Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 63, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF POSITIVE SOLUTIONS TO THE NONLINEAR CHOQUARD EQUATION WITH COMPETING POTENTIALS

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Communicated by Paul H. Rabinawitz

Abstract. This article concerns the existence of positive solutions of the nonlinear Choquard equation

$$-\Delta u + a(x)u = b(x)\Big(\frac{1}{|x|} * |u|^2\Big)u, \quad u \in H^1(\mathbb{R}^3),$$

where the coefficients a and b are positive functions such that $a(x) \to \kappa_\infty$ and $b(x) \to \mu_\infty$ as $|x| \to \infty$. By comparing the decay rate of the coefficients a and b, we prove the existence of positive ground and bound stat solutions of Choquard equation.

1. Introduction

In this article studies the existence of positive solution of the nonlinear Choquard equation

$$-\Delta u + a(x)u = b(x)\Big(\frac{1}{|x|} * |u|^2\Big)u, \quad u \in H^1(\mathbb{R}^3), \tag{1.1}$$

where the coefficients $a(x)$ and $b(x)$ are positive functions such that $\lim_{|x|\to\infty} a(x) = \kappa_\infty > 0$ and $\lim_{|x|\to\infty} b(x) = \mu_\infty > 0$.

Equation (1.1) is called the nonlinear Choquard or Choquard-Pekar equation. It has several physical origins. Equation (1.1) first appeared as early as in 1954, in a work by Pekar describing the quantum mechanics of a polaron at rest [34]. In 1976, Choquard used (1.1) to describe an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one component plasma [22]. In 1996, Penrose proposed (1.1) as a model of self-gravitating matter, in a programme in which quantum state reduction is understood as a gravitational phenomenon [31]. In this context equation of type (1.1) is usually called the nonlinear Schrodinger-Newton equation. In general, many mathematicians study the existence of the solitary solutions of the nonlinear generalized Choquard equation

$$i\psi_t - \Delta\psi + K(x)\psi - b(x)\Big(\frac{1}{|x|^\alpha} * |\psi|^p\Big)|\psi|^{p-2}\psi = 0, \ (x,t) \in \mathbb{R}^N \times \mathbb{R}_+. \tag{1.2}$$

2010 *Mathematics Subject Classification.* 35J61, 35J20, 35Q55, 49J40.
Key words and phrases. Positive solutions; Choquard equation; competing coefficients; variational methods.

Submitted October 11, 2017. Published March 7, 2018.

where $N \geq 1$, $\alpha \in (0, N)$, $\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}$. To obtain the solitary solutions of (1.2), we set $\psi(t,x) = u(x)e^{i\omega t}$ (ω is a constant) in (1.2) and get the stationary equation of the form

$$-\Delta u + a(x)u = b(x)\Big(\frac{1}{|x|^{\alpha}} * |u|^{p}\Big)|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N), \tag{1.3}$$

where $a = K(x) - \omega$. Obviously, if $N = 3, \alpha = 2$ and $p = 2$ the equation (1.3) reduces to (1.1). In recent years, many papers are concerned with the existence of solutions of (1.3). Lieb [22] proved the existence and uniqueness of the ground state to (1.2). Lions [23] obtained the existence of a sequence of radially symmetric solutions for (1.3) by using variational methods. Papers [1, 38] showed the existence of multi-bump solutions of (1.3). Recently, papers [19, 26, 37] showed some partial uniqueness of the positive solutions of (1.3). Papers [18, 27] showed the existence of positive and nodal solution of (1.3). For more results on this direction one can refer to [5, 12, 13, 14, 15, 16, 17, 28, 29, 30, 32, 33] and the references therein.

It is worth to point out that in most of the papers mentioned above, the search for the positive ground state solutions to (1.3). In the present paper we consider a nonautonomous situation that has to be studied in a different way. We will find the positive solution which different from positive ground state solution. Here a solution u of (1.1) is nontrivial if $u \neq 0$. A solution of (1.1) is a *nontrivial bound state solution* if u is a nontrivial solution. A solution u with $u > 0$ is called a positive solution. A solution is called a *nontrivial ground state solution* (or *positive ground state solution*) if its energy is minimal among all the nontrivial solutions (or all the positive solutions) of (1.1). Here the energy functional corresponding to (1.1) is defined by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \big(|\nabla u|^2 + a(x)u^2\big) - \frac{1}{4}\int_{\mathbb{R}^3} b(x)\phi_u u^2, \quad u \in H^1(\mathbb{R}^3). \tag{1.4}$$

We set

$$a(x) = \kappa_\infty + \lambda\kappa(x), \quad b(x) = \mu_\infty + \mu(x), \tag{1.5}$$

where $\lambda \in \mathbb{R}^+$ and we assume

(A1) $\kappa \in L^{3/2}(\mathbb{R}^3)$, $\kappa \geq 0$, $\kappa \neq 0$, $\lim_{|x|\to\infty} \kappa(x) = 0$;
(A2) $\mu \in L^2(\mathbb{R}^3)$, $\mu \geq 0$, $\mu \neq 0$, $\lim_{|x|\to\infty} \mu(x) = 0$.

Hence, equation (1.1) can be rewritten as

$$-\Delta u + (\kappa_\infty + \lambda\kappa(x))u = (\mu_\infty + \mu(x))\phi_u(x)u, \ u \in H^1(\mathbb{R}^3). \tag{1.6}$$

The purpose of this paper is to describe some phenomena that can occur when the coefficients are competing. For each $\lambda \in [0, \infty)$, we prove the existence of positive ground state solution of (1.6) if $\kappa(x)$ decays faster than $\mu(x)$. Conversely, if $\mu(x)$ decays faster than $\kappa(x)$, we find the threshold value $\lambda^* > 0$ such that (1.6) has a ground state solution if $\lambda \in [0, \lambda^*)$, and no ground state solution for $\lambda \in [\lambda^*, \infty)$. Furthermore, we find the positive bound state solution of (1.6) if $\lambda \in [\lambda^*, \infty)$. Our study mainly motivated by the recent works [11, 10], while the authors study the existence of positive solutions of Schrödinger equation and Schrödinger-Poisson system with competing coefficients. Comparing to the previous works [11, 10], we encounter new difficulty in finding the positive solutions of (1.6). Precisely, we let u_0 denote the sign-changing solution of the Schödinger equation

$$-\Delta u + \kappa_\infty u = \mu_\infty |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N). \tag{1.7}$$

It is easy to check that $J_\infty(u_0) \geq 2k_\infty$ (see [11]), where

$$k_\infty = \inf J_\infty \quad \text{and} \quad J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \kappa_\infty u^2) - \frac{\mu_\infty}{p}\int_{\mathbb{R}^N} |u|^p. \tag{1.8}$$

In [11, 10], this fact was play an important role in recovering the compactness and finding the bound state solution. However, the situation is totally different in our case. In fact, consider the Choquard equation

$$-\Delta u + \kappa_\infty u = \mu_\infty \phi_u(x) u, \quad u \in H^1(\mathbb{R}^3). \tag{1.9}$$

According to [18], we know that the energy of sign-changing solution of (1.9) is strictly less than two times the least energy level of (1.9). This brings the difficulty in recovering the compactness. We shall use the idea of [25, 36] and consider our problem in convex set $H^1_+(\mathbb{R}^3)$ to overcome this difficult, where $H^1_+(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) : u \geq 0\}$.

Now we are ready to give the main results of the paper. We first state the results when $\kappa(x)$ decays faster than $\mu(x)$.

Theorem 1.1. *For $\tau \in (0,1)$, we assume that* (A1), (A2) *hold and*

(A3) $\lim_{|x|\to\infty} \kappa(x)|x|e^{\frac{2\tau}{1-\tau}\sqrt{\kappa_\infty}|x|} = 0, \quad \lim_{|x|\to\infty} \mu(x)e^{\frac{2\tau}{1+\tau}\sqrt{\kappa_\infty}|x|} = +\infty.$

Then for all $\lambda \in \mathbb{R}^+$, Equation (1.6) *always has a positive ground state solution.*

Next we study the case when $\mu(x)$ decays faster than $\kappa(x)$.

Theorem 1.2. *Assume that* (A1), (A2) *hold, and for some $\tau \in (0,1)$, $\sigma \in (0,\kappa_\infty)$, and $c_1, c_2 > 0$, we have*

(A4) $\liminf_{|x|\to\infty} \kappa(x)e^{\frac{4\tau}{1+\tau}\sqrt{\sigma}|x|} \geq c_1$ *and* $\limsup_{|x|\to\infty} \mu(x)e^{\frac{4\tau}{1-\tau}\sqrt{\sigma}|x|} \leq c_2$.

Then there exist a number $\lambda^* > 0$, such that for all $\lambda \in [0,\lambda^*)$, Equation (1.6) *has a positive ground state solution, while if $\lambda \in [\lambda^*, +\infty)$, Equation* (1.6) *has no positive ground state solution. Additionally, if we assume that*

(A5) $\limsup_{|x|\to\infty} \kappa(x)|x|^2 e^{2\sqrt{\kappa_\infty}|x|} \leq c_3$ *for some $c_3 > 0$,*

then for $\lambda \in [\lambda^*, +\infty)$, Equation (1.6) *has a positive solution.*

Remark 1.3. To the best our knowledge, this is the first results on the existence of positive solution of Choquard equation (1.6) with competing coefficients. We believe our arguments can also work on the generalized Choquard equation (1.3) and other nonlocal problems. This is an interesting issue that can be pursued in the future.

2. Preliminary results

Throughout this article we shall use the following notation.

- The scalar product in $H^1(\mathbb{R}^3)$ is defined by
$$(u,v) = \int_{\mathbb{R}^3} [\nabla u \nabla v + \kappa_\infty uv]$$
and the norm is defined by $\|u\| = \sqrt{(u,u)}$, where $\kappa_\infty > 0$ is given in (1.6);
- the norm of $D^{1,2}(\mathbb{R}^3)$ defined by $\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2$;
- c^* or c, c_i denote different positive constants;
- the norm in $L^p(\mathbb{R}^3)$ defined by $|u|^p_p = \int_{\mathbb{R}^3} |u|^p$.

In this part we given some basic knowledge which will be used in the later. Considering for all $u \in H^1(\mathbb{R}^3)$, the linear functional J_u defined in $D^{1,2}(\mathbb{R}^3)$ by

$$J_u(v) = \int_{\mathbb{R}^3} u^2 v.$$

We infer from the Hölder inequality that

$$|J_u(v)| \le C|u|_{12/5}^2 \|v\|_{D^{1,2}}. \tag{2.1}$$

By the Lax-Milgram theorem, we know that there exists unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla \phi_u \nabla v = \int_{\mathbb{R}^3} u^2 v \quad \forall v \in D^{1,2}(\mathbb{R}^3). \tag{2.2}$$

So, ϕ_u is a weak solution of $-\Delta\phi = u^2$ and the following formula holds

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy = \frac{1}{|x|} * u^2. \tag{2.3}$$

Moreover, $\phi_u > 0$ when $u \neq 0$.

We recall the following classical Hardy-Littlewood-Sobolev inequality (see [21, Theorem 4.3]). Assume that $f \in L^p(\mathbb{R}^3)$ and $g \in L^q(\mathbb{R}^3)$. Then one has

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f(x)g(y)}{|x-y|^t} dxdy \le c(p,q,t)|f|_p|g|_q, \tag{2.4}$$

where $1 < p, q < \infty$, $0 < t < N$ and $\frac{1}{p} + \frac{1}{q} + \frac{t}{3} = 2$. By (2.4) we know that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dxdy \le c|u|_{12/5}^4 \le c\|u\|^4. \tag{2.5}$$

It is well-known that solutions of (1.6) correspond to critical points of the energy functional

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (\kappa_\infty + \lambda\kappa(x))u^2\right) - \frac{1}{4}\int_{\mathbb{R}^3} (\mu_\infty + \mu(x))\phi_u u^2, \tag{2.6}$$

for $u \in H^1(\mathbb{R}^3)$. From (2.5), we know that I_λ is well defined, and that

$$I'_\lambda(u)[v] = \int_{\mathbb{R}^3} [\nabla u \nabla v + (\kappa_\infty + \lambda\kappa(x))uv] - \int_{\mathbb{R}^3} (\mu_\infty + \mu(x))\phi_u uv, \tag{2.7}$$

for all $v \in H^1(\mathbb{R}^3)$. We define the operator $\Phi : H^1(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3)$ as

$$\Phi[u] = \phi_u.$$

From [10, Proposition 2.2-2.3] we know that Φ has the following properties.

Lemma 2.1. (1) *Φ is continuous;*

(2) *Φ maps bounded sets into bounded sets;*

(3) *$\Phi[tu] = t^2\Phi[u]$ for all $t \in \mathbb{R}$;*

(4) *If $u_n \rightharpoonup u \in H^1(\mathbb{R}^3)$ then $\Phi[u_n] \to \Phi[u]$ in $D^{1,2}(\mathbb{R}^3)$. Moreover,*

$$\int_{\mathbb{R}^3} \mu(x)\phi_{u_n}(x)u_n^2 \to \int_{\mathbb{R}^3} \mu(x)\phi_u(x)u^2,$$

$$\int_{\mathbb{R}^3} \mu(x)\phi_{u_n}(x)u_n\phi \to \int_{\mathbb{R}^3} \mu(x)\phi_u(x)u\phi,$$

for all $\phi \in H^1(\mathbb{R}^3)$.

It is easy to verify that, whatever $\lambda \in \mathbb{R}$ is, the function I_λ is bounded neither from above nor from below. Hence, it is convenient to consider I_λ restricted to a natural constraint, the Nehari manifold. We set

$$\mathscr{N}_\lambda := \left\{u \in H^1(\mathbb{R}^3)\backslash\{0\} : I_\lambda'(u)u = 0\right\}. \tag{2.8}$$

The next lemma contains the statement of the main properties of $\mathscr{N}_\lambda$.

Lemma 2.2. *Assume that* (A1), (A2) *hold. Then for all $\lambda \in \mathbb{R}^+$, we have*

(i) *$\mathscr{N}_\lambda$ is a C^1 regular manifold diffeomorphic to the sphere of $H^1(\mathbb{R}^N)$;*
(ii) *I_λ is bounded from below on $\mathscr{N}_\lambda$ by a positive constant;*
(iii) *u is a free critical point of I_λ if and only if u is a critical point of I_λ constrained on $\mathscr{N}_\lambda$.*

Proof. (i) Let $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ be such that $\|u\| = 1$. We claim that there exists a unique $t \in (0, +\infty)$ for which $tu \in \mathscr{N}_\lambda$. In fact, considering the equation

$$I_\lambda'(tu)[tu] = t^2\Big[\int_{\mathbb{R}^3} |\nabla u|^2 + (k_\infty + \lambda k(x))u^2 - t^2 \int_{\mathbb{R}^3} (\mu_\infty + \mu(x))\phi_u u^2\Big] = 0. \tag{2.9}$$

It is clear that it admits a unique positive solution $t_\lambda(u) > 0$ and that corresponding point $t_\lambda(u)u \in \mathscr{N}_\lambda$, the projection of u on $\mathscr{N}_\lambda$, is such that

$$I_\lambda(t_\lambda(u)u = \max_{t\geq 0} I_\lambda(tu).$$

Similar to the proof of (2.5), we infer from $u \in \mathscr{N}_\lambda$ and (A_1)-(A_2) that

$$\|u\|^2 \leq \int_{\mathbb{R}^3} |\nabla u|^2 + (k_\infty + \lambda k(x))u^2 = \int_{\mathbb{R}^3} (\mu_\infty + \mu(x))\phi_u u^2 \leq c\|u\|^4. \tag{2.10}$$

This implies that

$$\|u\| \geq c > 0. \tag{2.11}$$

Set $G_\lambda(u) := I_\lambda'(u)[u]$. By the regularity of I_λ we know that $G_\lambda \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$. Moreover, by using (2.11), we obtain that

$$G_\lambda'(u)[u] = -2\int_{\mathbb{R}^3} |\nabla u|^2 + (k_\infty + \lambda k(x))u^2 \leq -2c < 0. \tag{2.12}$$

(ii) For all $u \in \mathscr{N}_\lambda$, one sees that

$$I_\lambda(u) = \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u|^2 + (k_\infty + \lambda k(x))u^2 \geq \frac{1}{4}\|u\|^2 \geq C > 0. \tag{2.13}$$

(iii) If $u \neq 0$ is a critical point of I_λ, then $I_\lambda'(u) = 0$ and then $u \in \mathscr{N}_\lambda$. On the other hand, if u is a critical point of I_λ' constrained on $\mathscr{N}_\lambda$, then there exist $\ell \in \mathbb{R}$ such that

$$0 = I_\lambda'(u)[u] = G_\lambda(u) = \ell G_\lambda'(u)[u].$$

We infer from (2.12) that $\ell = 0$. $\square$

Next we consider the limit functional $I_\infty : H^1(\mathbb{R}^3) \to \mathbb{R}$, defined as

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + k_\infty u^2\right) - \frac{1}{4}\int_{\mathbb{R}^3} \mu_\infty \phi_u(x)u^2, \quad u \in H^1(\mathbb{R}^3).$$

and the related natural constraint

$$\mathscr{N}_\infty := \left\{u \in H^1(\mathbb{R}^3)\backslash\{0\} : I_\infty'(u)u = 0\right\}.$$

Obviously, critical points of I_∞ are solutions of the limit problem at infinity

$$\begin{gathered} -\Delta u+\kappa_\infty u=\mu_\infty \phi_u(x)u, \quad \text{in } \mathbb{R}^3,\\ -\Delta\phi=u^2, \quad u\in H^1(\mathbb{R}^3). \end{gathered} \tag{2.14}$$

Clearly, the conclusions of Lemma 2.2 hold for I'_∞ and $\mathscr{N}_\infty$. Furthermore, for any $u\in H^1(\mathbb{R}^3)\backslash\{0\}$, it is easy to see that there exists unique $t(u)>0$ such that $t(u)u\in\mathscr{N}_\infty$. Set

$$m_\infty:=\inf\{I_\infty(u),u\in\mathscr{N}_\infty\}. \tag{2.15}$$

From [27, 26], we know that m_∞ is achieved by a radially symmetric function w, unique up to translations, and decreasing when the radial coordinate increases. Precisely, there exists a constant $c^*>0$ such that

$$\lim_{|x|\to+\infty}|w(x)||x|^{1-c^*}e^{\sqrt{\kappa_\infty}|x|}=\text{constant}. \tag{2.16}$$

In what follows, for any $y\in\mathbb{R}^3$, we use the translation symbol

$$w_y:=w(\cdot-y). \tag{2.17}$$

Set

$$m_\lambda:=\inf\{I_\lambda(u),u\in\mathscr{N}_\lambda\}. \tag{2.18}$$

Then the following properties of m_λ and m_∞ hold.

Lemma 2.3. *Suppose that* (A1), (A2) *hold. Then for* $\lambda\ge 0$ *we have*

$$0<m_\lambda\le m_\infty. \tag{2.19}$$

Proof. Let $\lambda\ge0$ be fixed. The first inequality of (2.19) is a straight consequence of (2.13). In order to show the second inequality we should construct a sequence $\{u_n\}\subset\mathscr{N}_\lambda$ and $\lim_n I_\lambda(u_n)=m_\infty$. To this aim, let us consider $(y_n)_n$, with $y_n\in\mathbb{R}^3,|y_n|\to+\infty$, as $n\to+\infty$ and we set $u_n=t_nw_{y_n}$, where w_{y_n} is defined in (2.17) and $t_n=t_\lambda(w_{y_n})$ such that $u_n=t_nw_{y_n}\in\mathscr{N}_\lambda$. We observe that

$$\begin{aligned} &I_\lambda(u_n)\\ &=\frac{t_n^2}{2}\int_{\mathbb{R}^3}|\nabla w_{y_n}|^2+(\kappa_\infty+\lambda\kappa(x))w_{y_n}^2-\frac{t_n^4}{4}\int_{\mathbb{R}^3}(\mu_\infty+\mu(x))\phi_{w_{y_n}}(x)w_{y_n}^2\\ &=\frac{t_n^2}{2}\Big[\|w\|^2+\lambda\int_{\mathbb{R}^3}\kappa(x+y_n))w^2\Big]-\frac{t_n^4}{4}\int_{\mathbb{R}^3}(\mu_\infty+\mu(x+y_n))\phi_w(x+y_n)w^2. \end{aligned} \tag{2.20}$$

Moreover, from $t_nw_{y_n}\in\mathscr{N}_\lambda$ it follows that

$$t_n^2=\frac{\|w\|^2+\lambda\int_{\mathbb{R}^3}\kappa(x+y_n)w^2}{\int_{\mathbb{R}^3}\mu_\infty\phi_ww^2+\int_{\mathbb{R}^3}\mu(x+y_n)\phi_w(x+y_n)w^2}.$$

It is clear that

$$\begin{gathered} \lim_{n\to\infty}\int_{\mathbb{R}^3}\kappa(x+y_n)w^2=0,\\ \lim_{n\to\infty}\int_{\mathbb{R}^3}\mu(x+y_n)\phi_w(x+y_n)w^2=0. \end{gathered}$$

Thus, we infer that

$$t_n\to1,\quad I_\lambda(u_n)\to m_\infty,\quad \text{as } n\to+\infty. \tag{2.21}$$

□

By apply the well-known concentration-compactness principle[24] and maximum principle[39], we have the following results for m_λ.

Lemma 2.4. *If the strictly inequality*
$$m_\lambda < m_\infty \tag{2.22}$$
holds, then m_λ is achieved by a positive function. Moreover, all the minimizing sequences are relatively compact.

Lemma 2.5. *Assume that $\lambda = 0$, (A1), (A2) hold. Then (1.6) has a positive ground state solution.*

Proof. Note that
$$\begin{gathered}
I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} (\mu_\infty + \mu(x))\phi_u(x)u^2, \\
I_\infty(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} \mu_\infty\phi_u(x)u^2.
\end{gathered}$$
Thus, we infer that $m_0 \le m_\infty$. To complete the proof we only need to show that $m_0 < m_\infty$. Assume, by contradiction, that $w \in \mathscr{N}_\infty$ and $I_\infty(w) = m_\infty = m_0$. Then there exists $t_n > 0$ such that $t_n w_{y_n} \in \mathscr{N}_0$, and $t_n \to 1$, as $n \to \infty$. This implies that
$$m_\infty = m_0 \le I_0(t_n w_{y_n}) < I_\infty(w) = m_\infty.$$
This is impossible. □

The next lemma analyzes the behavior of some sequences of $\{u_n\} \subset \mathscr{N}_{\lambda_n}$.

Lemma 2.6. *Suppose that (A1), (A2) hold. Let $(\lambda_n)_n$ be a sequence of positive numbers, for all $n \in \mathbb{N}$, and $u_n \in \mathscr{N}_{\lambda_n}$ be such that $I_{\lambda_n}(u_n) \le C$. Then $\{u_n\}_n$ is bounded in $H^1(\mathbb{R}^3)$.*

Proof. We infer from $\{u_n\} \subset \mathscr{N}_{\lambda_n}$ that
$$I_{\lambda_n}(u_n) = \frac{1}{4}\Big(\|u_n\|^2 + \lambda_n \int_{\mathbb{R}^3} \kappa(x)u_n^2\Big) \le C. \tag{2.23}$$
Thus the conclusion holds. □

Lemma 2.7. *Assume that (A1), (A2) hold. Let $u_n \in \mathscr{N}_{\lambda_n}$ be such that $I_{\lambda_n}(u_n) \le C$, and $\lambda_n \to \infty$, as $n \to \infty$. Then, for all $R > 0$,*
$$\begin{gathered}
u_n|_{B_R} \to 0, \ \text{in } L^2(B_R), \quad \lambda_n \int_{\mathbb{R}^3} \kappa(x)u_n^2 \le C, \\
\int_{\mathbb{R}^3} \mu(x)\phi_{u_n}u_n^2 \to \ 0, \quad \text{as } n \to +\infty.
\end{gathered} \tag{2.24}$$

Proof. Since $\{u_n\}$ satisfies the inequality (2.23), one can check that the first two conclusions of (2.24) are true. Next we prove the third one. By (A_2), we know that for any $\varepsilon > 0$, there exists $R > 0$ such that for all $x \in \mathbb{R}^3 \setminus B_R$, $\mu(x) < \varepsilon$. Thus we infer from Lemma 2.6 that
$$\int_{\mathbb{R}^3\setminus B_R} \mu(x)\phi_{u_n}u_n^2 \le \varepsilon \int_{\mathbb{R}^3\setminus B_R} \phi_{u_n}(x)u_n^2 \le \varepsilon C. \tag{2.25}$$

On the other hand, by Hardy-Littlewood-Sobolev inequality, one sees that

$$\int_{B_R} \mu(x)\phi_{u_n} u_n^2 \le C \int_{B_R} \phi_{u_n} u_n^2 \le C \left(\int_{B_R} u_n^{\frac{12}{5}}\right)^{\frac{5}{6}} \to 0, \ as\ n \to \infty. \tag{2.26}$$

From (2.25)-(2.26), we know that the conclusions hold. $\square$

3. Properties of the map $\lambda \mapsto m_\lambda$

In this section we shall show that the monotonicity property of the map $\lambda \mapsto m_\lambda$.

Lemma 3.1. *Assume that* (A1), (A2) *hold. Then the map $\lambda \mapsto m_\lambda$ is nondecreasing.*

Proof. For $u \in H^1(R^3)\backslash\{0\}$, $\lambda \in \mathbb{R}^+$ and $t_\lambda(u)u \in \mathscr{N}_\lambda$, we have

$$[t_\lambda(u)]^2 = \frac{\|u\|^2 + \lambda \int_{\mathbb{R}^3} \kappa(x)u^2}{\int_{\mathbb{R}^3}[\mu_\infty + \mu(x)]\phi_u u^2}.$$

If $\lambda_1, \lambda_2 \in \mathbb{R}^+$ such that $\lambda_1 < \lambda_2$, then $t_{\lambda_1}(u) \le t_{\lambda_2}(u)$. Moreover, $t_{\lambda_1}(u) = t_{\lambda_2}(u)$ if and only if $\int_{\mathbb{R}^3} \kappa(x)u^2 = 0$. So, we obtain

$$\begin{aligned} I_{\lambda_1}(t_{\lambda_1}u) &= \frac{t_{\lambda_1}^2}{4}\Big(\|u\|^2 + \int_{\mathbb{R}^3} \lambda_1\kappa(x)u^2\Big) \\ &\le \frac{t_{\lambda_2}^2}{4}\Big(\|u\|^2 + \int_{\mathbb{R}^3} \lambda_2\kappa(x)u^2\Big) \\ &= I_{\lambda_2}(t_{\lambda_1}u). \end{aligned} \tag{3.1}$$

Therefore, by the arbitrariness of u, we obtain that $m_{\lambda_1} \le m_{\lambda_2}$. $\square$

Remark 3.2. We observe that if $u \in H^1(R^3)\backslash\{0\}$, and $\lambda_1 < \lambda_2$, λ_1, $\lambda_2 \in \mathbb{R}^+$, then

$$t_{\lambda_1}(u) = t_{\lambda_2}(u) \iff \int_{\mathbb{R}^3} \kappa(x)u^2 = 0.$$

Next we prove some properties for m_λ and m_∞.

Lemma 3.3. *Assume that* (A1), (A2) *hold. If there exists $\nu \in \mathbb{R}^+$ such that $m_\nu = m_\infty$, then we have $m_\lambda = m_\infty$ for all $\lambda > \nu$. Moreover, m_λ is not achieved.*

Proof. By Lemmas 2.4 and 3.1, we deduce that

$$m_\infty = m_\nu \le m_\lambda \le m_\infty.$$

Thus, we obtain $m_\lambda = m_\infty$.

Next we shall prove that m_λ is not achieved. Arguing by contradiction, we assume that there exists $u_\lambda \in \mathscr{N}_\lambda$ such that $I_\lambda(u_\lambda) = m_\lambda = m_\infty$. Let $t_\nu = t_\nu(u_\lambda) > 0$ be such that $t_\nu u_\lambda \in \mathscr{N}_\nu$. By using the same arguments as in Lemma 3.1 and Remark 3.2, we can get $t_\nu < 1$, and thus

$$m_\infty = m_\nu \le I_\nu(t_\nu u_\lambda) < I_\lambda(u_\lambda) = m_\infty.$$

This is a contradiction. $\square$

As a consequence of Lemma 3.3, we have the following results.

Corollary 3.4. *Assume that* (A1), (A2) *hold. Then there exists at most one number $\nu \in \mathbb{R}^+$ such that $m_\nu = m_\infty$ and it is achieved.*

Let us define

$$\lambda^* := \sup\{\lambda \in \mathbb{R}^+ : m_\lambda < m_\infty\}. \tag{3.2}$$

Then the following lemma states the role of λ^*.

Proposition 3.5. *Suppose that $\lambda^* < +\infty$. Then*

$$m_{\lambda^*} = m_\infty. \tag{3.3}$$

and

$$\sup\{\lambda \in \mathbb{R}^+ : m_\lambda < m_\infty\} = \min\{\lambda \in \mathbb{R}^+ : m_\lambda = m_\infty\}. \tag{3.4}$$

Proof. We use the contradiction method. If $m_{\lambda^*} < m_\infty$, then there exists $u^* \in \mathcal{N}_{\lambda^*}$ such that $I_{\lambda^*}(u_{\lambda^*}) = m_{\lambda^*}$. Let $(\lambda_n)_n$ be a sequence of number such that $\lambda_n \searrow \lambda^*$. By Lemma 3.3, we know that $m_{\lambda_n} = m_\infty$. Moreover, there exist $t_n := t_{\lambda_n}(u_{\lambda^*})$ such that $t_n u_{\lambda^*} \in \mathcal{N}_{\lambda_n}$. By the definition of t_n, we obtain $t_n \to 1$, as $n \to \infty$. So, we obtain

$$\begin{aligned} m_\infty = m_{\lambda_n} \le I_{\lambda_n}(t_n u_{\lambda^*}) &= \frac{1}{4} t_n^2 [\|u_{\lambda^*}\|^2 + \int_{\mathbb{R}^3} \lambda_n \kappa(x) u_{\lambda^*}^2], \\ \xrightarrow[n \to +\infty]{} \frac{1}{4}[\|u_{\lambda^*}\|^2 &+ \int_{\mathbb{R}^3} \lambda^* \kappa(x) u_{\lambda^*}^2] = I_{\lambda^*}(u_{\lambda^*}) = m^* < m_\infty. \end{aligned}$$

This is a contradiction. Finally, (3.4) is easily obtain from (3.3). $\square$

Next we show the continuity of the map $\lambda \mapsto m_\lambda$.

Lemma 3.6. *Let* (A1), (A2) *be satisfied. Then the map $\lambda \mapsto m_\lambda$ is continuous for $\lambda \in \mathbb{R}^+$.*

Proof. We divide into the following two cases to prove the results.

Case 1. $\lambda^* = \infty$. By the definition of λ^*, we infer that for each $\lambda \in \mathbb{R}^+$, $m_\lambda < m_\infty$. By Lemma 2.4, there exists $u_\lambda \in \mathcal{N}_\lambda$ such that $I_\lambda(u_\lambda) = m_\lambda$. Let $\{\lambda_n\}$ be such that $\lambda_n \to \lambda$, and $t_n := t_{\lambda_n}(u_\lambda)$ be such that $t_n u_\lambda \in \mathcal{N}_{\lambda_n}$. Then, by using the definition of t_n, we obtain that $t_n \to 1$ as $n \to \infty$. Hence, one sees that

$$\begin{aligned} m_{\lambda_n} \le I_{\lambda_n}(t_n u_{\lambda_n}) &= \frac{1}{4} t_n^2 [\|u_\lambda\|^2 + \lambda_n \int_{\mathbb{R}^3} \kappa(x) u_{\lambda_n}^2] \\ &\xrightarrow[n \to +\infty]{} \frac{1}{4} [\|u_\lambda\|^2 + \lambda \int_{\mathbb{R}^3} \kappa(x) u_\lambda^2] \\ &= I_\lambda(u_\lambda) = m_\lambda. \end{aligned} \tag{3.5}$$

This implies

$$\limsup_n m_{\lambda_n} \le m_\lambda. \tag{3.6}$$

Since $m_{\lambda_n} < m_\infty$ for all $n \in \mathbb{N}$, we deduce from Lemma 2.4 that there exist $u_n \in \mathcal{N}_{\lambda_n}$ such that $I_{\lambda_n}(u_n) = m_{\lambda_n}$. Moreover, one infers from Lemma 2.7 that the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

Let $\tilde{t}_n := t_\lambda(u_n)$ be such that $\tilde{t}_n u_n \in \mathcal{N}_\lambda$. Since

$$1 = \frac{\|u_n\|^2 + \lambda_n \int_{\mathbb{R}^3} \kappa(x) u_n^2}{\int_{\mathbb{R}^3} (\mu_\infty + \mu(x)) \phi_{u_n} u_n^2}, \quad (\tilde{t}_n)^2 = \frac{\|u_n\|^2 + \lambda \int_{\mathbb{R}^3} \kappa(x) u_n^2}{\int_{\mathbb{R}^3} (\mu_\infty + \mu(x)) \phi_{u_n} u_n^2}.$$

we deduce that $\tilde{t}_n \to 1$ and $|I_\lambda(\tilde{t}_n u_n) - m_{\lambda_n}| \to 0$, as $n \to +\infty$. Thus, we obtain

$$m_\lambda \le \limsup_n \ m_{\lambda_n}. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain the conclusion as required.

Case 2. $\lambda^* \in \mathbb{R}^+$. For any $\lambda \in (0, \lambda^*)$, we can use the same arguments as Case 1 to obtain the conclusion. On the other hand, for any $\lambda \in (\lambda^*, +\infty)$, $\lambda \mapsto m_\lambda$ is a constant map. Therefor, we just need to prove the continuity when $\lambda = \lambda^*$.

Let $\{\lambda_n\}$ be a sequence of number and $\lambda_n \to \lambda^*$ By (3.3), if $\lambda_n \searrow \lambda^*$, the result is trivial. So, in the following we study the case $\lambda_n \nearrow \lambda^*$. By the definition of $\mathcal{N}_{\lambda^*}$, for fixed $\varepsilon > 0$, there exists $u_\varepsilon \in \mathcal{N}_{\lambda^*}$ such that $I_{\lambda^*}(u_\varepsilon) < m_{\lambda^*} + \varepsilon$. Let $t_{n,\varepsilon} := t_{\lambda_n}(u_\varepsilon)$ be such that $t_{n,\varepsilon} u_\varepsilon \in \mathscr{N}_{\lambda_n}$. On can easy to deduce that $t_{n,\varepsilon} \to 1$, as $n \to +\infty$. Moreover, we have

$$
\begin{aligned}
m_{\lambda_n} \le I_{\lambda_n}(t_{n,\varepsilon} u_\varepsilon) &= \frac{1}{4} t_{n,\varepsilon}^2 [\|u_\varepsilon\|^2 + \lambda_n \int_{\mathbb{R}^3} \kappa(x) u_\varepsilon^2] \\
&\overrightarrow{n \to \infty} \frac{1}{4} [\|u_\varepsilon\|^2 + \lambda^* \int_{\mathbb{R}^3} \kappa(x) u_\varepsilon^2] = I_{\lambda^*}(u_\varepsilon) < m_{\lambda^*} + \varepsilon.
\end{aligned}
$$

Thus, we obtain

$$
\limsup_n m_{\lambda_n} \le m_{\lambda^*} + \varepsilon.
$$

By the arbitrariness of ε, we can obtain

$$
\limsup_n \ m_{\lambda_n} \le m_{\lambda^*}.
$$

On the other hand, for all n, $m_{\lambda_n} < m_\infty$, we can use the same arguments as Case 1 to showing that

$$
\liminf_n \ m_{\lambda_n} \le m_{\lambda^*}.
$$

This completes the proof. □

Remark 3.7. By the continuity of the map $\lambda \mapsto m_\lambda$ and the fact that $m_0 < m_\infty$, we infer that $\lambda^* > 0$.

4. Two kinds of possible situations for λ^*

In this section we study the properties of λ^* according to the decay of the functions $\kappa(x)$ and $\mu(x)$. Let us first consider the case when $\kappa(x)$ decays faster than $\mu(x)$.

Lemma 4.1. *Assume that* (A1)–(A3) *hold. Then we have that* $\lambda^* = +\infty$*, where* λ^* *is defined in* (3.2).

Proof. First, we infer from Lemma 2.5 that $m_0 < m_\infty$. So, in the following we only need to consider the case $\lambda > 0$. For fixed $\lambda > 0$, we choose t_n such that $u_n = t_n w_{y_n} \in \mathscr{N}_\lambda$, where y_n and t_n are chose as in the proof of Lemma 2.3. Moreover, as in (2.21), we infer that $t_n \ge c > 0$. Thus, we obtain that

$$
\begin{aligned}
m_\lambda \le I_\lambda(u_n) &= I_\infty(t_n w) + \frac{t_n^2}{2}\Big[\lambda \int_{\mathbb{R}^3} \kappa(x+y_n)) w^2 - \frac{t_n^2}{2} \int_{\mathbb{R}^3} \mu(x+y_n))\phi_w w^2\Big] \\
&\le I_\infty(w) + \frac{t_n^2}{2}\Big[\lambda \int_{\mathbb{R}^3} \kappa(x+y_n)) w^2 - c \int_{\mathbb{R}^3} \mu(x+y_n))\phi_w w^2\Big] \\
&= m_\infty + \frac{t_n^2}{2}\Big[\lambda \int_{\mathbb{R}^3} \kappa(x+y_n)) w^2 - c \int_{\mathbb{R}^3} \mu(x+y_n))\phi_w w^2\Big].
\end{aligned} \tag{4.1}
$$

Hence, we obtain the conclusion if we show that, for large n,

$$
\int_{\mathbb{R}^3} [\lambda \kappa(x+y_n) w^2 - C\mu(x+y_n)\phi_w w^2] < 0. \tag{4.2}
$$

This is equivalent to prove that, for large n,

$$\begin{aligned} I_1 &:= \int_{\mathbb{R}^3\setminus B_{\tau|y_n|}} \Big[\frac{\lambda}{2}\kappa(x+y_n)w^2 - C\mu(x+y_n)\phi_w w^2\Big] \\ &< I_2 := \int_{B_{\tau|y_n|}} \Big[C\mu(x+y_n)\phi_w w^2 - \frac{\lambda}{2}\kappa(x+y_n)w^2\Big] \end{aligned}$$

To estimate I_1, from (2.16) we have

$$\begin{aligned} I_1 &< \int_{\mathbb{R}^3\setminus B_{\tau|y_n|}} \lambda\kappa(x+y_n)w^2 \\ &< \lambda\Big[\int_{\mathbb{R}^3\setminus B_{\tau|y_n|}} |\kappa(x+y_n)|^{\frac{3}{2}}\Big]^{2/3}\Big[\int_{\mathbb{R}^3\setminus B_{\tau|y_n|}} |w|^6\Big]^{1/3} \\ &< ce^{-2\tau\sqrt{\kappa_\infty}|y_n|}. \end{aligned} \tag{4.3}$$

Now we estimate the I_2 term. By (A3), for all $\varepsilon > 0$ and $M > 0$, there exists $n_0 \geq 1$ such that, for all $n \geq n_0$ and for all $x \in B_{\tau|y_n|}$,

$$\kappa(x+y_n) \leq \varepsilon(1-\tau)^{-1}|y_n|^{-1}e^{-2\tau\sqrt{\kappa_\infty}|y_n|}, \quad \mu(x+y_n) \geq Me^{-2\tau\sqrt{\kappa_\infty}|y_n|}. \tag{4.4}$$

By [20, Lemmas 2.3 and 2.6], we know that

$$\phi_w(x) = \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|}dy \sim \frac{1}{|x|}, \quad \text{as } |x| \to \infty. \tag{4.5}$$

Thus, we infer that, for n sufficiently large and for all $x \in B_{\tau|y_n|}$,

$$C\phi_w(x) - \frac{\lambda\kappa(x+y_n)}{\mu(x+y_n)} > \frac{C}{2}\phi_w(x).$$

Hence, one sees that

$$\begin{aligned} I_2 &= \int_{B_{\tau|y_n|}} \mu(x+y_n)w^2[C\phi_w(x) - \frac{\lambda\kappa(x+y_n)}{\mu(x+y_n)}] \\ &> \frac{C}{2}\int_{B_{\tau|y_n|}} \mu(x+y_n)\phi_w w^2 > CMe^{-2\tau\sqrt{\kappa_\infty}|y_n|}\int_{B_1} \phi_w w^2 \\ &> CMe^{-2\tau\sqrt{\kappa_\infty}|y_n|}. \end{aligned} \tag{4.6}$$

Combining (4.3)-(4.6), together with the arbitrariness of M, we can conclude that $I_1 < I_2$. This completes the proof. $\square$

Next we consider the case when $\kappa(x)$ decays slower than $\mu(x)$.

Lemma 4.2. *Suppose that* (A1), (A2), (A4) *hold. Then* $\lambda^* \in \mathbb{R}^+$*, where* λ^* *is defined in* (3.2).

Proof. We use the contradiction method. Assume that for all $\lambda \in \mathbb{R}^+$, $m_\lambda < m_\infty$. By proposition 3.5. Let $\{\lambda_n\}$ be a diverging sequence. From Lemma 2.4, there exist $\{u_n\}$ such that for all $n \in N$,

$$u_n > 0, \quad u_n \in \mathscr{N}_{\lambda_n}, \quad I_{\lambda_n}(u_n) = m_{\lambda_n} < m_\infty, \quad I'_{\lambda_n}(u_n) = 0.$$

We infer from Lemma 2.6 that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Let $\theta_n = \theta_{u_n}$ be such that $\theta_n u_n \in \mathscr{N}_\infty$. A direct computation show that

$$\theta_n^2 = \frac{\|u_n\|^2}{\mu_\infty\int_{\mathbb{R}^3}\phi_u(x)u^2}. \tag{4.7}$$

We claim that

$$I_{\lambda_n}(\theta_n u_n) < I_\infty(\theta_n u_n). \tag{4.8}$$

Otherwise, one sees that

$$m_\infty \le I_\infty(\theta_n u_n) \le I_{\lambda_n}(\theta_n u_n) \le I_{\lambda_n}(u_n) = m_{\lambda_n} < m_\infty. \tag{4.9}$$

This is impossible. So, the claim (4.8) holds. Moreover, we deduce from the boundedness of $\|u_n\|$ that there exist two numbers $c, C > 0$ such that

$$c \le \theta_n \le C. \tag{4.10}$$

From (4.8) we infer that

$$\frac{\lambda_n}{2}\int_{\mathbb{R}^3} \kappa(x)u_n^2 - \frac{\theta_n^2}{4}\int_{\mathbb{R}^3} \mu(x)\phi_{u_n}(x)u_n^2 < 0. \tag{4.11}$$

We deduce from (2.24) and (4.10) that

$$\frac{\lambda_n}{2}\int_{\mathbb{R}^3} \kappa(x)u_n^2 \to 0, \quad \text{as } n \to \infty. \tag{4.12}$$

Since $u_n \in \mathscr{N}_{\lambda_n}$ and $\theta_n u_n \in \mathscr{N}_\infty$, we deduce from (2.5) that

$$\begin{gathered}
c\Big(\int_{\mathbb{R}^3} \phi_{u_n}(x)u_n^2\Big)^{1/2} \le \|u_n\|^2 = \mu_\infty \int_{\mathbb{R}^3} \phi_{u_n}(x)u_n^2, \\
\mu_\infty(\theta_n^2 - 1)\int_{\mathbb{R}^3} \phi_{u_n}(x)u_n^2 = \int_{\mathbb{R}^3} \mu(x)\phi_{u_n}(x)u_n^2 - \lambda_n \int_{\mathbb{R}^3} \kappa(x)u_n^2 = o(1).
\end{gathered} \tag{4.13}$$

These together with (2.24) imply that

$$\lim_n \theta_n = 1. \tag{4.14}$$

Hence, one infers from (2.24) and (4.12) that

$$\begin{aligned}
m_\infty &> I_\lambda(u_n) = I_{\lambda_n}(\theta_n u_n) + o(1) \\
&= I_\infty(\theta_n u_n) + \frac{\theta_n \lambda_n}{2}\int_{\mathbb{R}^3} \kappa(x)u_n^2 - \frac{\theta_n^2}{4}\int_{\mathbb{R}^3} \mu(x)\phi_{u_n}(x)u_n^2 \\
&= I_\infty(\theta_n u_n) + o(1) \\
&\ge m_\infty + o(1).
\end{aligned} \tag{4.15}$$

This implies

$$I_\infty(\theta_n u_n) \to m_\infty, \quad \text{as } n \to +\infty. \tag{4.16}$$

By the uniqueness of the family of minimizers of I_∞ on $\mathscr{N}_\infty$, there exists sequence $\{y_n\}$ such that $y_n \in \mathbb{R}^3$ and

$$\theta_n u_n - w_{y_n} \to 0 \quad \text{in } H^1(\mathbb{R}^3) \quad \text{as } n \to +\infty,$$

where w is given by (2.16). Set $v_n(x) = u_n(x + y_n)$. We infer from (4.14) that

$$v_n \to w \quad \text{in } H^1(\mathbb{R}^3) \quad \text{as } n \to +\infty.$$

Since, for all n, v_n is a solution of

$$-\Delta u + (\kappa_\infty + \lambda\kappa(x + y_n)u = (\mu_\infty + \mu(x + y_n)\phi_u(x)u. \tag{4.17}$$

By the Schauder interior(see [35]), we know that $v_n \to w$ in $C^2_{\rm loc}(\mathbb{R}^3)$. Moreover, from the decay estimates (see [27]), one deduces that for some $\sigma \in (0, \sqrt{\kappa_\infty})$

$$|v_n(x)| \le ce^{-\sqrt{\sigma}|x|}. \tag{4.18}$$

By (4.11), it suffices to show that for n large enough,

$$\frac{\lambda_n}{2}\int_{\mathbb{R}^3}\kappa(x+y_n)v_n^2-\frac{3}{8}\int_{\mathbb{R}^3}\mu(x+y_n)\phi_{v_n}v_n^2>0. \tag{4.19}$$

That is, we need to prove that for $\tau\in(0,1)$,

$$\begin{aligned}I_1 &:= \int_{B_{\tau|y_n|}}[\frac{\lambda_n}{2}\kappa(x+y_n)v_n^2-\frac{3}{8}\mu(x+y_n)\phi_{v_n}v_n^2]\\ &< I_2 := \int_{\mathbb{R}^3\setminus B_{\tau|y_n|}}[\frac{3}{8}\mu(x+y_n)\phi_{v_n}v_n^2-\frac{\lambda_n}{2}\kappa(x+y_n)v_n^2]\end{aligned} \tag{4.20}$$

To estimate I_1, from (A4) we have that for all $x\in B_{\tau|y_n|}$,

$$\kappa(x+y_n)\ge c_1e^{-4\tau\sqrt{\sigma}|y_n|}\quad\text{and}\quad \mu(x+y_n)\le c_2e^{-4\tau\sqrt{\sigma}|y_n|}.$$

Thus, for any $C>0$ and $x\in B_{\tau|y_n|}$, we infer that if n is large enough,

$$\frac{\lambda_n}{2}-\frac{c\kappa(x+y_n)}{\mu(x+y_n)}>\frac{\lambda_n}{4}.$$

Since $v_n\to w$ in $C^2_{\rm loc}(\mathbb{R}^3)$, it follows that

$$\begin{aligned}I_1 &\ge \int_{B_{\tau|y_n|}}\kappa(x+y_n)v_n^2\Big(\frac{\lambda_n}{2}-\frac{c\mu(x+y_n)}{\kappa(x+y_n)}\Big)\\ &> c\lambda_n\int_{B_{\tau|y_n|}}\kappa(x+y_n)v_n^2\\ &> c\lambda_ne^{-4\tau\sqrt{\sigma}|y_n|}\int_{B_1}v_n^2>c\lambda_ne^{-4\tau\sqrt{\sigma}|y_n|}.\end{aligned} \tag{4.21}$$

On the other hand, from (4.18) one infers that

$$I_2<\int_{\mathbb{R}^3\setminus B_{\tau|y_n|}}\frac{3}{8}\mu(x+y_n)\phi_{v_n}v_n^2\le c\Big(\int_{\mathbb{R}^3\setminus B_{\tau|y_n|}}v_n^{\frac{12}{5}}\Big)^{5/3}<ce^{-4\tau\sqrt{\sigma}|y_n|}. \tag{4.22}$$

Hence, by the divergence of λ_n, for large n, we can conclude that $I_2<I_1$. This completes the proof. $\square$

Proof of Theorem1.1. By Lemmas 2.4 and 4.1, we know that the conclusions of Theorem 1.1 hold. $\square$

5. Proof of Theorem 1.2

As we already pointed out in the introduction, new difficulty arises here. That is, according to [18], we know that any sign-changing solution u of (2.14) such that $I_\infty(u)<2m_\infty$. From this we can not prove that I_λ satisfies the $(PS)_c$-condition for $c\in(m_\infty,2m_\infty)$. Motivated by [25, 36], we shall consider our problem in convex set $H^1_+(\mathbb{R}^3)$ to overcome the difficult, where $H^1_+(\mathbb{R}^3):=\{u\in H^1(\mathbb{R}^3):u\ge0\}$.

For any point $u\in H^1_+(\mathbb{R}^3)$, we define

$$J(u)=\sup_{u_1\in H^1_+(\mathbb{R}^3),\ \|u-u_1\|<1}\langle\, I'(u),u-u_1\rangle. \tag{5.1}$$

It is easy to check that J is continuous on $H^1_+(\mathbb{R}^3)$. We define

$$\mathscr{N}_\lambda^+=\mathscr{N}_\lambda\cap H^1_+(\mathbb{R}^3), \tag{5.2}$$

$$d=\inf_{u\in\mathscr{N}_\lambda}I(u),\quad d^+=\inf_{u\in\mathscr{N}_\lambda^+}I(u). \tag{5.3}$$

Next we study the properties of the Palais-Smale sequence of (1.6) on $H^1_+(\mathbb{R}^3)$ at level c, for $c \in (m_\infty, 2m_\infty)$.

Lemma 5.1. *Suppose that* (A1), (A2) *hold. Let* $\{u_n\} \subseteq \mathcal{N}^+_\lambda$ *be a sequence such that* $I_\lambda(u_n)$ *is bounded, and* $J(u_n) \to 0$ *strongly in* $H^1_+(\mathbb{R}^3)$. *Then, up to a subsequence, there exist a solution* $\bar{u}$ *of* (I_λ), *a number* $k \in \mathbb{N} \cup \{0\}$, k *functions* $u^1, \dots, u^k$ *of* $H^1_+(\mathbb{R}^3)$ *and* k *sequence of points* $(y^j_n), y^j_n \in \mathbb{R}^3$, $0 \le j \le k$ *such that, as* $n \to +\infty$,

$$
\begin{gathered}
u_n - \sum_{j=1}^k u^j(\cdot - y^j_n) \to \bar{u} \text{ in } H^1_+(\mathbb{R}^3), \quad I_\lambda(u_n) \to \sum_{j=1}^k I_\infty(u^j) + I_\lambda(\bar{u}), \\
|y^j_n| \to +\infty, \quad |y^j_n - y^j_n| \to +\infty \ (\text{if } i \neq j), \\
\text{and } u^j \text{ are weak solutions of } (2.14).
\end{gathered}
$$

Moreover, we notice that in the case $k = 0$, *the above holds without* u^j.

Proof. We claim that $I'(u_n) \to 0$ as $n \to \infty$. To prove this we first prove that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. We assume that $\|u_n\| \to \infty$ as $n \to \infty$. It is very easy to see that

$$
y_n = u_n \pm \frac{u_n}{1 + \|u_n\|} \in H^1_+(\mathbb{R}^3). \tag{5.4}
$$

So, we infer from $J(u_n) \to 0$ as $n \to \infty$, that

$$
\langle I'(u_n), \frac{u_n}{1 + \|u_n\|} \rangle \to 0 \quad \text{as } n \to \infty. \tag{5.5}
$$

Thus, we obtain

$$
\begin{aligned}
0 \leftarrow \frac{I(u_n) - \frac{1}{4}\langle I'(u_n), u_n \rangle}{1 + \|u_n\|} &= \frac{\frac{1}{4}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \lambda \kappa(x) u_n^2}{1 + \|u_n\|} \\
&\ge \frac{\frac{1}{4}\|u_n\|^2}{1 + \|u_n\|} \to \infty.
\end{aligned} \tag{5.6}
$$

as $n \to \infty$. This is contradiction. So, $\|u_n\|$ is bounded. Moreover, as in (5.4)-(5.6) we obtain that $I'_\lambda(u_n)u_n \to 0$ as $n \to \infty$.

We now use an idea from [25, Theorem 7] to claim that $I'(u_n) \to 0$ as $n \to \infty$. Since $\|u_n\|$ is bounded, without loss of generality we assume that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^3)$, $u_n \to u_0$ in $L^p_{\rm loc}(\mathbb{R}^N)(\forall p \in (2, 2^*))$, and $u_n(x) \to u_0(x)$ a.e., in $\mathbb{R}^3$, where $u_0 \ge 0$. From the assumption we can infer that $J(u_n) = o_n(1)$, where $o_n(1) \to 0$ as $n \to \infty$. Let $\varepsilon_n > 0$ be such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $\lim_{n\to\infty} o_n(1)\varepsilon_n^{-1} = 0$. For any $g_1 \in C_0^\infty(\mathbb{R}^3)$, we set

$$
u_{1,n} = u_n + \varepsilon_n g_1 + g_{1,\varepsilon_n} \in H^1_+(\mathbb{R}^3), \tag{5.7}
$$

where $g_{1,\varepsilon_n} = -\min\{0, u_n + \varepsilon_n g_1\} \ge 0$. By the definition of J we know that

$$
\langle I'(u_n), u_n - u_{1,n} \rangle \le J(u_n) = o_n(1). \tag{5.8}
$$

So,

$$
\langle I'(u_n), g_1 \rangle \ge -\varepsilon_n^{-1} \langle I'(u_n), g_{1,\varepsilon_n} \rangle + \varepsilon_n^{-1} o_n(1). \tag{5.9}
$$

By a direct computations we can show that

$$\begin{aligned}
&-\langle I'(u_n), g_{1,\varepsilon_n}\rangle \\
&= -\int_{\mathbb{R}^3} (\nabla u_n \nabla g_{1,\varepsilon_n} + (\kappa_\infty + \lambda\kappa(x))u_n g_{1,\varepsilon_n} + (\mu_\infty + \mu(x))\phi_{u_n} u_n g_{1,\varepsilon_n}) \\
&= \int_{\Omega_n} (\nabla u_n \nabla(u_n + \varepsilon_n g_1) + (\kappa_\infty + \lambda\kappa(x))u_n(u_n + \varepsilon_n g_1)) \\
&\quad - \int_{\Omega_n} (\mu_\infty + \mu(x))\phi_{u_n} u_n(u_n + \varepsilon_n g_1) \\
&\geq \varepsilon_n \int_{\Omega_n} (\nabla u_n \nabla g_1 + (\kappa_\infty + \lambda\kappa(x))u_n g_1) - (\mu_\infty + \mu(x))\phi_{u_n} u_n g_1) \\
&\quad - \int_{\Omega_n} (\mu_\infty + \mu(x))\phi_{u_n} u_n^2 \\
&\geq \varepsilon_n \int_{\Omega_n} (\nabla u_n \nabla g_1 + (\kappa_\infty + \lambda\kappa(x))u_n g_1) - (\mu_\infty + \mu(x))\phi_{u_n} u_n g_1) \\
&\quad - \varepsilon_n^2 \int_{\Omega_n} (\mu_\infty + \mu(x))\phi_{u_n} g_1^2.
\end{aligned} \tag{5.10}$$

where $\Omega_n := \{x \in \mathbb{R}^3 : u_n(x) + \varepsilon_n g_1 < 0\}$. Form $\|u_n\|$ is bounded, we infer that $|\int_{\Omega_n} (\mu_\infty + \mu(x))\ \phi_{u_n} g_1^2|$ and

$$|\int_{\Omega_n} (\nabla u_n \nabla g_1 + (\kappa_\infty + \lambda\kappa(x))u_n g_1) - (\mu_\infty + \mu(x))\phi_{u_n} u_n g_1)\,|$$

are bounded. Moreover, since $|\Omega_n| \to 0$ as $n \to \infty$, we can obtain that:

$$-\langle I'(u_n), g_{1,\varepsilon_n}\rangle \geq o(\varepsilon_n). \tag{5.11}$$

By letting $n \to \infty$, we infer from (5.9) and (5.11) that

$$\lim_{n\to\infty} \langle I'(u_n), g_1\rangle \geq 0, \quad \forall g_1 \in C_0^\infty(\mathbb{R}^3). \tag{5.12}$$

Reversing the sign of g_1 and since $C_0^\infty(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$, We infer that $\lim_{n\to\infty}\langle I'(u_n), g_1\rangle = 0$, for all $g_1 \in H^1(\mathbb{R}^3)$. So, $I'(u_n) \to 0$ as $n \to \infty$ and the claim holds.

The rest of proof is similar to [3, Theorem 4.1], and we omit the details here. □

Now we are ready to prove the compactness condition for the functional I_λ.

Lemma 5.2. *Assume that* (A1), (A2) *hold on* $H_+^1(\mathbb{R}^3)$. *If* $m_\lambda = m_\infty$, *then the functional* I_λ *satisfies the (PS) condition at level* c, *for* $c \in (m_\infty, 2m_\infty)$.

Proof. Let $\{u_n\}$ be a Palais-Smale sequence of I_λ constrained on $\mathcal{N}_\lambda^+$ at level c, for $c \in (m_\infty, 2m_\infty)$. Applying Lemma 5.1 we can get that for any solution of (2.14) satisfies $u \geq 0$ and $I_\infty \geq m_\infty$. Moreover, any critical point $\bar{u}$ of (I_λ) is such that $I_\lambda(\bar{u}) \geq m_\lambda = m_\infty$. Thus, we know that k must be zero, and the conclusion of this lemma holds. □

Let us now recall the barycenter definition of a function $u \in H_+^1(\mathbb{R}^3) \setminus \{0\}$, which has introduced in [6]. Set

$$\widehat{\mu}(u)(x) = \frac{1}{|B_1(0)|} \int_{|B_1(x)|} |u(y)|,$$

which belongs to $L^\infty(\mathbb{R}^3)$ and is continuous; and set

$$\widehat{u}(x) = \Big[\widehat{\mu}(u)(x) - \frac{1}{2}\max \widehat{\mu}(x)\Big]^+, \quad \widehat{u} \in C_0(\mathbb{R}^3).$$

We define that $\beta : H^1_+(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}^3$ as

$$\beta(u) = \frac{1}{|\widehat{u}|_1} \int_{\mathbb{R}^3} x\widehat{u}(x) \in \mathbb{R}^3.$$

Since $\widehat{u}$ has compact support, β is well defined. Moreover, the following properties hold

(a) β is continuous in $H^1_+(\mathbb{R}^3)$;
(b) if u is a radial function, $\beta(u) = 0$;
(c) for all $t \neq 0$ and $u \in H^1_+(\mathbb{R}^3) \setminus \{0\}$, $\beta(tu) = \beta(u)$;
(d) given $z \in \mathbb{R}^3$ and setting $u_z(x) = u(x - z)$, $\beta(u_z) = \beta(u) + z$.

Let

$$\mathscr{B}_0^\lambda := \inf\{I_\lambda(u) : u \in \mathscr{N}_\lambda^+, \beta(u) = 0\}.$$

Lemma 5.3. *Assume that* (A1), (A2) *hold. If* $\lambda \geq 0$ *be fixed, and let* $m_\lambda = m_\infty$ *be not achieved. then*

$$m_\lambda = m_\infty < \mathscr{B}_0^\lambda.$$

Proof. We use the contradiction method. Let $\{u_n\} \subseteq \mathscr{N}_\lambda^+$ be such that $\beta(u_n) = 0$ and $I_\lambda(u_n) = m_\infty + o_n(1)$. From the Ekeland variational principle(see [39] or [25]), we can obtain there exist a sequence of functions $\{v_n\}$ such that

$$\begin{gathered} v_n \in \mathscr{N}_+, \quad I_\lambda(v_n) = m_\infty + o_n(1), \quad J(v_n) \to 0, \\ |\beta(v_n) - \beta(u_n)| = o_n(1). \end{gathered} \tag{5.13}$$

Since m_λ is not achieved, $(v_n)_n$ can not be relatively compact, by Lemma 5.1, the equality

$$v_n = w_{y_n} + o(1).$$

must be true with $|y_n| \to +\infty$, which contradicts (5.13). $\square$

Let $\xi \in \mathbb{R}^3$ with $|\xi| = 1$ and $\Sigma = \partial B_2(\xi)$. We define

$$\mathbf{w} = \frac{w}{\big(\int_{\mathbb{R}^3} \phi_w w^2\big)^{1/4}} \tag{5.14}$$

and for any $y \in \mathbb{R}^3$, $\mathbf{w}_y = \mathbf{w}(\cdot - y)$. Observing that $\mathbf{w}$ satisfies

$$-\Delta\mathbf{w} + \kappa_\infty \mathbf{w} = \mathbf{M}\phi_{\mathbf{w}}\mathbf{w}, \tag{5.15}$$

and by a direct computation we obtain that

$$\mathbf{M} = 2m_\infty^{1/2}\mu_\infty^{1/2}. \tag{5.16}$$

For any $\rho > 0$ and $(z, s) \in \Sigma \times [0, 1]$, we define

$$\psi_\rho(z, s) = (1 - s)\mathbf{w}_{\rho z} + s\mathbf{w}_{\rho\xi}.$$

Let $\Psi_\rho : \Sigma \times [0, 1] \to \mathscr{N}_\lambda^+$ be defined by

$$\Psi_\rho(z, s) = t^\lambda_{z,s}\psi_\rho(z, s),$$

where $t^\lambda_{z,s} > 0$ be such that $t^\lambda_{z,s}\psi_\rho(z, s) \in \mathscr{N}_\lambda^+$. Then we have the following results to describe the property of $\mathscr{B}_0^\lambda$.

Lemma 5.4. *Assume that* (A1), (A2) *hold and let* $\lambda > 0$ *be fixed. Then for all* $\rho > 0$ *we have*

$$\mathscr{B}_0^\lambda \leq \mathscr{T}_\rho^\lambda := \max_{\Sigma\times[0,1]} I_\lambda(\Psi_\rho(z,s)).$$

Proof. Since $\beta(\Psi_\rho(z,0)) = \rho z$, we assert that $\beta \circ \Psi_\rho(\Sigma \times \{0\})$ is homotopically equivalent in $\mathbb{R}^3\backslash\{0\}$ to $\rho\Sigma$, then, we can find $(\overline{z}, \overline{s}) \in \Sigma \times [0, 1]$ and satisfied $\beta(\Psi_\rho(\overline{z}, \overline{s})) = 0$, and, naturally,

$$\mathscr{B}_0^\lambda \leq I_\lambda(\Psi_\rho(\overline{z}, \overline{s})) \leq \mathscr{T}_\rho^\lambda.$$

This completes the proof. $\square$

Lemma 5.5. *Let assumptions* (A1), (A2), (A4) *hold. Then there exist* $\rho_0 > 0$ *such that for* $\rho > \rho_0$,

$$\mathscr{T}_\rho^\lambda < 2m_\infty.$$

Proof. The idea of the proof is similar to that used in [7, 8], and we just sketch it here for reader's convenience. Observing that

$$I_\lambda(\Psi_\rho(z,s)) = \frac{1}{4}\Big\{\frac{\|\psi_\rho(z,s)\|^2 + \lambda\int_{\mathbb{R}^3}\kappa(x)\psi_\rho^2(z,s)}{[\int_{\mathbb{R}^3}(\mu_\infty + \mu(x))\phi_{\psi_\rho(z,s)}(x)\psi_\rho^2(z,s)]^{1/2}}\Big\}^2$$

Let us first evaluate

$$\begin{aligned}
&\mathscr{N}_\rho^\lambda(z,s)\\
&:= \|\psi_\rho(z,s)\|^2 + \lambda\int_{\mathbb{R}^3}\kappa(x)\psi_\rho^2(z,s)\\
&= (1-s)^2\|\mathbf{w}_{\rho z}\|^2 + 2s(1-s)(\mathbf{w}_{\rho z}, \mathbf{w}_{\rho\xi})_{H_+^1} + s^2\|\mathbf{w}_{\rho\xi}\|^2\\
&\quad + \lambda\Big[(1-s)^2\int_{\mathbb{R}^3}\kappa(x)\mathbf{w}_{\rho z}^2 + 2s(1-s)\int_{\mathbb{R}^3}\kappa(x)\mathbf{w}_{\rho z}\mathbf{w}_{\rho\xi} + s^2\int_{\mathbb{R}^3}\kappa(x)\mathbf{w}_{\rho\xi}^2\Big].
\end{aligned}$$

Since $\mathbf{w}$ satisfies (5.15), it follows that $\|\mathbf{w}_{\rho z}\|^2 = \|\mathbf{w}_{\rho\xi}\|^2 = \mathbf{M}$, and

$$(\mathbf{w}_{\rho z}, \mathbf{w}_{\rho\xi})_{H^1} = \mathbf{M}\int_{\mathbb{R}^3}\phi_{\mathbf{w}_{\rho z}}\mathbf{w}_{\rho z}\mathbf{w}_{\rho\xi} = \mathbf{M}\int_{\mathbb{R}^3}\phi_{\mathbf{w}_{\rho\xi}}\mathbf{w}_{\rho\xi}\mathbf{w}_{\rho z}.$$

Then, from [4, Proposition 1.2] or [2, Lemma 3.7], and (4.5) and (A_5) and the facts $|z| \geq 1$ and $c^* > 0$, we infer that

$$\begin{gathered}
\varepsilon_\rho = \int_{\mathbb{R}^3}\phi_{\mathbf{w}_{\rho z}}\mathbf{w}_{\rho z}\mathbf{w}_{\rho\xi} = \int_{\mathbb{R}^3}\phi_{\mathbf{w}_{\rho\xi}}\mathbf{w}_{\rho\xi}\mathbf{w}_{\rho z} \sim |2\rho|^{2c^*-1}e^{-2\rho\sqrt{\kappa_\infty}},\\
\int_{\mathbb{R}^3}\kappa(x)\mathbf{w}_{\rho z}^2 \leq c|\rho z|^{2c^*-2}\log|\rho z|e^{-2|\rho z|\sqrt{\kappa_\infty}} < c\log(3\rho)\rho^{2c^*-2}e^{-2\rho\sqrt{\kappa_\infty}} = o(\varepsilon_\rho),\\
\int_{\mathbb{R}^3}\kappa(x)\mathbf{w}_{\rho\xi}^2 \leq c|\rho\xi|^{2c^*-2}\log|\rho\xi|e^{-2|\rho\xi|\sqrt{\kappa_\infty}} < c\rho^{2c^*-2}e^{-2\rho\sqrt{\kappa_\infty}}\log\rho = o(\varepsilon_\rho),\\
\int_{\mathbb{R}^3}\kappa(x)\mathbf{w}_{\rho z}\mathbf{w}_{\rho\xi} \leq c\left(\int_{\mathbb{R}^3}\kappa(x)\mathbf{w}_{\rho z}^2 + \int_{\mathbb{R}^3}\kappa(x)\mathbf{w}_{\rho\xi}^2\right) = o(\varepsilon_\rho).
\end{gathered}$$

So

$$\mathscr{N}_\rho^\lambda(z,s) = [(1-s)^2 + s^2]\mathbf{M} + 2s(1-s)\mathbf{M}\varepsilon_\rho + o(\varepsilon_\rho).$$

Moreover, by [9, lemma 2.7], we obtain

$$\mathscr{D}_\rho^\lambda(z,s) := \int_{\mathbb{R}^3}(\mu_\infty + \mu(x))\phi_{\psi_\rho(z,s)}(x)\psi_\rho^2(z,s)$$

$$\geq [(1-s)^4 + s^4]\mu_\infty + 3[(1-s)^3 s + (1-s)s^3]\mu_\infty \varepsilon_\rho.$$

Hence

$$\frac{\mathscr{N}_\rho^\lambda(z,s)}{(\mathscr{D}_\rho^\lambda(z,s))^{1/2}} \leq \frac{1}{\mu_\infty^{1/2}}\Big\{\frac{[(1-s)^2+s^2]\mathbf{M}}{[(1-s)^4+s^4]^{1/2}} + 2\gamma(s)\mathbf{M}\varepsilon_\rho + o(\varepsilon_\rho)\Big\}$$

where

$$\gamma(s) = \frac{(1-s)s}{[(1-s)^4+s^4]^{\frac{1}{2}}}\Big(\frac{1}{4} - \frac{3s^2(1-s)^2}{2(1-s)^4+2s^4}\Big).$$

By a direct computation we obtain that $\gamma(1/2) < 0$, hence, there exists $\mathscr{I}_{\frac{1}{2}}$, neighborhood of 1/2, satisfied $\gamma(1/2) < c < 0$ for all $t \in \mathscr{I}_{\frac{1}{2}}$. Hence, for ρ enough large,

$$\max\Big\{\frac{\mathscr{N}_\rho^\lambda(z,s)}{\mathscr{D}_\rho^\lambda(z,s)^{1/2}} | z \in \Sigma, s \in \mathscr{I}_{\frac{1}{2}}\Big\} \leq \frac{\frac{1}{4}\mathbf{M} + 2c\mathbf{M}\varepsilon_\rho + o(\varepsilon_\rho)}{\mu_\infty^{1/2}} < \frac{1}{4}\mu_\infty^{-1/2}\mathbf{M};$$

On the another hand, we have

$$\begin{aligned}
&\lim_{\rho\to+\infty} \max\Big\{\frac{\mathscr{N}_\rho^\lambda(z,s)}{\mathscr{D}_\rho^\lambda(z,s)^{1/2}} | z \in \Sigma, s \in [0,1]\backslash\mathscr{I}_{\frac{1}{2}}\Big\} \\
&\leq \mu_\infty^{-1/2}\mathbf{M}\max\Big\{\frac{[(1-s)^2+s^2]}{[(1-s)^4+s^4]^{1/2}} | s \in [0,1]\backslash\mathscr{I}_{\frac{1}{2}}\Big\} \\
&< \frac{1}{4}\mu_\infty^{-1/2}\mathbf{M}.
\end{aligned}$$

When ρ is large enough,

$$\max_{\Sigma\times[0,1]} \frac{\mathscr{N}_\rho^\lambda(z,s)}{\mathscr{D}_\rho^\lambda(z,s)^{1/2}} < \frac{1}{4}\mu_\infty^{-1/2}\mathbf{M}.$$

By (5.16), we have

$$\mathscr{T}_\rho^\lambda < \frac{1}{4}(2^{1/2}\mu_\infty^{-1/2}\mathbf{M})^2 = 2m_\infty.$$

This completes the proof. □

Lemma 5.6. *Let the assumptions of lemma 5.3 hold. Then for $\rho > 0$ sufficiently large,*

$$\mathscr{A}_\rho^\lambda := \max_\Sigma I_\lambda(\Psi_\rho(z,0)) < \mathscr{B}_0^\lambda.$$

Proof. From (5.14), (5.15) and (5.16), we have that for sufficiently large ρ,

$$\begin{aligned}
I_\lambda(\Psi_\rho(z,0)) &= \frac{1}{4}\Big\{\frac{\|\mathbf{w}_{\rho z}\|^2 + \lambda\int_{\mathbb{R}^3}\kappa(x)\mathbf{w}_{\rho z}^2}{[\int_{\mathbb{R}^3}(\mu_\infty + \mu(x))\phi_{\mathbf{w}_{\rho z}}\mathbf{w}_{\rho z}^2]^{1/2}}\Big\}^2 \\
&= \frac{1}{4}\Big[\mu_\infty^{-1/2}\mathbf{M} + o_\rho(1)\Big]^2 \\
&= m_\infty + o_\rho(1).
\end{aligned}$$

From lemma 5.3 the conclusion follows. □

Proof of Theorem1.2. Let λ^* be the number which has defined in (3.2). We infer from Proposition 4.2 that $\lambda^* \in \mathbb{R}^+$. Then, we deduce from Proposition 2.5 that if $\lambda < \lambda^*$, then $m_\lambda < m_\infty$. Furthermore, m_λ is achieved.

Next we consider the case $\lambda > \lambda^*$. From Lemma 3.3 and Proposition 3.5, one deduces that $m_\lambda = m_\infty$, and m_λ is not achieved. Thus, we can not use minimization to solve (1.6). However, we can prove that (1.6) has a higher energy than m_∞ exists. For any $c \in \mathbb{R}$, we let $I_\lambda^c := \{u \in \mathscr{N}_\lambda^+ : I_\lambda(u) \le c\}$. By Lemmas 5.4-5.6, we have the following inequalities

$$m_\infty \le \mathscr{A}_\rho^\lambda < \mathscr{B}_0^\lambda \le \mathscr{T}_\rho^\lambda < 2m_\infty.$$

We end the proof by showing that there exists a number $c^* \in [\mathscr{B}_0^\lambda, \mathscr{T}_\rho^\lambda]$ which is a critical level of $I_\lambda|_{\mathscr{N}_\lambda^+}$. We use the contradiction arguments. Assume that this is not the case. Then the Palais-Smale condition holds in $(m_\infty, 2m_\infty)$ by Lemma 5.2. We can apply usual deformation arguments(see [39]) and assert the existence of a number $\delta > 0$ and a continuous function $\eta : I_\lambda^{\mathscr{T}_\rho^\lambda} \to I_\lambda^{\mathscr{B}_0^\lambda - \delta}$ such that $\mathscr{B}_0^\lambda - \delta > \mathscr{A}_\rho^\lambda$ and $\eta(u) = u$ for all $u \in I_\lambda^{\mathscr{B}_0^\lambda - \delta}$. Thus, we see that

$$0 \notin \beta \circ \eta \circ \Psi_\rho(\Sigma, [0,1]). \tag{5.17}$$

On the other hand, since $\Psi_\rho(\Sigma, [0,1]) \subset I_\lambda^{\mathscr{A}_\rho^\lambda}$, $\beta \circ \eta \circ \Psi_\rho(\Sigma, [0,1])$ is homeomorphic to $\rho\Sigma$ in $\mathbb{R}^3 \backslash \{0\}$. So, one has

$$0 \in \beta \circ \eta \circ \Psi_\rho(\Sigma, [0,1]),$$

which contradicts (5.17).

Finally, because for any $\lambda \in \mathbb{R}^+$, we can find a solution u_λ of (1.6) with $I_\lambda(u_\lambda) < 2m_\infty$. Moreover, since we find the second solution in $H_+^1(\mathbb{R}^3)$, we conclude that it is positive. $\square$

Acknowledgements. This work was supported by NNSFC (Grants 11571140, 11671077), by the Fellowship of Outstanding Young Scholars of Jiangsu Province (BK20160063), by the Six big talent peaks project in Jiangsu Province (XYDXX-015), and by the NSF of Jiangsu Province (BK20150478).

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