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Viscous profiles for traveling waves of scalar balance laws: The uniformly hyperbolic case *

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Abstract

We consider a scalar hyperbolic conservation law with a nonlinear source term and viscosity ε. For $\varepsilon = 0$, there exist in general different types of heteroclinic entropy traveling waves. It is shown that for $\varepsilon > 0$ sufficiently small the viscous equation possesses similar traveling wave solutions and that the profiles converge in exponentially weighted L^1-norms as $\varepsilon \searrow 0$. The proof is based on a careful study of the singularly perturbed second-order equation that arises from the traveling wave ansatz.

1 Introduction

We are concerned with traveling wave solutions for scalar hyperbolic balance laws

$$u_t + f(u)_x = g(u), \quad x \in \mathbb{R},\ u \in \mathbb{R}. \tag{H}$$

The question whether these traveling waves can be obtained as the limit of traveling waves of the viscous balance law

$$u_t + f(u)_x = \varepsilon u_{xx} + g(u), \quad x \in \mathbb{R},\ u \in \mathbb{R} \tag{P}$$

when the viscosity parameter ε tends to zero is discussed in this article using singular perturbation theory.

Hyperbolic balance laws are extensions of hyperbolic conservation laws where a source term g is added. These reaction terms can model chemical reactions, combustion or other interactions [12], [1]. The source terms can dramatically change the long-time behaviour of the equation compared to hyperbolic conservation laws. While for conservation laws the only traveling wave solutions are shock waves, balance laws exhibit different types of traveling waves. A classification of the traveling waves in the case of a convex flow function f has been done by Mascia [10]. We summarize his results in section 2.1.

Since hyperbolic balance laws are often considered as a simplified model for some parabolic (*viscous*) equation with a very small viscosity, it is important

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to know, whether traveling wave solutions of the hyperbolic (*inviscid*) equation correspond to traveling waves of the viscous equation. If this is true in a sense to be specified below, we say that the traveling wave admits a *viscous profile*.

In this paper we prove that under mild assumptions on f and g some types of waves of the hyperbolic equation admit a viscous profile. In particular, it can be shown that in many situations one can choose initial conditions for the viscous problem such that the corresponding traveling wave solution is L^1-close to the traveling wave solution of the inviscid problem for every positive time. This is different from Kruzhkovs fundamental approximation result where the initial conditions are equal but for a fixed viscosity the solutions of the viscous and the inviscid equation are only close on some finite time interval.

However, there are situations where the profiles are only close in L^1 if one allows the traveling wave of the viscous and the inviscid equation to have a slightly different wave speed.

The paper is organized as follows: In chapter 2 we introduce the notion of entropy traveling waves, make the meaning of viscous profiles more precise and state the main result. There are three different types of traveling waves for which the classical geometrical singular perturbation theory of Fenichel can be applied. They involve only parts of the slow manifold which are uniformly hyperbolic with respect to the fast field. These cases are treated separately in chapters 3-5. The remaining types of traveling waves involve a study of trajectories that pass near points on the slow manifold where the fast field is not hyperbolic. These cases are discussed elsewhere [6].

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2 Entropy Traveling Waves

We assume the following about f and g:

(F) f is strictly convex: $f \in C^2$, $f''(u) > 0$

(G) $g \in C^1$ and g has finitely many simple zeroes

We denote the zeroes of g with u_i where $i \in \{2, 3, \ldots, n\}$. For notational convenience we set $u_1 := -\infty$ and $u_{n+1} := +\infty$.

It is straightforward to generalize all results to the case when g has infinitely many isolated zeroes.

The set of all zeroes is called $\mathcal{Z}(g)$. Depending on the sign of g' the zeroes of g are divided into two sets :

$$\begin{aligned} \mathcal{R}(g) &:= \{u_i \in \mathcal{Z}(g) : g'(u_i) > 0\} \\ \mathcal{A}(g) &:= \{u_i \in \mathcal{Z}(g) : g'(u_i) < 0\} \end{aligned}$$

Like hyperbolic conservation laws, balance laws (H) do in general not possess global smooth solutions. Since passing to weak solutions destroys the uniqueness, an entropy condition has to be given which chooses the "correct" solution among all weak solutions. Here we define directly for traveling waves what is meant by such an entropy solution.

Definition 2.1 *An* **entropy traveling wave** *is a solution of the hyperbolic balance law (H) which is of the form $u(x,t) = u(\xi)$ with $\xi = x - st$ for some wave speed $s \in \mathbb{R}$ and which has the following properties:*

(i) *$u \in BV(\mathbb{R})$ is of bounded variation and u is piecewise C^1.*

(ii) *At points where u is continuously differentiable it satisfies the ordinary differential equation*

$$(f'(u(\xi)) - s)\ u'(\xi) = g(u(\xi)). \tag{1}$$

(iii) *At points of discontinuity the one-sided limits $u(\xi+)$ and $u(\xi-)$ of u satisfy both the Rankine-Hugoniot condition*

$$s\,(u(\xi+) - u(\xi-)) = f(u(\xi+)) - f(u(\xi-))$$

and the entropy condition

$$u(\xi+) \leq u(\xi-).$$

Due to the convexity assumption (F), for any $u \in \mathbb{R}$ and any speed s there is at most one other value $h(u,s)$ which satisfies the Rankine-Hugoniot condition

$$\frac{f(u) - f(h(u,s))}{u - h(u,s)} = s.$$

If there is no such $h(u,s)$ we set

$$h(u,s) := \begin{cases} -\infty & \text{for} \quad f'(u) - s > 0 \\ +\infty & \text{for} \quad f'(u) - s < 0 \end{cases}$$

Definition 2.2 *A traveling wave u is said to be a* **heteroclinic wave** *if*

$$\lim_{\xi \to -\infty} u(\xi) = u_i \quad \textit{and} \quad \lim_{\xi \to +\infty} u(\xi) = u_j$$

for some u_i, $u_j \in \mathbb{R}$.

Remark 2.3 *From (1) we can immediately conclude that $g(u_i) = g(u_j) = 0$. For this reason, we say that there is a (heteroclinic) connection between the equilibria u_i and u_j.*

2.1 Heteroclinic waves of the hyperbolic equation

Mascia [10] has classified the heteroclinic waves that occur for convex f. We collect here the results of [10, theorems 2.3-2.5] but sort them in a different way and make the statements on wave speeds more precise. To this end we distinguish three types of waves:

- Heteroclinic waves which exist for a whole interval of wave speeds s
- waves which can be found only if the speed s takes one of the discrete values $f'(u_i)$ for some i and
- undercompressive waves which do also show up only for exceptional shock speeds.

Proposition 2.4 *Heteroclinic connections from u_i to u_j that exist for a range of wave speeds are of the following types:*

(A1) Continuous monotone waves that connect adjacent equilibria

(i) $j = i+1$, $u_i \in \mathcal{A}(g)$ *and* $s \geq f'(u_{i+1})$

(ii) $j = i-1$, $u_i \in \mathcal{A}(g)$ *and* $s \geq f'(u_i)$

(iii) $j = i+1$, $u_i \in \mathcal{R}(g)$ *and* $s \leq f'(u_i)$

(iv) $j = i-1$, $u_i \in \mathcal{R}(g)$ *and* $s \leq f'(u_{i-1})$

(A2) Discontinuous heteroclinic waves

(i) $i > j$, $u_i \in \mathcal{A}(g)$, $u_j \in \mathcal{R}(g)$ $u_i \in (h(u_{j+1}, s), h(u_{j-1}, s))$ *with wave speed* $s \in \left(\frac{f(u_{j-1})-f(u_i)}{u_{j-1}-u_i}, \frac{f(u_{j+1})-f(u_i)}{u_{j+1}-u_i}\right)$

(ii) $i > j$, $u_i \in \mathcal{R}(g)$, $u_j \in \mathcal{R}(g)$, $(h(u_{j+1}, s), h(u_{j-1}, s)) \cap (u_{i-1}, u_{i+1}) \neq \emptyset$

(iii) $i > j$, $u_i \in \mathcal{R}(g)$, $u_j \in \mathcal{A}(g)$, $h(u_j, s) \in (u_{i-1}, u_{i+1})$.

Proposition 2.5 *Heteroclinic connections from u_i to u_j that exist only for a particular wave speed are of the following types:*

(B1) Continuous, monotone increasing waves

$j = i+2$, $u_i, u_j \in \mathcal{A}(g)$ *and* $s = f'(u_{i+1})$

(B2) Continuous, monotone decreasing waves

$j = i-2$, $u_i, u_j \in \mathcal{R}(g)$ *and* $s = f'(u_{i-1})$

(B3) *(i)* $i \geq j$, $u_i \in \mathcal{A}(g)$, $u_j \in \mathcal{A}(g)$, $s = f'(u_{i+1})$ *and* $h(u_j, s) < u_{i+2}$,

(ii) $i > j$, $u_i \in \mathcal{A}(g)$, $u_j \in \mathcal{R}(g)$, $s = f'(u_{i+1})$ *and* $h(u_{j-1}, s) < u_{i+2}$,

(iii) $i > j$, $u_i \in \mathcal{R}(g)$, $u_j \in \mathcal{A}(g)$, $s = f'(u_{j-1})$ *and* $h(u_{j-2}, s) < u_{i-1}$,

(iv) $i \geq j$, $u_i \in \mathcal{A}(g)$, $u_j \in \mathcal{A}(g)$, $s = f'(u_{j-1})$ *and* $h(u_i, s) < u_{j-2}$.

(B4) *(i) Discontinuous waves that connect* u_i *to* u_{i+2} *with speed* $s = f'(u_{i+1})$,

(ii) Discontinuous waves that connect u_i *to* u_{i+1} *with speed* $s = f'(u_{i+1})$.

(C) Undercompressive shocks: $i > j$, u_i, $u_j \in \mathcal{A}(g)$, $s = \dfrac{f(u_i) - f(u_j)}{u_i - u_j}$.

Note that (B1) contains a one-parameter family of different waves. The (non-negative) parameter is the length of the interval where the profile takes the value u_2. Similarly, (B3) waves comprise a large number of different entropy traveling waves.

2.2 The main result

Our goal is to find traveling wave solutions of (P) that correspond to the traveling waves of (H) as $\varepsilon \searrow 0$. Unlike for viscosity solutions of hyperbolic conservation laws, we cannot get rid of the viscosity parameter ε by a simple scaling but have to discuss the full singularly perturbed system (P).

With the traveling wave ansatz $u(x,t) = u(x - st)$ we get from (P) the equation

$$\varepsilon u'' = (f'(u) - s)u' - g(u). \tag{2}$$

Here the prime denotes differentiation with respect to a new coordinate $\xi := x - st$. We are now able to define what we mean by a viscosity traveling wave solution.

Definition 2.6 *A traveling wave solution* u_0 *of (H) is called a* **viscosity traveling wave solution** *with wave speed* s_0 *if there is a sequence* (u_{ε_n}) *of solutions of (2) such that* $\varepsilon_n \searrow 0$, $s_n \to s_0$ *and* $\|u_{\varepsilon_n} - u_0\|_{L^1(\mathbb{R})} \to 0$. *In this case, the heteroclinic wave of the hyperbolic equation is said to admit a viscous profile.*

So, we are able to approximate a traveling wave profile of the hyperbolic equation by traveling wave profiles of the viscous equation. The price we may have to pay for this uniform approximation, however, is, that the wave speeds might differ slightly.

In the present paper, we will prove admissibility for some of the heteroclinic waves. This implies that the traveling waves found in the simpler, hyperbolic model do have counterparts in the viscous equation provided the viscosity is small enough.

Our main result is the following:

Theorem 2.7 *The heteroclinic waves of type (A1), (A2) and (C) admit a viscous profile.*

We concentrate on these types of traveling waves, since they fit into the classical setting of geometrical singular perturbation theory and can be treated in a similar way.

2.3 Weighted L^1-spaces

Although our main interest is in L^1-convergence, we will be more general and prove convergence in spaces with exponentially weighted norms. To this end, we define for $\beta \geq 0$ the norm

$$\|u\|_{L^1_\beta} := \int_{\mathbb{R}} (1 + e^{\beta|\xi|})|u(\xi)|\, d\xi$$

and the space

$$L^1_\beta := \{u \in L^1, \|u\|_{L^1_\beta} < \infty\}.$$

Obviously, the choice $\beta = 0$ is equivalent to the usual L^1-norm. The following lemma will simplify the later proofs.

Lemma 2.8 *For $\varepsilon \geq 0$, consider a family of functions $u_\varepsilon \in C^0(\mathbb{R})$. Assume that there exist limiting states*

$$u_\pm = \lim_{\xi \to \pm\infty} u_\varepsilon(\xi)$$

independent of ε and constants C, $c > 0$, $-\infty < \xi_- < \xi_+ < \infty$ such that the following conditions are satisfied:

(i) $|u_\varepsilon(\xi) - u_-| \leq Ce^{c\xi}$ *for all* $\xi \leq \xi_-$ *and all* $\varepsilon \geq 0$,

(ii) $|u_\varepsilon(\xi) - u_+| \leq Ce^{-c\xi}$ *for all* $\xi \geq \xi_+$ *and all* $\varepsilon \geq 0$,

(iii) $\lim_{\varepsilon \searrow 0} \int_a^b |u_\varepsilon(\xi) - u_0(\xi)|\, d\xi = 0$ *holds for any* $-\infty < a < b < +\infty$.

Then for any weight $0 \leq \beta < c$

$$\lim_{\varepsilon \searrow 0} \|u_\varepsilon - u_0\|_{L^1_\beta} = 0.$$

Proof: Given any integer n, we can find $a_n < \xi_-$ such that

$$\int_{-\infty}^{a_n} (1 + e^{\beta|\xi|})Ce^{-c\xi}\, d\xi \leq \frac{1}{5n}.$$

Using (i), we get by comparison

$$\int_{-\infty}^{a_n} (1 + e^{\beta|\xi|})|u_\varepsilon(\xi) - u_-|\, d\xi \leq \frac{1}{5n}.$$

Similarly, by (ii), we can find $b_n > \xi_+$ with

$$\int_{b_n}^{+\infty} (1 + e^{\beta|\xi|})|u_\varepsilon(\xi) - u_+|\, d\xi \leq \frac{1}{5n}.$$

Using (iii), we can choose ε sufficiently small such that

$$\int_{a_n}^{b_n} |u_\varepsilon(\xi) - u_0(\xi)|\, d\xi \leq \frac{1}{5n(1 + \max\{e^{\beta|a_n|}, e^{\beta|b_n|}\})}$$

and estimate the L^1_β-norm of $u_0 - u_\varepsilon$ as

$$\begin{aligned}
&\|u_0 - u_\varepsilon\|_{L^1_\beta} \\
&= \int_{-\infty}^{a_n} (1+e^{\beta|\xi|})|u_- - u_\varepsilon(\xi)|\,d\xi + \int_{-\infty}^{a_n} (1+e^{\beta|\xi|})|u_- - u_0(\xi)|\,d\xi \\
&\quad + \int_{a_n}^{b_n} (1+e^{\beta|\xi|})|u_0(\xi) - u_\varepsilon(\xi)|\,d\xi \\
&\quad + \int_{b_n}^{\infty} (1+e^{\beta|\xi|})|u_+ - u_\varepsilon(\xi)|\,d\xi + \int_{b_n}^{+\infty} (1+e^{\beta|\xi|})|u_+ - u_0(\xi)|\,d\xi \\
&\le \frac{1}{n}
\end{aligned}$$

which completes the proof of the lemma since n was arbitrary. $\diamondsuit$

This lemma shows the key ingredients in the convergence proofs. Typically, (i) and (ii) will be consequences of the hyperbolicity of some fixed points, while (iii) is the point where one has to do some work.

2.4 Singular Perturbations

We return now to the study of the viscous balance law. A convenient way to write the second-order equation (2) as a first-order system is the Liénard plane

$$\begin{aligned}
\varepsilon u' &= v + f(u) - su \\
v' &= -g(u).
\end{aligned} \tag{3}$$

From this "slow-fast"-system two limiting systems can be derived which both capture a part of the behavior that is observed for $\varepsilon > 0$.

One is the "slow" system obtained by simply putting $\varepsilon = 0$:

$$\begin{aligned}
0 &= v + f(u) - su \\
v' &= -g(u).
\end{aligned} \tag{4}$$

The flow is confined to a curve

$$\mathcal{C}_s := \{(u,v) : v + f(u) - su = 0\}$$

that we call the **singular curve**. The other, "fast" system originates in a different scaling. With $\xi =: \varepsilon\eta$ and a dot denoting differentiation with respect to η we arrive at

$$\begin{aligned}
\dot{u} &= v + f(u) - su \\
\dot{v} &= -\varepsilon g(u).
\end{aligned} \tag{5}$$

In the limit $\varepsilon = 0$, equation (5) defines a vector field for which the singular curve $\mathcal{C}_s$ consists of equilibrium points only. This vector field is called the "fast" system. It points to the left below the curve $\mathcal{C}_s$ and to the right above.

Trajectories of the fast system connect only points for which $v + f(u) - su$ has the same values. This is exactly the Rankine-Hugoniot condition for waves propagating with speed s. Moreover the direction of the fast vector field is in accordance with the Oleinik entropy condition.

Geometric singular perturbation theory in the spirit of Fenichel [2] makes precise statements how the slow and the fast equations together describe the dynamics of (3) for small $\varepsilon > 0$. It is a strong tool in regions where the singular curve is normally hyperbolic, i.e. where the points on $\mathcal{C}_s$ are hyperbolic with respect to the fast field.

The only non-hyperbolic point on $\mathcal{C}_s$ is the fold point where $f'(u) = s$. The heteroclinic waves of type (A1), (A2) and (C) stay away from these points and hence fit into the classical framework. The other cases involving non-hyperbolic points on the singular curve are more subtle and will be treated by blow-up techniques in a forthcoming paper [6].

We collect some of the properties of (3) which will prove useful later. The steady states of system (3) are exactly the points

$$\{(u,v):\ (u,v) \in \mathcal{C}_s,\ u \in \mathcal{Z}(g)\} = \{(u,v);\ u = u_i \text{ for some } i,\ v + f(u_i) - su_i = 0\}.$$

For this reason, we will often speak of the equilibrium u_i when we mean the steady state $(u_i, -f(u_i) + su_i)$ of (3). The linearization of (3) in such a steady state $(u_i, -f(u_i) + su_i)$, possesses the eigenvalues

$$\lambda_i^{\pm} = \frac{f'(u_i) - s \pm \sqrt{(f'(u_i) - s)^2 - 4\varepsilon g'(u_i)}}{2\varepsilon} \tag{6}$$

which are real except when

$$g'(u_i) > 0 \quad \text{and} \quad (f'(u_i) - s)^2 < 4\varepsilon g'(u_i).$$

Note that in (s,ε)-parameter space any point on the axis $\varepsilon = 0$ can be approximated by a sequence (s_n, ε_n) such that the eigenvalues of all the equilibrium states are all real.

Another interesting feature of system (3) is the fact that there are some narrow regions near $\mathcal{C}_s$ which are invariant for small ε.

Lemma 2.9

(i) Assume that $f'(u_i) < f'(u_{i+1}) < s$. Then there exists a (large) positive number k such that for all ε sufficiently small the region

$$P_i := \left\{ (u,v);\ u_i \le u \le u_{i+1},\ |v + f(u) - su - \varepsilon \frac{g(u)}{f'(u) - s}| \le k\varepsilon^2 |g(u)| \right\}$$

is positively invariant.

(ii) If $s < f'(u_i) < f'(u_{i+1})$ then there exists a number k such that for all ε sufficiently small the region

$$N_i := \left\{ (u,v);\ u_i \le u \le u_{i+1},\ |v + f(u) - su - \varepsilon \frac{g(u)}{f'(u) - s}| \le k\varepsilon^2 |g(u)| \right\}$$

is negatively invariant.

Proof:
(i): This is a refined version of lemma 3.5 in [4]. Let $\sigma :=$ sign $g(u)$ for $u \in (u_i, u_{i+1})$ and

$$v_1(u) := \frac{g(u)}{f'(u) - s}. \tag{7}$$

The scalar product of the outer normal vector with the vector field (3) along the upper boundary $v + f(u) - su = \varepsilon v_1(u) + k\varepsilon^2 \sigma g(u)$ of P_i is

$$\begin{aligned} & \begin{pmatrix} f'(u) - s - \varepsilon v_1' - k\sigma\varepsilon^2 g'(u) \\ 1 \end{pmatrix}^T \cdot \begin{pmatrix} \varepsilon^{-1}(v + f(u) - su) \\ -g(u) \end{pmatrix} \\ = \quad & -\varepsilon g(u) \left(\frac{(f'(u) - s)g'(u) - g(u)f''(u)}{(f'(u) - s)^3} + k\sigma(f'(u) - s) \right) + \mathcal{O}(\varepsilon^2) \\ < \quad & 0 \end{aligned}$$

whenever k is sufficiently large and ε is sufficiently small, since $\sigma g(u)(f'(u) - s)$ is negative on (u_i, u_{i+1}).

An analogous calculation for the lower boundary of P completes the proof that P is positively invariant.

(ii) can be proved in the same way. ♢

3 Traveling waves between adjacent equilibria

In this chapter we will prove the first statement of theorem 2.7. We concentrate on waves of type (A1)(i) since the other cases can be treated similarly. Since $u_i \in \mathcal{A}(g)$, we know from (6) that u_i is of saddle type, u_{i+1} is a sink and $g(u) < 0$ for $u \in (u_i, u_{i+1})$. The wave speed of the hyperbolic traveling wave will be denoted by s_0. Since we want to apply lemma 2.8 we need to find a family (u_ε) of candidates for a viscous profile, i.e. a family of heteroclinic orbits of system (3) with ε small. It turns out that such a family can be found by varying only ε while keeping s fixed at the value s_0 of the hyperbolic entropy traveling wave.

Lemma 3.1 *For ε sufficiently small, there exists a monotone heteroclinic connection from u_i to u_{i+1} in (3) with $s = s_0$.*

Proof: To establish the existence of a heteroclinic connection, we show that a branch of the unstable manifold $W^u(u_i)$ of u_i enters the invariant region P_i found in lemma 2.9 provided that k is large enough. The eigenvector associated with the positive eigenvalue λ_i^+ of u_i is

$$e_i^+ = \begin{pmatrix} 2 \\ \sqrt{(f'(u_i) - s_0)^2 - 4\varepsilon g'(u_i)} - (f'(u_i) - s_0) \end{pmatrix}$$

Expanding the square root with respect to ε one obtains for the asymptotic slope of $W^u(u_i)$ in u_i the expression

$$-(f'(u_i)-s_0)-\frac{\varepsilon g'(u_i)}{f'(u_i)-s_0}-\frac{\varepsilon^2 g'(u_i)^2}{4(f'(u_i)-s_0)^3}+\mathcal{O}(\varepsilon^3).$$

It is easily checked that this expression coincides up to order ε with the slope of the boundaries of P_i at u_i. Now by choosing k larger, if necessary, we can achieve that a branch of the unstable manifold $W^u(u_i)$ lies in P_i while P_i is still positively invariant. Since all trajectories in P_i are monotone, $W^u(u_i)$ has to be a heteroclinic orbit u_ε from u_i to the only other equilibrium u_{i+1} on the boundary of P_i. Monotonicity of u_ε follows from the fact that it lies above the singular curve $\mathcal{C}_{s_0}$ where $u'>0$. ♢

Lemma 3.2 *The heteroclinic orbits found in lemma 3.1 provide a viscous profile for the entropy traveling waves of type (A1). If $s>f'(u_{i+1})$ then*

$$\lim_{\varepsilon\searrow 0}\|u_\varepsilon-u_0\|_{L^1_\beta}=0 \quad \textit{for}\quad 0\le\beta<\min\{\left|\frac{g'(u_i)}{f'(u_i)-s_0}\right|,\left|\frac{g'(u_{i+1})}{f'(u_{i+1})-s_0}\right|\}.$$

Proof: Two cases have to be distinguished, depending on the smoothness of the hyperbolic wave. We begin with the case $s_0>f'(u_{i+1})$ where the profile is differentiable and we can prove convergence in L^1_β. The case $s_0=f'(u_{i+1})$ where the profile u_0 is continuous but not differentiable is discussed later.

I. $s_0>f'(u_{i+1})$**:** As indicated above, we want to apply lemma 2.8. First we parametrize all the heteroclinic orbits $u_\varepsilon(\xi)$ of the viscous problem (2) and the heteroclinic orbit $u_0(\xi)$ of the hyperbolic problem (1) in a way such that

$$u_0(0):=u_\varepsilon(0):=\frac{u_i+u_{i+1}}{2}.$$

Then we fix some $c\in\left(\beta,\min\{\left|\frac{g'(u_i)}{f'(u_i)-s_0}\right|,\left|\frac{g'(u_{i+1})}{f'(u_{i+1})-s_0}\right|\}\right)$. Since

$$c<\frac{g'(u_i)}{f'(u_i)-s_0}=\left.\frac{d}{du}\right|_{u=u_i}\frac{g(u)}{f'(u)-s_0u}$$

it is possible to choose $\delta>0$ and $\varepsilon_1>0$ small with the property that for $u_i\le u\le u_i+\delta$ and all $\varepsilon\le\varepsilon_1$ we have

$$-\frac{g(u)}{f'(u)-s_0}-k\varepsilon g(u)\le -c(u-u_i). \tag{8}$$

Similarly, we require for $u_{i+1}-\delta\le u\le u_{i+1}$

$$-\frac{g(u)}{f'(u)-s_0}-k\varepsilon g(u)\le -c\,(u_{i+1}-u). \tag{9}$$

Let

$$\begin{aligned}\xi_- &:= \inf_{0\le\varepsilon\le\varepsilon_1}\{\xi: u_\varepsilon(\xi)=u_i+\delta\}\\ \xi_+ &:= \sup_{0\le\varepsilon\le\varepsilon_1}\{\xi: u_\varepsilon(\xi)=u_{i+1}-\delta\}.\end{aligned}$$

Both $\xi_- > -\infty$ and $\xi_+ < +\infty$ follow from the fact that

$$u'_\varepsilon = \frac{g(u_\varepsilon)}{f'(u_\varepsilon) - s_0} + \mathcal{O}(\varepsilon)$$

is bounded away from 0 on the interval $[u_i+\delta, u_{i+1}-\delta]$ independent of $\varepsilon \in [0, \varepsilon_1]$. From a comparison argument and (8), (9) it follows that

$$|u_\varepsilon(\xi) - u_i| \le \delta e^{-c(\xi_- - \xi)} \quad \text{for} \quad \xi < \xi_-$$

and

$$|u_\varepsilon(\xi) - u_{i+1}| \le \delta e^{-c(\xi - \xi_+)} \quad \text{for} \quad \xi > \xi_+$$

and all $\varepsilon \in [0, \varepsilon_1]$. This implies that assumptions (i) and (ii) of lemma 2.8 are met with $C := \delta$, $u_- = u_i$ and $u_+ = u_{i+1}$. It remains to show that for arbitrary $a < b$

$$\int_a^b |u_\varepsilon(\xi) - u_0(\xi)|\, d\xi \to 0 \quad \text{for all} \ -\infty < a < b < +\infty.$$

From (2), (1) and the fact that u_ε lies within P_i we know

$$|u'_\varepsilon(\xi) - u'_0(\xi)| \le \left| \frac{g(u_\varepsilon)}{f'(u_\varepsilon) - s_0} - \frac{g(u_0)}{f'(u_0) - s_0} \right| + k\varepsilon |g(u_\varepsilon)|,$$

because u_ε lies within the narrow strip P_i. Hence

$$\begin{aligned} |u_\varepsilon(\xi) - u_0(\xi)| &\le \left| \int_0^\xi u'_\varepsilon(\eta) - u'_0(\eta)\, d\eta \right| \\ &\le \int_0^\xi |v_1(u_\varepsilon(\eta)) - v_1(u_0(\eta))| + \varepsilon k |g(u_\varepsilon(\eta))|\, d\eta \\ &\le \int_0^\xi \left(L|u_\varepsilon(\eta) - u_0(\eta)| + \varepsilon k \sup |g(u)| \right)\, d\eta \end{aligned}$$

where L is a Lipschitz constant for the function v_1 from (7) on the interval $[u_i, u_{i+1}]$ and the sup is taken over the same interval. In particular, this estimate is independent of a and b. Applying the Gronwall inequality we get

$$|u_\varepsilon(\xi) - u_0(\xi)| \le \varepsilon \frac{k \sup |g|}{L} \left(e^{L|\xi|} - 1 \right)$$

for $\xi \in [a, b]$. Hence

$$\int_a^b |u_0(\xi) - u_\varepsilon(\xi)|\, d\xi \le \int_a^b \varepsilon \frac{k \sup |g|}{L} \left(e^{L|\xi|} - 1 \right)\, d\xi \to 0$$

as $\varepsilon \searrow 0$. This is exactly assumption (iii) of lemma 2.8. As a consequence of this lemma we conclude that u_ε converges to u_0 in L^1_β.

II. $s_0 = f'(u_{i+1})$**:** This limiting case has to be treated seperately because the traveling wave u_0 of the hyperbolic equation is only continuous but not C^1. Fixing a parametrization we have $u_0(\xi) \equiv u_{i+1}$ for $\xi \geq 0$ while for $\xi \leq 0$ u_0 solves the differential equation

$$u_0'(\xi) = \begin{cases} \dfrac{g(u_0)}{f'(u_0) - s_0} & \text{for } \; u_0 \neq u_{i+1} \\[2ex] \dfrac{g'(u_{i+1})}{f''(u_{i+1})} & \text{for } \; u_0 = u_{i+1} \end{cases}$$

with $u_0(0) = u_{i+1}$. Assume for the moment that we can approximate s_0 by a sequence s_n with $s_n \searrow s_0$ such that the corresponding traveling waves $u_0^{(n)}$ of (H) satisfy

$$\|u_0^{(n)} - u_0\|_{L^1} \leq \frac{1}{2n}. \tag{10}$$

Since for each s_n the inequality of case I is satisfied, there exists ε_n with $\varepsilon_n \searrow 0$ such that the corresponding heteroclinic wave u_{ε_n} of (P) from u_i to u_{i+1} with speed s_n satisfies

$$\|u_{\varepsilon_n} - u_0^{(n)}\|_{L^1} \leq \frac{1}{2n}.$$

Together with (10), this estimate shows that the heteroclinic wave u_0 admits a viscous profile.

We still have to show that (10) can be satisfied by an appropriate sequence $(u_0^{(n)})_{n\in\mathbb{N}}$. Note that in this step of the proof only traveling waves of the hyperbolic equation (H), although with different speed s, are involved. To this end, we fix some small number σ and derive estimates which hold for all wave speeds $s \in [s_0, s_0 + \sigma]$.

Let u_0^s be the entropy traveling wave of (H) with speed $s > s_0$ which connects u_i to u_{i+1}. From (1) we know that u_0^s solves the ordinary differential equation

$$u' = \frac{g(u)}{f'(u) - s}. \tag{11}$$

First, we determine δ_- such that

(i) $g(u) \leq \frac{1}{2} g'(u_i)(u - u_i)$ for all $u \in [u_i, u_i + \delta_-]$ and

(ii)

$$\delta_- \int_{-\infty}^0 e^{\frac{g'(u_i)}{2(f'(u_i) - s_0 - \sigma)}\xi} \, d\xi \leq \frac{1}{10n}.$$

Since $|f'(u) - s| \leq |f'(u_i) - s_0 - \sigma|$ holds for all $u \in [u_i, u_i + \delta_-]$ and all $s \in [s_0, s_0 + \sigma]$ we conclude from (i) that

$$\frac{g(u)}{f'(u) - s} \geq \frac{g'(u_i)(u - u_i)}{2(f'(u_i) - s_0 - \sigma)}$$

for all $u \in [u_i, u_i + \delta_-]$ and all $s \in [s_0, s_0 + \sigma]$

With this δ_-, define $\xi_- < 0$ such that $u_0(\xi_-) = u_i + \delta_-$ and parametrize the other traveling waves u_0^s by

$$u_0^s(\xi_-) := u_0(\xi_-) = u_i + \delta_-.$$

By comparison, we know that

$$|u_0^s(\xi) - u_i| \le \delta_- \exp\left(\frac{g'(u_i)(u-u_i)}{2(f'(u_i) - s_0 - \sigma)}(\xi - \xi_-)\right)$$

From the uniform decay estimate (i) we conclude that

$$\begin{aligned}\int_{-\infty}^{\xi_-} |u_0(\xi) - u_0^s(\xi)|\, d\xi &\le \int_{-\infty}^{\xi_-} |u_0^s(\xi) - u_i|\, d\xi + \int_{-\infty}^{\xi_-} |u_0(\xi) - u_i|\, d\xi \\ &\le \frac{1}{10n} + \frac{1}{10n} \le \frac{1}{5n}\end{aligned}$$

independent of $s \in [s_0, s_0 + \sigma]$ by (ii).

In a next step we can determine δ_+ such that $|f'(u_{i+1} - \delta_+) - s_0 - \sigma| \le 1/2$, $g(u) \le \frac{g'(u_{i+1})}{2}(u - u_{i+1})$ for $u \in [u_{i+1} - \delta_+, u_{i+1}]$ and

$$\delta_+ \int_0^\infty e^{-g'(u_{i+1})\xi}\, d\xi \le \frac{1}{10n}.$$

From this estimate we get uniformly for $u \in [u_{i+1} - \delta_+, u_{i+1}]$. and $s \in [s_0, s_0 + \sigma]$ the estimate

$$\frac{g(u)}{f'(u) - s} \ge -g'(u_{i+1})(u - u_{i+1}) \ge 0.$$

Since $\frac{g(u)}{f'(u)-s} > c_0 > 0$ for $u \in [u_i + \delta_-, u_{i+1} - \delta_+]$ and $s \in [s_0, s_0 + \sigma]$, we can find ξ_+ with the property that $u_0^s(\xi_+) \in [u_{i+1} - \delta_+, u_{i+1}]$ for all $s \in [s_0, s_0 + \sigma]$.

By comparison we get

$$|u_0^s(\xi) - u_{i+1}| \le \delta_+ e^{-g'(u_{i+1})(\xi - \xi_+)} \text{ for } \xi \ge \xi_+$$

and therefore

$$\begin{aligned}\int_{\xi_+}^{\infty} |u_0(\xi) - u_0^s(\xi)|\, d\xi &\le \int_{\xi_+}^{\infty} |u_0^s(\xi) - u_{i+1}|\, d\xi + \int_{\xi_+}^{\infty} |u_0(\xi) - u_{i+1}| d\xi \\ &\le \frac{1}{10n} + \frac{1}{10n} \le \frac{1}{5n}\end{aligned}$$

independent of $s \in [s_0, s_0 + \sigma]$.

To get estimates on the intermediate part $[\xi_-, \xi_+]$ we define

$$\bar{u} := \sup\{u_0^s(\xi_+),\ s \in [s_0, s_0 + \sigma]\}.$$

This implies immediately that $u_0^s(\xi) \in [u_i + \delta_-, \bar{u}]$ for $\xi \in [\xi_-, \xi_+]$ and all $s \in [s_0, s_0 + \sigma]$. We can now estimate

$$\begin{aligned} |u_0(\xi) - u_0^s(\xi)| &= \int_{\xi_-}^{\xi} \left| \frac{g(u_0)}{f'(u_0) - s_0} - \frac{g(u_0^s)}{f'(u_0^s) - s} \right| d\xi \\ &= \int_{\xi_-}^{\xi} \left| \frac{g(u_0)}{f'(u_0) - s_0} - \frac{g(u_0^s)}{f'(u_0^s) - s_0} \cdot \frac{1}{1 + \frac{s_0 - s}{f'(u_0^s) - s_0}} \right| d\xi \\ &\leq \int_{\xi_-}^{\xi} L|u_0(\xi) - u_0^s(\xi)| + \mathcal{O}(|s - s_0|)\, d\xi \end{aligned}$$

where L is a Lipschitz constant for $\frac{g(u)}{f'(u)-s}$ on $[u_i + \delta_-, \bar{u}]$.

Using the Gronwall inequality, we find that

$$|u_0(\xi) - u_0^s(\xi)| = \mathcal{O}(|s - s_0|) \text{ for } \xi \in [\xi_-, \xi_+]$$

and hence

$$\int_{\xi_-}^{\xi_+} |u_0(\xi) - u_0^s(\xi)|\, d\xi = \mathcal{O}(|s - s_0|).$$

This proves (10) and concludes thereby the proof that all type (A1) entropy traveling waves are admissible. ♢

4 Discontinuous waves

This chapter is devoted to the heteroclinic waves of type (A2). We distinguish two cases depending on type of the equilibria involved. In the "Lax"-like situation (A2)(i) and (A2)(iii) a connection from a saddle to a sink or from a source to a saddle is considered. In contrast the waves of type (A2)(ii) connect a source to a sink. This is analogous to the case of overcompressive shock waves of hyperbolic conservation laws.

4.1 The "Lax" case

The heteroclinic waves of type (A2)(i) and (A2)(iii) are related via the symmetry $\xi \mapsto -\xi$, so we treat only waves of type (A2)(i). We restrict ourselves to the case $u_j < h(u_i, s_0)$, see figure 1 as the case $u_j > h(u_i, s_0)$ can be treated in a similar way.

Lemma 4.1 *For ε small and the wave speed s_0 identical to that of the hyperbolic entropy traveling wave, a branch of the unstable manifold of u_i is a heteroclinic orbit from u_i to u_j.*

Proof: Fix some small number $\delta > 0$. The unstable manifold $W^u(u_i)$ of the equilibrium u_i is a single trajectory $(u_\varepsilon(\xi), v_\varepsilon(\xi))$. Since $W^u(u_i)$ is $\mathcal{O}(\varepsilon)$-close to the unstable eigenspace, we can parametrize the trajectory such that

$u_\varepsilon(0) = u_i - \delta$ and $v_\varepsilon(0) = -f(u_i) + s_0 u_i + \mathcal{O}(\varepsilon)$. Outside a neighborhood of the singular curve $\mathcal{C}_{s_0}$ the vector field (3) has a horizontal component of order $\mathcal{O}(\varepsilon^{-1})$ and a vertical component of order $\mathcal{O}(1)$. Hence, following the unstable manifold, a cross-section $u = h(u_i, s_0) + \delta$ near the other branch of $\mathcal{C}_{s_0}$ is reached at $(u_\varepsilon(\xi_1), v_\varepsilon(\xi_1))$ where

$$\begin{aligned} u_\varepsilon(\xi_1) &= h(u_i, s_0) + \delta, \\ v_\varepsilon(\xi_1)) &= -f(u_i) + s_0 u_i + \mathcal{O}(\varepsilon) \\ \xi_1 &= \mathcal{O}(\varepsilon). \end{aligned}$$

Near the singular curve the vector field can be transformed to a normal form due to Takens [13]. By calculations analogous to those in [5] it can then be shown that it takes a "time" ξ of order $\mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$ until the trajectory enters a positively invariant region

$$P := \left\{ (u, v);\ u_j \le u \le h(u_i, s_0) + \delta,\ \left| v + f(u) - s_0 u - \varepsilon \frac{g(u)}{f'(u) - s_0} \right| \le k\varepsilon^2 |g(u)| \right\}$$

of width $2k\varepsilon^2 g(u)$. As in lemma 3.1 it can be shown that for k sufficiently large and all small ε any trajectory in this region converges to u_j. The unstable manifold $W^u(u_i)$ enters this region at a point $(u_\varepsilon(\xi_2), v_\varepsilon(\xi_2))$ where $\xi_2 = \mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$ and

$$|v_\varepsilon(\xi_2) + f(u_i) - s_0 u_i| = \mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon}).$$

Another way to find this asymptotic behaviour can be found in Mishchenko and Rozovs book [11]. In particular, $W^u(u_i)$ converges to u_j and is therefore a heteroclinic orbit u_ε. $\diamondsuit$

Remark 4.2 *For entropy traveling waves of type (A2)(iii) one needs to establish a negatively invariant region N near $\mathcal{C}_{s_0}$ similar to the positively invariant region P. The heteroclinic connection is then found by following the stable manifold of u_j backward.*

Lemma 4.3 *The heteroclinic orbits found in lemma 4.1 satisfy*

$$\lim_{\varepsilon \searrow 0} \|u_\varepsilon - u_0\|_{L^1_\beta} = 0 \quad \textit{for} \ \ 0 \le \beta < \left| \frac{g'(u_j)}{f'(u_j) - s_0} \right|.$$

In particular, they provide a viscous profile for the type (A2)(i) entropy traveling waves.

Proof: We use lemma 2.8 again. To this end we fix $c \in (\beta, \frac{-g'(u_j)}{f'(u_j) - s_0})$ and determine some small $\delta > 0$ and $\varepsilon_1 > 0$ with the property that

$$\left| \frac{g(u)}{f'(u) - s_0} - k\varepsilon \right| > c \cdot |u - u_j| \quad \text{for all } |u - u_j| < \delta; ;\text{and} \ \ 0 \le \varepsilon \le \varepsilon_1$$

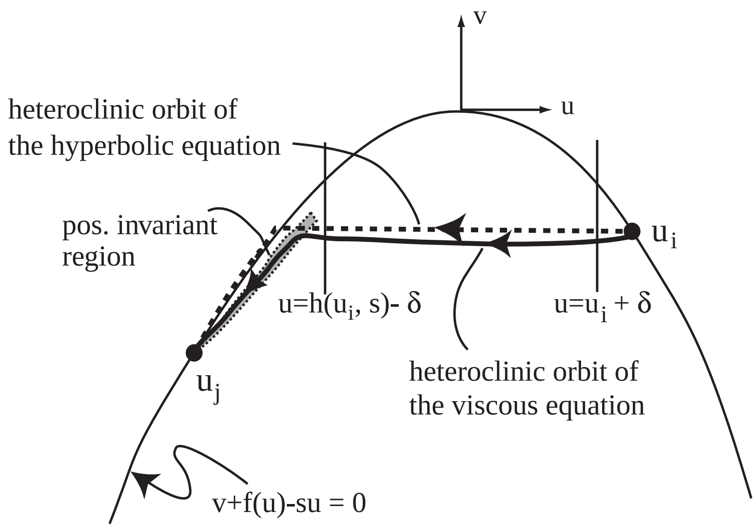

Figure 2: A "Lax" heteroclinic traveling wave (dashed) and its viscous counterpart

where k is the constant related to the width of the invariant region P. We parametrize u_ε as in lemma 4.1 by $u_\varepsilon(0) = u_i - \delta$ and $u_0(\xi)$ by

$$u_0(\xi) = u_i \iff \xi \le 0.$$

As in lemma 3.2 we can find

$$\xi_+ := \sup_{0 \le \varepsilon \le \varepsilon_1} \{\xi : u_\varepsilon(\xi) = u_j + \delta\}$$

independent of $\varepsilon \in [0, \varepsilon_1]$ by decreasing ε_1 if necessary. A comparison argument shows that for $\xi > \xi_+$ and $C := \delta$ assumption (ii) of lemma 2.8 is satisfied.

Since the linearization of (3) in u_i has a positive eigenvalue of order $\mathcal{O}(1/\varepsilon)$, we know that there is a constant $M > 0$ such that

$$|u_\varepsilon(\xi) - u_i| \le \delta e^{M/\varepsilon \xi} \le \delta e^{c\xi}$$

for all $\xi < 0$ and $\varepsilon \le \varepsilon_1$. With $\xi_- = 0$, $C = \delta$ and $u_- = u_i$ this proves assumption (i) of lemma 2.8.

To check assumption (iii) of this lemma we fix a and b. Without restriction we may assume that $a < 0 < b$. Since $\xi_2 = \mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$ and u_0' is bounded, we know that $|u_0(\xi_2) - h(u_i, s_0)| = \mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$. Since v_ε' is also bounded we have $v_\varepsilon(\xi_2) = -f(u_i) + s_0 u_i + \mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$. This implies that $u_\varepsilon(\xi_2) = h(u_i, s_0) + \mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$ since $(u_\varepsilon(\xi_2), v_\varepsilon(\xi_2))$ lies on the boundary of the invariant region P. Together we have

$$|u_0(\xi_2) - u_\varepsilon(\xi_2)| \le |u_0(\xi_2) - h(u_i, s_0)| + |u_\varepsilon(\xi_2) - h(u_i, s_0)| = \mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon}).$$

By a Gronwall type estimate like in the proof of lemma 3.2 we can conclude from these facts that for $\xi \in [\xi_2, b]$

$$\begin{aligned}
|u_0(\xi) - u_\varepsilon(\xi)| &\leq |u_0(\xi_2) - u_\varepsilon(\xi_2)| + \int_{\xi_2}^{\xi} |u_0'(\xi) - u_\varepsilon'(\xi)|\, d\xi \\
&\leq |u_0(\xi_2) - u_\varepsilon(\xi_2)| + \int_{\xi_2}^{\xi} L|u_0(\xi) - u_\varepsilon(\xi)| + \mathcal{O}(\varepsilon)\, d\xi \\
\Rightarrow |u_0(\xi) - u_\varepsilon(\xi)| &\leq \mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})
\end{aligned}$$

where L is again a Lipschitz constant for $\frac{g(u)}{f'(u)-s_0}$ on the interval $[u_j+\delta, h(u_i, s_0)]$.

The last step consists of estimating

$$\begin{aligned}
& \int_a^b |u_0(\xi) - u_\varepsilon(\xi)|\, d\xi \\
= & \int_a^0 |u_0(\xi) - u_\varepsilon(\xi)|\, d\xi + \int_0^{\xi_2} |u_0(\xi) - u_\varepsilon(\xi)|\, d\xi + \int_{\xi_2}^{b} |u_0(\xi) - u_\varepsilon(\xi)|\, d\xi \\
\leq & \; \delta \int_a^0 e^{C\xi/\varepsilon}\, d\xi + +\mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon}) + C \int_{\xi_2}^{b} e^{L\xi} \varepsilon \ln \frac{1}{\varepsilon}\, d\xi \\
= & \; \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})
\end{aligned}$$

Therefore assumption (iii) of lemma 2.8 must hold and applying this lemma shows that all waves of type (A2)(i) are admissible. $\diamondsuit$

4.2 The "overcompressive" case

Similarly as for overcompressive shocks of conservation laws, for a fixed wave speed s_0 we have a whole one-parameter-family of heteroclinic waves of type (A2)(ii) with a shock at $\xi = 0$, where the jump values $u_0(0+)$ plays the role of a parameter. We pick one of these entropy traveling waves, call it u_0 and prove its admissibility. To find heteroclinic waves of the parabolic equation (P) which provide a viscous profile for such a heteroclinic wave, we define $(u_\varepsilon, v_\varepsilon)$ as the solution of (3) with

$$u_\varepsilon(0) = \frac{u_0(0+) + u_0(0-)}{2}$$

and

$$v_\varepsilon(0) = -f(u_0(0+)) + s_0 u_0(0+) = -f(u_0(0-)) + s_0 u_0(0-)$$

where $u(0+)$, $u(0-)$ are the one-sided limits of the hyperbolic wave at the shock.

Lemma 4.4 *For ε sufficiently small, $(u_\varepsilon(\xi), v_\varepsilon(\xi))$ is a heteroclinic orbit from u_i to u_j and the family of these heteroclinic orbits provides a viscous profile for the entropy traveling wave of type (A2)(ii).*

Proof: The ingredients of the proof that these overcompressive traveling waves admit a viscous profile are exactly the same as in the Lax case, so we will be very brief here. We can find $\xi_2 > 0$ of order $\mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$ such that the at $\xi = \xi_2 > 0$ the heteroclinic trajectory enters a positively invariant region near $\mathcal{C}_{s_0}$. Moreover, $|u_0(\xi_2) - u_\varepsilon(\xi_2)| = \mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$ and the vector fields u_0' and u_ε' are $\mathcal{O}(\varepsilon)$-close. Again it takes only a finite time interval $[\xi_2, \xi_+]$ independent of ε to reach a vicinity of u_j where exponential estimates apply. For the intermediate region $[\xi_2, \xi_+]$ again the Gronwall lemma is used. This gives all necessary estimates for $\xi > 0$. For $\xi < 0$, one has to go backward and find ξ_{-2} such that at ξ_{-2} the backward trajectory enters a negatively invariant region N near the singular curve $\mathcal{C}_{s_0}$. All backward trajectories remain in this negatively invariant region and reach a δ-neighborhood of u_i where exponential convergence to u_i with a rate holds. There exists ξ_- such that independent of ε we have $|u_i - u_\varepsilon(\xi)|$ for all $\xi < \xi_-$ and all $\varepsilon \geq 0$ sufficiently small. Applying the Gronwall estimate and the exponential convergence near u_i to the backward trajectories $u_0(\xi)$ and $u_\varepsilon(\xi)$ with $\xi < 0$ yields the necessary estimates to prove the lemma. ♢

5 Undercompressive Shocks

In this chapter we treat the simple shock waves of type (C) which are of the form

$$u(x,t) = \begin{cases} u_j \text{ for } \ x - s_0 t < 0 \\ u_i \text{ for } x - s_0 t > 0 \end{cases}$$

with shock speed $s_0 = \frac{f(u_i)-f(u_j)}{u_i - u_j}$ given by the Rankine-Hugoniot relation. Here the source term is involved only via the fact that shocks can connect only equilibrium states of the reaction dynamics. Since both equilibria are of saddle-type here, we call this shock undercompressive. In the traveling wave setting this correspond to an entropy solution

$$u(\xi) = \begin{cases} u_j \text{ for } \ \xi < 0 \\ u_i \text{ for } \xi > 0. \end{cases}$$

Lemma 5.1 *There exists a wave speed $s(\varepsilon)$ with $s(\varepsilon) \to s_0$ such that (3) possesses a heteroclinic orbit from u_i to u_j.*

Proof: We consider the unstable manifold of u_i and the stable manifold of u_j. For $s < s_0$ and ε sufficiently small the unstable manifold of u_i is almost a horizontal line and passes below the stable manifold of u_j in the u-v-plane. In fact $W^u(u_i)$ intersects the line $u = \frac{u_i+u_j}{2}$ at a height $f(u_i) - su_i + \mathcal{O}(\varepsilon)$ which is strictly smaller than the height $f(u_j) - su_j + \mathcal{O}(\varepsilon)$ where $W^s(u_j)$ intersects this line. For $s > s_0$ the situation is reversed and $W^u(u_i)$ lies above $W^s(u_j)$, so there exists a wave speed $s = s(\varepsilon)$ such that $W^u(u_i) \cap W^s(u_j) \neq \emptyset$. Since this intersection is one-dimensional, it must be a heteroclinic orbit u_ε. As for any fixed $s \neq s_0$ the unstable manifold of u_j and the stable manifold of u_i miss each other if ε is small enough , the limiting relation $\lim_{\varepsilon \searrow 0} s(\varepsilon) = s_0$ holds. ♢

Lemma 5.2 *For any $\beta \geq 0$*

$$\lim_{\varepsilon \searrow 0} \|u_\varepsilon - u_0\|_{L^1_\beta} = 0,$$

in particular, all entropy traveling waves of type (C) are admissible.

Proof: First, we choose a small number $\delta > 0$.

We parametrize the heteroclinic orbits u_ε in such a way that

$$u_\varepsilon(0) = \frac{u_i + u_j}{2}.$$

There are ξ_- and ξ_+ such that

$$u_\varepsilon(\xi_-) \geq u_i - \delta \text{ and } u_\varepsilon(\xi_+) \leq u_j + \delta$$

uniformly for all ε small. As $u'_\varepsilon \leq -\frac{m}{\varepsilon}$ for $u_\varepsilon \in [u_j + \delta, u_i - \delta]$ and some $m > 0$, we can conclude that $|\xi_+ - \xi_-| \leq \frac{\varepsilon|u_i - u_j|}{m}$.

Linearizing (3) at the equilibria u_i and u_j one finds eigenvalues of order $\mathcal{O}(1/\varepsilon)$. This implies that the convergence of u_ε to the equilibria is exponentially fast with a rate bigger than M/ε for some $M > 0$ as long as $u_\varepsilon \in [u_i - \delta, u_i)$ or $u_\varepsilon \in (u_j, u_j + \delta]$. In particular, assumptions (i) and (ii) of lemma 2.8 can be satisfied for any given β by making ε small enough.

To check assumption (iii) of this lemma we can assume without restriction that $a < \xi_- < \xi_+ < b$ and estimate

$$\begin{aligned}
&\int_a^b |u_0(\xi) - u_\varepsilon(\xi)|\, d\xi \\
&= \int_a^{\xi_-} |u_0(\xi) - u_\varepsilon(\xi)|\, d\xi + \int_{\xi_-}^{\xi_+} |u_0(\xi) - u_\varepsilon(\xi)|\, d\xi + \int_{\xi_+}^{b} |u_0(\xi) - u_\varepsilon(\xi)|\, d\xi \\
&\leq \delta \int_a^{\xi_-} e^{M\xi/\varepsilon}\, d\xi + \frac{\varepsilon|u_i - u_j|^2}{m} + \delta \int_a^{\xi_-} e^{M\xi/\varepsilon}\, d\xi \\
&= \mathcal{O}(\varepsilon).
\end{aligned}$$

The claim follows now directly from lemma 2.8. ♢

6 Discussion

Kruzhkovs classical result [9] states that solutions of the viscous equation are a good approximation for the solution of the hyperbolic equation with the same initial data as long as a fixed bounded time interval is considered. Here, we have taken a different approach and asked whether special solutions of the hyperbolic equation can be approximated on an unbounded time interval by solutions of the viscous equation which are of the same type, namely traveling wave solutions.

Using methods of classical singular perturbation theory, we have in this paper shown that several types of entropy traveling waves of scalar balance laws admit a viscous profile. This shows that they are close to solutions of the viscous balance law in the sense that their profiles are close to each other. The price one has to pay for this qualitative agreement is the change of the wave speed which makes $\|u_\varepsilon(\cdot,t)-u_0(\cdot,t)\|_{L^1}$ grow as $t\to\infty$.

However, not all heteroclinic waves admit a viscous profile: There are discontinuous waves with more than one discontinuity, which can be shown not to possess a viscous profile by a simple application of the Jordan curve theorem. This negative result will be treated in a forthcoming paper [6] together with some other cases.

There are many obvious generalizations. For instance, the question of existence and viscous admissibility of heteroclinic traveling waves can be asked for systems of balance laws, too. While the existence part seems to be quite straightforward, the existence of viscous profiles will lead to singularly perturbed equations with many fast and many slow variables.

An interesting question concerning traveling waves is always stability. To determine the linearized stability of the viscous traveling wave, one has to look at the equation

$$v_t + f''(u_\varepsilon(x))u_\varepsilon'(x)v + (f'(u_\varepsilon(x))-s)v_x = \varepsilon v_{xx} + g'(u_\varepsilon(x))v.$$

Writing the corresponding eigenvalue problem as a first order system

$$\begin{aligned} \varepsilon v_x &= w \\ w_x &= f''(u_\varepsilon(x))u_\varepsilon'(x)v + \frac{f'(u_\varepsilon(x))-s}{\varepsilon}w - g'(u_\varepsilon(x))v + \lambda v \end{aligned}$$

the linear stability problem is reduced to the study of the spectrum of

$$\mathcal{L} = \frac{d}{dx} + \begin{pmatrix} 0 & 1 \\ f''(u_\varepsilon(x))u_\varepsilon'(x) - g'(u_\varepsilon(x)) + \lambda & \dfrac{f'(u_\varepsilon(x))-s}{\varepsilon} \end{pmatrix}$$

where $\mathcal{L}$ is considered as an unbounded operator on $L^2(\mathbb{R},\mathbb{R}^2)$.

It is a well known result (see e.g. [7]) that the essential spectrum of $\mathcal{L}$ lies to the left of the spectrum of the operators

$$\mathcal{L}_\pm := \frac{d}{dx} + \begin{pmatrix} 0 & 1 \\ \lambda - g'(u_\pm) & \dfrac{f'(u_\pm)-s}{\varepsilon} \end{pmatrix}.$$

where $u_\pm$ are the asymptotic states of the traveling wave. By Fourier transform, one can easily check that the real part of the essential spectrum of $\mathcal{L}_\pm$ is bounded by $g'(u_\pm)$. This yields immediately (linear) instability of the overcompressive waves and of the monotone waves which connect adjacent equilibria except when f' vanishes at one of the asymptotic states. In the latter case, the essential spectrum touches the imaginary axis. Only for the undercompressive shocks the essential spectrum of $\mathcal{L}$ is contained in the open left half plane.

It remains to discuss eigenvalues of $\mathcal{L}$. As a consequence of the translation invariance, 0 is an eigenvalue with eigenfunction u_ε'. For monotone waves, Sturm-Liouville type arguments show that this is in fact the eigenvalue with the largest real part, which proves stability for the undercompressive shock waves at a fixed value of ε. To prove uniform exponential stability, the other eigenvalues must not approach zero as ε tends to 0. It seems possible to determine via an Evans function calculation whether there is a uniform upper bound for the second eigenvalue. For recent accounts on Evans function see the papers by Kapitula and Sandstede [8] and Gardner and Zumbrun [3].

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