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GENERALIZATIONS OF THE DRIFT LAPLACE EQUATION IN THE HEISENBERG GROUP AND GRUSHIN-TYPE SPACES

THOMAS BIESKE, KELLER BLACKWELL

Abstract. We find fundamental solutions to p-Laplace equations with drift terms in the Heisenberg group and Grushin-type planes. These solutions are natural generalizations of the fundamental solutions discovered by Beals, Gaveau, and Greiner for the Laplace equation with drift term. Our results are independent of the results of Bieske and Childers, in that Bieske and Childers consider a generalization that focuses on the p-Laplace-type equation while we primarily concentrate on a generalization of the drift term.

1. Introduction

When studying partial differential equations, one frequent problem under consideration concerns establishing a closed-form fundamental solution. While it is often not possible to do so, equations that possess such closed-form solutions spark further study and interest. One of the most well-known examples is the p-Laplace equation in (Euclidean) $\mathbb{R}^n$. In their seminal paper, Capogna, Danielli, and Garofalo [7] establish the closed-form fundamental solution to the p-Laplace equation in a class of sub-Riemannian spaces called groups of Heisenberg-type. The first author and Gong [6] found a closed-form fundamental solution to the p-Laplace equation in some Grushin-type spaces, which are sub-Riemannian spaces that lack an algebraic group law. Because of this deficiency, the closed-form only holds when the singularity is at certain points. (See Sections 3 and 4 for further discussion concerning the Heisenberg group and Grushin-type planes.)

Beals, Gaveau, and Greiner [1] establish a formula for the fundamental solution to the 2-Laplace equation with drift term in a large class of sub-Riemannian spaces. In [5] the first author and Childers expanded these results by invoking a p-Laplace generalization that encompasses the formulas of [1, 7, 6] by generalizing the p-Laplace operator. That paper also included a negative result [5, Theorems 4.1, 4.2]. In this paper, we focus on that negative result and produce a different natural generalization of the p-Laplace equation with drift term by focusing on generalizing the drift term. Our solutions are stable under limits when $p \to \infty$ and when the drift parameter $L \to 0$ (which is the standard p-Laplace equation).

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2. Motivating results

2.1. **Heisenberg Group.** In the Heisenberg group (See Section 3 for further details and discussion.) the following theorem establishing the fundamental solution of the p-Laplace equation in the Heisenberg group was proved by Capogna, Danielli, and Garofalo [7].

Theorem 2.1 ([7]). *Let $1 < p < \infty$. In the first Heisenberg group $\mathbb{H}^1$, let*

$$u(x_1, x_2, x_3) = (x_1^2 + x_2^2)^2 + 16x_3^2.$$

For $p \neq 4$, let

$$\eta_p = \frac{4-p}{4(1-p)},$$

and let

$$\zeta_p = \begin{cases} u(x_1, x_2, x_3)^{\eta_p} & p \neq 4 \\ \log u(x_1, x_2, x_3) & p = 4. \end{cases}$$

Then we have $\Delta_p \zeta_p = C\delta_0$ for some constant C in the sense of distributions.

Beals, Gaveau, and Greiner [1] extend this result by finding the fundamental solution to the 2-Laplace equation with a drift term, as shown in the following theorem (cf. [5, Theorem 3.4]).

Theorem 2.2 ([1]). *Let $L \in \mathbb{R}$, $|L| \neq 1$. Consider the constants*

$$\eta = \frac{L-1}{2} \quad \text{and} \quad \tau = \frac{-(L+1)}{2}$$

together with the functions

$$v(x_1, x_2, x_3) = (x_1^2 + x_2^2) - 4ix_3 \quad \text{and} \quad w(x_1, x_2, x_3) = (x_1^1 + x_2^2) + 4ix_3,$$

for defining our main function

$$u_{2,L}(x_1, x_2, x_3) = v(x_1, x_2, x_3)^{\eta} w(x_1, x_2, x_3)^{\tau}.$$

Then $\Delta_2 u_{2,L} + iL[X_1, X_2]u_{2,L} = C\delta_0$ for some constant C, in the sense of distributions.

2.2. **Grushin-type planes.** The first author and Gong [6] proved the following theorem establishing the fundamental solution to the p-Laplace equation in Grushin-type planes $\mathbb{G}_n$. (See Section 4 for further details and discussion.)

Theorem 2.3 ([6]). *Let $1 < p < \infty$ and define*

$$F(y_1, y_2) = c^2(y_1 - a)^{(2n+2)} + (n+1)^2(y_2 - b)^2.$$

For $p \neq n+2$, consider

$$\tau_p = \frac{n+2-p}{(2n+2)(1-p)}$$

so that in $\mathbb{G}_n$ we have the well-defined function

$$\psi_p = \begin{cases} F(y_1, y_2)^{\tau_p} & p \neq n+2 \\ \log F(y_1, y_2) & p = n+2. \end{cases}$$

Then $\Delta_p \psi_p = C\delta_0$ for some constant C, in the sense of distributions.

As in the Heisenberg environment, Beals, Gaveau and Greiner [1] extend this result by finding the fundamental solution to the 2-Laplace equation with a drift term, as shown in the following theorem (cf. [5, Theorem 3.2]).

Theorem 2.4 ([1]). *Let $L \in \mathbb{R}$, $|L| \neq 1$. Consider the quantities*

$$\alpha = \frac{-n}{(2n+2)}(1+L) \quad \text{and} \quad \beta = \frac{-n}{(2n+2)}(1-L).$$

We use these constants with the functions

$$g(y_1, y_2) = c(y_1 - a)^{n+1} + i(n+1)(y_2 - b),$$
$$h(y_1, y_2) = c(y_1 - a)^{n+1} - i(n+1)(y_2 - b)$$

for defining our main function

$$f_{2,L}(y_1, y_2) = g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta}.$$

Then $\Delta_2 f_{2,L} + iL[Y_1, Y_2] f_{2,L} = C\delta_0$ for some constant C, in the sense of distributions.

To motivate our study, we make the following key observation.

Observation. In the Heisenberg group $\mathbb{H}^1 \setminus \{0\}$, both the equation and solution of Theorem 2.1 when $p = 2$ coincides with the equation and solution of Theorem 2.2 when $L = 0$. In particular, $u_{2,0} = \zeta_2$. Similarly, in Grushin-type planes $\mathbb{G}_n \setminus \{(a, b)\}$, both the equation and solution of Theorem 2.3 when $p = 2$ coincides with the equation and solution of Theorem 2.4 when $L = 0$. In particular, $f_{2,0} = \psi_2$.

This observation then leads us to state our main question under consideration.

Main question. Can we extend the preceding relationship in $\mathbb{H} \setminus \{0\}$ and in $\mathbb{G}_n \setminus \{(a, b)\}$ from $p = 2$ to all p, $1 < p \leq \infty$?

Specifically, we have the following goals:

- In the case of the Heisenberg group, we wish to find a differential operator $\mathcal{H}_{p,L}$ and a function $u_{p,L}$ satisfying:

$$\mathcal{H}_{p,0} = \Delta_p \quad \text{and} \quad \mathcal{H}_{2,L} = \Delta_2 + iL[X_1, X_2]$$

  with $u_{p,0}$ being the solution of Theorem 2.1 and $u_{2,L}$ being the solution of Theorem 2.2 such that

$$\mathcal{H}_{p,L} u_{p,L}(q) = 0$$

  for $q \in \mathbb{H}^1 \setminus \{0\}$, $1 < p \leq \infty$, and $L \in \mathbb{R}$.
- In the case of the Grushin-type planes, we wish to find a differential operator $\mathcal{G}_{p,L}$ and a function $f_{p,L}$ satisfying:

$$\mathcal{G}_{p,0} = \Delta_p \quad \text{and} \quad \mathcal{G}_{2,L} = \Delta_2 + iL[Y_1, Y_2]$$

  with $f_{p,0}$ being the solution of Theorem 2.3 and $f_{2,L}$ being the solution of Theorem 2.4 such that

$$\mathcal{G}_{p,L} f_{p,L}(q) = 0$$

  for $q \in \mathbb{G}_n \setminus \{(a, b)\}$, $1 < p \leq \infty$, and $L \in \mathbb{R}$.
- Furthermore, we would like $f_{p,L}$ and $u_{p,L}$ to be the fundamental solutions to their respective equations.

3. Heisenberg group

3.1. **Properties.** We begin with $\mathbb{R}^3$ using the coordinates (x_1, x_2, x_3) and consider the linearly independent vector fields $\{X_1, X_2, X_3\}$, defined by:

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3}, \quad X_3 = \frac{\partial}{\partial x_3}$$

which obey the relation

$$[X_1, X_2] = X_3.$$

We then have a Lie Algebra denoted h_1 that decomposes as a direct sum $h_1 = V_1 \oplus V_2$ where $V_1 = \operatorname{span}[X_1, X_2]$ and $V_2 = \operatorname{span}[X_3]$. The Lie algebra is statified; i.e., $[V_1, V_1] = V_2$ and $[V_1, V_2] = 0$. We endow h_1 with an inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and related norm $\| \cdot \|_{\mathbb{H}}$ so that this basis is orthonormal.

The corresponding Lie Group is called the general Heisenberg group of dimension 1 and is denoted by $\mathbb{H}^1$. With this choice of vector fields the exponential map is the identity map, so that for any p, q in $\mathbb{H}^1$, written as $p = (x_1, x_2, x_3)$ and $q = (\widehat{x_1}, \widehat{x_2}, \widehat{x_3})$ the group multiplication law is given by

$$p \cdot q = \Big(x_1 + \widehat{x_1}, x_2 + \widehat{x_2}, x_3 + \widehat{x_3} + \frac{1}{2}(x_1\widehat{x_2} - x_2\widehat{x_1})\Big).$$

The natural metric on $\mathbb{H}^1$ is the Carnot-Carathéodory metric given by

$$d_C(p, q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\|_{\mathbb{H}}\, dt$$

where the set Γ is the set of all curves γ such that $\gamma(0) = p, \gamma(1) = q$ and $\gamma'(t) \in V_1$. By Chow's theorem (See, for example, [2].) any two points can be connected by such a curve, which makes $d_C(p, q)$ a left-invariant metric on $\mathbb{H}^1$.

Given a smooth function $u : \mathbb{H}^1 \to \mathbb{R}$, we define the horizontal gradient by

$$\nabla_0 u = (X_1 u, X_2 u).$$

Additionally, given a vector field $F = \sum_{i=1}^2 f_i X_i + f_3 X_3$, we define the Heisenberg divergence of F, denoted $\operatorname{div} F$, by

$$\operatorname{div} F = \sum_{i=1}^2 X_i f_i\, .$$

A quick calculation shows that when $f_3 = 0$, we have

$$\operatorname{div} F = \operatorname{div}_{\text{eucl}} F$$

where $\operatorname{div}_{\text{eucl}}$ is the standard Euclidean divergence. The main operator we are concerned with is the horizontal p-Laplacian for $1 < p < \infty$ defined by

$$\begin{aligned}\Delta_p u &= \operatorname{div}(\|\nabla_0 u\|_{\mathbb{H}}^{p-2} \nabla_0 u) = \sum_{i=1}^2 X_i \big(\|\nabla_0 u\|_{\mathbb{H}}^{p-2} X_i u\big) \\ &= \frac{p-2}{2}\|\nabla_0 u\|_{\mathbb{H}}^{p-4} \sum_{i=1}^2 X_i \|\nabla_0 u\|_{\mathbb{H}}^2 X_i u + \|\nabla_0 u\|_{\mathbb{H}}^{p-2} \sum_{i=1}^2 X_i X_i u.\end{aligned} \tag{3.1}$$

For an extensive treatment of the Heisenberg group, the interested reader is directed to [2, 4, 8, 9, 10, 11, 12, 13] and the references therein.

3.2. **Generalization in the Heisenberg group.** For the Heisenberg group $\mathbb{H}^1$, we consider the parameters

$$\eta = \frac{4-p+2L(1-p)}{4(1-p)} \quad \text{and} \quad \tau = \frac{4-p-2L(1-p)}{4(1-p)}$$

for $L \in \mathbb{R}$ with

$$L \neq \pm \frac{4-p}{2(1-p)}.$$

We use these parameters with the functions

$$v(x_1,x_2,x_3) = (x_1^2+x_2^2) - 4ix_3, w(x_1,x_2,x_3) = (x_1^2+x_2^2) + 4ix_3$$

to define our main function

$$u_{p,L}(y_1,y_2) = v(x_1,x_2,x_3)^\eta w(x_1,x_2,x_3)^\tau. \tag{3.2}$$

Using this equation, we have the following result.

Theorem 3.1. *Let $1 < p < \infty$. On $\mathbb{H}^1$, we have*

$$\mathcal{H}_{p,L}(u_{p,L}) := \Delta_p u_{p,L} + iL[X_1,X_2]\big(\|\nabla_0 u_{p,L}\|_{\mathbb{H}}^{p-2} u_{p,L}\big) = C\delta_0$$

for some constant C, in the sense of distributions.

Proof. Suppressing arguments and subscripts, we obtain the following:

$$\begin{aligned}
X_1 u &= 2v^{\eta-1}w^{\tau-1}\big((\eta w+\tau v)x_1 + (\eta w - \tau v)ix_2\big) \qquad (3.3)\\
\overline{X_1 u} &= 2v^{\tau-1}w^{\eta-1}\big((\eta v+\tau w)x_1 + (\eta v - \tau w)ix_2\big)\\
X_2 u &= 2v^{\eta-1}w^{\tau-1}\big((\eta w+\tau v)x_2 - (\eta w - \tau v)ix_1\big) \qquad (3.4)\\
\overline{X_2 u} &= 2v^{\tau-1}w^{\eta-1}\big((\eta v+\tau w)x_2 - (\eta v - \tau w)ix_1\big)\\
\text{and so } \|\nabla_0 u\|^2 &= 8(\eta^2+\tau^2)v^{\eta+\tau-1}w^{\eta+\tau-1}(x_1^2+x_2^2). \qquad (3.5)
\end{aligned}$$

Using the above we have

$$\begin{aligned}
X_1(X_1u) &= 2v^{\eta-2}w^{\tau-2}\Big(2\big((\eta w+\tau v)x_1^2 + (-\eta w - \tau v)ix_1x_2\big)\big((\eta-1)w + (\tau-1)v\big)\\
&\quad + 2i\big((\eta w+\tau v)x_2^2 + (\eta w - \tau v)ix_2^2\big)\big(-(\eta-1)w + (\tau-1)v)\big)\\
&\quad + vw\big(2(x_1^2+x_2^2)(\tau+\eta) + (\eta w + \tau v)\big)\Big),\\
X_2(X_2u) &= 2v^{\eta-2}w^{\tau-2}\Big(2\big((\eta w+\tau v)x_2^2 + (-\eta w + \tau v)ix_1x_2\big)\big((\eta-1)w + (\tau-1)v\big)\\
&\quad + 2i\big((\eta w+\tau v)x_1x_2 + (-\eta w + \tau v)ix_1^2\big)\big(-(\eta-1)w + (\tau-1)v)\big)\\
&\quad + vw\big(2(x_1^2+x_2^2)(\tau+\eta) + (\eta w + \tau v)\big)\Big).
\end{aligned}$$

In addition, we have

$$\begin{aligned}
X_1\|\nabla_0 u\|^2 &= 16(\eta^2+\tau^2)v^{\eta+\tau-2}w^{\eta+\tau-2}\\
&\quad \times \Big(vwx_1 + 2(\eta+\tau-1)(x_1^2+x_2^2)^2\big(x_1 - 4x_2x_3\big)\Big)
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
X_2\|\nabla_0 u\|^2 &= 16(\eta^2+\tau^2)v^{\eta+\tau-2}w^{\eta+\tau-2}\\
&\quad \times \Big(vwx_2 + 2(\eta+\tau-1)(x_1^2+x_2^2)^2\big(x_2 - 4x_1x_3\big)\Big)
\end{aligned} \tag{3.7}$$

so that

$$\begin{aligned}
&\sum_{j=1}^{2} X_j\|\nabla_0 u\|^2(X_j u)\\
&= 32(\eta^2+\tau^2)v^{2\eta+\tau-3}w^{\eta+2\tau-3}\Big((\eta w+\tau v)vw(x_1^2+x_2^2)\\
&\quad + 2(\eta+\tau-1)(x_1^2+x_2^2)^2\Big((\eta w+\tau v)(x_1^2+x_2^2)^2-4(\eta w-\tau v)ix_3\Big)\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\|\nabla_0 u\|^2\big(X_1X_1u+X_2X_2u\big)\\
&= 16(\eta^2+\tau^2)v^{2\eta+\tau-3}w^{\eta+2\tau-3}(x_1^2+x_2^2)\Big(2vw(\eta w+\tau v)+4vw(\eta+\tau)(x_1^2+x_2^2)\\
&\quad + 2\big((\eta-1)w+(\tau-1)v\big)(\eta w+\tau v)(x_1^2+x_2^2)\\
&\quad + 2\big(-(\eta-1)w+(\tau-1)v\big)(\eta w-\tau v)(x_1^2+x_2^2)\Big).
\end{aligned}$$

This yields

$$\begin{aligned}
\Delta_p u &= \|\nabla_0 u\|^{p-4}\Big(\frac{(p-2)}{2}\sum_{j=1}^{2} X_j\|\nabla_0 u\|^2(X_j u)+\|\nabla_0 u\|^2(X_1X_1f+X_2X_2u)\Big)\\
&= 2L\frac{(4-p)^{p-2}}{(1-p)^{p-2}}\Big(1+\frac{4L^2(1-p)^2}{(4-p)^2}\Big)^{\frac{p-2}{2}} v^{\frac12(p\eta+(p-2)\tau-p)}w^{\frac12((p-2)\eta+p\tau-p)}\\
&\quad\times (x_1^2+x_2^2)^{\frac{p-2}{2}}(-2L(x_1^2+x_2^2)+p4ix_3).
\end{aligned}$$

We then compute

$$\begin{aligned}
&iL[X_1,X_2](\|\nabla_0 u\|^{p-2}u)\\
&= iL\frac{(4-p)^{p-2}}{(1-p)^{p-2}}\Big(1+\frac{4L^2(1-p)^2}{(4-p)^2}\Big)^{\frac{p-2}{2}}(x_1^2+x_2^2)^{\frac{p-2}{2}}\\
&\quad\times\frac{\partial}{\partial x_3}v^{\frac12(p-2)(\eta+\tau-1)+\eta}w^{\frac12(p-2)(\eta+\tau-1)+\tau}\\
&= -2L\frac{(4-p)^{p-2}}{(1-p)^{p-2}}Big(1+\frac{4L^2(1-p)^2}{(4-p)^2}\Big)^{\frac{p-2}{2}}(x_1^2+x_2^2)^{\frac{p-2}{2}}\\
&\quad\times v^{\frac12(p\eta+(p-2)\tau-p)}w^{\frac12((p-2)\eta+p\tau-p)}(-2L(x_1^2+x_2^2)+p4ix_3)\\
&= -\Delta_p u
\end{aligned}$$

from which it follows that $\mathcal{H}_{p,L}u_{p,L}=0$ on $\mathbb{H}^1\setminus\{0\}$, away from the singularity. We now consider the normalization

$$\begin{aligned}
v_\varepsilon(x_1,x_2,x_3) &:= (x_1^2+x_2^2)+\varepsilon^2-4ix_3,\\
w_\varepsilon(x_1,x_2,x_3) &:= (x_1^2+x_2^2)+\varepsilon^2+4ix_3
\end{aligned}$$

so that

$$u_\varepsilon(x_1,x_2,x_3) := v_\varepsilon(x_1,x_2,x_3)^\eta w_\varepsilon(x_1,x_2,x_3)^\tau.$$

Suppressing arguments and computing similarly as before yields the distribution

$$\begin{aligned}
\mathcal{H}_{p,L}u_\varepsilon &= 2^{\frac{3p-2}{2}}\varepsilon^2\Big(\frac{p(4-p)}{4(1-p)}+L^2\Big)(\eta^2+\tau^2)^{\frac{p-2}{2}}(x_1^2+x_2^2)^{\frac{p-2}{2}}\\
&\quad\times v_\varepsilon^{\frac{\eta p+\tau(p-2)-p}{2}}w_\varepsilon^{\frac{\eta(p-2)+\tau p-p}{2}}.
\end{aligned}\tag{3.8}$$

By the argument in [1, Theorem 7.5, (c)], the distribution of (3.8) is determined by the density

$$\frac{2^{\frac{3p-2}{2}}\left(\frac{p(4-p)}{4(1-p)}+L^2\right)(\eta^2+\tau^2)^{\frac{p-2}{2}}\left((\frac{x_1}{\varepsilon})^2+(\frac{x_2}{\varepsilon})^2\right)^{\frac{p-2}{2}}dm\left(\frac{x_1^2+x_2^2}{\varepsilon^2}\right)d(\frac{x_3}{\varepsilon^2})\frac{1}{-2i}}{\left((\frac{x_1}{\varepsilon})^2+(\frac{x_2}{\varepsilon})^2+1-4i\frac{x_3}{\varepsilon^2}\right)^{-\frac{\eta p+\tau(p-2)-p}{2}}\left((\frac{x_1}{\varepsilon})^2+(\frac{x_2}{\varepsilon})^2+1+4i\frac{x_3}{\varepsilon^2}\right)^{-\frac{\eta(p-2)+\tau p-p}{2}}} \tag{3.9}$$

where dm denotes the Lebesgue measure in the complex plane. Then as $\varepsilon\to 0$ the distribution of (3.9) tends to the δ_0 distribution, up to a constant factor. $\square$

Observing that

$$L\neq\pm\frac{4-p}{2(1-p)}\quad\text{implies}\quad p\neq\Big|\frac{2L+4}{2L+1}\Big|,\Big|\frac{2L-4}{2L-1}\Big|$$

we have immediately the following corollary.

Corollary 3.2. *Let $p>\max\{\big|\frac{2L+4}{2L+1}\big|,\big|\frac{2L-4}{2L-1}\big|\}$. Then the function $u_{p,L}$ of (3.2) is a smooth solution to the Dirichlet problem*

$$\begin{gathered}\mathcal{H}_{tp,L}(u_{p,L}(q))=0\quad q\in\mathbb{H}^1\setminus\{0\}\\ 0\quad q=0.\end{gathered}$$

3.3. **Limit as $p\to\infty$.** Recall that the drift p-Laplace equation in the Heisenberg group $\mathbb{H}^1$ is given by:

$$\mathcal{H}_{p,L}(u):=\Delta_p u+iL[X_1,X_2]\big(\|\nabla_0 u\|_{\mathbb{H}}^{p-2}u\big)=0.$$

A routine expansion of the drift term yields the observation

$$\mathcal{H}_{p,L}(u)=\Delta_p u+iL\Big(\frac{p-2}{2}\|\nabla_0 u\|_{\mathbb{H}}^{p-4}\big(\frac{\partial}{\partial x_3}\|\nabla_0 u\|_{\mathbb{H}}^2\big)u+\|\nabla_0 u\|_{\mathbb{H}}^{p-2}\frac{\partial}{\partial x_3}u\Big)=0.$$

Dividing through by $\frac{p-2}{2}\|\nabla_0 u\|_{\mathbb{H}}^{p-4}$ and formally taking the limit $p\to\infty$, we obtain

$$\mathcal{H}_{\infty,L}(u)=\Delta_\infty u+iL[X_1,X_2](\|\nabla_0 u\|_{\mathbb{H}}^2)u.$$

Considering (3.2) and formally letting $p\to\infty$ yields

$$u_{\infty,L}(x_1,x_2,x_3)=v(x_1,x_2,x_3)^{\frac{1+2L}{4}}w(x_1,x_2,x_3)^{\frac{1-2L}{4}},$$

where we recall the functions

$$\begin{gathered}v(x_1,x_2,x_3)=(x_1^2+x_2^2)-4ix_3,\\ w(x_1,x_2,x_3)=(x_1^2+x_2^2)+4ix_3\,.\end{gathered}$$

Theorem 3.3. *The function $u_{\infty,L}$, defined above, is a smooth solution to the Dirichlet problem*

$$\begin{gathered}\mathcal{H}_{\infty,L}u_{\infty,L}(q)=0\quad q\in\mathbb{H}^1\setminus\{0\},\\ 0\quad q=0.\end{gathered}$$

Proof. We prove this theorem by letting $p\to\infty$ in (3.3), (3.4), (3.6), and (3.7), and invoking continuity (cf. Corollary 3.2). However, for completeness we compute it formally. We let

$$N=\frac{1+2L}{4}\quad\text{and}\quad T=\frac{1-2L}{4}\,.$$

Suppressing arguments and subscripts, we compute

$$
\begin{aligned}
X_1 u &= 2v^{N-1}w^{T-1}\big((Nw+Tv)x_1 + (Nw-Tv)ix_2\big),\\
X_2 u &= 2v^{N-1}w^{T-1}\big((Nw+Tv)x_2 - (Nw-Tv)ix_1\big),\\
\|\nabla_0 u\|^2 &= 8(N^2+T^2)v^{N+T-1}w^{N+T-1}(x_1^2+x_2^2),\\
X_1\|\nabla_0 u\|^2 &= 16(N^2+T^2)v^{N+T-2}w^{N+T-2}\\
&\quad\times\Big(vwx_1 + 2(N+T-1)(x_1^2+x_2^2)^2\big(x_1-4x_2x_3\big)\Big),\\
X_2\|\nabla_0 u\|^2 &= 16(N^2+T^2)v^{N+T-2}w^{N+T-2}\\
&\quad\times\Big(vwx_2 + 2(N+T-1)(x_1^2+x_2^2)^2\big(x_2-4x_1x_3\big)\Big),
\end{aligned}
$$

so that

$$
\begin{aligned}
\Delta_\infty u &= X_1\|\nabla_0 u\|^2 X_1 u + X_2\|\nabla_0 u\|^2 X_2 u\\
&= 32(N^2+T^2)v^{2N+T-3}w^{N+2T-3}\Big((Nw+Tv)vw(x_1^2+x_2^2)\\
&\quad + 2(N+T-1)(x_1^2+x_2^2)^2\Big((Nw+Tv)(x_1^2+x_2^2)^2 - 4(Nw-Tv)ix_3\Big)\Big)\\
&= 128iL(N^2+T^2)(x_1^2+x_2^2)x_3v^{2N+T-2}w^{N+2T-2}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
iL[X_1,X_2](\|\nabla_0 u\|^2)u &= iLv^Nw^T\frac{\partial}{\partial x_3}\|\nabla_0 f\|^2\\
&= -128iL(N^2+T^2)(x_1^2+x_2^2)x_3v^{2N+T-2}w^{N+2T-2}.
\end{aligned}
$$

The proof is complete. □

We notice that when $L=0$, this result was a part of the Ph.D. thesis of the first author [3]. In particular, combined with [3, 4], we have shown the following commutative diagram in $\mathbb{H}^1\setminus\{0\}$,

$$
\begin{array}{ccc}
\mathcal{H}_{p,L}(u_{p,L})=0 & \xrightarrow[p\to\infty]{} & \mathcal{H}_{\infty,L}(u_{\infty,L})=0\\
\big\downarrow{\scriptstyle L\to 0} & & \big\downarrow{\scriptstyle L\to 0}\\
\Delta_p u_{p,0}=0 & \xrightarrow[p\to\infty]{} & \Delta_\infty u_{\infty,0}=0
\end{array}
$$

4. Grushin-type planes

The Grushin-type planes differ from the Heisenberg group in that Grushin-type planes lack an algebraic group law. We begin with $\mathbb{R}^2$, possessing coordinates (y_1,y_2), $a\in\mathbb{R}$, $c\in\mathbb{R}\setminus\{0\}$ and $n\in\mathbb{N}$. We use them to construct the vector fields:

$$
Y_1 = \frac{\partial}{\partial y_1} \quad\text{and}\quad Y_2 = c(y_1-a)^n\frac{\partial}{\partial y_2}.
$$

For these vector fields, the only (possibly) nonzero Lie bracket is

$$
[Y_1,Y_2] = cn(y_1-a)^{n-1}\frac{\partial}{\partial y_2}.
$$

Because $n\in\mathbb{N}$, it follows that Hörmander's condition is satisfied by these vector fields.

We will put a (singular) inner product on $\mathbb{R}^2$, denoted $\langle\cdot,\cdot\rangle_{\mathbb{G}}$, with related norm $\|\cdot\|_{\mathbb{G}}$, so that the collection $\{Y_1, Y_2\}$ forms an orthonormal basis. We then have a sub-Riemannian space that we will call g_n, which is also the tangent space to a generalized Grushin-type plane $\mathbb{G}_n$. Points in $\mathbb{G}_n$ will also be denoted by $p = (y_1, y_2)$. The Carnot-Carathéodory distance on $\mathbb{G}_n$ is defined for points p and q as follows

$$d_{\mathbb{G}}(p,q) = \inf_{\Gamma} \int \|\gamma'(t)\|_{\mathbb{G}}\, dt\,,$$

with Γ the set of curves γ such that $\gamma(0) = p$, $\gamma(1) = q$ and

$$\gamma'(t) \in \operatorname{span}\{Y_1(\gamma(t)), Y_2(\gamma(t))\}.$$

By Chow's theorem, this is an honest metric.

We shall now discuss calculus on the Grushin-type planes. Given a smooth function f on $\mathbb{G}_n$, we define the horizontal gradient of f as

$$\nabla_0 f(p) = \big(Y_1 f(p), Y_2 f(p)\big).$$

Using these derivatives, we consider a key operator on $C^2_{\mathbb{G}}$ functions, namely the p-Laplacian for $1 < p < \infty$, given by

$$\begin{aligned}
\Delta_p f &= \operatorname{div}_{\mathbb{G}}(\|\nabla_0 f\|_{\mathbb{G}}^{p-2}\nabla_0 f) \\
&= Y_1\big(\|\nabla_0 f\|_{\mathbb{G}}^{p-2} Y_1 f\big) + Y_2\big(\|\nabla_0 f\|_{\mathbb{G}}^{p-2} Y_2 f\big) \\
&= \frac{p-2}{2}\|\nabla_0 f\|_{\mathbb{G}}^{p-4}\big(Y_1\|\nabla_0 f\|_{\mathbb{G}}^2 Y_1 f + Y_2\|\nabla_0 f\|_{\mathbb{G}}^2 Y_2 f\big) \\
&\quad + \|\nabla_0 f\|_{\mathbb{G}}^{p-2}\big(Y_1Y_1 f + Y_2Y_2 f\big).
\end{aligned} \tag{4.1}$$

4.1. **A Generalization in the Grushin plane.** For the Grushin-type planes, we consider the parameters

$$\alpha = \frac{n+2-p-Ln(1-p)}{2(n+1)(1-p)} \quad \text{and} \quad \beta = \frac{n+2-p+Ln(1-p)}{2(n+1)(1-p)}\,,$$

where $L \in \mathbb{R}$ with

$$L \neq \pm\frac{n+2-p}{n(1-p)}.$$

We use these constants with the functions

$$\begin{aligned}
g(y_1,y_2) &= c(y_1-a)^{n+1} + i(n+1)(y_2-b), \\
h(y_1,y_2) &= c(y_1-a)^{n+1} - i(n+1)(y_2-b)
\end{aligned}$$

to define our main function

$$f_{p,L}(y_1,y_2) = g(y_1,y_2)^{\alpha} h(y_1,y_2)^{\beta}. \tag{4.2}$$

Using this equation, we have the following theorem.

Theorem 4.1. *Let $1 < p < \infty$. On $\mathbb{G}_n$, we have*

$$\mathcal{G}_{p,L}(f_{p,L}) := \Delta_p f_{p,L} + iL[Y_1,Y_2](\|\nabla_0 f_{p,L}\|_{\mathbb{G}}^{p-2} f_{p,L}) = C\delta_0$$

for some constant C, in the sense of distributions.

Proof. Suppressing arguments and subscripts, we compute the following:

$$Y_1 f = c(n+1)(y_1-a)^n g^{\alpha-1} h^{\beta-1}(\alpha h+\beta g) \tag{4.3}$$
$$\overline{Y_1 f} = c(n+1)(y_1-a)^n g^{\beta-1} h^{\alpha-1}(\alpha g+\beta h)$$
$$Y_2 f = ic(n+1)(y_1-a)^n g^{\alpha-1} h^{\beta-1}(\alpha h-\beta g) \tag{4.4}$$
$$\overline{Y_2 f} = ic(n+1)(y_1-a)^n g^{\beta-1} h^{\alpha-1}(\alpha g-\beta h)$$
$$\|\nabla_0 f\|^2 = 2c^2(n+1)^2(y_1-a)^{2n} g^{\alpha+\beta-1} h^{\alpha+\beta-1}(\alpha^2+\beta^2). \tag{4.5}$$

Using the above we have

$$\begin{aligned} Y_1(Y_1 f) &= c(n+1)(y_1-a)^{n-1} g^{\alpha-2} h^{\beta-2} \\ &\quad \times \Big(ngh(\alpha h+\beta g) + c(n+1)(y_1-a)^{n+1} \\ &\quad \times \big((\alpha h+\beta g)((\alpha-1)h+(\beta-1)g) + gh(\alpha+\beta)\big)\Big), \\ Y_2(Y_2 f) &= -c^2(n+1)^2(y_1-a)^{2n} g^{\alpha-2} h^{\beta-2} \\ &\quad \times \big((\alpha h-\beta g)((\alpha-1)h-(\beta-1)g) - gh(\alpha+\beta)\big), \end{aligned}$$

$$\begin{aligned} Y_1\|\nabla_0 f\|^2 &= 4c^2(n+1)^2(\alpha^2+\beta^2)(y_1-a)^{2n-1} g^{\alpha+\beta-2} h^{\alpha+\beta-2} \\ &\quad \times \big(ngh + c^2(n+1)(\alpha+\beta-1)(y_1-a)^{2n+2}\big), \end{aligned} \tag{4.6}$$

$$\begin{aligned} Y_2\|\nabla_0 f\|^2 &= 4c^3(n+1)^4(\alpha^2+\beta^2)(y_1-a)^{3n}(y_2-b) \\ &\quad (\alpha+\beta-1) g^{\alpha+\beta-2} h^{\alpha+\beta-2} \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \sum_{i=1}^{2} Y_i\|\nabla_0 f\|^2 (Y_i f) &= 4c^3(n+1)^3(\alpha^2+\beta^2)(y_1-a)^{3n-1} g^{2\alpha+\beta-3} h^{\alpha+2\beta-3} \\ &\quad \times \Big((\alpha h+\beta g)\big(ngh + c^2(n+1)(\alpha+\beta-1)(y_1-a)^{2n+2}\big) \\ &\quad + ic(n+1)^2(y_1-a)^{n+1}(y_2-b)(\alpha+\beta-1)(\alpha h-\beta g)\Big), \\ \|\nabla_0 f\|^2(Y_1Y_1 f + Y_2Y_2 f) &= 2c^3(n+1)^3(\alpha^2+\beta^2)(y_1-a)^{3n-1} g^{2\alpha+\beta-3} h^{\alpha+2\beta-3} \\ &\quad \times \Big(ngh(\alpha h+\beta g) + 4c(n+1)(y_1-a)^{n+1} gh(\alpha\beta)\Big), \end{aligned}$$

so that

$$\begin{aligned} \Delta_p f &= \|\nabla_0 f\|^{p-4}\Big(\frac{(p-2)}{2}\sum_{j=1}^{2} Y_j\|\nabla_0 f\|^2(Y_j f) + \|\nabla_0 f\|^2(Y_1Y_1 f + Y_2Y_2 f)\Big) \\ &= -L2^{\frac{p-2}{2}} c^{p-1} n^2 (n+1)^{p-2}(y_1-a)^{n(p-1)-1}(\alpha^2+\beta^2)^{\frac{p-2}{2}} \\ &\quad \times g^{\frac12(\alpha p+\beta(p-2)-p)} h^{\frac12(\alpha(p-2)+\beta p-p)}(Lc(y_1-a)^{n+1} + i(1-p)(n+1)(y_2-b)). \end{aligned}$$

We then compute

$$\begin{aligned} iL[Y_1,Y_2](\|\nabla_0 f\|^{p-2} f) &= iL2^{\frac{p-2}{2}} c^{p-1} n(n+1)^{p-2}(y_1-a)^{n(p-1)-1}(\alpha^2+\beta^2)^{\frac{p-2}{2}} \\ &\quad \times \frac{\partial}{\partial y_2}\Big(g^{\frac12(\alpha p+\beta(p-2)-(p-2))} h^{\frac12(\alpha(p-2)+\beta p-(p-2))}\Big) \\ &= L2^{\frac{p-2}{2}} c^{p-1} n^2 (n+1)^{p-2}(y_1-a)^{n(p-1)-1}(\alpha^2+\beta^2)^{\frac{p-2}{2}} \end{aligned}$$

$$
\begin{aligned}
&\times g^{\frac{1}{2}(\alpha p+\beta(p-2)-p)} h^{\frac{1}{2}(\alpha(p-2)+\beta p-p)} \\
&\times (Lc(y_1-a)^{n+1} + i(1-p)(n+1)(y_2-b)) \\
&= -\Delta_p f
\end{aligned}
$$

from which it follows that $\mathcal{G}_{p,L} f_{p,L} = 0$ on $\mathbb{G}_n \setminus \{(a,b)\}$, away from the singularity. We now consider the normalization

$$
\begin{aligned}
g_\varepsilon(y_1,y_2) &:= c(y_1-a)^n + \varepsilon^2 + i(n+1)(y_2-b), \\
h_\varepsilon(y_1,y_2) &:= c(y_1-a)^n + \varepsilon^2 - i(n+1)(y_2-b)
\end{aligned}
$$

so that

$$
f_\varepsilon(y_1,y_2) := g_\varepsilon(y_1,y_2)^\alpha h_\varepsilon(y_1,y_2)^\beta .
$$

Suppressing arguments and computing similarly as before yields the distribution

$$
\begin{aligned}
\mathcal{G}_{p,L} f_\varepsilon = &-2^{\frac{p-2}{2}} \varepsilon^2 ((n+2-p) - nL^2) c^{p-1} n(n+1)^{p-2} (\alpha^2+\beta^2)^{\frac{p-2}{2}} \\
&\times (y_1-a)^{n(p-1)-1} g^{\frac{\alpha p+\beta(p-2)-p}{2}} h^{\frac{\alpha(p-2)+\beta p-p}{2}} .
\end{aligned} \tag{4.8}
$$

By the argument in [1, Theorem 7.5, (c)], the distribution of (4.8) is determined by the density

$$
\begin{aligned}
&-2^{\frac{p-2}{2}} \big((n+2-p) - nL^2\big) c^{p-1} n(n+1)^{p-2} (\alpha^2+\beta^2)^{\frac{p-2}{2}} \\
&\times \Big(\frac{y_1-a}{\varepsilon^{2/(n+1)}}\Big)^{n(p-1)-1} dm\Big(\frac{y_1-a}{\varepsilon^{2/(n+1)}}\Big) d\Big(\frac{y_2-b}{\varepsilon^2}\Big) \big(\frac{1}{-2i}\big) \\
&\times \Big(c\Big(\frac{y_1-a}{\varepsilon^{2/(n+1)}}\Big)^{n+1} + 1 + i(n+1)\frac{(y_2-b)}{\varepsilon^2}\Big)^{\frac{\alpha p+\beta(p-2)-p}{2}} \\
&\times \Big(c\Big(\frac{y_1-a}{\varepsilon^{2/(n+1)}}\Big)^{n+1} + 1 - i(n+1)\frac{(y_2-b)}{\varepsilon^2}\Big)^{\frac{\alpha(p-2)+\beta p-p}{2}}
\end{aligned} \tag{4.9}
$$

where dm denotes the Lebesgue measure in the complex plane. Then as $\varepsilon \to 0$ the distribution of (4.9) tends to the δ_0 distribution, up to a constant factor. $\square$

Observing that

$$
L \neq \pm \frac{n(p-1)}{n+2-p} \quad \text{implies} \quad p \neq \Big|\frac{L(n+2)+n}{n+L}\Big|, \Big|\frac{L(n+2)-n}{n-L}\Big|
$$

we have immediately the following corollary.

Corollary 4.2. *Let $p > \max\big\{\big|\frac{L(n+2)+n}{n+L}\big|, \big|\frac{L(n+2)-n}{n-L}\big|\big\}$. Then the function $f_{p,L}$ of Equation 4.2 is a smooth solution to the Dirichlet problem*

$$
\begin{gathered}
\mathcal{G}_{p,L}(f_{p,L}(q)) = 0 \quad q \in \mathbb{G}_n \setminus \{(a,b)\} \\
0 \quad q = (a,b).
\end{gathered}
$$

4.2. **Limit as $p \to \infty$.** Recall that the drift p-Laplace equation in the Grushin-type planes $\mathbb{G}_n$ is given by

$$
\mathcal{G}_{p,L}(f) := \Delta_p f + iL[Y_1,Y_2]\big(\|\nabla_0 f\|_{\mathbb{G}}^{p-2} f\big) = 0.
$$

A routine expansion of the drift term yields the observation

$$
\begin{aligned}
\mathcal{G}_{p,L}(f) = \Delta_p f + iLcn(y_1-a)^{n-1}\Big(&\frac{p-2}{2}\|\nabla_0 f\|_{\mathbb{G}}^{p-4}\big(\frac{\partial}{\partial y_2}\|\nabla_0 f\|_{\mathbb{G}}^2\big) f \\
&+ \|\nabla_0 f\|_{\mathbb{G}}^{p-2}\frac{\partial}{\partial y_2} f\Big) = 0.
\end{aligned}
$$

Dividing through by $\frac{p-2}{2}\|\nabla_0 f\|_{\mathbb{G}}^{p-4}$ and formally taking the limit $p\to\infty$, we obtain

$$\mathcal{G}_{\infty,L}(f) = \Delta_\infty f + iL[Y_1,Y_2](\|\nabla_0 f\|_{\mathbb{G}}^2)f.$$

Considering (4.2) and formally letting $p\to\infty$ yields

$$f_{\infty,L}(y_1,y_2) = g(y_1,y_2)^{\frac{1}{2(n+1)}(1-nL)} h(y_1,y_2)^{\frac{1}{2(n+1)}(1+nL)}$$

where we recall the functions

$$\begin{aligned} g(y_1,y_2) &= c(y_1-a)^{n+1} + i(n+1)(y_2-b),\\ h(y_1,y_2) &= c(y_1-a)^{n+1} - i(n+1)(y_2-b). \end{aligned}$$

Theorem 4.3. *The function $f_{\infty,L}$, defined above, is a smooth solution to the Dirichlet problem*

$$\begin{gathered} \mathcal{G}_{\infty,L} f_{\infty,L}(q) = 0 \quad q\in \mathbb{G}_n\setminus\{(a,b)\},\\ 0 \quad q=(a,b). \end{gathered}$$

Proof. We prove this theorem by letting $p\to\infty$ in (4.3), (4.4), (4.6), (4.7), and invoking continuity (cf. Corollary 4.2). However, for completeness we compute it formally. We let

$$A = \frac{1}{2(n+1)}(1-nL) \quad\text{and}\quad B = \frac{1}{2(n+1)}(1+nL)$$

and, suppressing arguments and subscripts, compute

$$\begin{gathered} Y_1 f = c(n+1)(y_1-a)^n g^{A-1}h^{B-1}(Ah+Bg),\\ Y_2 f = ic(n+1)(y_1-a)^n g^{A-1}h^{B-1}(Ah-Bg),\\ \|\nabla_0 f\|^2 = 2c^2(n+1)^2(y_1-a)^{2n} g^{A+B-1}h^{A+B-1}(A^2+B^2),\\ Y_1\|\nabla_0 f\|^2 = 4c^2(n+1)^2(A^2+B^2)(y_1-a)^{2n-1}g^{A+B-2}h^{A+B-2}\\ \times\big(ngh + c^2(n+1)(A+B-1)(y_1-a)^{2n+2}\big),\\ Y_2\|\nabla_0 f\|^2 = 4c^3(n+1)^4(A^2+B^2)(y_1-a)^{3n}(y_2-b)\\ (\alpha+\beta-1)g^{A+B-2}h^{A+B-2}, \end{gathered}$$

so that

$$\begin{aligned} \Delta_\infty f &= Y_1\|\nabla_0 f\|^2 Y_1 f + Y_2\|\nabla_0 f\|^2 Y_2 f\\ &= 4c^3(n+1)^3(A^2+B^2)(y_1-a)^{3n-1}g^{2A+B-3}h^{A+2B-3}\\ &\quad\times\Big((Ah+Bg)\big(ngh + c^2(n+1)(A+B-1)(y_1-a)^{2n+2}\big)\\ &\quad + ic(n+1)^2(y_1-a)^{n+1}(y_2-b)(A+B-1)(Ah-Bg)\Big)\\ &= 4iLc^3(n+1)^3n^2(A^2+B^2)(y_1-a)^{3n-1}(y_2-b)g^{2A+B-2}h^{A+2B-2}. \end{aligned}$$

We also compute

$$\begin{aligned} &iL[Y_1,Y_2](\|\nabla_0 f\|^2)f\\ &= iLg^Ah^B(cn(y_1-a)^{n-1}\frac{\partial}{\partial y_2}\|\nabla_0 f\|^2)\\ &= -4iLc^3(n+1)^3n^2(A^2+B^2)g^{2A+B-2}h^{A+2B-2}(y_1-a)^{3n-1}(y_2-b). \end{aligned}$$

The proof is complete. ∎

In particular, combining this with [6], we have shown that the following commutative diagram in $\mathbb{G}_n \setminus \{(a,b)\}$,

$$\begin{tikzcd} \mathcal{G}_{p,L} f_{p,L} = 0 \arrow[r, "p\to\infty"'] \arrow[d, "L\to 0"] & \mathcal{G}_{\infty,L} f_{\infty,L} = 0 \arrow[d, "L\to 0"] \\ \Delta_p f_{p,0} = 0 \arrow[r, "p\to\infty"'] & \Delta_\infty f_{\infty,0} = 0 \end{tikzcd}$$

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Thomas Bieske
Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
Email address: tbieske@usf.edu

Keller Blackwell
School of Engineering Stanford University, Stanford, CA 94305, USA
Email address: kellerb@stanford.edu