littlewood-Sobolev inequality, which leads to the lack of compactness. This will be an interesting issue to be pursued in the future.

Finally, we study the asymptotic behavior of the positive least energy solutions in the case $\beta \to 0^+$. Then we have the following result.

Theorem 1.7. *Assume that $N \geq 3$ and $\alpha \in (0,N) \cap (0,4)$. Let $\beta_n \in (0, \beta_1), n \in \mathbb{N}$, satisfy $\beta_n \to 0$ as $n \to +\infty$. Suppose that (u_n, v_n) is the positive least energy solutions of (1.4) with $\beta = \beta_n$ and (u_n, v_n) is radially symmetric, which exists by Theorem 1.5. Then passing to a subsequence, $(u_n, v_n) \to (\widehat{u}, \widehat{v})$ strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ as $n \to +\infty$, where $\widehat{u}$ is a positive least energy solution of*

$$-\Delta u + \nu_1 u = \mu_1 (\frac{1}{|x|^\alpha} * u^2) u, \quad u \in H^1(\mathbb{R}^N),$$

and $\widehat{v}$ is a positive least energy solution of

$$-\Delta v + \nu_2 v = \mu_2 (\frac{1}{|x|^\alpha} * v^2) v, \quad v \in H^1(\mathbb{R}^N).$$

The paper is organized as follows. Theorem 1.3 and Theorem 1.4 are proved in Section 2 and Section 3, respectively. In Section 4, we use the Nehari manifold approach and a mountain pass argument to prove Theorem 1.5. In Section 5, we study the limit behavior of the positive least energy solutions as $\beta \to 0^+$.

We give some notation here. Throughout this paper, we denote the norm of L^q by $|u|_q = (\int_{\mathbb{R}^N} |u|^q dx)^{\frac{1}{q}}$, the norm of $H^1(\mathbb{R}^N)$ by $\|u\|_\nu^2 = \int_{\mathbb{R}^N}(|\nabla u|^2+\nu u^2)$, the norm of H by $\|(u,v)\|_H^2 := \|u\|_{\nu_1}^2 + \|v\|_{\nu_2}^2$, $H_r := \{(u,v) \in H : u, v \text{ are radially symmetric}\}$, and positive constants (possibly different in different places) by C, C_1, C_2.

2. Proof of Theorem 1.3

By [24] we know that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + \nu u^2) \geq 2\sqrt{B}\Big(\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u^2)u^2\Big)^{1/2}, \quad \forall u \in H^1(\mathbb{R}^N), \tag{2.1}$$

where

$$B := \frac{1}{4}\int_{\mathbb{R}^N} |\nabla \varphi|^2 + \nu\varphi^2 = \frac{1}{4}\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * \varphi^2)\varphi^2, \tag{2.2}$$

and φ is a positive least energy solution of (1.9) with $\lambda = \nu$ and $\mu = 1$.

Conclusion of the proof of Theorem 1.3. Firstly, we prove (I) of Theorem 1.3. Since $0 < \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, it follows that the equation (1.11) has a solution (k, l) satisfying $k > 0, l > 0$. It is easy to see that $(\sqrt{k}\varphi, \sqrt{l}\varphi)$ is a nontrivial solution of (1.4). By (1.10) and (2.2) we have

$$A \leq E(\sqrt{k}\varphi, \sqrt{l}\varphi) = (k + l)B. \tag{2.3}$$

Let $(u_n, v_n) \subset \mathcal{N}$ be a minimizing sequence for A, that is, $E(u_n, v_n) \to A$. For simplicity of presentation, we set

$$\alpha_n = \Big(\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u_n^2)u_n^2\Big)^{1/2}, \quad \beta_n = \Big(\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * v_n^2)v_n^2\Big)^{1/2}.$$

Then, by (1.6) and (2.1) we have

$$\begin{aligned} 2\sqrt{B}\alpha_n &\leq \int_{\mathbb{R}^N} |\nabla u_n|^2 + \nu u_n^2 \\ &= \int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * u_n^2)u_n^2 + \beta(\frac{1}{|x|^\alpha} * u_n^2)v_n^2 \\ &\leq \mu_1\alpha_n^2 + \beta\alpha_n\beta_n, \end{aligned} \tag{2.4}$$

$$\begin{aligned} 2\sqrt{B}\beta_n &\leq \int_{\mathbb{R}^N} |\nabla v_n|^2 + \nu v_n^2 \\ &= \int_{\mathbb{R}^N} \mu_2(\frac{1}{|x|^\alpha} * v_n^2)v_n^2 + \beta(\frac{1}{|x|^\alpha} * v_n^2)u_n^2 \\ &\leq \mu_2\beta_n^2 + \beta\alpha_n\beta_n. \end{aligned} \tag{2.5}$$

Note that

$$E(u_n, v_n) = \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u_n|^2 + \nu u_n^2 + |\nabla v_n|^2 + \nu v_n^2.$$

Combining this with (2.4) and (2.5) we have

$$\begin{gathered} 2\sqrt{B}(\alpha_n + \beta_n) \leq 4E(u_n, v_n) = 4A + o(1) \leq 4(k + l)B + o(1), \\ \mu_1\alpha_n + \beta\beta_n \geq 2\sqrt{B}, \\ \mu_2\beta_n + \beta\alpha_n \geq 2\sqrt{B}. \end{gathered}$$

We deduce from (1.11) that the above three inequalities are equivalent to

$$
\begin{gathered}
(\alpha_n - 2k\sqrt{B}) + (\beta_n - 2l\sqrt{B}) \leq o(1), \\
\mu_1(\alpha_n - 2k\sqrt{B}) + \beta(\beta_n - 2l\sqrt{B}) \geq 0, \\
\beta(\alpha_n - 2k\sqrt{B}) + \mu_2(\beta_n - 2l\sqrt{B}) \geq 0.
\end{gathered}
$$

Therefore, $\alpha_n \to 2k\sqrt{B}$ and $\beta_n \to 2l\sqrt{B}$ as $n \to \infty$. Then

$$
4A = \lim_{n\to\infty} 4E(u_n, v_n) \geq \lim_{n\to\infty} 2\sqrt{B}(\alpha_n + \beta_n) = 4(k+l)B.
$$

Combining this with (2.3) we have

$$
A = (k+l)B = E(\sqrt{k}\varphi, \sqrt{l}\varphi). \tag{2.6}
$$

So $(\sqrt{k}\varphi, \sqrt{l}\varphi)$ is a positive least energy solution of (1.4).

Now we prove (II) of Theorem 1.3. Suppose that (u, v) is a nontrivial solution of (1.4) and satisfies $u \geq 0, v \geq 0$ in $\mathbb{R}^N$. By the strong maximum principle each of the functions u, v is strictly positive in $\mathbb{R}^N$. Repeating the proof of [7, Proposition 4.1], we know that the solutions of (1.4) which are in $H^1(\mathbb{R}^N)$ are also in $C^2(\mathbb{R}^N)$ and tend to zero as $|x| \to \infty$.

Next, we multiply the first equation in (1.4) by v, the second equation in (1.4) by u, and integrate the resulting equations over $\mathbb{R}^N$. Then we obtain

$$
\begin{gathered}
\int_{\mathbb{R}^N} (\nabla u \nabla v + \nu_1 uv) = \int_{\mathbb{R}^N} uv[\mu_1(\frac{1}{|x|^\alpha} * u^2) + \beta(\frac{1}{|x|^\alpha} * v^2)], \\
\int_{\mathbb{R}^N} (\nabla u \nabla v + \nu_2 uv) = \int_{\mathbb{R}^N} uv[\mu_2(\frac{1}{|x|^\alpha} * v^2) + \beta(\frac{1}{|x|^\alpha} * u^2)].
\end{gathered}
$$

Thus,

$$
\int_{\mathbb{R}^N} uv[(\nu_2 - \nu_1) + (\mu_1 - \beta)(\frac{1}{|x|^\alpha} * u^2) + (\beta - \mu_2)(\frac{1}{|x|^\alpha} * v^2)] = 0,
$$

which is in a contradiction with the positivity of u and v as long as the three constants $(\nu_2 - \nu_1), (\mu_1 - \beta), (\beta - \mu_2)$ are of the same sign or zero, and one of them is not zero. This implies that system (1.4) does not have a nontrivial solution with nonnegative components if $\nu_1 = \nu_2, \mu_1 \neq \mu_2$ and $\min\{\mu_1, \mu_2\} \leq \beta \leq \max\{\mu_1, \mu_2\}$. The proof is complete. □

3. Proof of Theorem 1.4

Conclusion of the proof of Theorem 1.4. Firstly, we consider the case in which $0 < \beta < \min\{\mu_1, \mu_2\}$. The following proof is inspired by [5]. Fix $\mu_1 > 0, \mu_2 > 0$ and $0 < \beta < \min\{\mu_1, \mu_2\}$. Let (u_1, v_1) be any a nontrivial least energy solution of (1.4), then $u_1, v_1 > 0$ in $\mathbb{R}^N$ by the strong maximum principle. Recalling $(\sqrt{k}\varphi, \sqrt{l}\varphi)$ in Theorem 1.3, first we claim that

$$
\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u_1^2) u_1^2 = k^2 \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * \varphi^2)\varphi^2. \tag{3.1}
$$

Observe that there exists $\delta > 0$ such that $0 < \beta < \min\{\mu, \mu_2\}$ for any $\mu \in (\mu_1 - \delta, \mu_1 + \delta)$. Then by Theorem 1.3, A is attained when μ_1 is replaced by μ. Recall the definition of $E, \mathcal{N}, A$, they all depend on μ and we use notation $E_\mu, \mathcal{N}_\mu, A(\mu)$ in this proof. Recall (1.11) and (2.6), we have

$$A(\mu) = \frac{\mu + \mu_2 - 2\beta}{\mu\mu_2 - \beta^2} B,$$

so $A'(\mu_1) := \frac{d}{d\mu} A(\mu)|_{\mu_1}$ exists. We define

$$\begin{aligned} f(t,s,\mu) &:= t\mu \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u_1^2) u_1^2 + s \int_{\mathbb{R}^N} \beta(\frac{1}{|x|^\alpha} * v_1^2) u_1^2 - \int_{\mathbb{R}^N} (|\nabla u_1|^2 + \nu u_1^2), \\ g(t,s,\mu) &:= s \int_{\mathbb{R}^N} \mu_2(\frac{1}{|x|^\alpha} * v_1^2) v_1^2 + t \int_{\mathbb{R}^N} \beta(\frac{1}{|x|^\alpha} * u_1^2) v_1^2 - \int_{\mathbb{R}^N} (|\nabla v_1|^2 + \nu v_1^2). \end{aligned}$$

It is easy to obtain that $f(1,1,\mu_1) = g(1,1,\mu_1) = 0$, and

$$\begin{gathered} \frac{\partial f}{\partial t}(1,1,\mu_1) = \mu_1 D(u_1^2, u_1^2), \quad \frac{\partial f}{\partial s}(1,1,\mu_1) = \beta D(u_1^2, v_1^2), \\ \frac{\partial g}{\partial t}(1,1,\mu_1) = \beta D(u_1^2, v_1^2), \quad \frac{\partial g}{\partial s}(1,1,\mu_1) = \mu_2 D(v_1^2, v_1^2). \end{gathered}$$

We define the matrix

$$G := \begin{pmatrix} \frac{\partial f}{\partial t}(1,1,\mu_1) & \frac{\partial f}{\partial s}(1,1,\mu_1) \\ \frac{\partial g}{\partial t}(1,1,\mu_1) & \frac{\partial g}{\partial s}(1,1,\mu_1) \end{pmatrix},$$

then we see from (1.6) that

$$\begin{aligned} \det(G) &= \mu_1\mu_2 D(u_1^2, u_1^2) D(v_1^2, v_1^2) - \beta^2 D^2(u_1^2, v_1^2) \\ &\geq (\mu_1\mu_2 - \beta^2) D(u_1^2, u_1^2) D(v_1^2, v_1^2) > 0. \end{aligned} \tag{3.2}$$

Therefore, by the implicit function theorem, functions $t(\mu)$ and $s(\mu)$ are well defined and class C^1 on $(\mu_1 - \delta_1, \mu_1 + \delta_1)$ for some $\delta_1 \leq \delta$. Moreover, $t(\mu_1) = s(\mu_1) = 1$, and so we may assume that $t(\mu), s(\mu) > 0$ for all $\mu \in (\mu_1 - \delta_1, \mu_1 + \delta_1)$ by choosing a small δ_1. Since $f(t(\mu), s(\mu), \mu) \equiv g(t(\mu), s(\mu), \mu) \equiv 0$, then we have $(\sqrt{t(\mu)} u_1, \sqrt{s(\mu)} v_1) \in \mathcal{N}_\mu$. By a direct computation we see that

$$t'(\mu_1) = -\frac{\mu_2 D(v_1^2, v_1^2) D(u_1^2, u_1^2)}{det(G)}, \quad s'(\mu_1) = \frac{\beta D(u_1^2, v_1^2) D(u_1^2, u_1^2)}{det(G)}.$$

Note that $t(\mu) = 1 + t'(\mu_1)(\mu - \mu_1) + o((\mu - \mu_1))$ and $s(\mu) = 1 + s'(\mu_1)(\mu - \mu_1) + o((\mu - \mu_1))$. Hence

$$A(\mu) \leq E_\mu(\sqrt{t(\mu)} u_1, \sqrt{s(\mu)} v_1) = A(\mu_1) + \frac{1}{4}\overline{B}(\mu - \mu_1) + o((\mu - \mu_1)),$$

where

$$\overline{B} := t'(\mu_1) \int_{\mathbb{R}^N} (|\nabla u_1|^2 + \nu u_1^2) + s'(\mu_1) \int_{\mathbb{R}^N} (|\nabla v_1|^2 + \nu v_1^2) = -D(u_1^2, u_1^2).$$

It follows that $\frac{A(\mu) - A(\mu_1)}{\mu - \mu_1} \geq \frac{\overline{B}}{4} + o(1)$, as $\mu \nearrow \mu_1$. So $A'(\mu_1) \geq \overline{B}/4$. Similarly, we have $A'(\mu_1) \leq \overline{B}/4$. Therefore, $A'(\mu_1) = \overline{B}/4 = -\frac{1}{4} \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u_1^2) u_1^2$. By Theorem 1.3, $(\sqrt{k}\varphi, \sqrt{l}\varphi)$ is also a positive least energy solution of (1.4). Hence, $A'(\mu_1) = -\frac{k^2}{4} \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * \varphi^2) \varphi^2$, and so (3.1) holds.

Similarly, by computing $A'(\mu_2)$ and $A'(\beta)$, respectively, we see that

$$\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * v_1^2) v_1^2 = l^2 \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * \varphi^2) \varphi^2, \quad \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u_1^2) v_1^2 = kl \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * \varphi^2) \varphi^2.$$

Therefore,

$$\int_{\mathbb{R}^N}(\frac{1}{|x|^\alpha}*u_1^2)v_1^2=\frac{l}{k}\int_{\mathbb{R}^N}(\frac{1}{|x|^\alpha}*u_1^2)u_1^2=\frac{k}{l}\int_{\mathbb{R}^N}(\frac{1}{|x|^\alpha}*v_1^2)v_1^2.$$

We define $(\overline{u},\overline{v}):=(\frac{1}{\sqrt{k}}u_1,\frac{1}{\sqrt{l}}v_1)$. Combining these with (1.11) and $(u_1,v_1)\in\mathcal{N}$, we obtain

$$\int_{\mathbb{R}^N}(|\nabla\overline{u}|^2+\nu\overline{u}^2)=\int_{\mathbb{R}^N}(\frac{1}{|x|^\alpha}*\overline{u}^2)\overline{u}^2,\quad \int_{\mathbb{R}^N}(|\nabla\overline{v}|^2+\nu\overline{v}^2)=\int_{\mathbb{R}^N}(\frac{1}{|x|^\alpha}*\overline{v}^2)\overline{v}^2. \tag{3.3}$$

Then, by (2.1) we have

$$\frac{1}{4}\int_{\mathbb{R}^N}(|\nabla\overline{u}|^2+\nu\overline{u}^2)\ge B,\quad \frac{1}{4}\int_{\mathbb{R}^N}(|\nabla\overline{v}|^2+\nu\overline{v}^2)\ge B.$$

Therefore,

$$\begin{aligned}A=(k+l)B&=\frac{1}{4}\int_{\mathbb{R}^N}(|\nabla u_1|^2+\nu u_1^2+|\nabla v_1|^2+\nu v_1^2)\\&=\frac{1}{4}k\int_{\mathbb{R}^N}(|\nabla\overline{u}|^2+\nu\overline{u}^2)+\frac{1}{4}l\int_{\mathbb{R}^N}(|\nabla\overline{v}|^2+\nu\overline{v}^2)\\&\ge(k+l)B.\end{aligned}$$

This implies that

$$\frac{1}{4}\int_{\mathbb{R}^N}(|\nabla\overline{u}|^2+\nu\overline{u}^2)=B,\quad \frac{1}{4}\int_{\mathbb{R}^N}(|\nabla\overline{v}|^2+\nu\overline{v}^2)=B.$$

Combining this with (3.3), it is easy to see that $\overline{u}$ and $\overline{v}$ are both positive least energy solutions of (1.9) with $\lambda=\nu$ and $\mu=1$. Since (u_1,v_1) satisfies (1.4), then we know that

$$-\Delta\overline{u}+\nu\overline{u}=\mu_1k(\frac{1}{|x|^\alpha}*\overline{u}^2)\overline{u}+\beta l(\frac{1}{|x|^\alpha}*\overline{v}^2)\overline{u}=(\frac{1}{|x|^\alpha}*\overline{u}^2)\overline{u},$$

that is, $\frac{1}{|x|^\alpha}*(\overline{u}^2-\overline{v}^2)\equiv 0$. It follows from lemma 1.2 that $\overline{u}=\overline{v}$. Denote $\varphi=\overline{u}$, then $(u_1,v_1)=(\sqrt{k}\varphi,\sqrt{l}\varphi)$, where φ is a positive least energy solution of (1.9) with $\lambda=\nu$ and $\mu=1$.

Next, we consider the case $\beta>\max\{\mu_1,\mu_2\}$. The following proof is inspired by [34]. First, we claim that if (u_2,v_2) is a least energy nontrivial solution of (1.4), then we obtain $v_2(x)=au_2(x)$, where $a=\sqrt{(\beta-\mu_1)/(\beta-\mu_2)}$ is a constant.

In fact, if this claim holds, it is easy to see that u_2 is a positive least energy solution of the equation

$$-\Delta u+\nu u=\frac{\beta^2-\mu_1\mu_2}{\beta-\mu_2}(\frac{1}{|x|^\alpha}*u^2)u, \tag{3.4}$$

and so $u_2=\sqrt{k}\varphi$, where φ is a positive least energy solution of (1.9) with $\lambda=\nu$ and $\mu=1$.

It suffices to prove this claim. Set $\widehat{u}_2(x)=a^{-1}v_2(x)$, then $(u_2,\widehat{u}_2)$ satisfies

$$\begin{gathered}-\Delta u_2+\nu u_2=\mu_1(\frac{1}{|x|^\alpha}*u_2^2)u_2+\beta a^2(\frac{1}{|x|^\alpha}*\widehat{u}_2^2)u_2,\quad x\in\mathbb{R}^N,\\-\Delta\widehat{u}_2+\nu\widehat{u}_2=\mu_2a^2(\frac{1}{|x|^\alpha}*\widehat{u}_2^2)\widehat{u}_2+\beta(\frac{1}{|x|^\alpha}*u_2^2)\widehat{u}_2,\quad x\in\mathbb{R}^N,\\u_2,\widehat{u}_2\ge 0,\quad u,v\in H^1(\mathbb{R}^N).\end{gathered}\tag{3.5}$$

By the proof of Theorem 1.3(II), we know that $u_2, \widehat{u}_2 \in C^2(\mathbb{R}^N)$ and tend to zero as $|x| \to \infty$. Let $\Omega_+ \equiv \{x \in \mathbb{R}^N | u_2(x) - \widehat{u}_2(x) > 0\}$. Then Ω_+ is a piecewise C^1 smooth domain. Multiplying the first equation in (3.5) by $\widehat{u}_2$ and the second equation in (3.5) by u_2, then integrating by parts on Ω_+ and subtracting together, we obtain the following integral identity

$$\int_{\partial\Omega_+} (\widehat{u}_2 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial \widehat{u}_2}{\partial n}) + \int_{\Omega_+} (\mu_1 - \beta) u_2 \widehat{u}_2 (\frac{1}{|x|^\alpha} * (u_2^2 - \widehat{u}_2^2)) = 0, \tag{3.6}$$

where n denotes the unit outward normal to $\partial\Omega_+$.

On the one hand, since $u_2(x) - \widehat{u}_2(x) > 0$ in Ω_+ and $u_2(x) - \widehat{u}_2(x) = 0$ on $\partial\Omega_+$, then we know that

$$\int_{\partial\Omega_+} (\widehat{u}_2 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial \widehat{u}_2}{\partial n}) = \int_{\partial\Omega_+} u_2 \frac{\partial (u_2 - \widehat{u}_2)}{\partial n} \leq 0. \tag{3.7}$$

On the other hand, since $\mu_1 - \beta < 0$ and $\frac{1}{|x|^\alpha} * (u_2^2 - \widehat{u}_2^2) \geq 0$ in Ω_+, then we have

$$\int_{\Omega_+} (\mu_1 - \beta) u_2 \widehat{u}_2 (\frac{1}{|x|^\alpha} * (u_2^2 - \widehat{u}_2^2)) \leq 0. \tag{3.8}$$

Therefore, from (3.6)-(3.8) we have $\Omega_+ = \emptyset$. Similarly, we may prove that the set $\Omega_- \equiv \{x \in \mathbb{R}^N | u_2(x) - \widehat{u}_2(x) < 0\}$ is also an empty set. It follows that $u_2(x) = \widehat{u}_2(x)$ in $\mathbb{R}^N$. Therefore, $v_2(x) = a u_2(x)$, where $a = \sqrt{(\beta - \mu_1)/(\beta - \mu_2)}$.

Finally, based on the above arguments, we can obtain the uniqueness of positive least energy solutions of system (1.4) when $0 < \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$ due to the uniqueness of positive solutions to (1.9) for $N = 3, 4, 5$, $\alpha = N - 2$ (see [33]). We completed the proof. □

4. Proof of Theorem 1.5

Multiply the equation for u in (1.4) by v, the equation for v by u, and integrate over $\mathbb{R}^N$, which yields

$$\int_{\mathbb{R}^N} uv[(\nu_2 - \nu_1) + (\mu_1 - \beta)(\frac{1}{|x|^\alpha} * u^2) + (\beta - \mu_2)(\frac{1}{|x|^\alpha} * v^2)] = 0.$$

Hence, (3) of Theorem 1.5 holds.

Firstly, we show the proof of (1) in Theorem 1.5. Similarly to (2.1), we get that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + \nu_i u^2) \geq 2\sqrt{\mu_i B_i} \Big(\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u^2) u^2 \Big)^{1/2}, \quad \forall u \in H^1(\mathbb{R}^N), \tag{4.1}$$

where

$$B_i := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \omega_i|^2 + \nu_i \omega_i^2) - \frac{1}{4} \int_{\mathbb{R}^N} \mu_i (\frac{1}{|x|^\alpha} * \omega_i^2) \omega_i^2, \tag{4.2}$$

and ω_i is a positive least energy solution of (1.9) with $\lambda = \nu_i$ and $\mu = \mu_i$, $i = 1, 2$. We define

$$\beta_3 := \min \Big\{ \sqrt{\mu_1 \mu_2 \frac{B_1}{B_2}}, \sqrt{\mu_1 \mu_2 \frac{B_2}{B_1}} \Big\}. \tag{4.3}$$

Then we have the following estimate.

Lemma 4.1. *For any $\beta \in (0, \beta_3)$, it holds*

$$A < B_1 + B_2. \tag{4.4}$$

Proof. Note that $(\sqrt{t_1}\omega_1, \sqrt{t_2}\omega_2) \in \mathcal{N}$ for some $t_1, t_2 > 0$ is equivalent to $t_1, t_2 > 0$ satisfying

$$\begin{aligned}
\int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * \omega_1^2)\omega_1^2 &= \int_{\mathbb{R}^N} |\nabla\omega_1|^2 + \nu_1\omega_1^2 \\
&= t_1 \int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * \omega_1^2)\omega_1^2 + t_2 \int_{\mathbb{R}^N} \beta(\frac{1}{|x|^\alpha} * \omega_1^2)\omega_2^2, \\
\int_{\mathbb{R}^N} \mu_2(\frac{1}{|x|^\alpha} * \omega_2^2)\omega_2^2 &= \int_{\mathbb{R}^N} |\nabla\omega_2|^2 + \nu_2\omega_2^2 \\
&= t_2 \int_{\mathbb{R}^N} \mu_2(\frac{1}{|x|^\alpha} * \omega_2^2)\omega_2^2 + t_1 \int_{\mathbb{R}^N} \beta(\frac{1}{|x|^\alpha} * \omega_1^2)\omega_2^2.
\end{aligned}$$

That is,

$$\begin{gathered}
t_1 = \frac{\mu_2 D(\omega_2^2,\omega_2^2)[\mu_1 D(\omega_1^2,\omega_1^2) - \beta D(\omega_1^2,\omega_2^2)]}{\mu_1\mu_2 D(\omega_1^2,\omega_1^2)D(\omega_2^2,\omega_2^2) - \beta^2 D^2(\omega_1^2,\omega_2^2)}, \\
t_2 = \frac{\mu_1 D(\omega_1^2,\omega_1^2)[\mu_2 D(\omega_2^2,\omega_2^2) - \beta D(\omega_1^2,\omega_2^2)]}{\mu_1\mu_2 D(\omega_1^2,\omega_1^2)D(\omega_2^2,\omega_2^2) - \beta^2 D^2(\omega_1^2,\omega_2^2)}.
\end{gathered}$$

Meanwhile, we deduce from (1.6) and $0 < \beta < \beta_3 \le \sqrt{\mu_1\mu_2}$ that

$$\begin{aligned}
\beta D(\omega_1^2,\omega_2^2) &< \sqrt{\mu_1\mu_2 \frac{B_1}{B_2}} D^{1/2}(\omega_1^2,\omega_1^2) D^{1/2}(\omega_2^2,\omega_2^2) \\
&= \mu_1 D(\omega_1^2,\omega_1^2).
\end{aligned}$$

Similarly, we have

$$\beta D(\omega_1^2,\omega_2^2) < \mu_2 D(\omega_2^2,\omega_2^2), \quad \mu_1\mu_2 D(\omega_1^2,\omega_1^2)D(\omega_2^2,\omega_2^2) - \beta^2 D^2(\omega_1^2,\omega_2^2) > 0.$$

So $t_1, t_2 > 0$ and $(\sqrt{t_1}\omega_1, \sqrt{t_2}\omega_2) \in \mathcal{N}$. Then

$$\begin{aligned}
A &\le E(\sqrt{t_1}\omega_1, \sqrt{t_2}\omega_2) \\
&= \frac{t_1}{4}\int_{\mathbb{R}^N} (|\nabla\omega_1|^2 + \nu_1\omega_1^2) + \frac{t_2}{4}\int_{\mathbb{R}^N} (|\nabla\omega_2|^2 + \nu_2\omega_2^2) \\
&= \frac{t_1}{4}\int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * \omega_1^2)\omega_1^2 + \frac{t_2}{4}\int_{\mathbb{R}^N} \mu_2(\frac{1}{|x|^\alpha} * \omega_2^2)\omega_2^2 \\
&< \frac{t_1}{4}\int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * \omega_1^2)\omega_1^2 + \beta(\frac{1}{|x|^\alpha} * \omega_1^2)\omega_2^2 \\
&\quad + \frac{t_2}{4}\int_{\mathbb{R}^N} \mu_2(\frac{1}{|x|^\alpha} * \omega_2^2)\omega_2^2 + \beta(\frac{1}{|x|^\alpha} * \omega_1^2)\omega_2^2 \\
&= \frac{1}{4}\int_{\mathbb{R}^N} (|\nabla\omega_1|^2 + \nu_1\omega_1^2) + \frac{1}{4}\int_{\mathbb{R}^N} (|\nabla\omega_2|^2 + \nu_2\omega_2^2) \\
&= B_1 + B_2.
\end{aligned}$$

□

The following lemma plays a crucial role in the proof of our main results.

Lemma 4.2. *Let $\alpha \in (0, N) \cap (0, 4)$. If $(u_j, v_j) \subset H_r$ be a sequence converging weakly to some $(u, v) \in H_r$ as $j \to \infty$, then*

$$\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u_j^2)v_j^2 \to \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u^2)v^2 \quad \text{as } j \to \infty, \tag{4.5}$$

$$\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u_j^2) u_j^2 \to \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u^2) u^2 \quad \text{as } j \to \infty. \tag{4.6}$$

Proof. Note that

$$\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u_j^2) v_j^2 = \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u^2) v^2 + \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u_j^2)(v_j^2 - v^2) + \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * (u_j^2 - u^2)) v^2,$$

and

$$u_j^2 \to u^2 \text{ in } L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N) \text{ as } j \to \infty, \quad v_j^2 \to v^2 \text{ in } L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N) \text{ as } j \to \infty.$$

Combining these with (1.7), we know that (4.5) and (4.6) hold. □

The following proposition shows the role of A.

Proposition 4.3. *If A is attained by a couple $(u, v) \in \mathcal{N}$, then this couple is a solution of* (1.4)*, provided* $0 < \beta < \sqrt{\mu_1 \mu_2}$.

Based upon (1.6), the proof of the above proposition is similar to that of [28, Proposition 1.1], and so we omit it. Before proceeding, we recall some facts about spherical rearrangement (see [16]).

Proposition 4.4. *Assume $N \geq 3$ and $\alpha \in (0, N) \cap (0, 4)$. Suppose that $u_1, u_2 \in H^1(\mathbb{R}^N)$ and let u_1^*, u_2^* be the symmetric-decreasing rearrangement of u_1, u_2. Then*

$$\|u_i^*\|_{\nu_i} \leq \|u_i\|_{\nu_i}, \quad \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * (u_1^*)^2)(u_2^*)^2 \geq \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u_1^2) u_2^2.$$

Let κ_1 be the smaller root of the equation

$$\beta^2 - \sqrt{\mu_1 \mu_2} \Big(\sqrt{\frac{B_1}{B_2}} + 2 \sqrt{\frac{B_2}{B_1}} \Big) \beta + \mu_1 \mu_2 = 0,$$

and κ_2 be the smaller root of the equation

$$\beta^2 - \sqrt{\mu_1 \mu_2} \Big(\sqrt{\frac{B_2}{B_1}} + 2 \sqrt{\frac{B_1}{B_2}} \Big) \beta + \mu_1 \mu_2 = 0.$$

Set

$$\beta_1 := \min \Big\{ \beta_3, \frac{\sqrt{\mu_1 \mu_2 B_1 B_2}}{B_1 + B_2}, \ \kappa_1, \ \kappa_2 \Big\}, \tag{4.7}$$

where β_3 is defined in (4.3).

The proof of (1) in Theorem 1.5. Assume that $\beta \in (0, \beta_1)$. The ideas of the following proof mainly come from [28]. Take a minimizing sequence $\{(u_n, v_n)\} \subset \mathcal{N}$ for A, then $\{(u_n, v_n)\}$ is bounded in H. By Proposition 4.4 the sequence of rearrangements $\{(u_n^*, v_n^*)\}$ is bounded in H. Up to a subsequence, we may assume that $(u_n^*, v_n^*) \to (u^*, v^*)$ weakly in H and strongly in $L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N) \times L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N)$. The proof is divided into three steps.

Step 1. We show that $u^* \not\equiv 0, v^* \not\equiv 0$. Define

$$a_n = D^{1/2}((u_n^*)^2, (u_n^*)^2), \quad b_n = D^{1/2}((v_n^*)^2, (v_n^*)^2).$$

By (1.6), (4.1) and Proposition 4.4, we have

$$\begin{aligned} 2\sqrt{\mu_1 B_1} a_n &\leq \int_{\mathbb{R}^N} |\nabla u_n^*|^2 + \nu_1 (u_n^*)^2 \\ &\leq \int_{\mathbb{R}^N} |\nabla u_n|^2 + \nu_1 u_n^2 \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N} \mu_1 (\frac{1}{|x|^\alpha} * u_n^2) u_n^2 + \beta (\frac{1}{|x|^\alpha} * u_n^2) v_n^2 \\
&\leq \int_{\mathbb{R}^N} \mu_1 (\frac{1}{|x|^\alpha} * (u_n^*)^2)(u_n^*)^2 + \beta (\frac{1}{|x|^\alpha} * (u_n^*)^2)(v_n^*)^2 \\
&\leq \mu_1 a_n^2 + \beta a_n b_n,
\end{aligned}
$$

and

$$
\begin{aligned}
2\sqrt{\mu_2 B_2} b_n &\leq \int_{\mathbb{R}^N} |\nabla v_n^*|^2 + \nu_2 (v_n^*)^2 \\
&\leq \int_{\mathbb{R}^N} |\nabla v_n|^2 + \nu_2 v_n^2 \\
&= \int_{\mathbb{R}^N} \mu_2 (\frac{1}{|x|^\alpha} * v_n^2) v_n^2 + \beta (\frac{1}{|x|^\alpha} * v_n^2) u_n^2 \\
&\leq \int_{\mathbb{R}^N} \mu_2 (\frac{1}{|x|^\alpha} * (v_n^*)^2)(v_n^*)^2 + \beta (\frac{1}{|x|^\alpha} * (v_n^*)^2)(u_n^*)^2 \\
&\leq \mu_2 b_n^2 + \beta a_n b_n.
\end{aligned}
$$

Note that

$$
E(u_n, v_n) = \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \nu_1 u_n^2 + |\nabla v_n|^2 + \nu_2 v_n^2.
$$

We deduce from lemma 4.4 that

$$
\begin{gathered}
\mu_1 a_n + \beta b_n \geq 2\sqrt{\mu_1 B_1}, \\
\beta a_n + \mu_2 b_n \geq 2\sqrt{\mu_2 B_2}, \\
\sqrt{\mu_1 B_1} a_n + \sqrt{\mu_2 B_2} b_n \leq 2(B_1 + B_2) + o(1).
\end{gathered} \tag{4.8}
$$

We would like to infer from (4.8) that there exists $C_2 > C_1 > 0$ such that

$$
C_1 < a_n, b_n < C_2.
$$

For this it is sufficient to show that each two of the lines

$$
\begin{gathered}
l_1 = \big\{ z = (x, y) \in \mathbb{R}^2 : \sqrt{\mu_1 B_1} x + \sqrt{\mu_2 B_2} y = 2(B_1 + B_2) \big\}, \\
l_2 = \big\{ z \in \mathbb{R}^2 : \mu_1 x + \beta y = 2\sqrt{\mu_1 B_1} \big\}, \\
l_3 = \big\{ z \in \mathbb{R}^2 : \beta x + \mu_2 y = 2\sqrt{\mu_2 B_2} \big\},
\end{gathered}
$$

meet, and their crossing points have strictly positive coordinates (these lines are determined by the parameters in (4.8)). Indeed, for large n the point (a_n, b_n) is arbitrarily close to the triangle (or segment, or point) between these crossing points. Let (x_0, y_0) be the crossing points of l_2 and l_3, by direct computation we have

$$
x_0 = \frac{2\mu_2 \sqrt{\mu_1 B_1} - 2\beta \sqrt{\mu_2 B_2}}{\mu_1 \mu_2 - \beta^2}, \quad y_0 = \frac{2\mu_1 \sqrt{\mu_2 B_2} - 2\beta \sqrt{\mu_1 B_1}}{\mu_1 \mu_2 - \beta^2}.
$$

Since $\beta < \beta_1 \leq \sqrt{\mu_1 \mu_2}$, we see that we have to verify the following inequalities

$$
\sqrt{\frac{B_1}{\mu_1}} < \frac{B_1 + B_2}{\sqrt{\mu_1 B_1}} < \frac{\sqrt{\mu_2 B_2}}{\beta}, \tag{4.9}
$$

$$
\sqrt{\frac{B_2}{\mu_2}} < \frac{B_1 + B_2}{\sqrt{\mu_2 B_2}} < \frac{\sqrt{\mu_1 B_1}}{\beta}, \tag{4.10}
$$

$$\sqrt{\mu_1 B_1}x_0 + \sqrt{\mu_2 B_2}y_0 \le 2(B_1 + B_2). \tag{4.11}$$

Inequalities (4.9) and (4.10) can be recast as $\beta < \frac{\sqrt{\mu_1\mu_2 B_1 B_2}}{B_1+B_2}$, which is true by the definition of β_1. Elementary computations show that (4.11) also holds. Finally, from lemma 4.2 and (4.8) we infer that there exists $C_1 > 0$ such that

$$C_1 \le \Big(\int_{\mathbb{R}^N}(\frac{1}{|x|^\alpha} * (u^*)^2)(u^*)^2\Big)^{1/2} \le \frac{2\sqrt{\mu_2}B_1}{\sqrt{\mu_1\mu_2 B_1} - \beta\sqrt{B_2}} < \frac{2\sqrt{\mu_2 B_2}}{\beta}, \tag{4.12}$$

$$C_1 \le \Big(\int_{\mathbb{R}^N}(\frac{1}{|x|^\alpha} * (v^*)^2)(v^*)^2\Big)^{1/2} \le \frac{2\sqrt{\mu_1}B_2}{\sqrt{\mu_1\mu_2 B_2} - \beta\sqrt{B_1}} < \frac{2\sqrt{\mu_1 B_1}}{\beta}, \tag{4.13}$$

where C_1 depends on $\mu_1, \mu_2, \beta, B_1, B_2$. Therefore $u^* \not\equiv 0$ and $v^* \not\equiv 0$.

Step 2. We show that there exist $t_1 > 0$ and $t_2 > 0$ such that $(\sqrt{t_1}u^*, \sqrt{t_2}v^*) \in \mathcal{N}$. By Lemma 4.2 and Proposition 4.4, we know that

$$\begin{aligned}
\|u^*\|_{\nu_1}^2 &\le \liminf_{n\to\infty}\|u_n^*\|_{\nu_1} \le \liminf_{n\to\infty}\|u_n\|_{\nu_1} \\
&= \liminf_{n\to\infty}\int_{\mathbb{R}^N}\mu_1(\frac{1}{|x|^\alpha} * u_n^2)u_n^2 + \beta(\frac{1}{|x|^\alpha} * u_n^2)v_n^2 \\
&\le \liminf_{n\to\infty}\int_{\mathbb{R}^N}\mu_1(\frac{1}{|x|^\alpha} * (u_n^*)^2)(u_n^*)^2 + \beta(\frac{1}{|x|^\alpha} * (u_n^*)^2)(v_n^*)^2 \\
&= \int_{\mathbb{R}^N}\mu_1(\frac{1}{|x|^\alpha} * (u^*)^2)(u^*)^2 + \beta(\frac{1}{|x|^\alpha} * (u^*)^2)(v^*)^2,
\end{aligned}$$

that is,

$$\|u^*\|_{\nu_1}^2 \le \int_{\mathbb{R}^N}\mu_1(\frac{1}{|x|^\alpha} * (u^*)^2)(u^*)^2 + \beta(\frac{1}{|x|^\alpha} * (u^*)^2)(v^*)^2. \tag{4.14}$$

Similarly, we have

$$\|v^*\|_{\nu_2}^2 \le \int_{\mathbb{R}^N}\beta(\frac{1}{|x|^\alpha} * (u^*)^2)(v^*)^2 + \mu_2(\frac{1}{|x|^\alpha} * (v^*)^2)(v^*)^2, \tag{4.15}$$

$$E(u^*, v^*) \le \liminf_{n\to\infty} E(u_n^*, v_n^*) \le \liminf_{n\to\infty} E(u_n, v_n) = A. \tag{4.16}$$

Let t_1, t_2 be the solutions of the linear system

$$\begin{gathered}
\Big(\int_{\mathbb{R}^N}\mu_1(\frac{1}{|x|^\alpha} * (u^*)^2)(u^*)^2\Big)t_1 + \Big(\int_{\mathbb{R}^N}\beta(\frac{1}{|x|^\alpha} * (u^*)^2)(v^*)^2\Big)t_2 = \|u^*\|_{\nu_1}^2, \\
\Big(\int_{\mathbb{R}^N}\beta(\frac{1}{|x|^\alpha} * (u^*)^2)(v^*)^2\Big)t_1 + \Big(\int_{\mathbb{R}^N}\mu_2(\frac{1}{|x|^\alpha} * (v^*)^2)(v^*)^2\Big)t_2 = \|v^*\|_{\nu_2}^2.
\end{gathered}$$

We claim that the solution of the above equations satisfies $t_1 > 0, t_2 > 0$ if $\beta \in (0, \beta_1)$. In fact, by a direct computation we see that

$$\begin{aligned}
t_1 &= \frac{\|u^*\|_{\nu_1}^2\int_{\mathbb{R}^N}\mu_2(\frac{1}{|x|^\alpha} * (v^*)^2)(v^*)^2 - \|v^*\|_{\nu_2}\int_{\mathbb{R}^N}\beta(\frac{1}{|x|^\alpha} * (u^*)^2)(v^*)^2}{\mu_1\mu_2 D((u^*)^2, (u^*)^2)D((v^*)^2, (v^*)^2) - \beta^2 D^2((u^*)^2, (v^*)^2)}, \\
t_2 &= \frac{\|v^*\|_{\nu_2}^2\int_{\mathbb{R}^N}\mu_1(\frac{1}{|x|^\alpha} * (u^*)^2)(u^*)^2 - \|u^*\|_{\nu_1}\int_{\mathbb{R}^N}\beta(\frac{1}{|x|^\alpha} * (u^*)^2)(v^*)^2}{\mu_1\mu_2 D((u^*)^2, (u^*)^2)D((v^*)^2, (v^*)^2) - \beta^2 D^2((u^*)^2, (v^*)^2)}.
\end{aligned}$$

Next, we prove that $t_1 > 0$. We need to show that

$$\|u^*\|_{\nu_1}^2\int_{\mathbb{R}^N}\mu_2(\frac{1}{|x|^\alpha} * (v^*)^2)(v^*)^2 > \|v^*\|_{\nu_2}^2\int_{\mathbb{R}^N}\beta(\frac{1}{|x|^\alpha} * (u^*)^2)(v^*)^2. \tag{4.17}$$

By (1.6), (4.1), (4.12), (4.13) and (4.15), inequality (4.17) is implied by

$$\begin{aligned}
&\mu_2\|u^*\|_{\nu_1}^2 D^{1/2}((v^*)^2,(v^*)^2) > \beta\|v^*\|_{\nu_2}^2 D^{1/2}((u^*)^2,(u^*)^2)\\
&\Leftarrow \quad \mu_2 D^{1/2}((v^*)^2,(v^*)^2) > \frac{\beta}{2\sqrt{\mu_1 B_1}}\|v^*\|_{\nu_2}^2\\
&\Leftarrow \quad 1 > \frac{\beta}{2\sqrt{\mu_1 B_1}}\Big(\frac{\beta}{\mu_2}D^{1/2}((u^*)^2,(u^*)^2) + D^{1/2}((v^*)^2,(v^*)^2)\Big)\\
&\Leftarrow \quad 1 > \frac{\beta}{2\sqrt{\mu_1 B_1}}\Big(\frac{\beta}{\mu_2}\frac{2\sqrt{\mu_2 B_2}}{\beta} + \frac{2\sqrt{\mu_1}B_2}{\sqrt{\mu_1\mu_2 B_2}-\beta\sqrt{B_1}}\Big).
\end{aligned}$$

Note that the last inequality above can be recast as

$$\sqrt{\frac{B_1B_2}{\mu_2}}\beta^2 - (\sqrt{\mu_1}B_1 + 2\sqrt{\mu_1}B_2)\beta + \mu_1\sqrt{\mu_2 B_1 B_2} > 0. \tag{4.18}$$

Therefore, by the definition of β_1, we know the inequality (4.17) is true, and so $t_1 > 0$. Similarly, we can prove that $t_2 > 0$. Moreover, we have $(\sqrt{t_1}u^*, \sqrt{t_2}v^*) \in \mathcal{N}$.

Step 3. We show that $(|u^*|,|v^*|)$ is a positive solution of (1.4) and $E(|u^*|,|v^*|) = A$. Note that $(\sqrt{t_1}u^*, \sqrt{t_2}v^*) \in \mathcal{N}$. Then

$$\begin{aligned}
A &\le E(\sqrt{t_1}u^*, \sqrt{t_2}v^*) = \frac{t_1}{4}\|u^*\|_{\nu_1}^2 + \frac{t_2}{4}\|v^*\|_{\nu_2}^2\\
&\le \frac{t_1}{4}\int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * (u^*)^2)(u^*)^2 + \beta(\frac{1}{|x|^\alpha} * (u^*)^2)(v^*)^2\\
&\quad + \frac{t_2}{4}\int_{\mathbb{R}^N} \beta(\frac{1}{|x|^\alpha} * (u^*)^2)(v^*)^2 + \mu_2(\frac{1}{|x|^\alpha} * (v^*)^2)(v^*)^2\\
&= \frac{1}{4}\|u^*\|_{\nu_1}^2 + \frac{1}{4}\|v^*\|_{\nu_2}^2\\
&\le \frac{1}{4}\liminf_{n\to\infty}(\|u_n\|_{\nu_1}^2 + \|v_n\|_{\nu_2}^2) = A,
\end{aligned}$$

which implies that (4.14) and (4.15) are equalities. Thus, $(u^*,v^*) \in \mathcal{N}$. Combining this with (4.16), one has that $A = E(u^*,v^*)$. Therefore, $(|u^*|,|v^*|) \in \mathcal{N}$ and $A = E(|u^*|,|v^*|)$. By Proposition 4.3 and the maximum principle, we see that $(|u^*|,|v^*|)$ is a positive least energy solution of (1.4). $\square$

It remains to prove (2) of Theorem 1.5. Assume that $\beta > 0$. Without loss of generality, we may assume that $\nu_1 \le \nu_2$. We define

$$\mathcal{A} := \inf_{h\in\Gamma}\max_{t\in[0,1]} E(h(t)), \tag{4.19}$$

where $\Gamma = \{h \in C([0,1],H) : h(0) = (0,0), E(h(1)) < 0\}$. By (1.8), we know that for any $(u,v) \in H$, and $(u,v) \neq (0,0)$,

$$\begin{aligned}
\max_{t>0} E(tu,tv) &= E(t_{u,v}u, t_{u,v}v)\\
&= \frac{1}{4}t_{u,v}^2\int_{\mathbb{R}^N}(|\nabla u|^2 + \nu_1 u^2 + |\nabla v|^2 + \nu_2 v^2),
\end{aligned} \tag{4.20}$$

where $t_{u,v} > 0$ satisfies

$$t_{u,v}^2 = \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 + \nu_1 u^2 + |\nabla v|^2 + \nu_2 v^2)}{\int_{\mathbb{R}^N}\mu_1(\frac{1}{|x|^\alpha} * u^2)u^2 + 2\beta(\frac{1}{|x|^\alpha} * u^2)v^2 + \mu_2(\frac{1}{|x|^\alpha} * v^2)v^2}.$$

Note that $(t_{u,v}u, t_{u,v}v) \in \mathcal{N}'$, where

$$\begin{aligned}\mathcal{N}' := \Big\{(u,v) \in H\backslash\{(0,0)\}, F(u,v) := \int_{\mathbb{R}^N} (|\nabla u|^2 + \nu_1 u^2 + |\nabla v|^2 + \nu_2 v^2) \\ - \int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * u^2)u^2 + 2\beta(\frac{1}{|x|^\alpha} * u^2)v^2 + \mu_2(\frac{1}{|x|^\alpha} * v^2)v^2 = 0\Big\},\end{aligned} \tag{4.21}$$

it is standard to see that

$$\begin{aligned}\mathcal{A} &= \inf_{H \ni (u,v) \neq (0,0)} \max_{t>0} E(tu, tv) \\ &= \inf_{(u,v)\in\mathcal{N}'} E(u,v) \\ &= \inf_{H\backslash\{(0,0)\}} \frac{\big(\|u\|_{\nu_1}^2 + \|v\|_{\nu_2}^2\big)^2}{4\int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * u^2)u^2 + 2\beta(\frac{1}{|x|^\alpha} * u^2)v^2 + \mu_2(\frac{1}{|x|^\alpha} * v^2)v^2}.\end{aligned} \tag{4.22}$$

Note that $\mathcal{N} \subseteq \mathcal{N}'$, one has that $\mathcal{A} \le A$. Denote

$$\beta_2 := \max\Big\{\frac{\mu_1(2 + \frac{\nu_2 - \nu_1}{\nu_1})^2 - (\mu_1 + \mu_2)}{2},\ \frac{3\mu_2 - \mu_1}{2},\ 0\Big\}. \tag{4.23}$$

Then we have the following lemma.

Lemma 4.5. *Assume that $N \ge 3$ and $\alpha \in (0, N) \cap (0, 4)$. Then for any $\beta > \beta_2$ it holds $\mathcal{A} < \min\{B_1, B_2\}$.*

Proof. It follows from (4.22) and (4.23) that

$$\begin{aligned}\mathcal{A} &\le \frac{\big(\|\omega_1\|_{\nu_1}^2 + \|\omega_1\|_{\nu_2}^2\big)^2}{4\int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * \omega_1^2)\omega_1^2 + 2\beta(\frac{1}{|x|^\alpha} * \omega_1^2)\omega_1^2 + \mu_2(\frac{1}{|x|^\alpha} * \omega_1^2)\omega_1^2} \\ &\le \frac{\mu_1(2 + \nu_2 - \nu_1)^2}{\mu_1 + \mu_2 + 2\beta} B_1 < B_1.\end{aligned} \tag{4.24}$$

Similarly, we see that

$$\begin{aligned}\mathcal{A} &\le \frac{\big(\|\omega_2\|_{\nu_1}^2 + \|\omega_2\|_{\nu_2}^2\big)^2}{4\int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * \omega_2^2)\omega_2^2 + 2\beta(\frac{1}{|x|^\alpha} * \omega_2^2)\omega_2^2 + \mu_2(\frac{1}{|x|^\alpha} * \omega_2^2)\omega_2^2} \\ &\le \frac{4\mu_2}{\mu_1 + \mu_2 + 2\beta} B_2 < B_2.\end{aligned} \tag{4.25}$$

By (4.24) and (4.25), we know that $\mathcal{A} < \min\{B_1, B_2\}$. □

Proof of (2) in Theorem 1.5. Assume that $\beta > \beta_2$. Since the functional E has a mountain pass structure, then by the mountain pass theorem (see [35]) there exists $\{(u_n, v_n)\}$ such that

$$\lim_{n\to\infty} E(u_n, v_n) = \mathcal{A}, \quad \lim_{n\to\infty} E'(u_n, v_n) = 0.$$

It is standard to see that $\{(u_n, v_n)\}$ is bounded in H. By Proposition 4.4 the sequence of rearrangements $\{(u_n^*, v_n^*)\}$ is bounded in H. Up to a subsequence, we may assume that $(u_n^*, v_n^*) \to (u^*, v^*)$ weakly in H and strongly in $L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N) \times$

$L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N)$. Similarly to the proof of (1) in Theorem 1.5, we know that

$$
\begin{aligned}
&\|(u^*,v^*)\|_H^2 \\
&\le \int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * (u^*)^2)(u^*)^2 + 2\beta(\frac{1}{|x|^\alpha} * (u^*)^2)(v^*)^2 + \mu_2(\frac{1}{|x|^\alpha} * (v^*)^2)(v^*)^2, \\
&\qquad E(u^*,v^*) \le \liminf_{n\to\infty} E(u_n^*,v_n^*) \le \liminf_{n\to\infty} E(u_n,v_n) = \mathcal{A}.
\end{aligned} \tag{4.26}
$$

Step 1. We show that $(u^*,v^*) \neq (0,0)$. We define

$$
c_n = \Big(\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * (u_n^*)^2)(u_n^*)^2\Big)^{1/2}, \quad d_n = \Big(\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * (v_n^*)^2)(v_n^*)^2\Big)^{1/2}.
$$

It follows from (4.1) and (4.26) that

$$
2\sqrt{B_1}c_n + 2\sqrt{B_2}d_n \le \mu_1 c_n^2 + 2\beta c_n d_n + \mu_2 d_n^2.
$$

By lemma 4.2 we have

$$
\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * (u^*)^2)(u^*)^2 + \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * (v^*)^2)(v^*)^2 \ge C > 0,
$$

which implies $(u^*,v^*) \neq (0,0)$.

Step 2. We show that $(|u^*|,|v^*|)$ is a positive solution of (1.4) and $E(|u^*|,|v^*|) = \mathcal{A}$. If inequality (4.26) is an equality, then $(u^*,v^*) \in \mathcal{N}'$ and $\mathcal{A}$ is attained by (u^*,v^*). If not, that is,

$$
\begin{aligned}
\|(u^*,v^*)\|_H^2 < \int_{\mathbb{R}^N} &\mu_1(\frac{1}{|x|^\alpha} * (u^*)^2)(u^*)^2 + 2\beta(\frac{1}{|x|^\alpha} * (u^*)^2)(v^*)^2 \\
&+ \mu_2(\frac{1}{|x|^\alpha} * (v^*)^2)(v^*)^2,
\end{aligned}
$$

take $s \in (0,1)$ such that $(u_1,v_1) = s(u^*,v^*) \in \mathcal{N}'$. Therefore,

$$
\begin{aligned}
\mathcal{A} \le \frac{1}{4}\|(u_1,v_1)\|_H^2 < \frac{1}{4}\|(u^*,v^*)\|_H^2 &\le \frac{1}{4}\liminf_{n\to\infty} \|(u_n^*,v_n^*)\|_H^2 \\
&\le \liminf_{n\to\infty} \|(u_n,v_n)\|_H^2 = \mathcal{A},
\end{aligned}
$$

which is a contradiction. Thus $(u^*,v^*) \in \mathcal{N}'$ and $E(u^*,v^*) = \mathcal{A}$. Therefore, $E(|u^*|,|v^*|) = \mathcal{A}$ and $(|u^*|,|v^*|) \in \mathcal{N}'$. So there exists a Lagrange multiplier $L \in \mathbb{R}$ such that

$$
E'(|u^*|,|v^*|) - LG'(|u^*|,|v^*|) = 0,
$$

where

$$
\begin{aligned}
G(u,v) = \|(u,v)\|_H^2 - \int_{\mathbb{R}^N} &\mu_1(\frac{1}{|x|^\alpha} * u^2)u^2 \\
&+ 2\beta(\frac{1}{|x|^\alpha} * u^2)v^2 + \mu_2(\frac{1}{|x|^\alpha} * v^2)v^2.
\end{aligned}
$$

Since $E'(|u^*|,|v^*|)(|u^*|,|v^*|) = G(|u^*|,|v^*|) = 0$ and

$$
\begin{aligned}
&G'(|u^*|,|v^*|)(|u^*|,|v^*|) \\
&= -2\int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * (u^*)^2)(u^*)^2 + 2\beta(\frac{1}{|x|^\alpha} * (u^*)^2)(v^*)^2 + \mu_2(\frac{1}{|x|^\alpha} * (v^*)^2)(v^*)^2 \neq 0,
\end{aligned}
$$

then we get that $L = 0$ and $E'(|u^*|, |v^*|) = 0$, that is $(|u^*|, |v^*|)$ is a solution of (1.4). It follows from lemma 4.5 that $|u^*| \not\equiv 0$ and $|v^*| \not\equiv 0$. This means $(|u^*|, |v^*|) \in \mathcal{N} \subset \mathcal{N}'$, and so $E(|u^*|, |v^*|) = \mathcal{A} = A$. Then using the strong maximum principle, we see that $|u^*|, |v^*| > 0$ in $\mathbb{R}^N$. Therefore, $(|u^*|, |v^*|)$ is a positive least energy solution of (1.4). The proof is complete. □

5. Proof of Theorem 1.7

Recall the definitions of $E, \mathcal{N}, A$, they all depend on β, and we use notation $E_\beta, \mathcal{N}_\beta, A_\beta$ in this section.

Lemma 5.1. *Assume that $N \geq 3$ and $\alpha \in (0, N) \cap (0, 4)$. Let $\beta \in (0, \beta_1)$ and (u_β, v_β) be the positive least energy solution of (1.4) which exists by Theorem 1.5. Then it holds*

$$\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^\alpha} * u_\beta^2\Big) u_\beta^2 \geq \Big(\frac{2\sqrt{\mu_1\mu_2 B_1 B_2} - 2\beta(B_1 + B_2)}{\mu_1\sqrt{\mu_2 B_2} - \beta\sqrt{\mu_1 B_1}}\Big)^2, \tag{5.1}$$

$$\int_{\mathbb{R}^N} \Big(\frac{1}{|x|^\alpha} * v_\beta^2\Big) v_\beta^2 \geq \Big(\frac{2\sqrt{\mu_1\mu_2 B_1 B_2} - 2\beta(B_1 + B_2)}{\mu_2\sqrt{\mu_1 B_1} - \beta\sqrt{\mu_2 B_2}}\Big)^2. \tag{5.2}$$

Proof. Note that $(u_\beta, v_\beta) \in \mathcal{N}_\beta$ with $E(u_\beta, v_\beta) \leq 4(B_1 + B_2)$. We denote

$$D_1 = \Big(\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u_\beta^2) u_\beta^2\Big)^{1/2}, \quad D_2 = \Big(\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * v_\beta^2) v_\beta^2\Big)^{1/2}.$$

Similarly to the proof of (1) in Theorem 1.5, we obtain

$$\mu_1 D_1 + \beta D_2 \geq 2\sqrt{\mu_1 B_1}, \tag{5.3}$$

$$\beta D_1 + \mu_2 D_2 \geq 2\sqrt{\mu_2 B_2}, \tag{5.4}$$

$$\sqrt{\mu_1 B_1} D_1 + \sqrt{\mu_2 B_2} D_2 \leq 2(B_1 + B_2). \tag{5.5}$$

Therefore, we deduce from (5.3)-(5.5) that

$$\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * u_\beta^2) u_\beta^2 \geq \Big(\frac{2\sqrt{\mu_1\mu_2 B_1 B_2} - 2\beta(B_1 + B_2)}{\mu_1\sqrt{\mu_2 B_2} - \beta\sqrt{\mu_1 B_1}}\Big)^2,$$
$$\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * v_\beta^2) v_\beta^2 \geq \Big(\frac{2\sqrt{\mu_1\mu_2 B_1 B_2} - 2\beta(B_1 + B_2)}{\mu_2\sqrt{\mu_1 B_1} - \beta\sqrt{\mu_2 B_2}}\Big)^2.$$

□

Conclusion of the proof of Theorem 1.7. Let $0 < \beta_n < \beta_1, n \in \mathbb{N}$, satisfy $\beta_n \to 0$ as $n \to +\infty$. Suppose that (u_n, v_n) is the positive least energy solutions of (1.4) with $\beta = \beta_n$, and (u_n, v_n) is radially symmetric. By Lemma 4.1, we know that $\{(u_n, v_n)\}$ is uniformly bounded in H. Passing to a subsequence, we may assume that

$$u_n \rightharpoonup \widehat{u}, \quad v_n \rightharpoonup \widehat{v} \quad \text{weakly in } H^1(\mathbb{R}^N),$$
$$u_n \to \widehat{u}, \quad v_n \to \widehat{v} \quad \text{almost everywhere in } \mathbb{R}^N.$$

Combining these with lemma 4.2 and lemma 5.1 we obtain

$$\int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * \widehat{u}^2)\widehat{u}^2 \geq 2\sqrt{\frac{B_1}{\mu_1}}, \quad \int_{\mathbb{R}^N} (\frac{1}{|x|^\alpha} * \widehat{v}^2)\widehat{v}^2 \geq 2\sqrt{\frac{B_2}{\mu_2}}.$$

Hence, $\widehat{u} \not\equiv 0, \widehat{v} \not\equiv 0$ and $\widehat{u}(x), \widehat{v}(x) \geq 0$ a.e. $x \in \mathbb{R}^N$. We multiply the equation for u in (1.4) by $\widehat{u}$ and integrate over $\mathbb{R}^N$, which implies

$$\int_{\mathbb{R}^N} \nabla u_n \nabla \widehat{u} + \nu_1 u_n \widehat{u} = \int_{\mathbb{R}^N} \mu_1 (\frac{1}{|x|^\alpha} * u_n^2) u_n \widehat{u} + \beta_n (\frac{1}{|x|^\alpha} * v_n^2) u_n \widehat{u}. \tag{5.6}$$

We claim that

$$\int_{\mathbb{R}^N} (|x|^{-\alpha} * u_n^2) u_n \widehat{u} \to \int_{\mathbb{R}^N} (|x|^{-\alpha} * \widehat{u}^2) \widehat{u}^2, \quad \text{as } n \to \infty. \tag{5.7}$$

Note that

$$u_n^2 \to \widehat{u}^2 \text{ strongly in } L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N), \text{ as } n \to \infty.$$

By the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$. Then we obtain

$$|x|^{-\alpha} * u_n^2 \to |x|^{-\alpha} \widehat{u}^2 \quad \text{strongly in } L^{\frac{2N}{\alpha}}(\mathbb{R}^N), \text{ as } n \to \infty.$$

Combining this with the fact that

$$u_n \to \widehat{u} \quad \text{strongly in } L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N), \text{ as } n \to \infty,$$

we have

$$(|x|^{-\alpha} * u_n^2) u_n \to (|x|^{-\alpha} * \widehat{u}^2) \widehat{u} \quad \text{strongly in } L^{\frac{4N}{2N+\alpha}}(\mathbb{R}^N), \text{ as } n \to \infty.$$

Therefore, (5.7) holds. It follows from (4.1), (5.6) and (5.7) that

$$\int_\Omega (|\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2) \leq \int_{\mathbb{R}^N} \mu_1 (\frac{1}{|x|^\alpha} * \widehat{u}^2) \widehat{u}^2 \leq (4B_1)^{-1} \Big(\int_{\mathbb{R}^N} (|\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2) \Big)^2, \tag{5.8}$$

and so

$$\int_{\mathbb{R}^N} (|\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2) \geq 4B_1. \tag{5.9}$$

Similarly we have

$$\int_{\mathbb{R}^N} (|\nabla \widehat{v}|^2 + \nu_2 \widehat{v}^2) \geq 4B_2. \tag{5.10}$$

Combining these with lemma 4.1 we know that

$$\begin{aligned}
& B_1 + B_2 \\
& \geq \lim_{n\to\infty} A_{\beta_n} \\
& = \frac{1}{4} \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \nu_1 u_n^2 + |\nabla v_n|^2 + \nu_2 v_n^2 \\
& = \frac{1}{4} \int_{\mathbb{R}^N} |\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2 + \frac{1}{4} \int_{\mathbb{R}^N} |\nabla \widehat{v}|^2 + \nu_2 \widehat{v}^2 \\
& \quad + \frac{1}{4} \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla (u_n - \widehat{u})|^2 + \nu_1 |u_n - \widehat{u}|^2 + |\nabla (v_n - \widehat{v})|^2 + \nu_2 |v_n - \widehat{v}|^2 \\
& \geq B_1 + B_2 + \frac{1}{4} \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla (u_n - \widehat{u})|^2 + \nu_1 |u_n - \widehat{u}|^2 \\
& \quad + \frac{1}{4} \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla (v_n - \widehat{v})|^2 + \nu_2 |v_n - \widehat{v}|^2 \\
& \geq B_1 + B_2.
\end{aligned} \tag{5.11}$$

This means that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla (u_n - \widehat{u})|^2 + \nu_1 |u_n - \widehat{u}|^2 + |\nabla (v_n - \widehat{v})|^2 + \nu_2 |v_n - \widehat{v}|^2 = 0,$$

and so

$$(u_n, v_n) \to (\widehat{u}, \widehat{v}) \quad \text{strongly in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \text{ as } n \to +\infty.$$

Moreover, $\int_{\mathbb{R}^N}(|\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2) = 4B_1$, and so we see from (5.8) that

$$\int_{\mathbb{R}^N}(|\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2) = \int_{\mathbb{R}^N} \mu_1(\frac{1}{|x|^\alpha} * \widehat{u}^2)\widehat{u}^2 = 4B_1.$$

Therefore, $\widehat{u}$ is a positive least energy solution of

$$-\Delta u + \nu_1 u = \mu_1(\frac{1}{|x|^\alpha} * u^2)u, \quad u \in H^1(\mathbb{R}^N).$$

Similarly, we know that $\widehat{v}$ is a positive least energy solution of

$$-\Delta v + \nu_2 v = \mu_2(\frac{1}{|x|^\alpha} * v^2)v, \quad v \in H^1(\mathbb{R}^N).$$

The proof is complete □

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Song You
Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
Email address: `yousong@mail.tsinghua.edu.cn`

Peihao Zhao
School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, China
Email address: `zhaoph@lzu.edu.cn`

Qingxuan Wang
Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China
Email address: `wangqx@zjnu.edu.cn`