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Gradient method in Sobolev spaces for nonlocal boundary-value problems *

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Abstract

An infinite-dimensional gradient method is proposed for the numerical solution of nonlocal quasilinear boundary-value problems. The iteration is executed for the boundary-value problem itself (i.e. on the continuous level) in the corresponding Sobolev space, reducing the nonlinear boundary-value problem to auxiliary linear problems. We extend earlier results concerning local (Dirichlet) boundary-value problems. We show linear convergence of our method, and present a numerical example.

1 Introduction

The object of this paper is to study the numerical solution to the nonlocal quasilinear boundary-value problem

$$T(u) \equiv -\operatorname{div} f(x, \nabla u) + q(x, u) = g(x) \quad \text{in } \Omega$$
$$Q(u) \equiv f(x, \nabla u) \cdot \nu + \int_{\partial\Omega} \varphi(x, y) u(y)\, d\sigma(y) = 0 \quad \text{on } \partial\Omega$$

on a bounded domain $\Omega \subset \mathbb{R}^N$. The exact conditions on the domain Ω and the functions f, q, g and φ will be given in Section 2.

The nonlocal boundary condition allows the normal component of the nonlinearity $f(x, \nabla u)$ to depend on a nonlocal expression of u, in contrast to a function of $u(x)$ in the usual case of mixed boundary conditions (or especially 0 in the case of Neumann problems). This kind of boundary condition has been analysed in detail e.g. in [13, 21]. Most often the studied nonlocal expression depends on a composite function of u, this boundary condition arises e.g. in plasma physics. General theoretical results on existence and uniqueness of weak solutions to such problems have been proved in [23] and [22] for linear and nonlinear equations, respectively. In this paper we consider the case when the

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nonlocal expression involves an integral for all the values of $u_{|\partial\Omega}$ (cf. [13]). (The weak formulation of our problem will also be given in Section 2.)

The usual way of the numerical solution of elliptic equations is to discretize the problem and use an iterative method for the solution of the arising nonlinear system of algebraic equations (see e.g. [12, 16]). However, the condition number of the Jacobians of these systems can be arbitrarily large when discretization is refined. This phenomenon would yield very slow convergence of iterative methods, hence suitable nonlinear preconditioning technique has to be used [2].

Our approach is opposite to the above: the iteration can be executed for the boundary-value problem itself (i.e. on the continuous level) directly in the corresponding Sobolev space, reducing the nonlinear boundary-value problem to auxiliary linear problems. Then discretization may be used for these auxiliary problems. This approach can be regarded as infinite-dimensional preconditioning, and yields automatically a fixed ratio of convergence for the iteration, namely, that which is explicitly obtained from the coefficients f, q and g. Concerning this, we note that the method in question is related to the Sobolev gradient technique, developed in [17, 18, 19]. Especially, in [17] nonlocal boundary conditions are discussed in connection with Sobolev gradients.

The theoretical background of this approach is the generalization of the gradient method to Hilbert spaces. This was first developed by Kantorovich for linear equations (see [11]). For the numerous results so far, we refer e.g. to [3, 5, 7, 20, 24]; the investigations of the author have included non-differentiable operators [9] and non-uniformly monotone operators [10]. The mentioned results focus on partial differential operators. Concerning numerical realization to local (Dirichlet) boundary-value problems relying on the Hilbert space gradient method, we refer to [6, 7].

This paper consists of three parts. The exact formulation of the problem is given in Section 2. The gradient method for the nonlocal boundary value problem is constructed and its linear convergence is proven in Section 3. The numerical realization is illustrated in Section 4.

2 Formulation of the problem

The exact formulation of the nonlocal boundary condition requires the following notion. (Therein and throughout the paper σ denotes Lebesgue measure on the boundary.)

Definition 2.1 Let $\Omega \subset \mathbb{R}^N$, $\partial\Omega \in C^1$. A function $\varphi : \partial\Omega^2 \to \mathbb{R}$ is called

(i) a *positive kernel* if it fulfills

$$\varphi(x, y) = \int_{\partial\Omega} \psi(x, z)\psi(z, y)\, d\sigma(z) \quad (x, y \in \partial\Omega)$$

with some $\psi \in L^2(\partial\Omega^2)$ satisfying $\psi(x, y) = \psi(y, x)$ $(x, y \in \partial\Omega)$;

(ii) *regular* if the function $x \mapsto \int_{\partial\Omega} \varphi(x,z)\,d\sigma(z)$ does not a.e. vanish on $\partial\Omega$.

The following properties are elementary to prove.

Proposition 2.1 *A positive kernel φ fulfills $\varphi \in L^2(\partial\Omega^2)$ and $\varphi(x,y) = \varphi(y,x)$ $(x, y \in \partial\Omega)$.*

Proposition 2.2 *Consider the linear integral operator $A : L^2(\partial\Omega) \to L^2(\partial\Omega)$,*

$$(Au)(x) = \int_{\partial\Omega} \varphi(x,y)u(y)\,d\sigma(y)\,. \tag{1}$$

(i) If φ is a positive kernel then A is a positive operator, i.e.

$$\int_{\partial\Omega} (Au)u \geq 0 \quad (u \in L^2(\partial\Omega)).$$

(ii) If φ is regular then A does not carry constants to the (a.e.) zero function.

Definition 2.2 Let φ be a regular positive kernel and $m > 0$. Then we define

$$\langle u, v\rangle \equiv \int_\Omega \nabla u \cdot \nabla v + \frac{1}{m}\iint_{\partial\Omega^2} \varphi(x,y)u(y)v(x)\,d\sigma(y)\,d\sigma(x)\,. \tag{2}$$

Proposition 2.3 *Formula (2) defines an inner product on $H^1(\Omega)$.*

The above inner product will be used in $H^1(\Omega)$ (with $m > 0$ to be defined in condition (C3) below) throughout the paper, and the corresponding norm will be denoted by $\|\,.\,\|$. We note that if $u \in H^2(\Omega)$ and $\frac{\partial u}{\partial \nu} + A(u) = 0$ on $\partial\Omega$, then the divergence theorem yields

$$\langle u, v\rangle = \int_\Omega (-\Delta u)v \tag{3}$$

with $m = 1$. (This is a special case of Remark 2.4 below with $T = -\Delta$.)

We will use notation ν for the outward normal vector on $\partial\Omega$, and dot product to denote the inner product in $\mathbb{R}^N$.

Now the nonlocal boundary-value problem can be formulated.

We consider the problem

$$\begin{aligned} &T(u) \equiv -\operatorname{div} f(x, \nabla u) + q(x,u) = g(x) \quad \text{in } \Omega \\ &Q(u) \equiv f(x,\nabla u)\cdot\nu + \int_{\partial\Omega} \varphi(x,y)u(y)\,d\sigma(y) = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{4}$$

with the following conditions:

(C1) $\Omega \subset \mathbb{R}^N$ is bounded, $\partial\Omega \in C^1$; $f \in C^1(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N)$, $q \in C^1(\overline{\Omega} \times \mathbb{R}^N)$, $g \in L^2(\Omega)$;

(C2) φ is a regular positive kernel;

(C3) there exist constants $m' \geq m > 0$ such that for all $(x,\eta) \in \overline{\Omega} \times \mathbb{R}^N$ the Jacobians $\dfrac{\partial f(x,\eta)}{\partial \eta} \in \mathbf{R}^{N\times N}$ are symmetric and their eigenvalues λ fulfill

$$m \leq \lambda \leq m'\,;$$

further, there exist constants $\kappa, \beta \geq 0$ such that for all $(x,u) \in \overline{\Omega} \times \mathbb{R}$

$$0 \leq \frac{\partial q(x,u)}{\partial u} \leq \kappa + \beta |u|^{p-2}$$

where $2 \leq p$ if $N = 2$ and $2 \leq p < \frac{2N}{N-2}$ if $N > 2$.

Remark 2.1 It is worth mentioning the following special cases of f.

(a) $f(x,\nabla u) = p(x,\nabla u)\nabla u$ where $p \in C^1(\overline{\Omega} \times \mathbb{R}^N)$. Then the boundary condition takes the form

$$p(x,\nabla u)\frac{\partial u}{\partial \nu} + \int_{\partial\Omega} \varphi(x,y)u(y)\,d\sigma(y) = 0\,.$$

(b) $f(x,\nabla u) = a(|\nabla u|)\nabla u$ where $a \in C^1[0,\infty)$ (a special case of (a)). The corresponding type of operator T arises e.g. in elasto-plasticity theory or in the study of magnetic potential [8, 15].

Remark 2.2 The assumption $2 \leq p$ (if $N = 2$), $2 \leq p < \frac{2N}{N-2}$ (if $N > 2$) in condition (C3) yields [1] that there holds the Sobolev embedding

$$H^1(\Omega) \subset L^p(\Omega)\,. \tag{5}$$

Remark 2.3 The condition that φ is a regular kernel is required to avoid the lack of injectivity when $f(x,0) = 0$ (e.g. in the cases of Remark 2.1). Namely, there would otherwise hold $Q(c) = 0$ on $\partial\Omega$ for constant functions c as in the case of Neumann boundary condition.

Proposition 2.4 *For any $u, v \in H^1(\Omega)$ let*

$$\langle F(u), v\rangle \equiv \int_\Omega \big(f(x,\nabla u)\cdot\nabla v + q(x,u)v\big) + \iint_{\partial\Omega^2} \varphi(x,y)u(y)v(x)\,d\sigma(y)\,d\sigma(x)\,. \tag{6}$$

Then formula (6) defines an operator $F : H^1(\Omega) \to H^1(\Omega)$.

Proof Condition (C3) implies that for all $i, j = 1, .., N$ and $(x, \eta) \in \overline{\Omega} \times \mathbb{R}^N$

$$\left| \frac{\partial f_i}{\partial \eta_j}(x, \eta) \right| \leq m' .$$

Lagrange's inequality yields that for all $(x, \eta) \in \overline{\Omega} \times \mathbb{R}^N$ we have

$$|f_i(x, \eta)| \leq |f_i(x, 0)| + m' N^{1/2} |\eta| , \quad |q(x, u)| \leq |q(x, 0)| + \kappa |u| + \beta |u|^{p-1} .$$

Consequently, the integral on Ω in (6) can be estimated by

$$\begin{aligned} &\int_\Omega \Big(\sum_{i=1}^N \big(|f_i(x, 0)| + m' N^{1/2} |\nabla u| \big) |\partial_i v| + \big(|q(x, 0)| + \kappa |u| \big) |v| + \beta |u|^{p-1} |v| \Big) \\ &\quad \leq \; \big(\|f(x, 0)\|_{L^2(\Omega)^N} + m' N \|\nabla u\|_{L^2(\Omega)^N} \big) \|\nabla v\|_{L^2(\Omega)^N} \\ &\qquad + \big(\|q(x, 0)\|_{L^2(\Omega)} + \kappa \|u\|_{L^2(\Omega)} \big) \|v\|_{L^2(\Omega)} + \beta \|u\|^{p-1}_{L^p(\Omega)} \|v\|_{L^p(\Omega)} . \end{aligned}$$

Using (2) and (5), we obtain the following estimate for the right side of (6):

$$\begin{aligned} \Big(\|f(x, 0)\|_{L^2(\Omega)^N} + m' N \|\nabla u\|_{L^2(\Omega)^N} + K_{2,\Omega} \big(\|q(x, 0)\|_{L^2(\Omega)} \\ + \kappa \|u\|_{L^2(\Omega)} \big) + \beta K_{p,\Omega} \|u\|^{p-1}_{L^p(\Omega)} + \|u\| \Big) \|v\| , \end{aligned}$$

where $K_{p,\Omega}$ $(p \geq 2)$ is the embedding constant in the inequality

$$\|u\|_{L^p(\Omega)} \leq K_{p,\Omega} \|u\| \quad (u \in H^1(\Omega)) \tag{7}$$

corresponding to (5). Hence for all fixed $u \in H^1(\Omega)$ Riesz's theorem ensures the existence of $F(u) \in H^1(\Omega)$. $\diamondsuit$

Definition 2.2 A *weak solution* of problem (4) is defined in the usual way as a function $u^* \in H^1(\Omega)$ satisfying

$$\langle F(u^*), v \rangle = \int_\Omega g v \quad (v \in H^1(\Omega)). \tag{8}$$

Remark 2.4 For any $u \in H^2(\Omega)$ with $Q(u) = 0$ on $\partial\Omega$, we have

$$\langle F(u), v \rangle = \int_\Omega T(u) v \quad (v \in H^1(\Omega)).$$

This follows from the divergence theorem:

$$\int_\Omega T(u) v = \int_\Omega \big(f(x, \nabla u) \cdot \nabla v + q(x, u) v \big) - \int_{\partial\Omega} \big(f(x, \nabla u) \cdot \nu \big) v \, d\sigma .$$

Consequently (as usual), a solution of (4) is a weak solution, and a weak solution $u^* \in H^2(\Omega)$ with $Q(u^*) = 0$ on $\partial\Omega$ satisfies (4).

3 Construction and convergence of the gradient method in Sobolev space

The construction of the gradient method relies on the following property of the generalized differential operator.

Theorem 3.1 *Let $F : H^1(\Omega) \to H^1(\Omega)$ be defined in (6). Then F is Gateaux differentiable and F' satisfies*

$$m\|h\|^2 \le \langle F'(u)h, h\rangle \le M(\|u\|)\|h\|^2 \quad (u, h \in H^1(\Omega)), \tag{9}$$

where

$$M(r) = m' + \kappa K_{2,\Omega}^2 + \beta K_{p,\Omega}^p r^{p-2} \tag{10}$$

with $K_{p,\Omega}$ defined in (7).

Proof For any $u \in H^1(\Omega)$ let $S(u) : H^1(\Omega) \to H^1(\Omega)$ be the bounded linear operator defined by

$$\begin{aligned}\langle S(u)h, v\rangle \quad \equiv \quad & \int_\Omega \Big(\frac{\partial f}{\partial \eta}(x, \nabla u)\nabla h \cdot \nabla v + \frac{\partial q}{\partial u}(x, u)hv\Big) \\ & + \iint_{\partial\Omega^2} \varphi(x, y)h(y)v(x)\, d\sigma(y)\, d\sigma(x)\,,\end{aligned} \tag{11}$$

for all $u, h, v \in H^1(\Omega)$. The existence of $S(u)$ is provided by Riesz's theorem similarly as in Proposition 2.4, now using the estimate

$$\Big(m' + \kappa K_{2,\Omega}^2 + \beta K_{p,\Omega}^2 \|u\|_{L^p(\Omega)}^{p-2}\Big)\, \|h\|\|v\|$$

for the integral term on Ω. We will prove that

$$F'(u) = S(u) \quad (u \in H^1(\Omega)) \tag{12}$$

in Gateaux sense. Therefore, let $u, h \in H^1(\Omega)$ and $\mathcal{E} := \big\{v \in H^1(\Omega) : \|v\| = 1\big\}$. Then

$$\begin{aligned} D_{u,h}(t) \quad &\equiv \quad \frac{1}{t}\|F(u + th) - F(u) - tS(u)h\| \\ &= \quad \frac{1}{t} \sup_{v\in\mathcal{E}} \langle F(u + th) - F(u) - tS(u)h, v\rangle \\ &= \quad \frac{1}{t} \sup_{v\in\mathcal{E}} \int_\Omega \Big[\Big(f(x, \nabla u + t\nabla h) - f(x, \nabla u) - t\frac{\partial f}{\partial \eta}(x, \nabla u)\nabla h\Big) \cdot \nabla v \\ &\qquad + \Big(q(x, u + th) - q(x, u) - t\frac{\partial q}{\partial u}(x, u)h\Big)v\Big] \\ &= \quad \sup_{v\in\mathcal{E}} \int_\Omega \Big[\Big(\frac{\partial f}{\partial \eta}(x, \nabla u + t\theta\nabla h) - \frac{\partial f}{\partial \eta}(x, \nabla u)\Big)\nabla h \cdot \nabla v \\ &\qquad + \Big(\frac{\partial q}{\partial u}(x, u + t\theta h) - \frac{\partial q}{\partial u}(x, u)\Big)hv\Big] \end{aligned}$$

$$
\begin{aligned}
\leq \quad & \sup_{v\in\mathcal{E}} \Big[\big\| \big(\frac{\partial f}{\partial \eta}(x, \nabla u + t\theta \nabla h) - \frac{\partial f}{\partial \eta}(x, \nabla u) \big) \nabla h \big\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\
& + \big\| \big(\frac{\partial q}{\partial u}(x, u + t\theta h) - \frac{\partial q}{\partial u}(x, u) \big) h \big\|_{L^q(\Omega)} \|v\|_{L^p(\Omega)} \Big] ,
\end{aligned}
$$

where $p^{-1} + q^{-1} = 1$. Here $\|\nabla v\|_{L^2(\Omega)} \leq \|v\| \leq 1$ and $\|v\|_{L^2(\Omega)} \leq K_{2,\Omega}\|v\| \leq K_{2,\Omega}$. Further, $|t\theta\nabla h| \to 0$ and $|t\theta h| \to 0$ (as $t \to 0$) a.e. on Ω, hence the continuity of $\frac{\partial f}{\partial \eta}$ and $\frac{\partial q}{\partial u}$ implies that the integrands tend to 0 as $t \to 0$. For $|t| \leq t_0$ the integrands are majorated by $(2m'|\nabla h|)^2 \in L^1(\Omega)$ and $(2\kappa + \beta(|u + t_0h|^{p-2} + |u|^{p-2})h)^q \leq const.\cdot(2\kappa + \beta(|u + t_0h|^{(p-2)q} + |u|^{(p-2)q})h^q) \in L^1(\Omega)$. (The latter holds since $u, h \in L^p(\Omega)$ implies $u^{(p-2)q} \in L^{\frac{p}{(p-2)q}}(\Omega)$ and $h^q \in L^{\frac{p}{q}}(\Omega)$, and here $\frac{(p-2)q}{p} + \frac{q}{p} = 1$ from $p^{-1} + q^{-1} = 1$.) Hence Lebesgue's theorem yields that the obtained expression tends to 0 (as $t \to 0$), thus

$$
\lim_{t\to 0} D_{u,h}(t) = 0 \,.
$$

Now the inequality (9) is left to prove. From (12) and (11) we have for any $u, h \in H^1(\Omega)$

$$
\begin{aligned}
\langle F'(u)h, h\rangle \quad = \quad & \int_\Omega \big(\frac{\partial f}{\partial \eta}(x, \nabla u)\nabla h \cdot \nabla h + \frac{\partial q}{\partial u}(x, u)h^2 \big) \\
& + \iint_{\partial\Omega^2} \varphi(x, y)h(y)h(x)\, d\sigma(y)\, d\sigma(x) \,.
\end{aligned}
$$

From condition (C3) we have

$$
m|\nabla h|^2 \leq \frac{\partial f}{\partial \eta}(x, \nabla u)\nabla h \cdot \nabla h \leq m'|\nabla h|^2 \,,
$$

which, together with $\frac{\partial q}{\partial u} \geq 0$, implies directly that

$$
m\|h\|^2 \leq \langle F'(u)h, h\rangle \,.
$$

Further,

$$
\begin{aligned}
\langle F'(u)h, h\rangle \quad \leq \quad & \int_\Omega \big[m'|\nabla h|^2 + (\kappa + \beta|u|^{p-2})h^2 \big] \\
& + \iint_{\partial\Omega^2} \varphi(x, y)h(y)h(x)\, d\sigma(y)\, d\sigma(x) \\
\leq \quad & m'\|h\|^2 + \kappa\|h\|^2_{L^2(\Omega)} + \beta\|u\|^{p-2}_{L^p(\Omega)}\|h\|^2_{L^p(\Omega)} \\
\leq \quad & (m' + \kappa K^2_{2,\Omega} + \beta K^p_{p,\Omega}\|u\|^{p-2})\|h\|^2 \,,
\end{aligned}
$$

i.e. the right side of (9) is also satisfied. $\diamondsuit$

Now we quote an abstract result on the gradient method in Hilbert space, which in this form follows from [10] (Theorem 2 and Corollary 1).

Theorem 3.2 *Let H be a real Hilbert space, $b \in H$ and let $F : H \to H$ satisfy the following properties:*

(i) F is Gateaux differentiable;

(ii) for any $u, k, w, h \in H$ the mapping $s, t \mapsto F'(u + sk + tw)h$ is continuous from $\mathbb{R}^2$ to H;

(iii) for any $u \in H$ the operator $F'(u)$ is self-adjoint;

(iv) there exists $m > 0$ and an increasing function $M : [0, \infty) \to (0, \infty)$ such that for all $u, h \in H$

$$m\|h\|^2 \leq \langle F'(u)h, h\rangle \leq M(\|u\|)\|h\|^2 .$$

Then

(1) the equation $F(u) = b$ has a unique solution $u^* \in H$.

(2) Let $u_0 \in H$, $M_0 := M\left(\|u_0\| + \frac{1}{m}\|F(u_0) - b\|\right)$. Then the sequence

$$u_{n+1} = u_n - \frac{2}{M_0 + m}(F(u_n) - b) \quad (n \in \mathbb{N})$$

converges linearly to u^*, namely,

$$\|u_n - u^*\| \leq \frac{1}{m}\|F(u_0) - b\| \left(\frac{M_0 - m}{M_0 + m}\right)^n \quad (n \in \mathbb{N}) .$$

Now we are in position for constructing the gradient method for (4) in $H^1(\Omega)$ and to verify its convergence.

Theorem 3.3 *(1) Problem (4) has a unique weak solution $u^* \in H^1(\Omega)$.*

(2) Let $b \in H^1(\Omega)$ such that

$$\langle b, v\rangle = \int_\Omega gv \quad (v \in H^1(\Omega)),$$

and let F denote the generalized differential operator as in (6). Let $u_0 \in H^1(\Omega)$, $M_0 := M\left(\|u_0\| + \frac{1}{m}\|F(u_0) - b\|\right)$, where $M(r) = m' + \kappa K_{2,\Omega}^2 + \beta K_{p,\Omega}^p r^{p-2}$. Then the sequence

$$u_{n+1} = u_n - \frac{2}{M_0 + m}(F(u_n) - b) \quad (n \in \mathbb{N}) \tag{13}$$

converges linearly to u^*, namely,

$$\|u_n - u^*\| \leq \frac{1}{m}\|F(u_0) - b\| \left(\frac{M_0 - m}{M_0 + m}\right)^n \quad (n \in \mathbb{N}) .$$

Proof Our task is to verify conditions (i)-(iv) of Theorem 3.2 for (4) in $H^1(\Omega)$. Conditions (i) and (iv) have been proved in Theorem 3.1. The hemicontinuity of F' follows similarly to the differentiability of F if in the proof of Theorem 3.1 we examine $\tilde{D}_{u,k,w,h}(t) \equiv \|(F'(u+sk+tw)-F'(u))h\|$ instead of $D_{u,h}(t)$. Finally, the symmetry of $F'(u)$ follows immediately from (12), (11) and the symmetry of φ and of the Jacobians $\frac{\partial f}{\partial \eta}(x,\eta)$. ♢

Remark 3.1 Assume that u_n is constructed. Then

$$u_{n+1} = u_n - \frac{2}{M_0+m} z_n \ ,$$

where $z_n \in H^1(\Omega)$ satisfies

$$\langle z_n, v\rangle = \langle F(u_n), v\rangle - \int_\Omega gv \quad (v \in H^1(\Omega)).$$

That is, in order to find z_n we need to solve the auxiliary linear variational problem

$$\begin{aligned} \int_\Omega \nabla z_n \cdot \nabla v + \frac{1}{m} \iint_{\partial\Omega^2} \varphi(x,y) z_n(y) v(x)\, d\sigma(y)\, d\sigma(x) \\ = \quad \langle F(u_n), v\rangle - \int_\Omega gv \quad (v \in H^1(\Omega)). \end{aligned} \tag{14}$$

Remark 3.2 If there hold the regularity properties $u_n \in H^2(\Omega)$ and $z_n \in H^2(\Omega)$, then the auxiliary problem (14) can be written in strong form as follows. Using the divergence theorem, we obtain from (14) that

$$\begin{aligned} &\int_\Omega (-\Delta z_n) v + \int_{\partial\Omega} \left(\frac{\partial z_n}{\partial \nu}(x) + \frac{1}{m} \int_{\partial\Omega} \varphi(x,y) z_n(y)\, d\sigma(y) \right) v(x)\, d\sigma(x) \\ = &\int_\Omega (T(u_n) - g) v + \int_{\partial\Omega} \left(f(x, \nabla u_n)\cdot \nu + \int_{\partial\Omega} \varphi(x,y) u_n(y)\, d\sigma(y) \right) v(x)\, d\sigma(x) \end{aligned}$$

holds for all $v \in H^1(\Omega)$. If especially all $v \in H_0^1(\Omega)$ are considered, then we obtain

$$-\Delta z_n = T(u_n) - g \, .$$

Hence for all $v \in H^1(\Omega)$ the boundary integral terms coincide, which implies that

$$\begin{aligned} &\frac{\partial z_n}{\partial \nu} + \frac{1}{m} \int_{\partial\Omega} \varphi(x,y) z_n(y)\, d\sigma(y) \\ = \quad & f(x, \nabla u_n)\cdot \nu + \int_{\partial\Omega} \varphi(x,y) u_n(y)\, d\sigma(y) = \quad Q(u_n). \end{aligned}$$

Consequently, in this case z_n is the solution of the linear boundary-value problem

$$\begin{aligned} -\Delta z_n = T(u_n) - g \, , \\ \frac{\partial z_n}{\partial \nu} + \frac{1}{m} \int_{\partial\Omega} \varphi(x,y) z_n(y)\, d\sigma(y) = Q(u_n). \end{aligned} \tag{15}$$

(In the general case – without regularity of z_n and u_n – (14) is the weak formulation of (15).)

Remark 3.3 Consider the semilinear special case $T(u) \equiv -\Delta u + q(x,u)$ and assume that u_0 is chosen to satisfy $Q(u_0) = 0$, further, that $z_n \in H^2(\Omega)$ for all $n \in \mathbf{N}$. Then $m = 1$ and the boundary condition in (15) is $Q(z_n) = Q(u_n)$. Hence by induction $Q(z_n) = Q(u_n) = 0$ ($n \in \mathbb{N}$), i.e. in each step homogeneous boundary condition is imposed on the auxiliary problem.

Remark 3.4 The construction of the method requires an estimate for the embedding constants $K_{p,\Omega}$. For this we can rely on the exact constants for the embedding of $H^1(\Omega)$ into $L^p(\Omega)$ obtained in [4]. When the lower order term of the equation has at most linear growth (or is not present at all), then only $K_{2,\Omega}$ is needed, which can be estimated, as usual, using a suitable Cauchy-Schwarz inequality. (The numerical example in the following section includes a direct estimation of the required constants.)

4 Numerical example

The summary of the result in the previous section is as follows. The Sobolev space gradient method reduces the solution of the nonlinear boundary value problem (4) to auxiliary linear problems given by (14). The ratio of convergence of the iteration is the number $\frac{M_0-m}{M_0+m}$, which is determined by the original coefficients f, q, g and φ and is independent of the numerical method used for the solution of the auxiliary linear problems.

The numerical realization of the obtained gradient method is established by choosing a suitable numerical method for the solution of the auxiliary problems (14). The latter method may be a finite difference or finite element discretization. In this case the advantage of having executed the iteration for the original problem (4) in the Sobolev space lies in the fact that the numerical questions concerning discretization arise only for the linear problems (14) instead of the nonlinear one (4), whereas the convergence of the iteration is guaranteed as mentioned in the preceding paragraph. This kind of coupling the Sobolev space gradient method with discretization of the auxiliary problems has been developed for local (Dirichlet) boundary-value problems [6, 7]. It is plausible that this coupling may have a similarly effective realization for our nonlocal boundary-value problem (4). Nevertheless, we prefer another situation for giving a numerical example, namely, when the auxiliary linear problems can be solved directly (without discretization).

The model problem. Let $\Omega = [0,\pi]^2 \subset \mathbb{R}^2$, and

$$g(x,y) = \frac{2\cos x \cos y}{\pi(2 - 0.249\cos 2x)(2 - 0.249\cos 2y)}\,.$$

We consider the semilinear problem

$$\begin{gathered} -\Delta u + u^3 = g(x,y) \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \int_{\partial\Omega} u(y)\, d\sigma(y) = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{16}$$

The calculations will be made up to accuracy 10^{-4}.

The function $g(x,y)$ is approximated by its cosine Fourier partial sum

$$\tilde g(x,y) = \sum_{\substack{k,l \text{ are odd} \\ k+l\le 6}} a_{kl} \cos kx \cos ly\,, \qquad a_{kl} = 2.9200 \cdot 4^{-(k+l)} \tag{17}$$

which yields $\|g-\tilde g\|_{L^2(\Omega)} \le 0.0001$. We consider instead of (16) the equation $-\Delta u + u^3 = \tilde g(x,y)$ with the given boundary condition, and denote its solution by $\tilde u$.

The main idea of the numerical realization is the following. Let

$$\mathcal{P} = \{ \sum_{\substack{k,l \text{ are odd} \\ k+l\le m}} c_{kl} \cos kx \cos ly :\ m \in \mathbb{N}^+,\ c_{kl} \in \mathbb{R}\}.$$

Then T is invariant on $\mathcal{P}$, i.e. $u \in \mathcal{P}$ implies $T(u) \in \mathcal{P}$. Hence also $T(u) - \tilde g \in \mathcal{P}$. Further, any $u \in \mathcal{P}$ fulfills the considered boundary condition (in fact, there even holds $\frac{\partial u}{\partial \nu} = \int_{\partial\Omega} u\, d\sigma = 0$). Hence for any $h \in \mathcal{P}$ the solution of the problem

$$\begin{gathered} -\Delta z = h \quad \text{in } \Omega \\ \frac{\partial z}{\partial \nu} + \int_{\partial\Omega} z\, d\sigma = 0 \quad \text{on } \partial\Omega \end{gathered}$$

fulfills $z \in \mathcal{P}$, namely, if

$$h(x,y) = \sum_{\substack{k,l \text{ are odd} \\ k+l\le m}} c_{kl} \cos kx \cos ly$$

then

$$z(x,y) = \sum_{\substack{k,l \text{ are odd} \\ k+l\le m}} \frac{c_{kl}}{k^2+l^2} \cos kx \cos ly\,.$$

(That is, the inversion of the Laplacian is now elementary.) Summing up: using Remark 3.3, we obtain that for any $u_0 \in \mathcal{P}$ the GM iteration

$$\begin{gathered} -\Delta z_n = T(u_n) - \tilde g\,, \quad \tfrac{\partial z_n}{\partial \nu} + \textstyle\int_{\partial\Omega} z_n\, d\sigma = 0\,; \\ u_{n+1} = u_n - \frac{2}{M_0+m} z_n \end{gathered} \tag{18}$$

fulfills $u_n \in \mathcal{P}$ for all $n \in \mathbb{N}^+$, and in each step u_{n+1} is elementary to obtain from u_n.

Now our remaining task is to choose an initial approximation $u_0 \in \mathcal{P}$ and to determine the corresponding ellipticity constants M_0 and m. For simplicity, we choose

$$u_0 \equiv 0.$$

Using the notations of conditions (C1)-(C3) in Section 2, the coefficients are

$$f(x,\eta) = \eta, \quad q(x,u) = u^3 \quad \text{and } \varphi \equiv 1.$$

Hence we have

$$m = m' = 1, \quad \kappa = 0, \quad \beta = 3 \quad \text{and } p = 4.$$

Thus Theorem 3.1 yields

$$M(r) = 1 + 3K_{4,\Omega}^4 r^2, \tag{19}$$

and from Theorem 3.3 we obtain

$$M_0 = M(\|b\|) = 1 + 3K_{4,\Omega}^4 \|b\|^2 \tag{20}$$

where $b \in H^1(\Omega)$ such that

$$\langle b, v\rangle = \int_\Omega \tilde{g} v \quad (v \in H^1(\Omega)).$$

We recall that now, owing to $m = 1$ and $\varphi \equiv 1$, the inner product (2) on $H^1(\Omega)$ is

$$\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v + \left(\int_{\partial\Omega} u\, d\sigma\right)\left(\int_{\partial\Omega} v\, d\sigma\right). \tag{21}$$

Proposition 4.1 *There holds*

$$b(x,y) = \sum_{\substack{k,l \ \text{are odd} \\ k+l \le m}} \frac{a_{kl}}{k^2 + l^2} \cos kx \cos ly\,,$$

where (from (17))

$$a_{kl} = 2.92 \cdot 4^{-(k+l)}.$$

Proof We have $-\Delta b = \tilde{g}$, hence (3) yields

$$\langle b, v\rangle = \int_\Omega (-\Delta b) v = \int_\Omega \tilde{g} v \quad (v \in H^1(\Omega)).$$

Corollary 4.1 *Since* $\int_{\partial\Omega} b\, d\sigma = 0$*, therefore (21) yields*

$$\|b\|^2 = \int_\Omega |\nabla b|^2 = \left(\frac{\pi}{2}\right)^2 \sum_{\substack{k,l \ \text{are odd} \\ k+l \le m}} \frac{a_{kl}^2}{k^2 + l^2} = 0.1014\,.$$

Remark 4.1 In the same way as above, we have for all $u \in \mathcal{P}$

$$\|u\|^2 = \int_\Omega |\nabla u|^2. \tag{22}$$

In order to find now an estimate for $K_{4,\Omega}$, we note that its value is only required for the (closure of the) subspace $\mathcal{P}$ where (u_n) runs. That is, it suffices to determine $\tilde{K}_{4,\Omega}$ satisfying

$$\|u\|_{L^4(\Omega)} \le \tilde{K}_{4,\Omega}\|u\| \quad (u \in \mathcal{P}).$$

Proposition 4.2 *There holds* $\tilde{K}^4_{4,\Omega} \le 10.3776$.

The proof of this proposition consists of some calculations sketched in the Appendix.

Substituting in (20), we obtain M_0.

Corollary 4.2 *The ellipticity constants are* $m = 1$ *and* $M_0 = 4.1569$.

The corresponding stepsize and convergence quotient are

$$\frac{2}{M_0 + m} = 0.3878, \quad \frac{M_0 - m}{M_0 + m} = 0.6122\,.$$

The algorithm (18) has been performed in MATLAB, which is convenient for the required elementary matrix operations determined by storing the functions u_n as matrices of coefficients. (In order to avoid the inconvenient growth of the matrix sizes, the high-index almost zero coefficients were dropped within a 10^{-4} error calculated from the square sum of the coefficients.)

The actual error $\|\tilde{u} - u_n\|$ was estimated using the residual

$$r_n = \|T(u_n) - \tilde{g}\|_{L^2(\Omega)}\,.$$

The connection between $\|\tilde{u} - u_n\|$ and r_n is based on the following propositions.

Proposition 4.3 *For any* $u \in \mathcal{P}$

$$\|u\|_{L^2(\Omega)} \le 2^{-1/2}\|u\|.$$

Proof Let

$$u(x,y) = \sum_{\substack{k,l \text{ are odd} \\ k+l \le m}} c_{kl} \cos kx \cos ly\,.$$

Then from (22)

$$\begin{aligned} \|u\|^2 &= \int_\Omega |\nabla u|^2 = \left(\frac{\pi}{2}\right)^2 \sum_{\substack{k,l \text{ are odd} \\ k+l \le m}} (k^2 + l^2) c_{kl}^2 \\ &\ge 2\left(\frac{\pi}{2}\right)^2 \sum_{\substack{k,l \text{ are odd} \\ k+l \le m}} c_{kl}^2 = 2\|u\|^2_{L^2(\Omega)}\,. \end{aligned}$$

Proposition 4.4 *For all* $u, v \in \mathcal{P}$

$$\|u - v\| \le 2^{-1/2}\|T(u) - T(v)\|_{L^2(\Omega)}\,.$$

Proof The uniform ellipticity of T implies

$$\begin{aligned} \|u-v\|^2 &\leq \int_\Omega (T(u)-T(v))(u-v) \\ &\leq \|T(u)-T(v)\|_{L^2(\Omega)}\|u-v\|_{L^2(\Omega)} \\ &\leq 2^{-1/2}\|T(u)-T(v)\|_{L^2(\Omega)}\|u-v\|\,. \end{aligned}$$

Corollary 4.3 *Let*

$$e_n = 2^{-1/2} r_n = 2^{-1/2}\|T(u_n)-\tilde{g}\|_{L^2(\Omega)} \quad (n \in \mathbb{N}). \tag{23}$$

Then, applying Proposition 4.4 to u_n and $\tilde{u}$, we obtain

$$\|\tilde{u}-u_n\| \leq e_n\,.$$

Based on these, the error was measured by e_n defined in (23). (Since $T(u_n)$ and $\tilde{g}$ are trigonometric polynomials, this only requires square summation of the coefficients.)

The following table contains the error e_n versus the number of steps n.

step n	1	2	3	4	5	6	7
error e_n	1.1107	0.6754	0.3992	0.2290	0.1288	0.0718	0.0402

step n	8	9	10	11	12	13	14
error e_n	0.0225	0.0127	0.0072	0.0042	0.0024	0.0014	0.0008

step n	15	16	17	18	19	20	21
error e_n	0.0005	0.0003	0.0003	0.0002	0.0002	0.0002	0.0001

Table 1.

Remark 4.2 We have determined above numerically, up to accuracy 10^{-4}, the solution $\tilde{u}$ of the approximated problem with $\tilde{g}$ instead of g. Since $\tilde{u}$ and u^* are in $\overline{\mathcal{P}}$, Proposition 4.4 yields

$$\|\tilde{u}-u^*\| \leq 2^{-1/2}\|\tilde{g}-g\|_{L^2(\Omega)} \leq 2^{-1/2}\cdot 0.0001\,.$$

5 Appendix

Proof of Proposition 4.2. The proof can be achieved through two lemmata.

Lemma 5.1 *For any $u \in \mathcal{P}$,*

$$\int_\Omega u^4 \leq \frac{1}{8}\left(\int_{\partial\Omega} u^2\,d\sigma + 8^{1/2}\|u\|^2\right).$$

Proof It is proved in [14] that for any $u \in H_0^1(\Omega)$

$$\int_\Omega u^4 \le 4\|u\|^2_{L^2(\Omega)}\|\partial_1 u\|_{L^2(\Omega)}\|\partial_2 u\|_{L^2(\Omega)} \le 2\|u\|^2_{L^2(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}\,.$$

Taking into account the boundary, we obtain in the same way that for any $u \in H^1(\Omega)$

$$\int_\Omega u^4 \le 2\left(\frac{1}{4}\int_{\partial\Omega} u^2\,d\sigma + \|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\right)^2.$$

This yields the desired estimate for any $u \in \mathcal{P}$, using Remark 4.1 and Proposition 4.3 for $\|u\|_{L^2(\Omega)}$ and $\|\nabla u\|_{L^2(\Omega)}$. $\diamondsuit$

Lemma 5.2 *For any $u \in \mathcal{P}$,*

$$\int_{\partial\Omega} u^2\,d\sigma \le 2\pi\|u\|^2.$$

Proof Let $\Gamma_1 = [0,\pi]\times\{0\}$, $\Gamma_2 = \{\pi\}\times[0,\pi]$, $\Gamma_3 = [0,\pi]\times\{\pi\}$, $\Gamma_4 = \{0\}\times[0,\pi]$. Then $\partial\Omega = \cup\{\Gamma_i : i = 1,\dots,4\}$. Now let $u \in \mathcal{P}$. For any $x, y \in [0,\pi]$ we have

$$u(x,\pi) - u(0,y) = \int_0^x \partial_1 u(s,y)ds + \int_y^\pi \partial_2 u(x,t)\,dt\,.$$

Raising to square and integrating over Ω, we obtain

$$\begin{aligned}
&\pi\left(\int_{\Gamma_3} u^2\,d\sigma + \int_{\Gamma_4} u^2\,d\sigma\right) - 2\left(\int_{\Gamma_3} u\,d\sigma\right)\left(\int_{\Gamma_4} u\,d\sigma\right)\\
&\quad\le\ 2\int_0^\pi\int_0^\pi\left[\left(\int_0^x \partial_1 u(s,y)ds\right)^2 + \left(\int_y^\pi \partial_2 u(x,t)dt\right)^2\right]dxdy\\
&\quad\le\ \pi^2\int_\Omega\left[(\partial_1 u)^2 + (\partial_2 u)^2\right],
\end{aligned}$$

where Cauchy-Schwarz inequality was used. We can repeat the same argument for the pairs of edges (Γ_1,Γ_2), (Γ_2,Γ_3) and (Γ_1,Γ_4) in the place of (Γ_3,Γ_4). Then, summing up and using $\partial\Omega = \cup\{\Gamma_i : i = 1,\dots,4\}$, we obtain

$$2\pi\int_{\partial\Omega} u^2\,d\sigma - 2\left(\int_{\Gamma_1\cup\Gamma_3} u\,d\sigma\right)\left(\int_{\Gamma_2\cup\Gamma_4} u\,d\sigma\right) \le 4\pi^2\int_\Omega |\nabla u|^2. \tag{24}$$

Using notations $\Gamma_x = \Gamma_1\cup\Gamma_3$ and $\Gamma_y = \Gamma_2\cup\Gamma_4$, there holds

$$\begin{aligned}
2\left(\int_{\Gamma_x} u\,d\sigma\right)\left(\int_{\Gamma_y} u\,d\sigma\right) &= \left(\int_{\Gamma_x\cup\Gamma_y} u\,d\sigma\right)^2 - \left(\int_{\Gamma_x} u\,d\sigma\right)^2 - \left(\int_{\Gamma_x} u\,d\sigma\right)^2\\
&\le \int_{\partial\Omega} u\,d\sigma = 0\,,
\end{aligned}$$

hence (24) yields

$$2\pi \int_{\partial\Omega} u^2 \, d\sigma \le 4\pi^2 \int_\Omega |\nabla u|^2 = 4\pi^2 \|u\|^2.$$

Proof of the proposition. Lemmata 1 and 2 yield

$$\|u\|^4_{L^4(\Omega)} \le \frac{1}{8}(2\pi + 8^{1/2})\|u\|^4 \,,$$

that is

$$\tilde{K}^4_{4,\Omega} \le \frac{1}{8}(2\pi + 8^{1/2}) = 10.3776$$

up to accuracy 10^{-4}.

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