Electronic Journal of Differential Equations, Vol. 2005(2005), No. 38, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

DEFICIENCY INDICES OF A DIFFERENTIAL OPERATOR SATISFYING CERTAIN MATCHING INTERFACE CONDITIONS

PALLAV KUMAR BARUAH, M. VENKATESULU

Abstract. A pair of differential operators with matching interface conditions appears in many physical applications such as: oceanography, the study of step index fiber in optical fiber communication, and one dimensional scattering in quantum theory. Here we initiate the study the deficiency index theory of such operators which precedes the study of the spectral theory.

1. Introduction

In the study of acoustic wave guides in the ocean, and of one dimensional time independent scattering in quantum theory, we come across of problems of the from

$$L_1 f_1 = \sum_{k=0}^{n} P_k \frac{df_1^k}{dt^k} = \lambda f_1$$

defined on an interval $I_1 = (a, c]$ and

$$L_1 f_1 = \sum_{k=0}^{n} P_k \frac{df_1^k}{dt^k} = \lambda f_1$$

defined on an interval $I_2 = [c, b)$, with $-\infty \le a < c < b \le +\infty$. Here λ is an unknown constant and the functions f_1, f_2 are required to satisfy certain mixed conditions at the interface $t = c$. In most cases, the complete set of physical conditions on the system give rise to selfadjoint spectral problems associated with the pair (L_1, L_2).

Initial-value problem and boundary-value problems for regular and singular cases for these equations have been discussed in publications such as [2, 5, 6, 7, 8, 9]. It is important to study the deficiency index theory of an operator before one embarks on the study of the spectral theory. Here we present a simple result on deficiency index of such operators. We take help of the results available in [3]; however the proof of the main theorem rendered here is new, not the same as that found in [3]. A similar study is found in a recent work of Orochko [4], where he has considered two arbitrary even ordered symmetric differential expressions degenerated at the point of interface. The operator depends on two parameters p, q and based on

2000 *Mathematics Subject Classification.* 34B10.
Key words and phrases. Ordinary differential operators; Green's formula; deficiency index; formal selfadjoint boundary-value problems; boundary form; deficiency space.

Submitted October 7, 2004. Published March 29, 2005.

certain relations between these parameters and the order of the expressions, the interface point is classified into penetrable or impenetrable. Whereas in this work we consider the the interface point to be regular and the functions to be sufficiently smooth.

Definitions and Notation. Let $I_1 = (a, c]$ and $I_2 = [c, b)$ where $-\infty \leq< a < c < b \leq +\infty$. For any non-negative integer n, let $C^n(I_i)$ denote the space of all complex valued n-times continuously differentiable functions defined on $I_i; i = 1, 2$. Let $C^\infty(I_i)$ denote the space of all infinitely many times differentiable complex valued functions defined on $I_i; i = 1, 2$. Let $A^n(I_i)$ denote the space of all functions in $C^{(n-1)}(I_i)$ such that $(n-1)^{th}$derivative is absolutely continuous over each compact subset of $I_i; i = 1, 2$. For a function f, $f^{(j)}$ denote the j^{th} derivative of f, if it exists. For any $m \times n$ matrix A, let A^* denote the adjoint of A. For a square matrix A, A^{-1} denotes the inverse of A, if it exists. For any two nonempty sets(topological spaces) V_1 and V_2, let $V_1 \times V_2$ denote the cartesian product (space equipped with product topology) of V_1 and V_2, taken in that order. Let $L_2(I_i)$ denote the space of all measurable complex-valued functions square integrable on $I_i, i = 1, 2$. Let the inner product in $L_2(I_i)$ be denoted by $\langle ., . \rangle$, $i = 1, 2$. Let $H^n(I_i)$ denote those functions f in $A^n(I_i)$ such that $f^{(n)}$ belongs to $L_2(I_i), i = 1, 2$. Let $H^n_0(I_i)$ denote the space of all functions f in $H^n(I_i)$ such that f vanishes in a neighbourhood of a and $f(c) = f'(c) = \dots\dots = f^{(n-1)}(c) = 0$. Let $H^n_0(I_2)$ denote the space of all functions f in $H^n(I_2)$ such that f vanishes in a neighbourhood of b and $f(c) = f'(c) = \dots = f^{(n-1)}(c) = 0$.

Let A and B be non singular $n \times n$ matrices with complex entries. For $f_i \in C^n(I_i)$, let $\tilde{f}_i(t) = column(f_i(t), f'_i(t), \dots, f^{(n-1)}(t)), t \in I_i, i = 1, 2$. Let $H^n(I_1 \times I_2)$ denote the space of all pairs $(f_1, f_2) \in H^n(I_1) \times H^n(I_2)$ such that $A\tilde{f}_1(c) = B\tilde{f}_2(c)$. Let $H^n_0(I_1 \times I_2)$ denote the space of all pairs $(f_1, f_2) \in H^n(I_1 \times I_2)$ such that f_1 vanishes in a neighbourhood of a and f_2 vanishes in a neighbourhood of b.

Let τ_1 and τ_2 be a pair of formal ordinary differential operators of order n defined on the intervals I_1 and I_2, respectively, of the form

$$\tau_1 = \sum_{k=0}^{n} a_k(t)(\frac{d}{dt})^k, \quad \tau_2 = \sum_{k=0}^{n} b_k(t)(\frac{d}{dt})^k$$

where the coefficients $a_k \in C^\infty(I_1)$, $b_k \in C^\infty(I_2)$ and $a_n(t) \neq 0$ and $b_n(t) \neq 0$ on I_1 and I_2 respectively. For $(f_1, f_2) \in A^n(I_1) \times A^n(I_2)$, let

$$(\tau_1, \tau_2)(f_1, f_2) = (\tau_1 f_1, \tau_2 f_2)$$

where

$$(\tau_1 f_1)(t) = \sum_{k=0}^{n} a_k(t) f_1^{(k)}(t), t \in I_1 ,$$

$$(\tau_2 f_2)(t) = \sum_{k=0}^{n} b_k(t) f_2^{(k)}(t), t \in I_2$$

We define $T_0(\tau_i)$,$T_1(\tau_i)$ in $L_2(I_i)$, $i=1,2$ and $T_0(\tau_1,\tau_2)$, $T_1(\tau_1,\tau_2)$ in $L_2(I_1)\times L_2(I_2)$ as follows:

$$D(T_0(\tau_i))=H_0^n(I_i), T_0(\tau_i)f_i=\tau_i f_i, f_i\in D(T_0(\tau_i)); i=1,2;$$
$$D(T_1(\tau_i))=H^n(I_i), T_1(\tau_i)f_i=\tau_i f_i, f_i\in D(T_1(\tau_i)); i=1,2;$$
$$D(T_0(\tau_1,\tau_2))=H_0^n(I_1\times I_2), T_0(\tau_1,\tau_2)(f_1,f_2)=(\tau_1 f_1,\tau_2 f_2);$$
$$D(T_1(\tau_1,\tau_2))=H^n(I_1\times I_2), T_1(\tau_1,\tau_2)(f_1,f_2)=(\tau_1 f_1,\tau_2 f_2).$$

We note that $T_0(\tau_i),T_1(\tau_i))$ are densely defined unbounded operators in $L_2(I_i)$, $i=1,2; T_0(\tau_1,\tau_2),T_1(\tau_1,\tau_2)$ are densely defined unbounded operators in $L_2(I_1)\times L_2(I_2)$. We also note that the matching conditions at the interface $t=c$, viz. $A\tilde{f}_1(c)=B\tilde{f}_2(c)$ are introduced into the domains of $T_0(\tau_1,\tau_2)$ and $T_1(\tau_1,\tau_2)$. It is true that $T_0(\tau_i)$, $i=1,2$, $T_0(\tau_1,\tau_2)$ are minimal unclosed operators and and $T_1(\tau_i)$, $i=1,2$; $T_1(\tau_1,\tau_2)$ are the maximal closed operators in the respective spaces. Moreover, $T_0(\tau_i)=T_0(\tau_i^*)^*$, where τ_i^* is the formal adjoint of τ_i, $i=1,2$. Under certain assumptions on the matrices A,B and the boundary matrices for τ_1,τ_2 at c, we shall prove in the next section that $T_1(\tau_1,\tau_2)=T_0(\tau_1^*,\tau_2^*)^*$. Thus if τ_1 and τ_2 are formally selfadjoint, then we have $T_1(\tau_i)=T_0(\tau_i)^*$, $i=1,2$ and $T_1(\tau_1,\tau_2)=T_0(\tau_1,\tau_2)^*$. The positive and negative deficiency spaces of $T_0(\tau_1),T_0(\tau_2)$ and $T_0(\tau_1,\tau_2)$ are defined as follows:

$$D'_+=\{f_1\in D(T_1(\tau_1))/\tau_1 f_1=if_1\},$$
$$D'_-=\{f_1\in D(T_1(\tau_1))/\tau_1 f_1=-if_1\},$$
$$D"_+=\{f_2\in D(T_1(\tau_2))/\tau_2 f_2=if_2\},$$
$$D"_-=\{f_2\in D(T_1(\tau_2))/\tau_2 f_2=-if_2\},$$
$$D_+=\{(f_1,f_2)\in D(T_1(\tau_1,\tau_2)),/(\tau_1,\tau_2)(f_1,f_2)=i(f_1,f_2)\},$$
$$D_-=\{(f_1,f_2)\in D(T_1(\tau_1,\tau_2)),/(\tau_1,\tau_2)(f_1,f_2)=-i(f_1,f_2)\},$$

and the following quantities

$$d'_+=\dim D'_+; d'_-=\dim D'_-\,,$$
$$d''_+=\dim D''_+; d''_-=\dim D''_-\,,$$
$$d_+=dim D_+; d_-=dim D_-$$

are called the positive and negative deficiencies of $T_0(\tau_1),T_0(\tau_1),T_0(\tau_1,\tau_2)$, respectively. Our main interest here is to prove the following theorem.

Theorem 1.1. *If τ_1,τ_2 are formally selfadjoint and*

$$(A^{-1})^*F_c(\tau_1)A^{-1}=(B^{-1})^*F_c(\tau_2)B^{-1} \tag{1.1}$$

where $F_c(\tau_i)$ is the boundary matrix of τ_i at $t=c, i=1,2$, then

$$d_+=d'_++d''_+-n \quad \text{and} \quad d_-=d'_-+d''_--n\,.$$

If $\tau_1=\tau_2$, that is the same differential operator is defined on I_1 and I_2, and $A=B=I$, (where I denotes the identity matrix) then [3, corollary (XIII).2.26] *becomes a special case of the above theorem. The proof of Theorem 1.1, that we present here is new and more appealing than the proof given in* [3], *for the special case $\tau_1=\tau_2$, and $A=B=I$.*

2. Preliminary results

In this section, we present a few definitions and results that are useful towards proving Theorem 1.1.

Let g_i be complex valued measurable function which is integrable over every compact subinterval of I_i, $i = 1, 2$. Consider the boundary-value problem (BVP)

$$(\tau_1, \tau_2)(f_1, f_2) = (g_1, g_2) \tag{2.1}$$

$$A\tilde{f}_1(c) = B\tilde{f}_2(c) \tag{2.2}$$

By a solution of problem (2.1)-(2.2), we mean a pair $(f_1, f_2) \in A^n(I_1) \times A^n(I_2)$ such that

(i) $(\tau_1 f_1)(t) = g_1(t)$ for almost all $t \in I_1$
(ii) $(\tau_2 f_2)(t) = g_2(t)$ for almost all $t \in I_2$
(iii) $A\tilde{f}_1(c) = B\tilde{f}_2(c)$.

Let $t_i \in I_i$, $i = 1, 2$ and $\{c_0, \dots, c_{n-1}\}, \{d_0, \dots, d_{n-1}\}$ be arbitrary set of complex numbers. Consider the initial conditions

$$f_1^{(i)}(t_1) = c_i, i = 0, 1, \dots, n-1, t_1 \in I_1, \tag{2.3}$$

$$f_2^{(i)}(t_2) = d_i, i = 0, 1, \dots, n-1, t_2 \in I_2, \tag{2.4}$$

The following results can be proved easily.

Lemma 2.1. *The initial boundary-value problem* (2.1)-(2.2)-(2.3) *(*(2.1)-(2.2)-(2.4)*) has a unique solution.*

Lemma 2.2. *If g_i has k continuous derivatives in I_i, then the component f_i of the solution (f_1, f_2) of* (2.1)-(2.2)-(2.3) *(*(2.1)-(2.2)-(2.4)*) has $(n+k)$ continuous derivatives in I_i, $i = 1, 2$.*

Lemma 2.3. *If $(g_1, g_2) = (0, 0)$ and $0 = c_0 = c_1 = \dots = c_{n-1}(0 = d_0 = d_1 = \dots = d_{n-1})$, then $(f_1, f_2) = (0, 0)$ is the only solution of* (2.1)-(2.2)-(2.3) *(*(2.1)-(2.2)-(2.4)*).*

We say the pairs $(f_{11}, f_{21}), \dots, (f_{1p}, f_{2p})$ are linearly independent on $I_1 \times I_2$ if

$$\sum_{k=1}^{p} \alpha_k f^{(j)}(t) = 0, \quad t \in I_i,\ j = 0, 1, \dots, n-1,\ i = 1, 2$$

where $\alpha_1, \dots, \alpha_p$ are scalars, then $\alpha_1 = \alpha_2 = \dots = \alpha_p = 0$.

The next result follows easily from Result 2.1.

Lemma 2.4. *The boundary-value problem*

$$(\tau_1, \tau_2)(f_1, f_2) = (0, 0) \tag{2.5}$$

$$A\tilde{f}_1(c) = B\tilde{f}_2(c) \tag{2.6}$$

has exactly n linearly independent solutions.

We now prove the Green's formula for the pair (τ_1, τ_2).

Theorem 2.5. *Let $I_1 = [a, c], I_2 = [c, b]$, $-\infty < a < c < b < +\infty$. Let relation (1.1) be true. Then for $(f_1, f_2)(g_1, g_2) \in H^n(I_1 \times I_2)$,*

$$\begin{aligned}&\int_a^c (\tau_1 f_1)(t)\bar{g}_1(t)dt + \int_c^b (\tau_2 f_2)(t)\bar{g}_2(t)dt \\ &= \int_a^c f_1(t)(\bar{\tau_1^*} g_1)(t)dt + \int_c^b f_2(t)(\bar{\tau_2^*} g_2)(t)dt + F_b(f_2, g_2) - F_a(f_1, g_1)\end{aligned}$$

where $F_t(f_i, g_i)$ is the boundary form for τ_i at $t \in I_i$.

Proof. Being the proof routine it suffices to verify that

$$F_c(f_1, g_1) = F_c(f_2, g_2).$$

To show this, we consider,

$$\begin{aligned}F_c(f_1, g_1) &= (\tilde{g}_1(c))^* F_c(\tau_1)\tilde{f}_1(c) \\ &= (\tilde{g}_1(c))^* A^*(A^{-1})^* F_c(\tau_1)A^{-1}A\tilde{f}_1(c) \\ &= (A\tilde{g}_1(c))^*(A^{-1})^* F_c(\tau_1)A^{-1}(A\tilde{f}_1(c)) \\ &= (B\tilde{g}_2(c))^*(B^{-1})^* F_c(\tau_2)B^{-1}(B\tilde{f}_2(c)) \\ &= (\tilde{g}_2(c))^* B^*(B^{-1})^* F_c(\tau_2)B^{-1}B\tilde{f}_2(c) \\ &= (\tilde{g}_2(c))^* F_c(\tau_2)\tilde{f}_2(c) \\ &= F_c(f_2, g_2)\end{aligned}$$

□

The following corollary is immediate.

Corrolary 2.6. *If I_1 and I_2 are arbitrary intervals and Relation (1.1) is true, then Green's formula is valid for $(f_1, f_2), (g_1, g_2) \in H^n(I_1 \times I_2)$ (or even $(f_1, f_2) \in H^n(I_1 \times I_2), (g_1, g_2) \in A^n(I_1) \times A^n(I_2)$ satisfying $A\tilde{g}_1(c) = B\tilde{g}_2(c)$) provided that either (f_1, f_2) or (g_1, g_2) vanishes outside a compact subcell of $I_1 \times I_2$.*

In the rest of the work, we assume Relation (1.1) to be true. The following results could be proved with suitable modifications as in [3, pp 1291-1295].

Lemma 2.7. *Let f_i be a function whose square is integrable over every compact subinterval of $I_i, i = 1, 2$. Suppose that*

$$\sum_{i=1}^{2} \int_{I_i} f_i(t)\bar{\tau_i^*} g_i(t)dt = 0, \quad \textit{for all } (g_1, g_2) \in H_0^n(I_1 \times I_2).$$

Then (after modification on a set of measure zero)

$$(f_1, f_2) \in C^\infty(I_1) \times C^\infty(I_2), A\tilde{f}_1(c) = B\tilde{f}_2(c) and (\tau_1, \tau_2)(f_1, f_2) = (0, 0).$$

Lemma 2.8. $T_1(\tau_1, \tau_2) = T_0(\tau_1^*, \tau_2^*)^*$.

From Lemma 2.8 it follows that $T_1(\tau_1, \tau_2)$ is a closed operator. Thus $T_1(\tau_1, \tau_2)$ is an extension of $T_0(\tau_1, \tau_2)$ and hence $T_0(\tau_1, \tau_2)$ has an minimal closed extension $\bar{T_0}(\tau_1, \tau_2)$.

Lemma 2.9. *If τ_1, τ_2 are formally selfadjoint then $T_0(\tau_1, \tau_2)$ is the restriction of $T_0(\tau_1, \tau_2)^*$. (that is $T_0(\tau_1, \tau_2)$ is symmetric).*

Lemma 2.10. *If τ_1, τ_2 are formally selfadjoint then $D'_+, D'_-; D''_+, D''_-; D_+, D_-$ consists precisely of those solutions of the equations $(\tau_1 - i)f_1 = 0$, $(\tau_1 + i)f_1 = 0$; $(\tau_2 - i)f_2 = 0$, $(\tau_2 + i)f_2 = 0$; $((\tau_1, \tau_2) + i)(f_1, f_2) = (0, 0)$, satisfying $A\tilde{f_1}(c) = B\tilde{f_2}(c)$, lying in $L_2(I_1), L_2(I_2), L_2(I_1) \times L_2(I_2)$, respectively.*

Lemma 2.11. *Let $J_1 \times J_2$ be a compact subcell of $I_1 \times I_2$. Then*

(i) *The space $H^n(J_1 \times J_2)$ is complete in the norm*

$$\| (f_1, f_2) \| = \max\Big(\sum_{i=0}^{n-1} \max_{t \in J_1} |f_1^{(i)}(t)|, \sum_{i=0}^{n-1} \max_{t \in J_2} |f_2^{(i)}(t)|\Big) + \Big(\sum_{i=1}^{2} \int_{J_i} |f_i^{(n)}(t)|^2 dt\Big)^{1/2}.$$

(ii) *$\{(f_{1n}, f_{2n})\}$ is a sequence in $H^n(I_1 \times I_2)$ such that $\{(f_{1n}, f_{2n})\}$ and $(\tau_1, \tau_2)\{(f_{1n}, f_{2n})\}$ converge (converge weakly) in $L_2(I_1) \times L_2(I_2)$, then the sequence $\{(f_{1n}, f_{2n})\}$ converges (converge weakly) in the topology of $H^n(J_1 \times J_2)$ defined by the above norm. For $(f_1, f_2), (g_1, g_2) \in L_2(I_1) \times L_2(I_2)$, the inner product in $L_2(I_1) \times L_2(I_2)$ is given by*

$$\big\langle (f_1, f_2), (g_1, g_2) \big\rangle = \langle f_1, f_2 \rangle + \langle g_1, g_2 \rangle$$

Since $T_1(\tau_1, \tau_2)$ is closed, $H^n(I_1 \times I_2) = D(T_1(\tau_1, \tau_2))$ becomes a Hilbert space upon introduction of the inner product

$$\big\langle (f_1, f_2), (g_1, g_2) \big\rangle^* = \big\langle (f_1, f_2), (g_1, g_2) \big\rangle + \big\langle (\tau_1, \tau_2)(f_1, f_2), (\tau_1, \tau_2)(g_1, g_2) \big\rangle$$

Definition. A boundary value for (τ_1, τ_2) is a continuous linear functional Θ on $D(T_1(\tau_1, \tau_2))$ which vanishes on $D(T_0(\tau_1, \tau_2))$. If $\Theta(f_1, f_2) = 0$ for each $(f_1, f_2) \in D(T_1(\tau_1, \tau_2))$ which vanishes in a neighbourhood of a, Θ is called a boundary value at a. A boundary value at b is defined similarly . An equation $\Theta(f_1, f_2) = 0$, when Θ is a boundary value for (τ_1, τ_2), is called a boundary condition for (τ_1, τ_2). A complete set of boundary values is a maximal linearly independent set of boundary values. Similarly a complete set of boundary values at $a(b)$ is a maximal linearly independent set of boundary values at $a(b)$.

Note: If τ_1, τ_2 formally selfadjoint, the boundary values for (τ_1, τ_2) coincides with [3, Definition (XII)4.20] of a boundary value for $T_0(\tau_1, \tau_2)$.

The following results can be provided with suitable modifications as in [3, pp: 1298-1301].

Lemma 2.12. *The space of boundary values for (τ_1, τ_2) is the direct sum of the space of boundary values for (τ_1, τ_2) at a and the space of boundary values for (τ_1, τ_2) at b.*

Lemma 2.13. *There exists a one to one linear mapping of the space of all boundary values for $\tau_1(\tau_2)$ at $a(b)$ on to the space of all boundary values for (τ_1, τ_2) at $a(b)$.*

Lemma 2.14. *$\tau_1(\tau_2)$ and (τ_1, τ_2) have the same number of linearly independent boundary conditions at $a(b)$.*

Lemma 2.15. *(τ_1, τ_2) has at most n linearly independent boundary values at $a(b)$.*

Lemma 2.16. *If $I_1 = [a, c]$, $-\infty < a(I_2 = [c, b], b < +\infty)$, then the functionals $\Theta_i(f_1, f_2) = f_1^{(i)}(a)(f_2^{(i)}(b)), i = 0, 1, \dots n-1$ form a complete set of boundary values for (τ_1, τ_2) at $a(b)$.*

Lemma 2.17. *If τ_1, τ_2 are formally selfadjoint and*
$$d = d_+ + d_-, \quad d' = d'_+ + d'_-, \quad d'' = d''_+ + d''_-$$
then $d = d' + d'' - 2n$.

3. Proof of Theorem 1.1

Proof. Let $(f_{11}, f_{21}), \dots, (f_{1d+}, f_{2d_+})$ be a basis for D_+ ; $g_{11}, \dots, g_{1d'_+}$ be basis for D'_+; $h_{21}, \dots, h''_{2d_+}$ be a basis for D''_+. Clearly, $\{(f_{1i}, f_{2i})\}$, $i = 1, 2 \dots, d_+$ are linearly independent and belong to $L_2(I_1) \times L_2(I_2)$; $\{g_{1i}\}, i = 1, \dots, d'_+$ are linearly independent and belong to $L_2(I_1)$; $\{h_{2i}\}, i = 1, \dots, d''_+$ are linearly independent and belong to $L_2(I_2)$. We have $d_+ \le d''_+, d_+ \le d'_+$.
Claim 1: At least $(d'_+ - d_+)$ number of g_{i1}s are linearly independent with respect to the set $S = \{f_{11}, \dots, f_{1d_+}\}$. For, if possible, let this number of g_{i1}s be strictly less than $(d'_+ - d_+)$. Then at least $(d_+ + 1)$ number of g_{i1}s shall be linearly dependent to S. Without loss of generality, we may assume that $g_{11}, \dots, g_{1d_+}$ are linearly independent to S. Then there exists scalars $\alpha_{ij}, i, j = 1, 2, dots, d_+$ and $\beta_1, \dots, \beta_{d_+}$ such that
$$\begin{gathered}
\alpha_{11} f_{11} + \dots + \alpha_{1d_+} f_{1d_+} = g_{11} \\
\alpha_{21} f_{11} + \dots + \alpha_{2d_+} f_{1d_+} = g_{12} \\
\vdots \\
\alpha_{d_+1} f_{11} + \dots + \alpha_{d_+d_+} f_{1d_+} = g_{1d_+}
\end{gathered} \tag{3.1}$$
and
$$\beta_1 f_{11} + \dots + \beta_{d_+} f_{1d_+} = g_{1d_++1} \tag{3.2}$$
Since $g_{11}, \dots, g_{1d_+}$ are linearly independent, the matrix
$$\begin{pmatrix} \alpha_{11} & \dots & \alpha_{1d_+} \\ \alpha_{21} & \dots & \alpha_{2d_+} \\ \vdots & & \vdots \\ \alpha_{d_+1} & \dots & \alpha_{d_+d_+} \end{pmatrix}$$
is nonsingular and consequently system (3.1) gives that each $f_{1i}, i = 1, \dots, d_+$ can be expressed as a linear combination of $g_{11}, \dots, g_{1d_+}$ and then substituting into equation (3.2), we get g_{1d_++1} is a linear combination of $g_{11}, \dots, g_{1d_+}$, a contradiction. Hence the claim is true.

Now, let $g_{1d_++1}, \dots, g_{1d'_+}$ be linearly independent with respect to S. Using Lemma 2.1, we can extend these functions to the pairs $(g_{1d_++1}, g_{2d_++1}), \dots, (g_{1d'_+}, g_{2d'_+})$ satisfying
$$((\tau_1, \tau_2) - i)(g_{1i}, g_{2i}) = (0, 0), \tag{3.3}$$
$$A\tilde{g}_{1i}(c) = B\tilde{g}_{2i}(c), i = d_+ + 1, \dots, d'_+ \tag{3.4}$$
Clearly, $(f_{11}, f_{21}), \dots, (f_{1d_+}, f_{2d_+}), (g_{1d_++1}, g_{2d_++1}), \dots, (g_{1d'_+}, g_{2d'_+})$ are linearly independent and $g_{2i} \notin L_2(I_2)$, for any $i = d_+ + 1, \dots, d'_+$.

Next, let $\tilde{S} = \{f_{21}, \dots, f_{2d_+}\}$. As in claim 1, we can prove at least $(d''_+ - d_+)$ number of h_{2i}s must be linearly independent with respect to $\tilde{S}$. Using Lemma 2.1, we can extend these functions to the pairs $(h_{1d_++1}, h_{2d_++1}), \dots \dots, (h_{1d''_+}, h_{2d''_+})$ satisfying

$$((\tau_1,\tau_2)-i)(h_{1i},h_{2i})=(0,0) \tag{3.5}$$
$$A\tilde{h}_{1i}(c)=B\tilde{h}_{2i}(c) \tag{3.6}$$

Clearly, $(f_{11},f_{21}),\dots,(f_{1d_+},f_{2d_+})$, $(h_{1d_++1},h_{2d_++1}),\dots,(h_{1d''_+},h_{2d''_+})$ are linearly independent and $h_{1i}\notin L_2(I_1)$, for any $i=d_++1,\dots,d''_+$.

Claim 2: $(f_{11},f_{21}),\dots,(f_{1d_+},f_{2d_+}),(g_{1d_++1},g_{2d_++1}),\dots,(g_{1d'_+},g_{2d'_+})$, $(h_{1d_++1},h_{2d_++1}),\dots,(h_{1d''_+},h_{2d''_+})$ are linearly independent solutions of

$$((\tau_1,\tau_2)-i)(f_1,f_2)=(0,0) \tag{3.7}$$
$$A\tilde{f}_1(c)=B\tilde{f}_2(c)\,. \tag{3.8}$$

It suffices to verify the linear independency of these pairs of functions. Again it suffices to show that g's and h's are mutually linear independent. If possible for some i, let

$$(g_{1i},g_{2i})=\alpha_1(h_{1d_++1},h_{2d_++1})+\dots+\alpha_{d''_+-d_+}(h_{1d''_+},h_{2d''_+})$$

for some scalars $\alpha_1,\dots,\alpha_{d''_+-d_+}$, not all zeros. Then

$$g_{2i}=\sum_{i=1}^{d''_+-d_+}\alpha_i h_{2i}$$

a contradiction, since the left-hand side is not in $L_2(I_2)$, whereas the right-hand side is in $L_2(I_2)$. Similarly, it can be proved that no (h_{1i},h_{2i}) is a linear combination of (g_{1i},g_{2i}), $i=d_++1,\dots,d'_+$. This proves claim 2.

Finally by Lemma 2.4, we have

$$d_++(d'_+-d_+)+(d''_+-d_+)\le n.$$

That is

$$d''_++d'_+-d_+\le n \tag{3.9}$$

Similarly we get,

$$d''_-+d'_--d_-\le n \tag{3.10}$$

Claim 3: $d''_++d'_+-d_+\le n$ and $d''_++d'_+-d_+\le n$ For if possible, let the strict inequality hold in either (3.9) or (3.10). Then, adding these two we get

$$(d''_++d''_-)+(d'_++d'_-)-(d_++d_-)<2n\,.$$

That is, $d''+d'-d<2n$ which is a contradiction to Lemma 2.17. This proves claim 3 and the proof of the theorem is complete. □

We remark that the operators of the form considered here occur in many physical situations such as acoustic wave guides in oceans; see [1, 5, 6, 7, 8, 9, 4].

References

[1] C. Allan Boyles; *Acoustic waveguides application to oceanic sciences*, Interscience, New York, 1984.

[2] Pallav Kumar Baruah and Dibya Jyoti Das; *Study of a pair of singular sturm liouville equations for an interface problem*, to appear, International Journal of Mathematical Sciences.

[3] N. Dunford and J. T. Schawrtz; *Linear operators Part II*, Interscience, New York 1963.

[4] Orochko, Yu. B.; *Impenetrability condition for the point of degeneration of an even-order one-term symmetric differential operator.* (Russian) Mat. Sb. 194 (2003), no. 5, 109–138; translation in Sb. Math. 194 (2003), no. 5-6, 745–774.

[5] M. Venkatesulu and T. Gnana Bhaskar: *Selfadjoint boundary value problems associated with a pair of mixed linear ordinary differential equations*, Math. Anal. & Appl. 144, No. 2, 1989, 322-341.

[6] M. Venkatesulu and T.Gnana Bhaskar; *Solutions of initial value problems associated with a pair of mixed linear ordinary differential equations*, Math. Anal. & Appl. 148, No.1, 1990, 63-78.

[7] M. Venkatesulu and Pallav Kumar Baruah; *Solution of initial value problem associated with a pair of first order system of singular ordinary differential equations with interface spatial conditions*, J. Appl. Math. and St. Anal. 9, Number 3, 1996, 303-314.

[8] M. Venkatesulu and Pallav Kumar Baruah; *Classical approach to eigenvalue problems associated with a pair of mixed regular Sturm-Liouville equations -I*, J. Appl.Math. and St. Anal. 14:2(2001), 205-214

[9] M. Venkatesulu and Pallav Kumar Baruah; *Classical approach to Eigenvalue problems associated with a pair of Mixed Regular Sturm-Liouville Equations -II*, J. Appl. Math. and St. Anal. 15:2(2002), 197-203.

Pallav Kumar Baruah
Department of Mathematics and Computer Science, Sri Sathya Sai Institute of Higher Learning, Prasanthinilayam, India
E-mail address: baruahpk@yahoo.com

M. Venkatesulu
Department of Computer Applications, Arulmigu Kalasalingam College of Engineering, Krishnankoil-626190. Virudhunagar (District), Tamil Nadu, India
E-mail address: venkatesulu_m2000@yahoo.co.in