Electronic Journal of Differential Equations, Vol. 2006(2006), No. 40, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ON THE ψ-DICHOTOMY FOR HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

PHAM NGOC BOI

Abstract. In this article we present some conditions for the ψ-dichotomy of the homogeneous linear differential equation $x' = A(t)x$. Under our condition every ψ-integrally bounded function f the nonhomogeneous linear differential equation $x' = A(t)x + f(t)$ has at least one ψ-bounded solution on $(0, +\infty)$.

1. Introduction

The problem of solutions being ψ-bounded and ψ-stable for systems of ordinary differential equations has been studied by many authors; see for example Akinyele [1], Avramescu [2], Constantin [3]. In particular, Diamandescu [6, 7] presented some necessary and sufficient conditions for existence of a ψ-bounded solution to the linear nonhomogeneous system $x' = A(t)x + f(t)$.

Denote by $\mathbb{R}^d$ the d-dimensional Euclidean space. Elements in this space are denoted by $x = (x_1, x_2, \dots, x_d)^T$ and their norm by $\|x\| = \max\{|x_1|, |x_2|, \dots, |x_d|\}$. For real $d \times d$ matrices, we define norm $|A| = \sup_{\|x\|\leqslant 1} \|Ax\|$. Let $\mathbb{R}_+ = [0, +\infty)$ and $\psi_i : \mathbb{R}_+ \to (0, \infty)$, $i = 1, 2, \dots, d$ be continuous functions. Set

$$\psi = \operatorname{diag}[\psi_1, \psi_2, \dots, \psi_d].$$

Definition 1.1 ([6]). A function $f : \mathbb{R}_+ \to \mathbb{R}^d$ is said to be

- ψ-bounded on $\mathbb{R}_+$ if $\psi(t)f(t)$ is bounded on $\mathbb{R}_+$.
- ψ-integrable on $\mathbb{R}_+$ if $f(t)$ is measurable and $\psi(t)f(t)$ is Lebesgue integrable on $\mathbb{R}_+$.

In $\mathbb{R}^d$, consider the following equations

$$x' = A(t)x + f(t) \tag{1.1}$$

$$x' = A(t)x \tag{1.2}$$

where $A(t)$ is continuous matrix on $\mathbb{R}_+$.

By solution of (1.1), (1.2), we mean an absolutely continuous function satisfying the system for all $t \in \mathbb{R}_+$. Let $Y(t)$ be fundamental matrix of (1.2) with $Y(0) = I_d$, the identity $d \times d$ matrix. By X_1 denote the subspace of $\mathbb{R}^d$ consisting of the initial values of all ψ-bounded solutions of equation (1.2) and let X_2 be the closed subspace

2000 *Mathematics Subject Classification.* 34A12, 34C11, 34D05.
Key words and phrases. ψ-bounded, ψ-integrable, ψ-integrally bounded, ψ-exponential.

Submitted December 18, 2005. Published March 26, 2006.

of $\mathbb{R}^d$, supplementary to X_1. Also let P_1, P_2 denote the corresponding projections of $\mathbb{R}^d$ on to X_1, X_2.

Definition 1.2. The equation (1.2) is said to has a ψ-exponential dichotomy if there exist positive constants K, L, α, β such that

$$|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)| \leqslant Ke^{-\alpha(t-s)} \quad for \quad 0 \leqslant s \leqslant t, \tag{1.3}$$

$$|\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)| \leqslant Ke^{\beta(t-s)} \quad for \quad 0 \leqslant t \leqslant s. \tag{1.4}$$

The equation (1.2) is said to be has a ψ-ordinary dichotomy if (1.3), (1.4) hold with $\alpha = \beta = 0$.

We say that (1.2) has ψ-bounded grow if for some fixed $h > 0$ there exists a constant $C \geqslant 1$ such that every solution $x(t)$ of (1.2) is satisfied

$$\|\psi(t)x(t)\| \leqslant C\|\psi(s)x(s)\| \text{ for } 0 \leqslant s \leqslant t \leqslant s+h. \tag{1.5}$$

Remark 1.3. For $\psi_i = 1$, $i = 1, 2, \dots, d$, we obtain the notion exponential and ordinary dichotomy [4, 5].

Diamandescu proved the following results.

Theorem 1.4 ([6]). *The equation* (1.1) *has at least one ψ-bounded solution on $\mathbb{R}_+$ for every ψ-integrable function f on $\mathbb{R}_+$ if and only if* (1.2) *has a ψ-ordinary dichotomy.*

Theorem 1.5 ([8]). *Let*

$$|\psi(t)A(t)\psi^{-1}(t)| \leqslant M \quad \text{for all } t \geqslant 0,$$

$$|\psi(t)\psi^{-1}(s)| \leqslant L \quad \text{for } 0 \leqslant s \leqslant t.$$

Then (1.1) *has at least one ψ-bounded solution on $\mathbb{R}_+$ for every ψ-bounded function f on $\mathbb{R}_+$ if and only if* (1.2) *has ψ-exponential dichotomy.*

In this paper we prove some condition of the ψ-dichotomy for a homogeneous linear differential equations and we concerted that with the preceding results. Finally, it is noted that the concept of ψ-dichotomy for linear differential equations remain valid in Banach spaces. In this case we need a few changes for the definition of ψ. It seems to us that the majority of the results of this paper remain true for Banach spaces.

2. Preliminaries

lemma 2.1. *The equation* (1.2) *has a ψ-exponential dichotomy if there exist positive constants K', L', T, α, β such that*

$$|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)| \leqslant K'e^{-\alpha(t-s)}, \quad for\ T \leqslant s \leqslant t \tag{2.1}$$

$$|\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)| \leqslant L'e^{\beta(t-s)}, \quad for\ T \leqslant s \leqslant t. \tag{2.2}$$

Proof. We will show that (1.3) holds. Using a lemma of Coppel [4],

$$|Y^{-1}(s)| \leqslant (2^d - 1)\frac{|Y(s)|^{d-1}}{|detY(s)|}.$$

On the other hand $Y(s)$ is continuous, we deduce $|Y^{-1}(s)| \leqslant N_1 < +\infty$ for $0 \leqslant s \leqslant T$. It follows from the continuity of $\psi(t)$, $\psi^{-1}(t)$, $Y(t)$, that $|\psi(t)|$, $|\psi^{-1}(t)|$, $|Y(t)|$are bounded on $[0,T]$. Thus $|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)| \leqslant N < +\infty$ for $0 \leqslant s \leqslant T$, $0 \leqslant t \leqslant T$. If $0 \leqslant s \leqslant T \leqslant t$, then

$$\begin{aligned}
&|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)| \\
&\leqslant |\psi(t)Y(t)P_1Y^{-1}(T)\psi^{-1}(T)||\psi(T)Y(T)Y^{-1}(s)\psi^{-1}(s)| \\
&\leqslant N|\psi(t)Y(t)P_1Y^{-1}(T)\psi^{-1}(T)| \\
&\leqslant NK'e^{-\alpha(t-T)} \leqslant NK'e^{\alpha T}e^{-\alpha(t-s)}.
\end{aligned}$$

If $0 \leqslant s \leqslant t \leqslant T$, then

$$\begin{aligned}
&|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)| \\
&\leqslant |\psi(t)Y(t)Y^{-1}(T)\psi^{-1}(T)||\psi(T)Y(T)P_1Y^{-1}(T)\psi^{-1}(T)| \\
&\quad |\psi(T)Y(T)Y^{-1}(s)\psi^{-1}(s)| \\
&\leqslant N^2K' \leqslant N^2K'e^{\alpha T}e^{-\alpha(t-s)}.
\end{aligned}$$

Thus the inequality (1.3) holds for $K = \max\{K', NK'e^{\alpha T}, N^2K'e^{\alpha T}\}$. Similarly, inequality (1.4) holds for $L = \max\{L', NL'e^{\alpha T}, N^2L'e^{\alpha T}\}$. □

lemma 2.2. *Equation* (1.2) *has a ψ-exponential dichotomy if only if following statements are satisfied*

$$\|\psi(t)Y(t)P_1\xi\| \leqslant K'e^{-\alpha(t-s)}\|\psi(s)Y(s)P_1\xi\|, \quad \textit{for all } \xi \in \mathbb{R}^d \textit{ and } t \geqslant s \geqslant 0 \quad (2.3)$$

$$\|\psi(t)Y(t)P_2\xi\| \leqslant L'e^{\beta(t-s)}\|\psi(s)Y(s)P_2\xi\|, \quad \textit{for all } \xi \in \mathbb{R}^d \textit{ and } s \geqslant t \geqslant 0 \quad (2.4)$$

$$|\psi(t)Y(t)P_1Y^{-1}(t)\psi^{-1}(t)| \leqslant M \quad \textit{for } t \geqslant 0 \quad (2.5)$$

where K', L', M are positive constants.

Proof. If (1.2) has a ψ-exponential dichotomy then for any vector $y \in \mathbb{R}^d$, we get

$$\|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)y\| \leqslant Ke^{-\alpha(t-s)}\|y\| \text{ for } 0 \leqslant s \leqslant t.$$

Choose $y = \psi(s)Y(s)P_1\xi$, we obtain (2.3). The proof of (2.2) is similar. Inequality (2.5) evidently holds. Conversely, if inequality (2.3), (2.4), (2.5) are true. For any vector $y \in \mathbb{R}^d$, putting $\xi = Y^{-1}(s)\psi^{-1}(s)y$ we get

$$\begin{aligned}
\|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)y\| &\leqslant K'e^{-\alpha(t-s)}\|\psi(s)Y(s)P_1Y^{-1}(s)\psi^{-1}(s)y\| \\
&\leqslant MK'e^{-\alpha(t-s)}\|y\| \quad \text{for } t \geqslant s \geqslant 0.
\end{aligned}$$

Thus, we have (1.3). The proof of (1.4) is similar. □

Remark 2.3. *By Lemma 2.1 and in the same way as in the proof of Lemma 2.2, we can show that* (1.2) *has ψ-exponential dichotomy if there exists positive constant Q such that*

$$\|\psi(t)Y(t)P_1\xi\| \leqslant K'e^{-\alpha(t-s)}\|\psi(s)Y(s)P_1\xi\|, \quad \textit{for all } \xi \in \mathbb{R}^d \textit{ and } t \geqslant s \geqslant Q, \quad (2.6)$$

$$\|\psi(t)Y(t)P_2\xi\| \leqslant L'e^{\beta(t-s)}\|\psi(s)Y(s)P_2\xi\|, \quad \textit{for all } \xi \in \mathbb{R}^d \textit{ and } s \geqslant t \geqslant Q, \quad (2.7)$$

$$|\psi(t)Y(t)P_1Y^{-1}(t)\psi^{-1}(t)| \leqslant M \quad \textit{for } t \geqslant Q. \quad (2.8)$$

lemma 2.4. *Equation* (1.2) *has ψ-bounded grow if and only if there exist positive constants K, γ such that*

$$|\psi(t)Y(t)Y^{-1}(s)\psi^{-1}(s)| \leqslant Ke^{\gamma(t-s)}, \quad \textit{for } t \geqslant s \geqslant 0. \quad (2.9)$$

Proof. Suppose that (1.2) has a ψ-bounded grow. For arbitrary vector $\xi \in \mathbb{R}^d$, we consider the solution $x(t)$ of (1.2), with $x(0) = Y^{-1}(s)\psi^{-1}(s)\xi$. Setting $n = [\frac{t-s}{h}]$, we get

$$\begin{aligned}\|\psi(t)x(t)\| &= \|\psi(nh+s)x(nh+s)\| \\ &\leqslant C\|\psi(nh+s-h)x(nh+s-h)\| \\ &\leqslant \dots \leqslant C^n\|\psi(s)x(s)\| \\ &\leqslant C^{\frac{t-s}{h}}\|\psi(s)x(s)\| \text{ for } 0 \leqslant s \leqslant t.\end{aligned}$$

Set $K = C$, $\gamma = h^{-1}\ln C$, we obtain

$$\|\psi(t)x(t)\| \leqslant Ke^{\gamma(t-s)}\|\psi(s)x(s)\|.$$

Therefore, $\|\psi(t)Y(t)Y^{-1}(t)\psi^{-1}(s)\xi\| \leqslant Ke^{\gamma(t-s)}|\xi\|$. It follows (2.9).

Conversely, if (2.9) is true, then we can take $C = Ke^{\gamma h}$. Thus (1.5) is satisfied. □

Remark 2.5. The preceding proof shows that the condition of ψ-bounded grow of (1.2) is independent of the choice of h.

3. The main results

Theorem 3.1. *If (1.2) has a ψ-exponential dichotomy, then for any $0 < \theta < 1$ there exists constants $T > 0$ such that every solution $x(t)$ of (1.2) satisfies*

$$\|\psi(t)x(t)\| \leqslant \theta \sup_{\|s-t\| \leqslant T} \|\psi(s)x(s)\| \quad \textit{for all } t \geqslant T. \tag{3.1}$$

Proof. Set $x_1(t) = Y(t)P_1Y^{-1}(t)x(t)$, $x_2(t) = Y(t)P_2Y^{-1}(t)x(t)$. Suppose that

$$\|\psi(s)x_2(s)\| \geqslant \|\psi(s)x_1(s)\|.$$

It follows from (2.3) that

$$\|\psi(s)x_1(s)\| \leqslant K'e^{-\alpha(t-s)}\|\psi(s)x_1(s)\| \leqslant K'e^{-\alpha(t-s)}\|\psi(s)x_2(s)\| \quad \text{for } 0 \leqslant s \leqslant t.$$

Applying (2.4) for $\xi = Y^{-1}(s)x_2(s)$,

$$\begin{aligned}\|\psi(t)x_2(t)\| &= \|\psi(t)Y(t)P_2Y^{-1}(s)x_2(s)\| \\ &\geqslant {L'}^{-1}e^{\beta(t-s)}\|\psi(s)Y(s)P_2Y^{-1}(s)x_2(s)\| \quad \text{for } 0 \leqslant s \leqslant t.\end{aligned}$$

Note that $x_2(t) = Y(t)P_2Y^{-1}(t)x_2(t)$. Thus

$$\|\psi(t)x_2(t)\| \geqslant {L'}^{-1}e^{\beta(t-s)}\|\psi(s)x_2(s)\| \quad \text{for } 0 \leqslant s \leqslant t.$$

Therefore,

$$\|\psi(t)x(t)\| \geqslant \frac{1}{2}[{L'}^{-1}e^{\beta(t-s)} - K'e^{-\alpha(t-s)}]\|\psi(s)x(s)\| \quad \text{for } 0 \leqslant s \leqslant t.$$

Similarly, if $\|\psi(s)x_1(s)\| \geqslant \|\psi(s)x_2(s)\|$, then

$$\|\psi(t)x(t)\| \geqslant \frac{1}{2}[{K'}^{-1}e^{\alpha(t-s)} - L'e^{-\beta(t-s)}]\|\psi(s)x(s)\| \quad \text{for } 0 \leqslant t \leqslant s.$$

For any $0 < \theta < 1$ we can choose $T > 0$ large so that

$${L'}^{-1}e^{\beta T} - K'e^{-\alpha T} \geqslant 2\theta^{-1} \quad \text{and} \quad {K'}^{-1}e^{\alpha T} - L'e^{-\beta T} \geqslant 2\theta^{-1}.$$

Thus for $t \geqslant T$,

$$\|\psi(t)x(t)\| \leqslant \max\{\theta\|\psi(t+T)x(t+T)\|, \theta\|\psi(t-T)x(t-T)\|\}.$$

Then (3.1) is satisfied. □

Definition 3.2. The function $f : \mathbb{R}_+ \to \mathbb{R}^d$ is said to be ψ-integrally bounded if it is measurable and Lebesgue integrals $\int_t^{t+1} \|\psi(u)f(u)\|du$ are uniformly bounded for any $t \in \mathbb{R}_+$.

Theorem 3.3. *Equation* (1.1) *has at least one ψ-bounded solution on $\mathbb{R}_+$ for every ψ-integrally bounded function f if and only if* (1.2) *has a ψ-exponential dichotomy.*

Proof. First we prove the "if" part. Suppose that (1.2) has a ψ-exponential dichotomy. Consider the function

$$\begin{aligned}\tilde{x}(t) &= \int_0^t \psi(t)Y(t)P_1Y^{-1}(s)f(s)ds - \int_t^\infty \psi(t)Y(t)P_2Y^{-1}(s)f(s)ds \\ &= \int_0^t \psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)\psi(s)f(s)ds \\ &\quad - \int_t^\infty \psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)\psi(s)f(s)ds\end{aligned}$$

for $t \geqslant 0$. The function $\tilde{x}(t)$ is bounded. In fact, suppose that

$$\int_t^{t+1} \|\psi(s)f(s)\|ds \leqslant c \quad \text{for } t \geqslant 0.$$

Then

$$\begin{aligned}\int_0^t e^{-\alpha(t-s)}\|\psi(s)f(s)\|ds &\leqslant c(1-e^{-\alpha})^{-1}, \\ \int_0^\infty e^{\beta(t-s)}\|\psi(s)f(s)\|ds &\leqslant c(1-e^{-\beta})^{-1},\end{aligned}$$

by using a Lemma in Massera and Schaffer. Set

$$x(t) = \psi^{-1}(t)\tilde{x}(t) = \int_0^t Y(t)P_1Y^{-1}(s)f(s)ds - \int_t^\infty Y(t)P_2Y^{-1}(s)f(s)ds.$$

Then $x(t)$ is the ψ-bounded and continuous function on $\mathbb{R}_+$.

$$\begin{aligned}x'(t) &= A(t)\Big[\int_0^t Y(t)P_1Y^{-1}(s)f(s)ds - \int_t^\infty Y(t)P_2Y^{-1}(s)f(s)ds\Big] \\ &\quad + Y(t)P_1Y^{-1}(t)f(t) + Y(t)P_2Y^{-1}(t)f(t) \\ &= A(t)x(t) + f(t).\end{aligned}$$

It follows that $x(t)$ is a solution of (1.1).

Now, we prove the "only part". We define the set

$$C_\psi = \{x : \mathbb{R}_+ \to \mathbb{R}^d; x \text{ is } \psi\text{-bounded and continuous on } \mathbb{R}_+\}.$$

It is well-known that C_ψ is real Banach space with the norm

$$\|x\|_{C_\psi} = \sup_{t \geqslant 0} \|\psi(t)x(t)\|.$$

First we show that (1.1) has a unique ψ-bounded solution $x(t)$ with $x(0) \in X_2$ for each $f \in C_\psi$. Further, there exists a positive constant r independent of f such that

$$\|x\|_{c_\psi} \leqslant r\|f\|_{c_\psi}. \tag{3.2}$$

We prove the existence. Suppose $f \in C_\psi$. By hypothesis, there exists a ψ-bounded solution $x(t)$ of (1.1). We denote by $y(t)$ the solution of the Cauchy problem

$$y' = A(t)y; \qquad y(0) = -P_1x(0).$$

This solution $y(t)$ is ψ-bounded by definition of the subset X_1. But then $z = x + y$ is a ψ-bounded solution of (1.1) for which

$$P_1z(0) = P_1x(0) - P_1^2x(0) = 0.$$

Thus $z(0) \in X_2$. Hence $z(t)$ is a ψ-bounded solution of (1.1) with $z(0) \in X_2$.

We prove the uniqueness. Let $x(t)$ and $y(t)$ be the ψ-bounded solutions of equation (1.1) with $x(0) \in X_2, y(0) \in X_2$. Hence $x - y$ is a ψ-bounded of (1.2) and $x(0) - y(0) \in X_2$. But $x(0) - y(0) \in X_1$. we obtain $x(0) = y(0)$, hence $x = y$.

We prove the inequality (3.2) Consider the map $T : c_\psi \to c_\psi$ which is defined $Tf = x$, where x is the ψ-bounded solution of (1.1) with $x(0) \in X_2$. We will show that T is continuous. Suppose that $x_n = Tf_n$, $f_n \to f$ and $x_n \to x$. For any fixed t, we have

$$\begin{aligned}\lim_{n\to\infty} \| \int_0^t [f_n(s) - f(s)]ds\| &\leqslant \lim_{n\to\infty} \int_0^t |\psi^{-1}(s)|\|\psi(s)f_n(s) - \psi(s)f(s)\|ds \\ &\leqslant \lim_{n\to\infty} \|f_n - f\|_{C_\psi} \int_0^t |\psi^{-1}(s)|ds = 0.\end{aligned} \tag{3.3}$$

On the other hand

$$\begin{aligned}&\lim_{n\to\infty} \| \int_0^t A(s)[x_n(s) - x(s)]ds\| \\ &\leqslant \lim_{n\to\infty} \int_0^t |A(s)\psi^{-1}(s)|\|\psi(s)x_n(s) - \psi(s)x(s)\|ds \\ &\leqslant \lim_{n\to\infty} \|x_n - x\|_{C_\psi} \int_0^t |A(s)\psi^{-1}(s)|ds = 0.\end{aligned} \tag{3.4}$$

From (3.3) and (3.4) we obtain

$$\begin{aligned}x(t) - x(0) &= \lim_{n\to\infty} (x_n(t) - x_n(0)) \\ &= \lim_{n\to\infty} \int_0^t [A(s)x_n(s) + x_n'(t) - A(s)x_n(s)]ds \\ &= \lim_{n\to\infty} \int_0^t [A(s)x_n(s) + f_n(s)]ds \int_0^t [A(s)x(s) + f(s)]ds.\end{aligned}$$

Thus $x(t)$ is a solution of (1.1). Since $x(t)$ is ψ-bounded and

$$x(0) = \lim_{n\to\infty} x_n(0) \in X_2$$

we have $x = Tf$. It follows from the Closed Graph Theorem that the linear map T is continuous. Hence (3.2) is proved. Now, put

$$G(t,s) = \begin{cases} Y(t)P_1Y^{-1}(s) & \text{for} 0 \leqslant s \leqslant t \\ -Y(t)P_2Y^{-1}(s) & \text{for} 0 \leqslant t \leqslant s. \end{cases}$$

If $\tilde{f} \in C_\psi$, $\tilde{f}(t) = 0$ for $t > t_1 > 0$, then

$$\tilde{x}(t) = \int_0^{t_1} G(t,s)\tilde{f}(s)ds \tag{3.5}$$

is a solution of (1.1). Moreover $\tilde{x} \in C_\psi$, since

$$\psi(t)\tilde{x}(t) = \int_0^{t_1} \psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)\psi(s)\tilde{f}(s)ds \quad \text{for } t \geqslant t_1.$$

On the other hand, $\tilde{x}(0) = -P_2\int_0^{t_1} Y^{-1}(s)\tilde{f}(s)ds \in X_2$. Thus

$$\|\tilde{x}\|_{c_\psi} \leqslant r\|\tilde{f}\|_{c_\psi}. \tag{3.6}$$

Let x is an nontrivial solution of (1.2) and let $\alpha(t)$ be any continuous real-valued function such that $0 \leqslant \alpha(t) \leqslant 1$ for all $t \geqslant 0$, $\alpha(t) = 0$ for $t \geqslant t_2$, $\alpha(t) = 1$ for $0 \leqslant t_0 \leqslant t \leqslant t_1 \leqslant t_2$. Set

$$\tilde{f}(t) = \alpha(t)x(t)\|\psi(t)x(t)\|^{-1}.$$

Then $\tilde{f} \in C_\psi$. From (3.5) and (3.6), we have

$$\|\int_{t_0}^{t_1} \psi(t)G(t,s)x(s)\|\psi(s)x(s)\|^{-1}ds\|_{c_\psi} = r \quad \text{for } t_1 \geqslant t_0 \geqslant 0. \tag{3.7}$$

By continuity, (3.7) remains true also in the case $t = s$. Choose $x(0) = P_1\xi, \xi \in \mathbb{R}^d$. By the arbitrary of t_1, from (3.7) we get

$$\|\psi(t)Y(t)P_1\xi\|\int_{t_0}^{t} \|\psi(u)Y(u)P_1\xi\|^{-1}du \leqslant r \quad \text{for } t \geqslant t_0 \geqslant 0.$$

Choose $x(0) = P_2\xi, \xi \in \mathbb{R}^d$. By the arbitrary of t_0, from (3.7) we get

$$\|\psi(t)Y(t)P_2\xi\|\int_{t}^{t_1} \|\psi(u)Y(u)P_2\xi\|^{-1}du \leqslant r \quad \text{for } 0 \leqslant t \leqslant t_1.$$

Next, putting $x_1(t) = Y(t)P_1Y^{-1}(s)x(s) = Y(t)P_1\xi$, we have

$$\|\psi(t)x_1(t)\|\int_{t_0}^{t} \|\psi(u)x_1(u)\|^{-1}du \leqslant r \quad \text{for } t \geqslant t_0 \geqslant 0. \tag{3.8}$$

Also putting $x_2(t) = Y(t)P_2Y^{-1}(s)x(s) = Y(t)P_2\xi$, we get

$$\|\psi(t)x_2(t)\|\int_{t}^{t_1} \|\psi(u)x_2(u)\|^{-1}du \leqslant r \quad \text{for } t_1 \geqslant t \geqslant 0. \tag{3.9}$$

It follows by integration that

$$\int_{t_0}^{s} \|\psi(u)x_1(u)\|^{-1}du \leqslant e^{-r^{-1}(t-s)}\int_{t_0}^{t} \|\psi(u)x_1(u)\|^{-1}du \quad \text{for } t \geqslant s \geqslant t_0. \tag{3.10}$$

$$\int_{s}^{t_1} \|\psi(u)x_2(u)\|^{-1}du \leqslant e^{r^{-1}(s-t)}\int_{t}^{t_1} \|\psi(u)x_2(u)\|^{-1}du \quad \text{for } t_1 \geqslant s \geqslant t. \tag{3.11}$$

Because a ψ-integrable function is ψ-locally integrable, by Theorem 1.4 there exists a positive constant K such that

$$\|\psi(t)x_1(t)\| \leqslant K\|\psi(s)x(s)\| \quad \text{for } 0 \leqslant s \leqslant t, \tag{3.12}$$

$$\|\psi(t)x_2(t)\| \leqslant K\|\psi(s)x(s)\| \quad \text{for } 0 \leqslant t \leqslant s. \tag{3.13}$$

Thus

$$rK^{-1}\|\psi(s)x(s)\|^{-1} \leqslant \int_{s}^{r+s} \|\psi(u)x_1(u)\|^{-1}du \quad \text{for } s \geqslant 0.$$

Using (3.10), replacing t_0 by s, s by $s+r$ we deduce

$$\begin{aligned}\int_s^{r+s} \|\psi(u)x_1(u)\|^{-1}du &\leqslant e^{-r^{-1}(t-r-s)} \int_s^t \|\psi(u)x_1(u)\|^{-1}du \\ &\leqslant ee^{-r^{-1}(t-s)} \int_s^t \|\psi(u)x_1(u)\|^{-1}du \quad \text{for } t \geqslant s+r.\end{aligned}$$

Hence

$$r\Big(\int_t^s \|\psi(u)x_1(u)\|^{-1}du\Big)^{-1} \leqslant eK\|\psi(s)x(s)\|e^{-r^{-1}(t-s)} \quad \text{for } t \geqslant s+r.$$

From (3.8), replacing t_0 by s, s by $s+r$, we get

$$\|\psi(t)x_1(t)\| \leqslant eK\|\psi(s)x(s)\|e^{-r^{-1}(t-s)} \text{ for } t \geqslant s+r.$$

It is easy to see that the inequality holds also for $s \leqslant t \leqslant s+r$. Since $x_1(t) = Y(t)P_1Y^{-1}(s)x(s)$, it follows that

$$\|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)\| \leqslant K'e^{-\alpha(t-s)} \quad \text{for } t \geqslant s \geqslant 0$$

where $K' = eK$, $\alpha = r^{-1}$. By the same way, using (3.9), (3.11), (3.13), we get

$$\|\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)\| \leqslant K'e^{\alpha(s-t)} \quad \text{for } s \geqslant t \geqslant 0.$$

The proof is complete. □

Now, we are going to show some conditions for (1.2) has a ψ-exponential dichotomy in the case it has ψ-bounded grow.

Theorem 3.4. *Suppose that* (1.2) *has ψ-bounded grow. Equation* (1.2) *has a ψ-exponential dichotomy if there exists constants $T > 0$, $0 < \theta < 1$ such that every solution of* (1.2) *satisfies* (3.1).

Proof. By Remark 2.3, we shall show that (2.6), (2.7), (2.8) are satisfied for some $Q > 0$. We may consider $x(t)$ is nontrivial solution of (1.2). The first we prove that every solution $x(t)$ of (1.2) with $x(0) \in X_1$ satisfies

$$\|\psi(t)x(t)\| \leqslant Ke^{-\alpha(t-s)}\|\psi(s)x(s)\| \quad \text{for } 0 \leqslant s \leqslant t.$$

By Remark 2.5 we can choose $h = T$, so that

$$\|\psi(t)x(t)\| \leqslant C\|\psi(s)x(s)\| \quad \text{for } 0 \leqslant s \leqslant t \leqslant s+T. \tag{3.14}$$

Hence $\|\psi(t)x(t)\| \leqslant \theta \sup_{u\geqslant s} \|\psi(u)x(u)\|$ for $s \geqslant 0$, $t \geqslant s+T$. Therefore,

$$\sup_{u\geqslant s} \|\psi(u)x(u)\| > \|\psi(t)x(t)\|$$

for $t \geqslant s+T$. It follow that

$$\sup_{u\geqslant s} \|\psi(u)x(u)\| = \sup_{s\leqslant\tau\leqslant s+T} \|\psi(\tau)x(\tau)\|. \tag{3.15}$$

Hence (3.14) and (3.15) yield $\|\psi(t)x(t)\| \leqslant C\|\psi(s)x(s)\|$ for $0 \leqslant s \leqslant t$. Set $n = [\frac{t-s}{T}]$ then

$$\begin{aligned}
&\|\psi(t)x(t)\| \\
&\leqslant \theta \sup_{\|u-t\|\leqslant T} \|\psi(u)x(u)\| \\
&\leqslant \theta \sup_{\|u-t\|\leqslant T} \{\theta \sup_{\|u-v\|\leqslant T} \|\psi(v)x(v)\|\} \leqslant \theta^2 \sup_{\|v-t\|\leqslant 2T} \|\psi(v)x(v)\| \\
&\leqslant \theta^n \sup_{\|v-t\|\leqslant nT} \|\psi(v)x(v)\| \; leqslant \theta^n C\|\psi(s)x(s)\| \leqslant \theta^{-1}C\theta^{\frac{t-s}{T}}\|\psi(s)x(s)\|.
\end{aligned}$$

Put $K = \theta^{-1}C > 1$, $\alpha = -T^{-1}ln\theta > 0$, we get

$$\|\psi(t)x(t)\| \leqslant Ke^{-\alpha(t-s)}\|\psi(s)x(s)\| \quad \text{for } 0 \leqslant s \leqslant t.$$

Now, for each $\xi \in \mathbb{R}^d$, consider the solution $x(t)$ of the equation (1) with $x(0) = P_1\xi$. Apply this inequality we deduce (2.6) for any $Q \geqslant 0$.

Now, suppose that $x(t)$ is any solution $x(t)$ of (1.2) with $x(0) \in X_2$. May be consider $\|\psi(0)x(0)\| = 1$. We can define sequence $t_n \to +\infty$ by

$$\|\psi(t_n)x(t_n)\| = \theta^{-n}C, \quad \|\psi(t)x(t)\| < \theta^{-n}C \quad \text{for } 0 \leqslant t \leqslant t_n.$$

Since $\|\psi(t)x(t)\| \leqslant C$ for $0 \leqslant t \leqslant T$ and $\|\psi(t_1)x(t_1)\| = C\theta^{-1} > C$ we get $T < t_1$. Consequently,

$$T < t_1 < t_2 < \cdots < t_n < \dots.$$

From

$$\|\psi(t_n)x(t_n)\| \leqslant \theta \sup_{0\leqslant u\leqslant t_n+T} \|\psi(u)x(u)\|$$

and

$$\|\psi(u)x(u)\| \leqslant \theta^{-1}\|\psi(t_n)x(t_n)\| \quad \text{for } 0 \leqslant u \leqslant t_n$$

we get $t_{n+1} < t_n + T$. Suppose that $0 \leqslant s \leqslant t$ and $t_m \leqslant t \leqslant t_{m+1}$, $t_n \leqslant s \leqslant t_{n+1}$ $(1 \leqslant m \leqslant n)$. Then

$$\begin{aligned}
\|\psi(t)x(t)\| < \theta^{-m-1}C \qquad\qquad &= \theta^{n-m}\|\psi(t_{n+1})x(t_{n+1})\| \\
&\leqslant C\theta^{-1}\theta^{n-m+1}\|\psi(s)x(s)\| \\
&\leqslant C\theta^{-1}\theta^{\frac{s-t}{T}}\|\psi(s)x(s)\|.
\end{aligned}$$

Thus $\|\psi(t)x(t)\| \leqslant Ke^{-\alpha(s-t)}\|\psi(s)x(s)\|$ for $t_1 \leqslant t \leqslant s$.

For any unit vector $\xi \in X_2$, let $x(t,\xi)$ be the solution of (1.2) with $\psi(0)x(0) = \xi$. Then $x(t,\xi)$ is unbounded, and hence there is a value $t = t_1(\xi)$ such that

$$\|\psi(t_1)x(t_1)\| = \theta^{-1}C.$$

We will show that the values $t_1(\xi)$ are bounded. In fact, otherwise there exists a sequence of unit vector $\xi_k \in X_2$ such that $t_1^k = t_1(\xi_k) \to +\infty$ as $k \to +\infty$. By the compactness of the unit sphere in X_2 we may suppose that $\xi_k \to \xi$ as $k \to +\infty$, where ξ is a unit vector. Then $x(t,\xi_k) \to x(t,\xi)$ for every $t \geqslant 0$. Since $\|\psi(t)x(t,\xi_k)\| < \theta^{-1}C$ for $0 \leqslant t \geqslant t_1^k$ and $t_1^k \to +\infty$ we get

$$\|\psi(t)x(t,\xi)\| \leqslant \theta^{-1}C \quad \text{for all } t \geqslant 0$$

which is a contradiction because $\xi \in X_2$. Thus there exists $Q > 0$ such that $t_1(\zeta)$ for all unit vector ζ and every solution $x(t)$ of equation (1.2) with $x(0) \in X_2$ satisfies

$$\|\psi(t)x(t)\| \leqslant Ke^{-\alpha(s-t)}\|\psi(s)x(s)\| \quad \text{for } Q \leqslant t \leqslant s.$$

Thus $|\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)| \leqslant Le^{\beta(t-s)}$, for $Q \leqslant t \leqslant s$. Thus (2.7) is proved. Note that (2.8) is proved in [8, Theorem 2.1, estimate (12)]. So the proof is cimplete. $\square$

From Theorem 3.1 and Theorem 3.4, we have the following result.

Corollary 3.5. *Suppose that* (1.2) *has ψ-bounded grow. Then equation* (1.2) *has a ψ-exponential dichotomy if and only if there exists constants $T > 0$, $0 < \theta < 1$ such that every solution of* (1.2) *is satisfied* (3.1).

Theorem 3.6. *Suppose that* (1.2) *has ψ-bounded grow. Then* (1.1) *has at least one ψ-bounded solution on $\mathbb{R}_+$ for every ψ-bounded function f on $\mathbb{R}_+$ if and only if* (1.2) *has ψ-exponential dichotomy.*

Proof. Diamandescu presented this Theorem. In the proof [8, Theorem 1.2], the author proved that $|\psi(t)A(t)\psi^{-1}(t)| \leqslant M$ for all $t \geqslant 0$ and $|\psi(t)\psi^{-1}(s)| \leqslant L$ for $t \geqslant s \geqslant 0$ deduce (2.9). Throughout the proof, he only used condition (2.9). By lemma 2.4, condition (2.9) is satisfied if and only if (1.2) has ψ-bounded grow. The proof is complete $\square$

Now, consider the perturbed equation

$$x'(t) = [A(t) + B(t)]x(t) \tag{3.16}$$

where $B(t)$ is a $d \times d$ continuous matrix function on $\mathbb{R}_+$. We have the following result.

Theorem 3.7. *(a) Suppose that* (1.2) *has a ψ-exponential dichotomy. If $\delta = \sup_{t\geqslant 0} |\psi(t)B(t)\psi^{-1}(t)|$ is sufficiently small, then* (3.16) *has a ψ-exponential dichotomy.*
(b) Suppose that (1.2) *has a ψ-exponential dichotomy or ψ-ordinary dichotomy. If $\int_0^\infty |\psi(t)B(t)\psi^{-1}(t)|dt < \infty$, then* (3.16) *has a ψ-ordinary dichotomy.*

Proof. (a) By Theorem 3.3 it suffices to show that the equation

$$x'(t) = [A(t) + B(t)]x(t) + f(t) \tag{3.17}$$

has at least a ψ-bounded solution for every ψ-integrally bounded f function. Denote $Y(t)$, P_1, P_2 as in the proof of the Theorem 3.3.

Consider the map $T : C_\psi \to C_\psi$ which is defined by

$$\begin{aligned} Tz(t) = &\int_0^t Y(t)P_1Y^{-1}(s)[B(s)z(s) + f(s)]ds \\ &- \int_t^\infty Y(t)P_2Y^{-1}(s)[B(s)z(s) + f(s)]ds. \end{aligned}$$

It is easy verified that $Tz \in C_\psi$. More ever if $z_1, z_2 \in C_\psi$ then

$$\begin{aligned}
&\|Tz_1 - Tz_2\| \\
&\leqslant \int_0^t |\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)||\psi(s)B(s)\psi^{-1}(s)|\|\psi(s)z_1(s) - \psi(s)z_2(s)\|ds \\
&\quad + \int_t^\infty |\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)||\psi(s)B(s)\psi^{-1}(s)|\|\psi(s)z_1(s) - \psi(s)z_2(s)\|ds \\
&\leqslant K\delta\|z_1 - z_2\|_{C_\psi} \int_0^t e^{-\alpha(t-s)}ds + L\delta\|z_1 - z_2\|_{C_\psi} \int_t^\infty e^{\beta(t-s)}ds \\
&\leqslant \delta(K\alpha^{-1} + L\beta^{-1})\|z_1 - z_2\|_{C_\psi}.
\end{aligned}$$

Hence, by the contraction principle, if $\delta(K\alpha^{-1} + L\beta^{-1}) < 1$, then the mapping T has a unique fixed point. Denoting this fixed point by z, we have

$$z(t) = \int_0^t Y(t)P_1Y^{-1}(s)[B(s)z(s)+f(s)]ds - \int_t^\infty Y(t)P_2Y^{-1}(s)[B(s)z(s)+f(s)]ds.$$

It follows that $z(t)$ is a solution of (3.17).
(b) We can assume that (1.2) has a ψ-ordinary dichotomy. By Theorem 1.4 it suffices to show that (3.17) has at least a ψ - bounded solution for every ψ-integrable f. From $\int_0^\infty |\psi(t)B(t)\psi^{-1}(t)|dt < \infty$, it follows that

$$k = K \int_T^\infty |\psi(t)B(t)\psi^{-1}(t)|dt < 1$$

for a sufficiently large and positive T. Let $C_{T,\psi}$ be the Banach space of all ψ - bounded and continuous functions $z(t)$ on $[T, \infty)$ equipped with the norm

$$\|z\|_{C_{T,\psi}} = \sup_{t \geqslant T} \|\psi(t)z(t)\|.$$

Consider the map $T : C_{T,\psi} \to C_{T,\psi}$ which is defined by

$$\begin{aligned}
&Tz(t) \\
&= \int_T^t Y(t)P_1Y^{-1}(s)[B(s)z(s) + f(s)]ds - \int_t^\infty Y(t)P_2Y^{-1}(s)[B(s)z(s) + f(s)]ds.
\end{aligned}$$

It is easy to check that $Tz \in C_{T,\psi}$. Moreover if $z_1, z_2 \in C_{T,\psi}$ then

$$\begin{aligned}
\|Tz_1 - Tz_2\|_{C_{T,\psi}} &\leqslant K \int_T^\infty |\psi(s)B(s)\psi^{-1}(s)|\|\psi(s)z_1(s) - \psi(s)z_2(s)\|ds \\
&\leqslant k\|z_1 - z_2\|_{C_{T,\psi}}.
\end{aligned}$$

It follows from the contraction principle that the equation $Tz = z$ has a unique solution $\tilde{z} \in C_{T,\psi}$. Denote by y the solution of (3.16), which is extension of $\tilde{z}$ on $\mathbb{R}_+$. Clearly y is a ψ - bounded solution of (3.16). The proof is complete. □

We remark that (1.2) has a ψ-ordinary dichotomy with $P_1 = I_d$ if and only if it is ψ-uniformly stable. Theorem 3.7 follows [7, Theorem 3.4].

Acknowledgment. The author wants to thank the anonymous referee for his critical review of the original manuscript and for the changes suggested. The author also wants thank Professor Diamandescu, at University of Craiova, for his suggestions.

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Pham Ngoc Boi

Department of Mathematics, Vinh University, Vinh City, Vietnam

E-mail address: pnboi_vn@yahoo.com