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Stability of strong detonation waves and rates of convergence *

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Abstract

In this article, we prove stability of strong detonation waves and find their rate of convergence for a combustion model. Our results read as follows: I) There exists a global solution that converges exponentially in time to a strong detonation wave, provided that the initial data is a small perturbation of a strong detonation wave that decays exponentially in $|x|$. II) When the initial perturbation decays algebraically in $|x|$, the solution converges algebraically in time. That is, the perturbation decays in t as 'fast' as the initial perturbation decays in $|x|$.

1 Introduction

Physical experimentation has shown that in a sufficiently insensitive mixture or in a typical condensed phase, explosive detonation waves approach a steady state as time goes by. The study of this steady state is a subject in explosive engineering and is based on measurements of pressure, velocity and other observables of detonation waves. To learn about the structure and the behavior of the steady state, we formulate questions such as: How does a detonation wave respond to a perturbation? How quickly is the steady state is attained? And what are the details of the flow as the steady solution is approached? In particular, hydrodynamic stability of the steady detonation is very interesting question, and has received a lot of attention. Fickett [3] studied the decay of small planar perturbations for strong steady detonation in a simple model. His work uses the linearization technique of hydrodynamic stability theory introduced by Erpenbeck [2]. Liu and Ying [13] proved that the strong detonation is stable for a combustion model, but did not show rates of convergence. In the present paper, we show that a perturbation to strong detonation wave in a combustion model decays in t as 'fast' as the initial perturbation decays in $|x|$.

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We study the dynamic combustion model

$$u_t + (f(u) - qz)_x = \epsilon u_{xx} \tag{1}$$
$$z_x = k\varphi(u)z\,, \tag{2}$$

where $u = u(x,t)$ and $z = z(x,t)$ are scalar functions representing the velocity or the temperature of the combustible gas, and the concentration of the unburnt gas; and the constants q, ϵ, and $k > 0$ represent the amount of heat released during the chemical reaction, the viscous coefficient, and the reaction rate, respectively. The reaction rate function has the form

$$\varphi(u) = \begin{cases} 0 & u \le u_i \\ \text{a smooth increasing function} & u_i < u \le 2u_i \\ 1 & u > 2u_i\,, \end{cases} \tag{3}$$

where $u_i \ge 0$ is a constant related to the ignition temperature.

Motivated by the study of shock waves for gas dynamics, and by the asymptotic analysis performed in [14], we require the that the flux f satisfy

$$f(0) = 0\,, \quad f'(0) > 0\,, \quad f''(u) > 0\,.$$

To make (1)-(2) a well-posed problem, the data are assumed to satisfy

$$u(x,0) = u_0(x)\,, \tag{4}$$
$$z(+\infty,t) = 1\,. \tag{5}$$

This model was derived by Rosales and Majda [14] under the assumptions of weak nonlinearity, high activation energy, and nearly sonic speed of the detonation wave. It describes the one-dimensional flow of a reactive gas with a high Mach number. It includes the two important physical mechanisms for this type of problem: the nonlinear transport and the chemical reaction through the energy release term.

Under appropriate conditions on the parameters q and $k_0 = \epsilon k$, this model predicted the qualitative internal structure of the strong detonation assumed by Zeldovich-von Neumann-Doring [14]. i.e., a detonation wave traveling at speed D has the internal structure of an ordinary precursor fluid dynamic shock wave traveling at speed D, followed by a reaction zone. The parameter k_0 measures the ratio of the width of the analogue of the fluid dynamic shock layer and the width of the reaction zone. The detonation wave has the form

$$(u(x,t), z(x,t)) = (\psi(x - Dt), Z(x - Dt)) = (\psi(\xi), Z(\xi))\,,$$

where $\xi = x - Dt$ is the traveling wave variable, and the pair $(\psi, Z)(\xi)$ is a solution to the system

$$-D\psi' + f'(\psi)\psi' = \epsilon\psi'' + qZ' \tag{6}$$
$$Z' = k\varphi(\psi)Z\,. \tag{7}$$

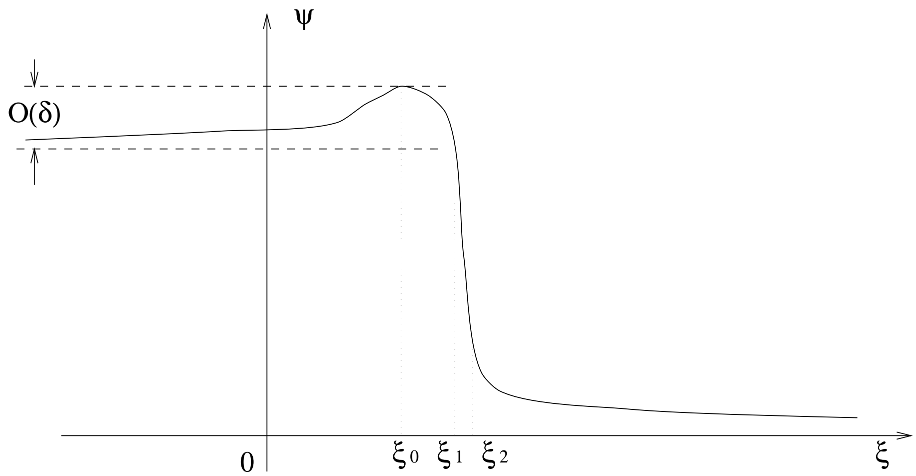

Figure 1: Traveling wave profile with $O(\delta) = O(q)$

When the boundary conditions are

$$\lim_{\xi\to-\infty}(\psi, Z)(\xi) = (u_l, 0) \tag{8}$$
$$\lim_{\xi\to+\infty}(\psi, Z)(\xi) = (0, 1)\,, \tag{9}$$

then the propagation speed D is determined by the boundary data,

$$D = \frac{f(\psi(+\infty)) - f(\psi(-\infty)) - qZ(+\infty)}{\psi(+\infty) - \psi(-\infty)}\,.$$

We will consider only strong detonation in this paper, that is,

$$f'(\psi(+\infty)) < D < f'(\psi(-\infty))\,. \tag{10}$$

The traveling wave solution has a non-monotone spike in the u-profile, which corresponds to a von Neumann spike. There is a maximum point ξ_0 on the profile, which separates the reaction zone and the viscous shock wave. See Figure 1. The traveling wave solution decays exponentially to its end state, as $|\xi| \to +\infty$.

We are concerned with the stability of the detonation waves described above. Stability of nonlinear problems for viscous shock waves for systems of nonlinear conservation laws have been studied by Goodman [5], Kawashima and Matsumura [6], Liu [11], and Liu and Xin [12].

By restricting our attention to strong detonation waves, we avoid dealing with the sonic point at the end of the reaction zone, which is present in the

Chapman-Jouguet detonation, and with the unsupported case. Studies on the stability of a Chapman-Jouguet detonation are presented in another article by the author, [10].

We establish the stability of strong detonation waves modeled by (1)-(2). Then, inspired by the work of Kawashima and Matsumura [6], we find rates of convergence. Our results are stated as follows:

(I) There exits a global solution to (1)-(2), (4)-(5), which converges exponentially in time to a traveling wave, provided the initial data is a small perturbation of the traveling wave that decays exponentially in $|x|$.

(II) If the rate of decay of the initial perturbation in the far fields is algebraic instead, the solution converges algebraically in time.

The weighted-energy method developed in [6] is used to obtain the rate of convergence. The characteristic-energy method used in [12, 13] is employed to deal with the difficulties arising from the non-monotonicity of the profile. Estimates involving z will be obtained through detailed L^2 and pointwise estimates.

In Section 2, we state our main results. Section 3 is a detailed stability analysis for the exponential decay. In Section 4, we study the algebraic decay.

2 Assumptions and main results

Due to the fact that (1) is a conservation law, we choose to work with the anti-derivative $v(x,t)$ of the perturbation $u(x,t)-\psi(x-Dt)$; see [5, 11]. The function $v(x,t)$ is defined as

$$v(x,t)=\int_{-\infty}^{x}(u(x,t)-\psi(x-Dt))\,dx\,. \tag{11}$$

We make the following assumptions on the data.

1.) Zero initial integral difference:

$$\int_{-\infty}^{+\infty}(u_0(x)-\psi(x))dx=0\,. \tag{12}$$

Note that (11), (12) imply

$$v(\pm\infty,t)=0\,. \tag{13}$$

This because (1) makes the integral $v(x,t)$ in (11) a conserved quantity, i.e.,

$$\frac{d}{dt}\int_{-\infty}^{+\infty}(u(x,t)-\psi(x-Dt))\,dx=0\,,$$

and

$$\int_{-\infty}^{+\infty}(u(x,t)-\psi(x-Dt))dx=\text{constant}$$

which equals zero due to our choice of data (12).

2.) Small heat release: $0 < q \ll \epsilon \ll 1$. So for $\xi \leq \xi_0$,

$$0 < \int_{-\infty}^{\xi} f'(\psi(\xi))_\xi \, d\xi = \delta_1 < Cq \ll \epsilon \ll 1 \,, \tag{14}$$

$$0 < f'(\psi(\xi))_\xi < \delta_2 < Cq \ll \epsilon \ll 1 \,. \tag{15}$$

These assumptions make the non-monotone spike of the strong detonation profile small (see Figure 1); so that the characteristic energy estimate can be obtained.

The smallness condition on the initial data and the stability analysis to be performed imply that there exist ξ_1 and ξ_2, $m > 0$ such that $\xi_0 < \xi_1 < \xi_2$ and

$$\varphi(\psi) = \varphi(u) = 0 \,, \quad \xi > \xi_2 \tag{16}$$

$$\varphi(\psi) = \varphi(u) = 1 \,, \quad \xi < \xi_1 \,. \tag{17}$$

Therefore,

$$-f'(\psi(\xi))_\xi > m > 0 \,, \quad \xi_1 < \xi < \xi_2 \tag{18}$$

$$f'(\psi(\xi)) - D > m > 0 \,, \quad \xi < \xi_1 \,, \tag{19}$$

where (19) holds because the detonation under consideration is strong (see (10)).

Again because the detonation wave is strong, we can find a $\xi_* \in (\xi_1, \xi_2)$ such that

$$f'(\psi(\xi_*)) = D \,. \tag{20}$$

See Figure 1.

Now we introduce some notation. Let

$$L^2 = \{v \mid \int_{-\infty}^{+\infty} v^2 dx < +\infty\}$$

and

$$H^2 = \{v \mid v \in L^2, v_x \in L^2, v_{xx} \in L^2\}.$$

Let $\omega(x) = \exp(\alpha\langle x - \xi_*\rangle)$ where $\langle x\rangle = (1 + x^2)^{1/2}$. Then we define the space

$$H^2_\omega = \{v \mid v e^{\frac{1}{2}\alpha\langle x - \xi_*\rangle} \in H^2\} \,,$$

the associated norm

$$\|v\|_{H^2_\omega} = \left(\int_{-\infty}^{\infty} \omega(v^2 + v_x^2 + v_{xx}^2) dx\right)^{1/2} .$$

Our main result of exponential decay is:

Theorem 1 *Suppose that $v_0 \in H^2_\omega$, $\|v_0\|_{H^2_\omega} \ll 1$, and Assumptions 1 and 2 from the previous section hold. Then there exists a global solution, $v(\cdot, t) \in H^2_\omega$, to (1)-(2), (4)-(5) satisfying*

$$\|v(\cdot, t)\|_{H^2_\omega} \leq \|v_0(\cdot)\|_{H^2_\omega} e^{-\beta t}. \tag{21}$$

Consequently,

$$\sup_{-\infty<x<+\infty} |u(x,t) - \psi(x - Dt)| \le Ce^{-\beta t/2}\,, \tag{22}$$

where β is a positive constant that depends on α, k, f, ϵ; and C depends only on initial data.

To state the algebraic decay result, we introduce the following notation. Let

$$L^2_\alpha = \{v \mid \langle x - \xi_*\rangle^{\alpha/2} v \in L^2\}\,,$$

with the associated norm

$$\|v\|_\alpha = \left(\int_{-\infty}^{+\infty} \langle x - \xi_*\rangle^\alpha |v|^2\,dx\right)^{1/2}.$$

Theorem 2 *Assume that $v_0 \in L^2_\alpha$, $v_{0,x} \in H^1$, $N_\alpha = \|v_{0,x}\|_1 + \|v_0\|_\alpha \ll 1$, and Assumptions 1 and 2 from the previous section hold. Then there exists a global solution $u(x,t)$ of problem (1)-(2), (4)-(5.) Moreover, this solution tends to a traveling wave solution at the rate $t^{-\alpha/2}$ as t tends to infinity, in the maximum norm. i.e.,*

$$\sup_{-\infty<x<+\infty} |u(x,t) - \psi(x - Dt)| \le C(1+t)^{-\alpha/2}(\|u_0 - \psi\|_1 + \|v_0\|_\alpha)\,,$$

where C is some positive constant depending on the initial data only.

3 Proof of stability: Exponential decay

In this section, we prove Theorem 1 by using weighted-energy estimates and the characteristic-energy method.

Assume *a priori* that

$$0 < \sup_{x,t} |v_x(x,t)| = \delta_3 \ll 1\,. \tag{23}$$

This assumption will be guaranteed by the smallness of initial data and the stability analysis to be performed.

Let

$$u(x,t) = \psi(x - Dt) + v_x(x,t)\,.$$

Then the anti-derivative of the perturbation is

$$v(x,t) = \int_{-\infty}^{x} (u(y,t) - \psi(y - Dt))\,dy\,.$$

Subtract (6), which is satisfied by $\psi(x - D_{CJ}t)$, from (1), which is satisfied by $u(x,t)$. Then integrate from $-\infty$ to x, and write the result in terms of the anti-derivative $v(x,t)$. In terms of the traveling-wave variable ξ, we have

$$v_t - Dv_\xi + f(\psi + v_\xi) - f(\psi) - q(z - Z) = \epsilon v_{\xi\xi}.$$

This expression can be rewritten as

$$v_t(\xi,t) + (f'(\psi) - D)v_\xi = \epsilon v_{\xi\xi} + qw + F(v_\xi, \psi)\,, \tag{24}$$

where $w(\xi,t) = z(\xi,t) - Z(\xi)$, and for $|v_\xi|$ small,

$$|F| = |-(f(\psi + v_\xi) - f(\psi) - f'(\psi)v_\xi)| \le C|v_\xi|^2\,.$$

Now, we prove a result that plays an important role in deriving weighted energy estimates, as in [3].

Lemma 1 *Let*

$$G_\alpha(\xi) = \begin{cases} -\frac{1}{2}f'(\psi(\xi))_\xi - \frac{\alpha}{2}\frac{(\xi-\xi_*)}{\langle\xi-\xi_*\rangle}(f'(\psi(\xi)) - D), & \xi_0 < \xi < +\infty \\ -\frac{\alpha}{2}\frac{(\xi-\xi_*)}{\langle\xi-\xi_*\rangle}(f'(\psi(\xi)) - D), & -\infty < \xi < \xi_0. \end{cases} \tag{25}$$

Then for some positive constant β,

$$G_\alpha(\xi) \ge \begin{cases} \beta - \frac{1}{4}f'(\psi(\xi))_\xi, & \xi_0 < \xi < +\infty \\ \beta, & -\infty < \xi < \xi_0\,. \end{cases} \tag{26}$$

Proof. We consider two cases. Recall that ξ_* satisfies (20).

Case i) When ξ is close to ξ_*, we have that $G_\alpha(\xi_*)$ is close to $-\frac{1}{2}f'(\psi(\xi_*))_\xi > m > 0$, see (18) and Figure 1. Choose β such that

$$0 < \beta < -\frac{1}{8}f'(\psi(\xi_*))_\xi\,.$$

Then

$$G_\alpha(\xi) \ge \beta - \frac{1}{8}f'(\psi(\xi_*))_\xi \ge \beta - \frac{1}{4}f'(\psi(\xi))_\xi\,.$$

Case ii) When ξ is away from ξ_*, say, $|\xi - \xi_*| > \delta_0$, then from (19) it follows that

$$G_\alpha(\xi) \ge \alpha cm > 0$$

for $-\infty < \xi < \xi_0$ and $c > 0$. For $\xi_0 < \xi < +\infty$, the convexity of f gives us

$$G_\alpha(\xi) \ge \alpha c\delta_0 - \frac{1}{4}f'(\psi(\xi))_\xi\,,$$

where c is some constant determined by the convexity of f. The desired inequality (26) follows by choosing β such that

$$0 < \beta < \min\{-\frac{1}{8}f'(\psi(\xi_*))_\xi, \alpha cm, \alpha c\delta_0\}.$$

Remark. The condition that the detonation is strong, (10), is the key condition in this lemma. For Chapman-Jouguet waves there is not such a result.

Now establish our main estimates. Multiplying (3.1) by $e^{\alpha\langle\xi-\xi_*\rangle}v$ and integrating, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} e^{\alpha\langle\xi-\xi_*\rangle}v^2\,d\xi \\
&+\int_{-\infty}^{+\infty} e^{\alpha\langle\xi-\xi_*\rangle}(f'(\psi(\xi))-D)vv_\xi\,d\xi-\epsilon\int e^{\alpha\langle\xi-\xi_*\rangle}vv_{\xi\xi}\,d\xi \\
&\quad=\int e^{\alpha\langle\xi-\xi_*\rangle}(qw+F(v_\xi,\psi))v\,d\xi\,.
\end{aligned}$$

Integrating by parts and using Lemma 1, we arrive at our main estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} e^{\alpha\langle\xi-\xi_*\rangle}v^2\,d\xi+\beta\int_{-\infty}^{+\infty} e^{\alpha\langle\xi-\xi_*\rangle}v^2\,d\xi \\
&+\frac{1}{2}\int_{-\infty}^{\xi_0}-|f'(\psi)_\xi|e^{\alpha\langle\xi-\xi_*\rangle}v^2\,d\xi+\frac{1}{4}\int_{\xi_0}^{+\infty}|f'(\psi)_\xi|e^{\alpha\langle\xi-\xi_*\rangle}v^2\,d\xi \\
&+\epsilon\int_{-\infty}^{+\infty} e^{\alpha\langle\xi-\xi_*\rangle}v_\xi^2\,d\xi+\left|\epsilon\int_{-\infty}^{+\infty}\alpha e^{\alpha\langle\xi-\xi_*\rangle}\frac{\xi-\xi_*}{\langle\xi-\xi_*\rangle}vv_\xi\,d\xi\right| \\
&\quad\le\left|\int_{-\infty}^{+\infty} e^{\alpha\langle\xi-\xi_*\rangle}(qw+F(v_\xi,\psi))v\,d\xi\right|\,.
\end{aligned}\tag{27}$$

To estimate the last term on the left hand side of (27), we make use of Schwarz's inequality to obtain

$$\left|\epsilon\alpha\int_{-\infty}^{+\infty}\frac{\xi-\xi_*}{\langle\xi-\xi_*\rangle}\omega(\xi)vv_\xi\,d\xi\right|\le\frac{\epsilon}{2}\int_{-\infty}^{+\infty}\omega(\xi)v_\xi^2\,d\xi+\frac{\alpha^2\epsilon}{2}\int_{-\infty}^{+\infty}\omega(\xi)v^2\,d\xi\,,$$

where $\omega(\xi)=e^{\alpha\langle\xi-\xi_*\rangle}$. Choose α such that $\frac{\beta}{2}\ge\frac{\alpha^2\epsilon}{2}$. Then our main estimate becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty}\omega(\xi)v^2\,d\xi+\frac{\beta}{2}\int_{-\infty}^{+\infty}\omega(\xi)v^2\,d\xi+\frac{1}{2}\int_{-\infty}^{\xi_0}-|f'(\psi)_\xi|\omega(\xi)v^2\,d\xi \\
&+\frac{1}{4}\int_{\xi_0}^{+\infty}|f'(\psi)_\xi|\omega(\xi)v^2\,d\xi+\frac{\epsilon}{2}\int_{-\infty}^{+\infty}\omega(\xi)v_\xi^2\,d\xi \\
&\quad\le\left|\int_{-\infty}^{+\infty}\omega(\xi)(qw+F(v_\xi,\psi))v\,d\xi\right|\,.
\end{aligned}$$

Now we use the characteristic-energy method to estimate the third term on the left hand side of (27), the bad term arising from the non-monotonicity of the profile. The idea is to integrate (24) for v along the characteristic direction to get v^2, and then plug it in the integration. The key condition here is that $|f'(\psi(\xi))_\xi|$ is small due to $q \ll 1$. See (15).

Let

$$S(\xi) = (f'(\psi(\xi)) - D)^{-1}\,.$$

Then (19) implies

$$0 < S(\xi) < \frac{1}{m}\,, \quad -\infty < \xi < \xi_0\,. \tag{28}$$

Multiplying (24) by vS, and then integrating from $-\infty$ to ξ, we obtain

$$\begin{aligned}
\frac{1}{2}v^2(\xi,t) &= \int_{-\infty}^{\xi} S(\eta)(-vv_t + \epsilon v v_{\eta\eta} + q\ w\ v + F(v_\eta,\psi)v)\,d\eta \\
&= \int_{-\infty}^{\xi} S(\eta)(-vv_t + qwv + F(v_\eta,\psi)v)\,d\eta + \epsilon\int_{-\infty}^{\xi} S(\eta)(-v_\eta^2)\,d\eta \\
&\quad + \epsilon S(\xi)(vv_\xi) + \epsilon\int_{-\infty}^{\xi} S^2(\eta)f'(\psi)_\eta(-vv_\eta)\,d\eta \\
&= \epsilon S(\xi)vv_\xi + \int_{-\infty}^{\xi} S(\eta)(-vv_t + qwv + F(v_\eta,\psi)v - \epsilon v_\eta^2)\,d\eta \\
&\quad + \epsilon\int_{-\infty}^{\xi} S^2(\eta)f'(\psi)_\eta(-vv_\eta)d\eta\,.
\end{aligned}$$

Multiplying the above inequality by $\omega(\xi)f'(\psi(\xi))_\xi$ and integrating from $-\infty$ to ξ_0, then using Schwarz's inequality and Fubini's theorem, we have

$$\begin{aligned}
&\int_{-\infty}^{\xi_0} \frac{1}{2}\omega(\xi)v^2(\xi,t)f'(\psi(\xi))_\xi\,d\xi \\
&\le \frac{1}{8}\int_{-\infty}^{\xi_0} f'(\psi(\xi))_\xi\omega(\xi)\frac{v^2}{2}\,d\xi + 8\delta_2\epsilon\int_{-\infty}^{\xi_0}\omega(\xi)v_\xi^2\,d\xi \\
&\quad + \int_{-\infty}^{\xi_0}\int_{\eta}^{\xi_0} f'(\psi(\xi))_\xi\omega(\xi)S(\eta)(-vv_t + qwv + F(v_\eta,\psi)v)\,d\xi\,d\eta \\
&\quad + \int_{-\infty}^{\xi_0}\int_{\eta}^{\xi_0} f'(\psi(\xi))_\xi\omega(\xi)S^2(\eta)f'(\psi(\eta))_\eta(-\epsilon vv_\eta)\,d\xi\,d\eta\,.
\end{aligned}$$

Now use the smallness assumptions (14), (15), and (23) to obtain

$$\begin{aligned}
&\int_{-\infty}^{\xi_0} \frac{1}{2}\omega(\xi)v^2(\xi,t)f'(\psi(\xi))_\xi\,d\xi \\
&\le \frac{1}{4}\int_{-\infty}^{\xi_0} f'(\psi(\xi))_\xi\omega(\xi)\frac{v^2}{2}\,d\xi + C(\delta_1+\delta_2)\epsilon\int_{-\infty}^{\xi_0}\omega(\xi)v_\xi^2\,d\xi \\
&\quad + C\delta_1\frac{d}{dt}\int_{-\infty}^{\xi_0} S(\eta)\omega(\eta)(-\frac{1}{2}v^2(\eta,t))\,d\eta + C\delta_1\int_{-\infty}^{\xi_0}\omega(\eta)qwv\,d\eta.
\end{aligned}$$

Using the argument that $a \leq \frac{1}{4}a+b$ implies $a \leq \frac{4}{3}b$, the above inequality implies that

$$
\begin{aligned}
&\int_{-\infty}^{\xi_0} f'(\psi(\xi))_\xi \omega(\xi) \frac{v^2(\xi,t)}{2}\, d\xi \\
&\leq \quad C\delta_1 \frac{d}{dt} - \int_{-\infty}^{\xi_0} \omega(\xi) \frac{1}{2} v^2(\xi,t)\, d\xi \\
&\qquad + C(\delta_1+\delta_2)\epsilon \int_{-\infty}^{\xi_0} \omega(\xi) v_\xi^2\, d\xi + C\delta_1 \int_{-\infty}^{\xi_0} \omega(\xi) qwv\, d\xi\,.
\end{aligned}
$$

Plugging this estimate into (27), noticing that δ_1, δ_2 are small, and using assumption (23), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{+\infty} \frac{1}{2}\omega(\xi) v^2\, d\xi + \frac{1}{2}\int_{-\infty}^{+\infty} |f'(\psi(\xi))_\xi| \omega(\xi) \frac{v^2(\xi,t)}{2}\, d\xi \\
&+ \frac{\beta}{2}\int_{-\infty}^{+\infty} \omega(\xi) v^2\, d\xi + \epsilon \int_{-\infty}^{+\infty} \omega(\xi) \frac{v_\xi^2}{2}\, d\xi \\
&\leq \quad C\left| \int_{-\infty}^{+\infty} \omega(\xi) qwv\, d\xi \right|.
\end{aligned}
$$

To estimate $|\int_{-\infty}^{+\infty} \omega(\xi) qvw\, d\xi|$, we divide the real line into three subintervals $(-\infty, \xi_1)$, (ξ_1, ξ_2), $(\xi_2, +\infty)$, and estimate the integral on each subinterval.

On the interval $(\xi_2, +\infty)$**.** According to assumption (16),

$$
\varphi(u) = \varphi(\psi) = 0 \quad \text{for } \xi > \xi_2\,.
$$

Hence

$$
\begin{aligned}
w(\xi,t) &= (z-Z)(\xi,t) \\
&= \exp(k\int_{\xi}^{+\infty} \varphi(u(\eta,t))\, d\eta) - \exp(k\int_{\xi}^{+\infty} \varphi(\psi(\eta))\, d\eta) \\
&= 0\,, \quad \text{for } \xi > \xi_2\,.
\end{aligned}
$$

So that

$$
\int_{\xi_2}^{+\infty} \omega(\xi) qwv\, d\xi = 0\,. \tag{29}
$$

On the interval (ξ_1, ξ_2)**.**

$$
|w(\xi,t)| = |\exp(k\int_{\xi}^{+\infty} \varphi(u(\eta,t))\, d\eta) - \exp(k\int_{\xi}^{+\infty} \varphi(\psi(\eta))\, d\eta)|
$$

$$\begin{aligned}
&= |\exp(k\int_{\xi}^{\xi_2}\varphi(u(\eta,t))d\eta)-\exp(k\int_{\xi}^{\xi_2}\varphi(\psi(\eta))\,d\eta)| \\
&= C\left|\int_{\xi}^{\xi_2}(\varphi(u(\eta,t))-\varphi(\psi(\eta)))\,d\eta\right| \\
&\le C\int_{\xi}^{\xi_2}|v_\eta|\,d\eta \\
&\le C(\int_{\xi_1}^{\xi_2}\omega(\eta)|v_\eta|^2\,d\eta)^{1/2}\,.
\end{aligned}$$

Using the Schwarz inequality, (18), and the above estimate for w, we have

$$\begin{aligned}
\left|\int_{\xi_1}^{\xi_2}\omega(\xi)qwv\,d\xi\right| &\le \frac{1}{2}\int_{\xi_1}^{\xi_2}\omega(\xi)qv^2\,d\xi+\frac{1}{2}\int_{\xi_1}^{\xi_2}\omega(\xi)qw^2\,d\xi \\
&\le Cq\int_{\xi_1}^{\xi_2}\omega(\xi)|f'(\psi)_\xi|v^2\,d\xi+Cq\int_{\xi_1}^{\xi_2}\omega(\xi)|v_\xi|^2\,d\xi\,.
\end{aligned}$$

Since $q\ll\epsilon\ll 1$, the terms on the right-hand side of (27) are under control.

On the interval $(-\infty,\xi_1)$**.** According to (17) and the result on the above subinterval, we have

$$|w(\xi,t)|=|w(\xi_1,t)|e^{-k(\xi_1-\xi)}\le C(\int_{\xi_1}^{\xi_2}\omega(\eta)|v_\eta|^2\,d\eta)^{1/2}.$$

An application of the Schwarz inequality yields

$$\begin{aligned}
\left|\int_{-\infty}^{\xi_1}\omega(\xi)qwv\,d\xi\right| &\le Cq\int_{-\infty}^{\xi_1}\omega(\xi)v^2e^{-k|\xi|}\,d\xi+Cq\int_{\xi_1}^{\xi_2}\omega(\xi)v_\xi^2\,d\xi \\
&:= I+II.
\end{aligned}$$

Since $q\ll\epsilon\ll 1$, II is under control in (27). For I, we find characteristic-energy estimates as we did for

$$\int_{-\infty}^{\xi_0}\omega(\xi)f'(\psi(\xi))_\xi\frac{v^2(\xi,t)}{2}\,d\xi$$

The result is

$$\begin{aligned}
&\int_{-\infty}^{\xi_1}e^{-k|\xi|}\omega(\xi)v^2(\xi,t)\,d\xi \\
&\quad\le C\int_{-\infty}^{\xi_1}\omega(\xi)|f'(\psi(\xi))_\xi|v^2\,d\xi+C\frac{d}{dt}\int_{-\infty}^{\xi_1}\omega(\xi)v^2(\xi,t)\,d\xi \\
&\qquad +C\int_{-\infty}^{\xi_1}\omega(\xi)v_\xi^2\,d\xi+C\int_{-\infty}^{\xi_1}\omega(\xi)qwv\,d\xi\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}\Big|\int_{-\infty}^{\xi_1}\omega(\xi)qwv\,d\xi\Big| &\leq Cq\Big\{C\int_{-\infty}^{\xi_2}\omega(\xi)v_\xi^2\,d\xi + C\frac{d}{dt}\int_{-\infty}^{\xi_1}\frac{1}{2}\omega(\xi)v^2(\xi,t)\,d\xi \\ &\quad + \int_{-\infty}^{\xi_1}\omega(\xi)|f'(\psi)_\xi|v^2\,d\xi\Big\}.\end{aligned}$$

Plugging the estimates of $|\int\omega(\xi)qwv\,d\xi|$ over the three intervals into (27) and noticing that $q\ll\epsilon\ll 1$, we have

$$\begin{aligned}0 &\geq \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty}\omega(\xi)v^2(\xi,t)\,d\xi + \frac{1}{4}\int_{-\infty}^{+\infty}|f'(\psi(\xi))_\xi|\omega(\xi)v^2\,d\xi \\ &\quad +\frac{\beta}{2}\int_{-\infty}^{+\infty}\omega(\xi)v^2\,d\xi + \frac{1}{4}\epsilon\int_{-\infty}^{+\infty}\omega(\xi)v_\xi^2\,d\xi\,.\end{aligned}$$

Similarly, we have estimates for the derivatives v_ξ and $v_{\xi\xi}$ of v.

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty}\omega(\xi)v_\xi^2(\xi,t)\,d\xi + \frac{\beta}{2}\int_{-\infty}^{+\infty}\omega(\xi)v_\xi^2\,d\xi + \frac{1}{4}\epsilon\int_{-\infty}^{+\infty}\omega(\xi)v_{\xi\xi}^2\,d\xi \leq 0\,,$$

and

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty}\omega(\xi)v_{\xi\xi}^2(\xi,t)\,d\xi + \frac{\beta}{2}\int_{-\infty}^{+\infty}\omega(\xi)v_{\xi\xi}^2\,d\xi + \frac{1}{4}\epsilon\int_{-\infty}^{+\infty}\omega(\xi)v_{\xi\xi\xi}^2\,d\xi \leq 0\,.$$

Combining these estimates, we have

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{-\infty}^{+\infty}\omega(\xi)v^2\,d\xi + \int_{-\infty}^{+\infty}\omega(\xi)v_\xi^2\,d\xi + \int_{-\infty}^{+\infty}\omega(\xi)v_{\xi\xi}^2\,d\xi\Big) \\ &\leq -\beta\Big(\int_{-\infty}^{+\infty}\omega(\xi)v^2\,d\xi + \int_{-\infty}^{+\infty}\omega(\xi)v_\xi^2\,d\xi + \int_{-\infty}^{+\infty}\omega(\xi)v_{\xi\xi}^2\,d\xi\Big)\,.\end{aligned}$$

By Gronwall's inequality, we have

$$\|v(\cdot,t)\|_{H^2_\omega} \leq \|v(\cdot,0)\|_{H^2_\omega}e^{-\beta t/2}\,.$$

Hence

$$\begin{aligned}|u(x,t)-\psi(x-Dt)| &= |v_x(x,t)| = \big(2\int_{-\infty}^{x}v_xv_{xx}(y,t)\,dy\big)^{1/2} \\ &\leq \big(\int_{-\infty}^{+\infty}v_x^2(x,t)\,dx + \int_{-\infty}^{+\infty}v_{xx}^2(x,t)\,dx\big)^{1/2} \\ &\leq \|v(\cdot,t)\|_{H^2_\omega} \\ &\leq Ce^{-\beta t/2}\,,\end{aligned}$$

which completes the proof of Theorem 1.

Remark: The above inequality guarantees that the *a priori* assumption (23) is satisfied.

4 Proof of stability: Algebraic decay

To prove Theorem 2, we use the iteration introduced by Kawashima and Matsumura [6], and weighted energy estimates. First we state a lemma similar to Lemma 1.

Lemma 2 *Let ξ_* be defined by (20), and*

$$A_\beta(\xi) = \begin{cases} \frac{1}{2}(\beta\frac{(\xi-\xi_*)}{\langle\xi-\xi_*\rangle}(D - f'(\psi(\xi))) - \langle\xi-\xi_*\rangle f'(\psi(\xi))_\xi\,, & \xi_0 < \xi < +\infty \\ \frac{1}{2}\beta\frac{(\xi-\xi_*)}{\langle\xi-\xi_*\rangle}(D - f'(\psi(\xi)))\,, & -\infty < \xi < \xi_0\,. \end{cases} \tag{30}$$

Then there exists a positive constant β such that

$$A_\beta(\xi) \geq \begin{cases} \beta - \frac{1}{4}f'(\psi(\xi))_\xi\,, & \xi_0 < \xi < +\infty \\ \beta\,, & -\infty < \xi < \xi_0\,. \end{cases} \tag{31}$$

The proof of this lemma is similar the proof of Lemma 1.

From the (24) it follows that the anti-derivative v of the perturbation $u - \psi$ satisfies

$$v_t(\xi,t) + (f'(\psi) - D)v_\xi = \epsilon v_{\xi\xi} + qw + F(v_\xi,\psi), \tag{32}$$

where $w(\xi,t) = z(\xi,t) - Z(\xi)$ and $|F(v_\xi,\psi)| \leq C|v_\xi|^2$ for small values of $|v_\xi|$.

Let

$$|v(\cdot,t)|_\beta^2 = \int_{-\infty}^{+\infty} \langle\xi-\xi_*\rangle^\beta v^2(\xi,t)\,d\xi\,.$$

Multiplying (32) by $(1+t)^\gamma\langle\xi-\xi_*\rangle^\beta v$, integrating by parts, and using Lemma 2, we obtain our main estimate,

$$\begin{aligned} &\frac{1}{2}(1+t)^\gamma|v(\cdot,t)|_\beta^2 + \beta\int_0^t (1+\tau)^\gamma|v(\cdot,t)|_{\beta-1}^2\,d\tau \\ &+\frac{1}{4}\int_0^t\int_{\xi_0}^{+\infty}(1+\tau)^\gamma|f'(\psi(\xi))_\xi|v^2\langle\xi-\xi_*\rangle^\beta d\xi\,d\tau \\ &-\frac{1}{2}\int_0^t\int_{-\infty}^{\xi_0}(1+\tau)^\gamma|f'(\psi(\xi))_\xi|v^2\langle\xi-\xi_*\rangle^\beta d\xi\,d\tau + \epsilon\int_0^t(1+\tau)^\gamma|v_\xi(\cdot,\tau)|_\beta^2\,d\tau \\ &\quad\leq\quad c|v_0|_\beta^2 + c\gamma\int_0^t(1+\tau)^{\gamma-1}|v(\cdot,\tau)|_\beta^2\,d\tau \\ &\qquad\quad +c\beta\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^\gamma\langle\xi-\xi_*\rangle^{\beta-2}\xi|vv_\xi|\,d\xi\,d\tau \\ &\qquad\quad +c\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^\gamma\langle\xi-\xi_*\rangle^\beta v(qw+F(v_\xi,\psi))\,d\xi\,d\tau\,. \end{aligned} \tag{33}$$

The third term on the right hand side of (33) can be estimated using twice the Schwarz inequality. Notice that

$$\begin{aligned}
&\beta\int_{-\infty}^{+\infty}\langle\xi-\xi_*\rangle^{\beta-1}(\xi)|vv_\xi|\,d\xi\\
&\le\ \frac{\beta}{2}\int_{-\infty}^{+\infty}\langle\xi-\xi_*\rangle^{\beta-1}v^2\,d\xi+\beta c\int_{-\infty}^{+\infty}\langle\xi-\xi_*\rangle^{\beta-1}v_\xi^2\,d\xi\\
&\le\ \frac{\beta}{2}\int_{-\infty}^{+\infty}\langle\xi-\xi_*\rangle^{\beta-1}v^2\,d\xi+\frac{\epsilon}{2}\int_{-\infty}^{+\infty}\langle\xi-\xi_*\rangle^{\beta}v_\xi^2\,d\xi+\beta c\int_{-\infty}^{+\infty}v_\xi^2\,d\xi\,.
\end{aligned}$$

To estimate the fourth term the left-hand side of (33), we use the characteristic energy method again. Since the non-monotonicity spike is small under our assumption $q\ll\epsilon\ll 1$, we use the method as in the previous section, with the weight function $\langle\xi-\xi_*\rangle^\beta$ instead of $\exp(\alpha\langle\xi-\xi_*\rangle)$.

Since the term

$$\int_0^t\int_{-\infty}^{\xi_0}qwv\langle\xi-\xi_*\rangle^\beta(1+\tau)^\gamma d\xi\,d\tau\,,$$

with $q\ll\epsilon\ll 1$, can be treated similarly as in the previous section, we omit the details of the calculations and just give the result here.

Combining the estimates for the term in (33) we obtain

$$\begin{aligned}
&\frac12(1+t)^\gamma|v(\cdot,t)|_\beta^2+\frac{\beta}{2}\int_0^t(1+\tau)^\gamma|v(\cdot,t)|_{\beta-1}^2\,d\tau \qquad (34)\\
&+\frac18\int_0^t\int_{\xi_0}^{+\infty}(1+\tau)^\gamma|f'(\psi(\xi))_\xi|v^2\langle\xi-\xi_*\rangle^\beta\,d\xi\,d\tau\\
&+\frac{\epsilon}{4}\int_0^t(1+\tau)^\gamma|v_\xi(\cdot,\tau)|_\beta^2\,d\tau\\
&\le\ c|v_0|_\beta^2+c\gamma\int_0^t(1+\tau)^{\gamma-1}|v(\cdot,\tau)|_\beta^2\,d\tau+c\beta\int_0^t(1+\tau)^\gamma\|v_\xi(\tau)\|^2\,d\tau\,.
\end{aligned}$$

Observing that the process for obtaining the above inequality also applies for $\beta=0$, we have

$$\begin{aligned}
&\frac12(1+t)^\gamma|v(\cdot,t)|^2+\frac14\int_0^t\int_{\xi_0}^{+\infty}(1+\tau)^\gamma|f'(\psi(\xi))_\xi|v^2\,d\xi\,d\tau\\
&+\frac{\epsilon}{2}\int_0^t(1+\tau)^\gamma|v_\xi(\cdot,\tau)|^2\,d\tau \qquad (35)\\
&\le\ c\Big(\|v_0\|^2+\gamma\int_0^t(1+\tau)^{\gamma-1}\|v(\cdot,\tau)\|^2\,d\tau\Big)\,.
\end{aligned}$$

Using the condition $N_\alpha = |v_0|_\alpha + \|v_{0,\xi}\|_1 \ll 1$, in the case of $\beta = 0, \gamma = 0$ we have that

$$\|v(t)\|_2^2 + \epsilon \int_0^t \|v_\xi(\tau)\|_2^2 d\tau \le c N_\alpha^2 \,. \tag{36}$$

Now, we prove the iteration lemma.

Lemma 3 *For γ in $[0, \alpha]$, we have*

$$\begin{aligned}(1+t)^\gamma |v(t)|^2_{\alpha-\gamma} + (\alpha-\gamma)\int_0^t (1+\tau)^\gamma |v(\tau)|^2_{\alpha-\gamma-1}\, d\tau + \int_0^t (1+\tau)^\gamma |v_\xi(\tau)|^2_{\alpha-\gamma}\, d\tau \\ \le \quad c N_\alpha^2 \,. \end{aligned} \tag{37}$$

Furthermore,

$$(1+t)^\gamma \|v(t)\|^2 + \epsilon \int_0^t (1+\tau)^\gamma \|v_\xi(\tau)\|^2 d\tau \le c N_\alpha^2. \tag{38}$$

Proof. First, we prove this lemma for γ integer in $[0, [\alpha]]$, by using the following steps.
Step 1. Let $\beta = 0, \gamma = 0$ in (34) and use (36) to get (37) with $\gamma = 0$. Therefore, the lemma is proved for $\alpha < 1$.
Step 2. If $\alpha \ge 1$, we use that (37) holds for $\gamma = 0$.

Let $\beta = 0, \gamma = 1$ in (34) and use (37) with $\gamma = 0$ to get (38) with $\gamma = 1$.

Let $\beta = \alpha - 1, \gamma = 1$ in (34) and use (37) with $\gamma = 0$ and (38) with $\gamma = 1$ to get (37) with $\gamma = 1$.

Therefore, the lemma is proved for $\alpha < 2$.
Step 3. $\alpha \ge 2$. Let $\beta = 0, \gamma = 2$ in (34) and use (35) with $\gamma = 1$ to get (38) with $\gamma = 2$.

Let $\beta = \alpha - 2, \gamma = 1$ in (34) and use (37) with $\gamma = 1$ and (38) with $\gamma = 2$ to get (37) with $\gamma = 2$.

The lemma is proved for $\alpha < 3$. And by an inductive argument we can prove this lemma for any α.

Similarly, for $l = 0, 1, 2$ we have

$$(1+t)^\gamma \|\partial_x^l v(t)\|^2 + \epsilon \int_0^t (1+\tau)^\gamma \|\partial_x^{l+1} v(\tau)\|^2\, d\tau \le c N_\alpha^2$$

Hence

$$(1+t)^\gamma \|v(t)\|_2^2 + \epsilon \int_0^t (1+\tau)^\gamma \|v_x(\tau)\|_2^2 d\tau \le c N_\alpha^2 \,,$$

which concludes the proof for γ integer in $[0, [\alpha]]$.

For $\gamma \in ([\alpha], \alpha]$, from (34) with $\beta = 0$ it follows that

$$\begin{aligned}
&(1+t)^{\gamma}\|v(\cdot,t)\|^2 + \epsilon \int_0^t (1+\tau)^{\gamma}\|v_\xi(\cdot,\tau)\|^2\,d\tau \\
&\quad\leq c\|v_0\|^2 + c\gamma \int_0^t (1+\tau)^{\gamma-1}\|v(\cdot,\tau)\|^2\,d\tau \\
&\quad\leq c\|v_0\|^2 + c\gamma \int_0^t (1+\tau)^{[\gamma]}\|v(\cdot,\tau)\|^2\,d\tau\,.
\end{aligned}$$

Combining the above results on integer exponents, we arrive at our conclusion for any $\gamma \in ([\alpha], \alpha]$. □

Finally, we obtain the estimate

$$\begin{aligned}
v_x^2(x,t) &= 2\int_{-\infty}^{x} v_x v_{xx}(y,t)\,dy \\
&\leq 2\Big(\int_{-\infty}^{+\infty} v_x^2(y,t)dy\Big)^{1/2}\Big(\int_{-\infty}^{+\infty} v_{xx}^2(y,t)\,dy\Big)^{1/2} \\
&\leq \int_{-\infty}^{+\infty} v_x^2(y,t)\,dy + \int_{-\infty}^{+\infty} v_{xx}^2(y,t)\,dy \\
&\leq cN_\alpha^2(1+t)^{-\gamma}\,.
\end{aligned}$$

Hence,

$$\sup_x |u(x,t) - \psi(x-Dt)| = \sup_x |v_x(x,t)| \leq CN_\alpha(1+t)^{-\gamma/2}\,,$$

which is the statement in Theorem 2.

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