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NONEXISTENCE RESULTS FOR HYPERBOLIC TYPE INEQUALITIES INVOLVING THE GRUSHIN OPERATOR IN EXTERIOR DOMAINS

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Abstract. We study the hyperbolic type differential inequality

$$u_{tt}(t,x,y)-\mathcal{L}_\ell u(t,x,y)\geq |u(t,x,y)|^p,\quad (t,x,y)\in(0,\infty)\times D_1\times D_2$$

under the boundary conditions

$$\begin{aligned}u(t,x,y)&\geq f(x),\quad (t,x,y)\in(0,\infty)\times\partial D_1\times D_2,\\ u(t,x,y)&\geq g(y),\quad (t,x,y)\in(0,\infty)\times D_1\times\partial D_2,\end{aligned}$$

where $p>1$, $D_k=\{z\in\mathbb{R}^{N_k}:|z|\geq 1\}$, $k=1,2$, $N_k\geq 2$, $f\in L^1(\partial D_1)$, $g\in L^1(\partial D_2)$, and $\mathcal{L}_\ell$, $\ell\in\mathbb{R}$, is the Grushin operator

$$\mathcal{L}_\ell u=\Delta_x u+|x|^{2\ell}\Delta_y u.$$

We obtain sufficient conditions depending on p, ℓ, N_1, N_2, f, and g, for which the considered problem admits no global weak solution. We discuss separately the four cases: $N_1=N_2=2$; $N_1=2$, $N_2\geq 3$; $N_1\geq 3$, $N_2=2$; $N_1,N_2\geq 3$.

1. Introduction

This article concerns the hyperbolic type differential inequality

$$\begin{gathered}u_{tt}(t,x,y)-\mathcal{L}_\ell u(t,x,y)\geq |u(t,x,y)|^p,\quad (t,x,y)\in(0,\infty)\times D_1\times D_2,\\ u(t,x,y)\geq f(x),\quad (t,x,y)\in(0,\infty)\times\partial D_1\times D_2,\\ u(t,x,y)\geq g(y),\quad (t,x,y)\in(0,\infty)\times D_1\times\partial D_2,\end{gathered}\tag{1.1}$$

where $p>1$, $D_1=\{x\in\mathbb{R}^{N_1}:|x|\geq 1\}$, $D_2=\{y\in\mathbb{R}^{N_2}:|y|\geq 1\}$, $N_1,N_2\geq 2$, $f\in L^1(\partial D_1)$, $g\in L^1(\partial D_2)$, and $\mathcal{L}_\ell$, $\ell\in\mathbb{R}$, is the Grushin operator of the form

$$\mathcal{L}_\ell u=\Delta_x u+|x|^{2\ell}\Delta_y u=\sum_{i=1}^{N_1}\frac{\partial^2 u}{\partial x_i^2}+|x|^{2\ell}\sum_{j=1}^{N_2}\frac{\partial^2 u}{\partial y_j^2}.\tag{1.2}$$

Namely, our aim is to derive sufficient conditions for which problem (1.1) admits no global weak solution.

Several works have been made to investigate the nonexistence of solutions for hyperbolic type differential inequalities. In [13], among other problems, Kato studied

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the hyperbolic inequality

$$u_{tt} - \Delta u \geq |u|^p, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N. \tag{1.3}$$

He proved that if the initial data satisfy some suitable positivity conditions, are compactly supported, and

$$1 < p \leq 1 + \frac{2}{N-1} \ (N \geq 2),$$

then no weak solution to (1.3) can exist in $(0,\infty) \times \mathbb{R}^N$. Véron and Pohozaev [23] studied the nonexistence of nontrivial global solutions to a wide class of nonlinear hyperbolic type inequalities of the form

$$u_{tt} \geq L_m(\varphi_p(u)) + |u|^q, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N, \tag{1.4}$$

where $p > 0$, φ_p is a locally bounded real valued function satisfying

$$|\varphi_p(r)| \leq c|r|^p$$

for certain $c > 0$, and $L_m(\zeta) = \sum_{|\alpha|=m} D^\alpha(a_\alpha(t,x)\zeta)$ is a homogeneous differential operator of order m in which the coefficients a_α are bounded measurable functions. By an appropriate choice of test functions and the dimensional analysis, it was shown that problem (1.4) admits no weak solution such that $\int_{\mathbb{R}^N} u_t(0,x)\,dx \geq 0$, provided that $q > \max\{1,p\}$ and either $2N - m \leq 0$ or $2N - m > 0$ and $\frac{N(q-p)}{q+1} \leq \frac{m}{2}$. In [10], the authors investigated the hyperbolic inequality

$$u_{tt} - \Delta u \geq |u|^p + |\nabla u|^q + f(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^N, \tag{1.5}$$

where $p, q > 1$ and $f \geq 0$, $f \not\equiv 0$. Namely, they derived general criteria for the nonexistence of global solutions to (1.5). In particular, when $N \geq 3$ and f depends only on the variable space, it was shown that (1.5) admits as Fujita critical exponent the real number

$$p^*(N,q) = \begin{cases} 1 + \frac{2}{N-2} & \text{if } q > 1 + \frac{1}{N-1}, \\ \infty & \text{if } q < 1 + \frac{1}{N-1}. \end{cases}$$

In all the above mentioned references the considered problems are posed in the whole space $\mathbb{R}^N$.

The study of hyperbolic type differential inequalities in other infinite domains was considered by some authors. In [16], among other problems, Laptev considered the hyperbolic inequality

$$u_{tt} - \Delta u \geq |u|^p, \quad (t,x) \in (0,\infty) \times K \tag{1.6}$$

under the Dirichlet type boundary condition

$$u(t,x) \geq 0, \quad (t,x) \in (0,\infty) \times \partial K, \tag{1.7}$$

where K is the cone defined by

$$K = \{(r,\omega) : r > 0,\ \omega \in \Omega\},$$

and Ω is a domain of S^{N-1}, $N \geq 3$. It was shown that, if

$$1 < p \leq 1 + \frac{2}{s^* + 1},$$

where

$$s^* = \frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda_1}$$

and λ_1 is the first eigenvalue of the Laplace Beltrami operator Δ_θ on Ω, then problem (1.6) under the boundary condition (1.7) has no nontrivial global weak solution. In [12] (see also [9]), motivated by Zhang [26], the authors investigated the nonexistence of global weak solutions for a system of inhomogeneous wave inequalities in exterior domains under three type boundary conditions: Dirichlet type, Neumann type and mixed boundary conditions. In particular, for the hyperbolic inequality

$$\begin{gathered} u_{tt} - \Delta u \ge |x|^a |u|^p, \quad (t,x) \in (0,\infty) \times \Omega^c, \\ u(t,x) \ge f(x), \quad (t,x) \in (0,\infty) \times \partial\Omega, \end{gathered} \tag{1.8}$$

where $a > -2$, Ω^c denotes the complement of Ω, Ω is a bounded smooth open set in $\mathbb{R}^N$ containing the origin, and $N \ge 3$, it was shown that, if $f \in L^1(\partial\Omega)$, $\int_{\partial\Omega} f\, d\sigma > 0$, and

$$1 < p < \frac{N+a}{N-2},$$

then problem (1.8) admits no global weak solution. Moreover, for $p > \frac{N+a}{N-2}$, problem (1.8) admits global solutions (namely, stationary solutions) for some $f > 0$. For other works related to differential inequalities in exterior domains, see e.g. [11, 20, 21] and the references therein.

A large amount of works have been made to study the Grushin operator $\mathcal{L}_\ell$ of the form (1.2) as well as the properties of the solutions to $-\mathcal{L}_\ell u = f$ (see [1, 6, 7, 8]). Capuzzo Dolcetta and Cutri [2] studied the differential inequality

$$-\mathcal{L}_\ell u \ge u^p, \quad u \ge 0, \quad x \in \mathbb{R}^{N_1}, \quad y \in \mathbb{R}^{N_2}. \tag{1.9}$$

It was shown that, if $\ell > 1$ and $1 < p \le \frac{Q}{Q-2}$, where $Q = N_1 + (\ell+1)N_2$, then (1.9) admits no nontrivial solution. D'Ambrosio and Lucente [4] investigated the differential inequality

$$L(x,y,D_x,D_y) \ge |x|^{\theta_1} |y|^{\theta_2} |u|^q, \quad (x,y) \in \mathbb{R}^d \times \mathbb{R}^k,$$

where L is a quasi-homogeneous differential operator including as special cases Tricomi or Grushin-type operators, $q > 1$, $\theta_1, \theta_2 \in \mathbb{R}$, and $k, d \ge 1$. Namely, they provided necessary conditions for existence of weak solutions to the considered inequality. For other nonexistence results for differential inequalities (stationary inequalities) involving Grushin type operators, see [3, 5, 14, 15, 17, 18, 19, 22, 24, 25, 27] and the references therein.

Motivated by the above mentioned contributions, our aim in this paper is to obtain sufficient conditions depending on p, ℓ, N_1, N_2, f and g, for which problem (1.1) not to admits global weak solutions.

The rest of the paper is organized as follows. In Section 2, we define global weak solutions to problem (1.1) and provide the main results of this paper. In Section 3, we establish some preliminary estimates that will be used in the proofs of our main results. In Section 4, we prove the main results of this paper. We discuss separately the cases: $N_1 = N_2 = 2$; $N_1 = 2$, $N_2 \ge 3$; $N_1 \ge 3$, $N_2 = 2$; $N_1, N_2 \ge 3$.

The symbols C or C_i denote always generic positive constants, which are independent of the scaling parameter R and the solution u. Their values could be changed from one line to another. We will use the notation $\mu \sim \nu$ for two positive functions or quantities, which satisfy $C_1\mu \le \nu \le C_2\mu$.

2. Main results

We first fix some notation that will be used throughout this paper. Let

$$D = D_1 \times D_2, \quad \Omega = (0,\infty) \times D,$$
$$\Gamma_1 = (0,\infty) \times \partial D_1 \times D_2, \quad \Gamma_2 = (0,\infty) \times D_1 \times \partial D_2.$$

We denote by $n_1 = n_1(x)$ the outward unit normal vector on ∂D_1 relative to D_1. Similarly, we denote by $n_2 = n_2(y)$ the outward unit normal vector on ∂D_2 relative to D_2.

We introduce the test function space

$$\Phi = \big\{\varphi \in C_c^2(\Omega) : \varphi \geq 0,\, \varphi|_{\partial D_1 \cup \partial D_2} = 0,\, \frac{\partial_x \varphi}{\partial n_1} \leq 0,\, \frac{\partial_y \varphi}{\partial n_2} \leq 0\big\}, \tag{2.1}$$

where $C_c^2(\Omega)$ denotes the space of C^2 functions compactly supported in Ω. Here,

$$\frac{\partial_x \varphi}{\partial n_1} = \nabla_x \varphi \cdot n_1 \quad \text{and} \quad \frac{\partial_y \varphi}{\partial n_2} = \nabla_y \varphi \cdot n_2.$$

Let us mention in which sense the solutions are considered.

Definition 2.1. Let $f \in L^1(\partial D_1)$ and $g \in L^1(\partial D_2)$. We say that

$$u \in L^p_{\rm loc}([0,\infty) \times D)$$

is a global weak solution to (1.1), if

$$\begin{aligned}
&\int_\Omega |u|^p \varphi \, dx\, dy\, dt - \int_{\Gamma_1} \frac{\partial_x \varphi}{\partial n_1} f(x)\, d\sigma_x\, dy\, dt - \int_{\Gamma_2} |x|^{2\ell} \frac{\partial_y \varphi}{\partial n_2} g(y)\, dx\, d\sigma_y\, dt \\
&\leq \int_\Omega u \left(\varphi_{tt} - \Delta_x \varphi - |x|^{2\ell} \Delta_y \varphi\right)\, dx\, dy\, dt
\end{aligned} \tag{2.2}$$

for every $\varphi \in \Phi$. Here, $d\sigma_x$ denotes the surface measure on ∂D_1, and $d\sigma_y$ denotes the surface measure on ∂D_2.

Our first main result is the following.

Theorem 2.2. *Let $N_1 = N_2 = 2$, $f \in L^1(\partial D_1)$, and $g \in L^1(\partial D_2)$.*

(I) *Let $\ell \leq -1$. If*

$$\int_{\partial D_1} f(x) d\sigma_x > 0 \ \text{or} \ \int_{\partial D_1} f(x) d\sigma_x = 0,\ \int_{\partial D_2} g(y) d\sigma_y > 0,$$

then for all $p > 1$, (1.1) admits no global weak solution.

(II) *Let $\ell > -1$. If*

$$\int_{\partial D_1} f(x) d\sigma_x > 0 \ \text{or} \ \int_{\partial D_2} g(y) d\sigma_y > 0,$$

then for all $p > 1$, (1.1) admits no global weak solution.

Remark 2.3. Let $N_1 = N_2 = 2$. From Theorem 2.2 we deduce that, if

$$\int_{\partial D_1} f(x) d\sigma_x > 0,$$

then for all $\ell \in \mathbb{R}$ and $p > 1$, (1.1) admits no global weak solution.

Clearly, Theorem 2.2 yields nonexistence results for the corresponding stationary problem

$$\begin{gathered} -\mathcal{L}_\ell u(x,y) \geq |u(x,y)|^p, \quad (x,y) \in D_1 \times D_2, \\ u(x,y) \geq f(x), \quad (x,y) \in \partial D_1 \times D_2, \\ u(x,y) \geq g(y), \quad (x,y) \in D_1 \times \partial D_2. \end{gathered} \tag{2.3}$$

Namely, we deduce the following result.

Corollary 2.4. *Let $N_1 = N_2 = 2$, $f \in L^1(\partial D_1)$, and $g \in L^1(\partial D_2)$.*

(I) *Let $\ell \leq -1$. If*

$$\int_{\partial D_1} f(x)d\sigma_x > 0 \text{ or } \int_{\partial D_1} f(x)d\sigma_x = 0,\ \int_{\partial D_2} g(y)d\sigma_y > 0,$$

then for all $p > 1$, (2.3) admits no weak solution.

(II) *Let $\ell > -1$. If*

$$\int_{\partial D_1} f(x)d\sigma_x > 0 \text{ or } \int_{\partial D_2} g(y)d\sigma_y > 0,$$

then for all $p > 1$, (2.3) admits no weak solution.

Remark 2.5. Consider the differential inequality

$$\begin{gathered} v_{tt} - \Delta v \geq v^p (v \geq 0), \quad (t,x) \in (0,\infty) \times D_1, \\ v(t,x) \geq f(x), \quad (t,x) \in (0,\infty) \times \partial D_1, \end{gathered} \tag{2.4}$$

where $N_1 = 2$ and $p > 1$. Let v be a possible solution to (2.4) and

$$u(t,x,y) = v(t,x), \quad (t,x,y) \in (0,\infty) \times D_1 \times D_2,$$

where $N_2 = 2$. Then for all $\ell \in \mathbb{R}$, u is a solution to (1.1) with $g \equiv 0$. Taking in consideration Remark 2.3, we deduce that, if $\int_{\partial D_1} f(x)d\sigma_x > 0$, then for all $p > 1$, (2.4) admits no solution.

Theorem 2.6. *Let $N_1 = 2$, $N_2 \geq 3$, $f \in L^1(\partial D_1)$, and $g \in L^1(\partial D_2)$.*

(I) *Let $\ell < -1$.*
  (i) *If $\int_{\partial D_1} f(x)d\sigma_x > 0$, then for all $p > 1$, (1.1) admits no global weak solution.*
  (ii) *If*

$$\int_{\partial D_1} f(x)d\sigma_x = 0 \quad \text{and} \quad \int_{\partial D_2} g(y)d\sigma_y > 0,$$

  then for all $1 < p < \frac{N_2}{N_2-2}$, (1.1) admits no global weak solution.

(II) *Let $\ell = -1$.*
  (i) *If $\int_{\partial D_1} f(x)d\sigma_x > 0$, then for all $p > 1$, (1.1) admits no global weak solution.*
  (ii) *If*

$$\int_{\partial D_1} f(x)d\sigma_x = 0 \quad \text{and} \quad \int_{\partial D_2} g(y)d\sigma_y > 0,$$

  then for all $1 < p \leq \frac{N_2}{N_2-2}$, (1.1) admits no global weak solution.

(III) *Let $-1 < \ell < 0$. If*

$$\int_{\partial D_1} f(x)d\sigma_x > 0 \quad \text{or} \quad \int_{\partial D_2} g(y)d\sigma_y > 0,$$

then for all $p > 1$, (1.1) admits no global weak solution.

(IV) *Let* $\ell \geq 0$.
   (i) *If* $\int_{\partial D_1} f(x)d\sigma_x > 0$, *then for all* $p > 1$, (1.1) *admits no global weak solution.*
   (ii) *If* $\int_{\partial D_2} g(y)d\sigma_y > 0$, *then for all* $1 < p < \frac{N_2}{N_2-2}$, (1.1) *admits no global weak solution.*

Remark 2.7. Let $N_1 = 2$ and $N_2 \geq 3$. By Theorem 2.6 we deduce that, if $\int_{\partial D_1} f(x)d\sigma_x > 0$, then for all $\ell \in \mathbb{R}$ and $p > 1$, (1.1) admits no global weak solution.

Remark 2.8. Let $N_1 = 2$, $N_2 \geq 3$, $\ell \geq 0$, $g \in L^1(\partial D_2)$, and $\int_{\partial D_2} g(y)d\sigma_y > 0$. Then by Theorem 2.6 (IV)-(ii), if

$$1 < p < \frac{N_2}{N_2 - 2}, \tag{2.5}$$

then (1.1) admits no global weak solution for all $f \in L^1(\partial D_1)$. Moreover, for $p > \frac{N_2}{N_2-2}$, we can can check easily that

$$u(t,x,y) = A|y|^{-\sigma}, \quad (t,x,y) \in (0,\infty) \times D_1 \times D_2,$$

where $A > 0$ is sufficiently small and $\frac{2}{p-1} < \sigma < N_2 - 2$, is a (stationary) solution to (1.1) with $f \equiv 0$ and $g \equiv A$. This shows that (2.5) is sharp.

Remark 2.9. As in the previous case (see Corollary 2.4), the nonexistence results given by Theorem 2.6 hold true for the stationary problem (2.3) in the case $N_1 = 2$ and $N_2 \geq 3$.

Remark 2.10. *Consider the differential inequality*

$$\begin{gathered} v_{tt} - \Delta v \geq v^p (v \geq 0), \quad (t,y) \in (0,\infty) \times D_2, \\ v(t,y) \geq g(y), \quad (t,y) \in (0,\infty) \times \partial D_2, \end{gathered} \tag{2.6}$$

where $N_2 \geq 3$. *Let* v *be a possible solution to* (2.6) *and*

$$u(t,x,y) = v(t,y), \quad (t,x,y) \in (0,\infty) \times D_1 \times D_2,$$

where $N_1 = 2$. *Then* u *is a solution to* (1.1) *with* $f \equiv 0$ *and* $\ell = 0$. *Taking in consideration Remark 2.8, we deduce that, if* $\int_{\partial D_2} g(y)d\sigma_y > 0$, *then for all* $1 < p < \frac{N_2}{N_2-2}$, (2.6) *admits no solution. We find* [12, Corollary 1.9] *for the case of positive solutions.*

Theorem 2.11. *Let* $N_1 \geq 3$, $N_2 = 2$, $f \in L^1(\partial D_1)$, *and* $g \in L^1(\partial D_2)$.
(I) *Let* $\ell \leq -\frac{N_1}{2}$. *If*

$$\int_{\partial D_1} f(x)d\sigma_x > 0 \quad \text{or} \quad \int_{\partial D_1} f(x)d\sigma_x = 0, \int_{\partial D_2} g(y)d\sigma_y > 0,$$

then for all $1 < p < \frac{N_1}{N_1-2}$, (1.1) *admits no global weak solution.*
(II) *Let* $-\frac{N_1}{2} < \ell < -1$.
   (i) *If* $\int_{\partial D_1} f(x)d\sigma_x > 0$, *then for all* $1 < p < \frac{N_1}{N_1-2}$, (1.1) *admits no global weak solution.*
   (ii) *If* $\int_{\partial D_2} g(y)d\sigma_y > 0$, *then for all* $1 < p < \frac{\ell}{\ell+1}$, (1.1) *admits no global weak solution.*
(III) *Let* $\ell \geq -1$.

(i) *If* $\int_{\partial D_1} f(x)d\sigma_x > 0$, *then for all* $1 < p < \frac{N_1}{N_1-2}$, (1.1) *admits no global weak solution.*

(ii) *If* $\int_{\partial D_2} g(y)d\sigma_y > 0$, *then for all* $p > 1$, (1.1) *admits no global weak solution.*

Remark 2.12. Let $N_1 \geq 3$, $N_2 = 2$, $f \in L^1(\partial D_1)$, and $\int_{\partial D_1} f(x)d\sigma_x > 0$. By Theorem 2.11 we deduce that for all $\ell \in \mathbb{R}$, $g \in L^1(\partial D_2)$, and

$$1 < p < \frac{N_1}{N_1 - 2}, \tag{2.7}$$

problem (1.1) admits no global weak solution. On the other hand, for $p > \frac{N_1}{N_1-2}$, we can check easily that

$$u(t,x,y) = A|x|^{-\sigma}, \quad (t,x,y) \in (0,\infty) \times D_1 \times D_2,$$

where $A > 0$ is sufficiently small and $\frac{2}{p-1} < \sigma < N_1 - 2$, is a (stationary) solution to (1.1) with $f \equiv A$ and $g \equiv 0$. This shows that (2.7) is sharp.

In the special case when $\int_{\partial D_1} f(x)d\sigma_x > 0$ and $\int_{\partial D_2} g(y)d\sigma_y > 0$, we deduce from Theorem 2.11 the following results.

Corollary 2.13. *Let* $N_1 \geq 3$, $N_2 = 2$, $f \in L^1(\partial D_1)$, *and* $g \in L^1(\partial D_2)$. *Suppose that*

$$\int_{\partial D_1} f(x)d\sigma_x > 0 \quad \text{and} \quad \int_{\partial D_2} g(y)d\sigma_y > 0.$$

(I) *Let* $\ell \leq -\frac{N_1}{2}$. *Then for all* $1 < p < \frac{N_1}{N_1-2}$, (1.1) *admits no global weak solution.*

(II) *Let* $-\frac{N_1}{2} < \ell < -1$. *Then for all* $1 < p < \frac{\ell}{\ell+1}$, (1.1) *admits no global weak solution.*

(III) *Let* $\ell \geq -1$. *Then for all* $p > 1$, (1.1) *admits no global weak solution.*

Remark 2.14. The nonexistence results given by Theorem 2.11 and Corollary 2.13 hold for the stationary problem (2.3) in the case $N_1 \geq 3$ and $N_2 = 2$.

Theorem 2.15. *Let* $N_1, N_2 \geq 3$, $f \in L^1(\partial D_1)$, *and* $g \in L^1(\partial D_2)$.

(I) *Let* $\ell \leq -\frac{N_1}{2}$.

(i) *If* $\int_{\partial D_1} f(x)d\sigma_x > 0$, *then for all* $1 < p < \frac{N_1}{N_1-2}$, (1.1) *admits no global weak solution.*

(ii) *If*

$$\ell < -\frac{N_1}{2}, \quad \int_{\partial D_1} f(x)d\sigma_x = 0, \quad \text{and} \quad \int_{\partial D_2} g(y)d\sigma_y > 0,$$

then for all $1 < p < \min\{\frac{N_1}{N_1-2}, \frac{N_2}{N_2-2}\}$, (1.1) *admits no global weak solution.*

(iii) *If*

$$\ell = -\frac{N_1}{2}, \quad \int_{\partial D_1} f(x)d\sigma_x = 0, \quad \text{and} \quad \int_{\partial D_2} g(y)d\sigma_y > 0,$$

then for all $1 < p < \min\{\frac{N_1}{N_1-2}, \frac{N_2}{N_2-2}\}$ *or* $p = \frac{N_2}{N_2-2} < \frac{N_1}{N_1-2}$, (1.1) *admits no global weak solution.*

(II) *Let* $-\frac{N_1}{2} < \ell < -1$.

(i) *If* $\int_{\partial D_1} f(x)d\sigma_x > 0$, *then for all* $1 < p < \frac{N_1}{N_1-2}$, (1.1) *admits no global weak solution.*
(ii) *If* $\int_{\partial D_2} g(y)d\sigma_y > 0$, *then for all* $1 < p < \frac{\ell}{\ell+1}$, (1.1) *admits no global weak solution.*

(III) *Let* $-1 \le \ell < 0$.
(i) *If* $\int_{\partial D_1} f(x)d\sigma_x > 0$, *then for all* $1 < p < \frac{N_1}{N_1-2}$, (1.1) *admits no global weak solution.*
(ii) *If* $\int_{\partial D_2} g(y)d\sigma_y > 0$, *then for all* $p > 1$, (1.1) *admits no global weak solution.*

(IV) *Let* $\ell \ge 0$.
(i) *If* $\int_{\partial D_1} f(x)d\sigma_x > 0$, *then for all* $1 < p < \frac{N_1}{N_1-2}$, (1.1) *admits no global weak solution.*
(ii) *If* $\int_{\partial D_2} g(y)d\sigma_y > 0$, *then for all* $1 < p < \frac{N_2}{N_2-2}$, (1.1) *admits no global weak solution.*

Remark 2.16. From Theorem 2.15, if $f \in L^1(\partial D_1)$ and $\int_{\partial D_1} f(x)d\sigma_x > 0$, then for all $\ell \in \mathbb{R}$, $g \in L^1(\partial D_2)$, and $1 < p < \frac{N_1}{N_1-2}$, (1.1) admits no global weak solution. We can check that the above condition is sharp (see Remark 2.12). Similarly, condition (IV)-(ii) is sharp (see Remark 2.8).

In the special case when $\int_{\partial D_1} f(x)d\sigma_x > 0$ and $\int_{\partial D_2} g(y)d\sigma_y > 0$, we deduce from Theorem 2.15 the following results.

Corollary 2.17. *Let* $N_1, N_2 \ge 3$, $f \in L^1(\partial D_1)$, *and* $g \in L^1(\partial D_2)$. *Suppose that*
$$\int_{\partial D_1} f(x)d\sigma_x > 0 \quad \textit{and} \quad \int_{\partial D_2} g(y)d\sigma_y > 0.$$

(I) *If* $\ell \le -\frac{N_1}{2}$, *then for all* $1 < p < \frac{N_1}{N_1-2}$, (1.1) *admits no global weak solution.*
(II) *If* $-\frac{N_1}{2} < \ell < -1$, *then for all* $1 < p < \frac{\ell}{\ell+1}$, (1.1) *admits no global weak solution.*
(III) *If* $-1 \le \ell < 0$, *then for all* $p > 1$, (1.1) *admits no global weak solution.*
(IV) *If* $\ell \ge 0$, *then for all* $1 < p < \max\left\{\frac{N_1}{N_1-2}, \frac{N_2}{N_2-2}\right\}$, (1.1) *admits no global weak solution.*

Remark 2.18. The nonexistence results given by Theorem 2.15 and Corollary 2.17 hold for the stationary problem (2.3) in the case $N_1, N_2 \ge 3$.

3. Preliminaries

Let $N_k \ge 2$, $k = 1, 2$. We introduce the following harmonic function defined in $D_k = \{z \in \mathbb{R}^{N_k} : |z| \ge 1\}$:
$$H_k(z) = \begin{cases} \ln|z| & \text{if } N_k = 2, \\ 1 - |z|^{2-N_k} & \text{if } N_k \ge 3. \end{cases}$$
We introduce two cut-off functions $\eta, \xi \in C^\infty([0,\infty))$ satisfying respectively
$$\eta \ge 0, \quad \eta \not\equiv 0, \quad \operatorname{supp}(\eta) \subset (0,1)$$
and
$$0 \le \xi \le 1, \quad \xi|_{[0,1]} \equiv 1, \quad \xi|_{[2,\infty)} \equiv 0.$$

For sufficiently large R and λ, let

$$\begin{aligned} a(t) &= \eta^\lambda\Big(\frac{t}{R}\Big), \quad t>0,\\ b(x) &= H_1(x)\xi^\lambda\Big(\frac{|x|}{R^\theta}\Big), \quad x\in D_1,\\ c(y) &= H_2(y)\xi^\lambda\Big(\frac{|y|}{R^\sigma}\Big), \quad y\in D_2, \end{aligned}$$

where $\theta,\sigma>0$ are constants to be chosen later. Consider

$$\varphi_R(t,x,y)=a(t)b(x)c(y), \quad (t,x,y)\in\Omega. \tag{3.1}$$

Proposition 3.1. *For sufficiently large R, the function φ_R belongs to the test function space Φ, where Φ is defined by* (2.1).

Proof. Clearly, we have

$$\varphi_R\in C_c^2(\Omega), \quad \varphi_R\geq 0, \quad \varphi_R|_{\partial D_1\cup\partial D_2}=0.$$

On the other hand,

$$\begin{aligned} \nabla_x\varphi_R(t,x,y) &= a(t)c(y)\nabla_x\Big(H_1(x)\xi^\lambda\big(\frac{|x|}{R^\theta}\big)\Big)\\ &= a(t)c(y)\Big[\xi^\lambda\big(\frac{|x|}{R^\theta}\big)\nabla_x H_1(x)+H_1(x)\nabla_x\xi^\lambda\big(\frac{|x|}{R^\theta}\big)\Big]. \end{aligned}$$

By the definition of H_1, for $x\in\partial D_1$, we obtain

$$\nabla_x H_1(x)=\begin{cases} x & \text{if } N_1=2,\\ (N_1-2)x & \text{if } N_1\geq 3.\end{cases}$$

By the properties of the function ξ, for $x\in\partial D_1$, we obtain (since R is sufficiently large)

$$\xi^\lambda\big(\frac{|x|}{R^\theta}\big)=1, \quad \big|\nabla_x\xi^\lambda\big(\frac{|x|}{R^\theta}\big)\big|=0.$$

Hence, for $(t,x,y)\in\Gamma_1$, we deduce that

$$\frac{\partial_x\varphi_R}{\partial n_1}(t,x,y)=\begin{cases} -a(t)c(y) & \text{if } N_1=2\\ -(N_1-2)a(t)c(y) & \text{if } N_1\geq 3\end{cases}\leq 0. \tag{3.2}$$

Similarly, for $(t,x,y)\in\Gamma_2$, we obtain

$$\frac{\partial_y\varphi_R}{\partial n_2}(t,x,y)=\begin{cases} -a(t)b(x) & \text{if } N_1=2\\ -(N_2-2)a(t)b(x) & \text{if } N_1\geq 3.\end{cases}\leq 0. \tag{3.3}$$

This shows that $\varphi_R\in\Phi$. $\square$

The following estimates follow from standard calculations.

Lemma 3.2. (i) *Let $\alpha\in\mathbb{R}$ and $\beta>-1$. As $R\to\infty$, we have*

$$\int_{z\in\mathbb{R}^2:1<|z|<R}|z|^\alpha(\ln|z|)^\beta\,dz\sim\begin{cases} 1 & \text{if } \alpha<-2,\\ (\ln R)^{\beta+1} & \text{if } \alpha=-2,\\ R^{\alpha+2}(\ln R)^\beta & \text{if } \alpha>-2.\end{cases}$$

(ii) *Let $\alpha,\beta\in\mathbb{R}$. As $R\to\infty$, we have*

$$\int_{z\in\mathbb{R}^2:R<|z|<2R}|z|^\alpha(\ln|z|)^\beta\,dz\sim R^{\alpha+2}(\ln R)^\beta.$$

Lemma 3.3. *Let $N \geq 3$.*

(i) *Let $\alpha \in \mathbb{R}$ and $\beta > -1$. As $R \to \infty$, we have*

$$\int_{z\in\mathbb{R}^N:1<|z|<R} |z|^\alpha \left(1-|z|^{2-N}\right)^\beta \,dz \sim \begin{cases} 1 & \text{if } \alpha < -N, \\ \ln R & \text{if } \alpha = -N, \\ R^{\alpha+N} & \text{if } \alpha > -N. \end{cases}$$

(ii) *Let $\alpha, \beta \in \mathbb{R}$. As $R \to \infty$, we have*

$$\int_{z\in\mathbb{R}^N:R<|z|<2R} |z|^\alpha \left(1-|z|^{2-N}\right)^\beta \,dz \sim R^{\alpha+N}.$$

Lemma 3.4. *Let $p > 1$. Then*

(i) $\int_0^\infty a(t)\,dt = CR$.

(ii) $\int_0^\infty a^{\frac{-1}{p-1}}(t)|a''(t)|^{\frac{p}{p-1}}\,dt = O\big(R^{1-\frac{2p}{p-1}}\big)$, *as $R \to \infty$.*

Proof. (i) is immediate, so we omit its proof. On the other hand, we have

$$|a''(t)| \leq CR^{-2}\eta^{\lambda-2}\Big(\frac{t}{R}\Big), \quad t \in (0,R),$$

which yields

$$a^{\frac{-1}{p-1}}(t)|a''(t)|^{\frac{p}{p-1}} \leq CR^{\frac{-2p}{p-1}}\eta^{\lambda-\frac{2p}{P-1}}\Big(\frac{t}{R}\Big), \quad t \in (0,R).$$

Then

$$\begin{aligned} \int_0^\infty a^{\frac{-1}{p-1}}(t)|a''(t)|^{\frac{p}{p-1}}\,dt &\leq CR^{\frac{-2p}{p-1}} \int_0^R \eta^{\lambda-\frac{2p}{P-1}}\Big(\frac{t}{R}\Big)\,dt \\ &= C\Big(\int_0^1 \eta^{\lambda-\frac{2p}{P-1}}(s)\,ds\Big)R^{1-\frac{2p}{p-1}}, \end{aligned}$$

which proves (ii). ∎

Lemma 3.5. *As $R \to \infty$, we have*

$$\int_{D_1} b(x)\,dx = \begin{cases} O\big(R^{2\theta}\ln R\big) & \text{if } N_1 = 2, \\ O\big(R^{\theta N_1}\big) & \text{if } N_1 \geq 3; \end{cases} \tag{3.4}$$

and

$$\int_{D_2} c(y)\,dy = \begin{cases} O\big(R^{2\sigma}\ln R\big) & \text{if } N_2 = 2, \\ O\big(R^{\sigma N_2}\big) & \text{if } N_2 \geq 3. \end{cases} \tag{3.5}$$

Proof. Let $N_1 = 2$. We have

$$\begin{aligned} \int_{D_1} b(x)\,dx &= \int_{|x|>1} H_1(x)\xi^\lambda\big(\frac{|x|}{R^\theta}\big)\,dx \\ &= \int_{1<|x|<2R^\theta} \ln|x|\xi^\lambda\big(\frac{|x|}{R^\theta}\big)\,dx \\ &\leq \int_{1<|x|<2R^\theta} \ln|x|\,dx. \end{aligned}$$

Hence, by Lemma 3.2 (with $\alpha = 0$ and $\beta = 1$), we obtain

$$\int_{D_1} b(x)\,dx \leq CR^{2\theta}\ln R.$$

For $N_1 \geq 3$, we have

$$\int_{D_1} b(x)\,dx \leq \int_{1<|x|<2R^\theta} \left(1-|x|^{2-N_1}\right)\,dx.$$

Using Lemma 3.3 (with $\alpha=0$ and $\beta=1$), we obtain

$$\int_{D_1} b(x)\,dx \leq CR^{\theta N_1}.$$

Therefore, (3.4) is proved. The same argument yields (3.5). □

Lemma 3.6. *As $R\to\infty$, we have*

$$\int_{D_1} b^{\frac{-1}{p-1}}|\Delta_x b|^{\frac{p}{p-1}}\,dx = \begin{cases} O\left(R^{\frac{-2\theta}{p-1}}\ln R\right) & \text{if } N_1=2,\\ O\left(R^{\frac{-2\theta p}{p-1}+\theta N_1}\right) & \text{if } N_1\geq 3; \end{cases} \tag{3.6}$$

and

$$\int_{D_2} c^{\frac{-1}{p-1}}|\Delta_y c|^{\frac{p}{p-1}}\,dx = \begin{cases} O\left(R^{\frac{-2\sigma}{p-1}}\ln R\right) & \text{if } N_2=2,\\ O\left(R^{\frac{-2\sigma p}{p-1}+\sigma N_2}\right) & \text{if } N_2\geq 3. \end{cases} \tag{3.7}$$

Proof. By the properties of the function b, we have

$$\int_{D_1} b^{\frac{-1}{p-1}}|\Delta_x b|^{\frac{p}{p-1}}\,dx = \int_{R^\theta<|x|<2R^\theta} b^{\frac{-1}{p-1}}|\Delta_x b|^{\frac{p}{p-1}}\,dx.$$

Let $N_1=2$. For $R^\theta<|x|<2R^\theta$, we obtain

$$\begin{aligned} \Delta_x b &= \Delta_x\Big((\ln|x|)\xi^\lambda\big(\frac{|x|}{R^\theta}\big)\Big)\\ &= \ln|x|\,\Delta_x\xi^\lambda\big(\frac{|x|}{R^\theta}\big) + 2\nabla_x(\ln|x|)\cdot\nabla_x\xi^\lambda\big(\frac{|x|}{R^\theta}\big)\\ &= \ln|x|\,\Delta_x\xi^\lambda\big(\frac{|x|}{R^\theta}\big) + 2R^{-\theta}\lambda\frac{1}{|x|^2}\xi^{\lambda-1}\big(\frac{|x|}{R^\theta}\big)x\cdot\nabla_x\xi\big(\frac{|x|}{R^\theta}\big), \end{aligned}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^{N_1}$, which yields

$$|\Delta_x b| \leq CR^{-2\theta}\ln|x|\xi^{\lambda-2}\big(\frac{|x|}{R^\theta}\big) + CR^{-\theta}|x|^{-1}\xi^{\lambda-1}\big(\frac{|x|}{R^\theta}\big)$$

and

$$\begin{aligned} &b^{\frac{-1}{p-1}}|\Delta_x b|^{\frac{p}{p-1}}\\ &\leq CR^{\frac{-2\theta p}{p-1}}(\ln|x|)\xi^{\lambda-\frac{2p}{p-1}}\big(\frac{|x|}{R^\theta}\big) + CR^{\frac{-\theta p}{p-1}}|x|^{\frac{-p}{p-1}}(\ln|x|)^{\frac{-1}{p-1}}\xi^{\lambda-\frac{p}{p-1}}\big(\frac{|x|}{R^\theta}\big)\\ &\leq C\Big(R^{\frac{-2\theta p}{p-1}}(\ln|x|) + R^{\frac{-\theta p}{p-1}}|x|^{\frac{-p}{p-1}}(\ln|x|)^{\frac{-1}{p-1}}\Big). \end{aligned}$$

Then, by Lemma 3.2, we deduce that

$$\begin{aligned} &\int_{D_1} b^{\frac{-1}{p-1}}|\Delta_x b|^{\frac{p}{p-1}}\,dx\\ &\leq C\Big(R^{\frac{-2\theta p}{p-1}}\int_{R^\theta<|x|<2R^\theta}\ln|x|\,dx + R^{\frac{-\theta p}{p-1}}\int_{R^\theta<|x|<2R^\theta}|x|^{\frac{-p}{p-1}}(\ln|x|)^{\frac{-1}{p-1}}\,dx\Big)\\ &\leq C\Big(R^{\frac{-2\theta p}{p-1}}R^{2\theta}\ln R + R^{\frac{-\theta p}{p-1}}R^{\theta\left(\frac{p-2}{p-1}\right)}(\ln R)^{\frac{-1}{p-1}}\Big)\\ &\leq CR^{\frac{-2\theta}{p-1}}\ln R. \end{aligned}$$

For $N_1 \geq 3$ and $R^\theta < |x| < 2R^\theta$, proceeding as above, and using Lemma 3.3, we obtain

$$b^{\frac{-1}{p-1}} |\Delta_x b|^{\frac{p}{p-1}} \leq C\Big(R^{\frac{-2\theta p}{p-1}} \left(1 - |x|^{2-N_1}\right) + R^{\frac{-\theta p}{p-1}} |x|^{\frac{(1-N_1)p}{p-1}} \left(1 - |x|^{2-N_1}\right)^{\frac{-1}{p-1}}\Big)$$

and

$$\begin{aligned}\int_{D_1} b^{\frac{-1}{p-1}} |\Delta_x b|^{\frac{p}{p-1}}\, dx &\leq CR^{\frac{-2\theta p}{p-1}} \int_{R^\theta<|x|<2R^\theta} \left(1 - |x|^{2-N_1}\right) dx \\ &\quad + CR^{\frac{-\theta p}{p-1}} \int_{R^\theta<|x|<2R^\theta} |x|^{\frac{(1-N_1)p}{p-1}} \left(1 - |x|^{2-N_1}\right)^{\frac{-1}{p-1}} dx \\ &\leq C\Big(R^{\frac{-2\theta p}{p-1}+\theta N_1} + R^{\frac{-\theta N_1}{p-1}}\Big) \\ &\leq CR^{\frac{-2\theta p}{p-1}+\theta N_1}.\end{aligned}$$

This proves (3.6). Similar calculations yield (3.7). □

The next Lemma follows immediately from Lemmas 3.2 and 3.3.

Lemma 3.7. (i) *Let $N_1 = 2$. As $R \to \infty$, we have*

$$\int_{D_1} |x|^{\frac{2\ell p}{p-1}} b(x)\, dx = \begin{cases} O(1) & \text{if } p(\ell+1) < 1, \\ O\big((\ln R)^2\big) & \text{if } p(\ell+1) = 1, \\ O\big(R^{2\theta(\frac{\ell p}{p-1}+1)} \ln R\big) & \text{if } p(\ell+1) > 1. \end{cases}$$

(ii) *Let $N_1 \geq 3$. As $R \to \infty$, we have*

$$\int_{D_1} |x|^{\frac{2\ell p}{p-1}} b(x)\, dx = \begin{cases} O(1) & \text{if } p(2\ell+N_1) < N_1, \\ O(\ln R) & \text{if } p(2\ell+N_1) = N_1, \\ O\big(R^{\theta(\frac{2\ell p}{p-1}+N_1)}\big) & \text{if } p(2\ell+N_1) > N_1. \end{cases}$$

Lemma 3.8. *As $R \to \infty$, we have*

$$\int_\Omega \varphi_R^{\frac{-1}{p-1}} |(\varphi_R)_{tt}|^{\frac{p}{p-1}}\, dy\, dx\, dt = \begin{cases} O\big(R^{1-\frac{2p}{p-1}+2\theta+2\sigma} (\ln R)^2\big) & \text{if } N_1 = N_2 = 2, \\ O\big(R^{1-\frac{2p}{p-1}+2\theta+\sigma N_2} \ln R\big) & \text{if } N_1 = 2, N_2 \geq 3, \\ O\big(R^{1-\frac{2p}{p-1}+\theta N_1+2\sigma} \ln R\big) & \text{if } N_1 \geq 3, N_2 = 2, \\ O\big(R^{1-\frac{2p}{p-1}+\theta N_1+2\sigma}\big) & \text{if } N_1, N_2 \geq 3. \end{cases}$$

Proof. By (3.1), we obtain

$$\begin{aligned}&\int_\Omega \varphi_R^{\frac{-1}{p-1}} |(\varphi_R)_{tt}|^{\frac{p}{p-1}}\, dy\, dx\, dt \\ &= \Big(\int_0^\infty a^{\frac{-1}{p-1}}(t) |a''(t)|^{\frac{p}{p-1}}\, dt\Big)\Big(\int_{D_1} b(x)\, dx\Big)\Big(\int_{D_2} c(y)\, dy\Big).\end{aligned}$$

Hence, using Lemmas 3.4 and 3.5, the desired estimates follow. □

Lemma 3.9. *As $R \to \infty$, we have*

$$\int_\Omega \varphi_R^{\frac{-1}{p-1}} |\Delta_x \varphi_R|^{\frac{p}{p-1}}\, dy\, dx\, dt = \begin{cases} O\big(R^{1-\frac{2\theta}{p-1}+2\sigma} (\ln R)^2\big) & \text{if } N_1 = N_2 = 2, \\ O\big(R^{1-\frac{2\theta}{p-1}+\sigma N_2} \ln R\big) & \text{if } N_1 = 2, N_2 \geq 3, \\ O\big(R^{1+2\sigma-\frac{2\theta p}{p-1}+\theta N_1} \ln R\big) & \text{if } N_1 \geq 3, N_2 = 2, \\ O\big(R^{1-\frac{2\theta p}{p-1}+\theta N_1+\sigma N_2}\big) & \text{if } N_1, N_2 \geq 3. \end{cases}$$

Proof. By (3.1), we have

$$\begin{aligned}
&\int_\Omega \varphi_R^{\frac{-1}{p-1}}|\Delta_x\varphi_R|^{\frac{p}{p-1}}\,dy\,dx\,dt\\
&=\Big(\int_0^\infty a(t)\,dt\Big)\Big(\int_{D_1} b^{\frac{-1}{p-1}}|\Delta_x b|^{\frac{p}{p-1}}\,dx\Big)\Big(\int_{D_2} c(y)\,dy\Big).
\end{aligned}$$

Using Lemmas 3.4, 3.5, and 3.6, the desired estimates follow. □

Lemma 3.10. *As $R\to\infty$, we have*

$$\int_\Omega |x|^{\frac{2\ell p}{p-1}}|\Delta_y\varphi|^{\frac{p}{p-1}}\varphi^{\frac{-1}{p-1}}\,dy\,dx\,dt=\begin{cases} O\big(R^{1-\frac{2\sigma}{p-1}}(\ln R)A(R)\big) & \text{if } N_1=N_2=2,\\ O\big(R^{1-\frac{2\sigma p}{p-1}+\sigma N_2}A(R)\big) & \text{if } N_1=2, N_2\geq 3,\\ O\big(R^{1-\frac{2\sigma p}{p-1}}(\ln R)B(R)\big) & \text{if } N_1\geq 3, N_2=2,\\ O\big(R^{1-\frac{2\sigma p}{p-1}+\sigma N_2}B(R)\big) & \text{if } N_1,N_2\geq 3,\end{cases}$$

where

$$\begin{gathered}
A(R)=\begin{cases}1 & \text{if } p(\ell+1)<1,\\ (\ln R)^2 & \text{if } p(\ell+1)=1,\\ R^{2\theta(\frac{\ell p}{p-1}+1)}\ln R & \text{if } p(\ell+1)>1;\end{cases}\\
B(R)=\begin{cases}1 & \text{if } p(2\ell+N_1)<N_1,\\ \ln R & \text{if } p(2\ell+N_1)=N_1,\\ R^{\theta(\frac{2\ell p}{p-1}+N_1)} & \text{if } p(2\ell+N_1)>N_1.\end{cases}
\end{gathered}\tag{3.8}$$

Proof. By (3.1), we have

$$\begin{aligned}
&\int_\Omega |x|^{\frac{2\ell p}{p-1}}|\Delta_y\varphi|^{\frac{p}{p-1}}\varphi^{\frac{-1}{p-1}}\,dy\,dx\,dt\\
&=\Big(\int_0^\infty a(t)\,dt\Big)\Big(\int_{D_1}|x|^{\frac{2\ell p}{p-1}}b(x)\,dx\Big)\Big(\int_{D_2} c^{\frac{-1}{p-1}}|\Delta_y c|^{\frac{p}{p-1}}\,dy\Big).
\end{aligned}$$

Hence, using Lemmas 3.4, 3.6, and 3.7, the desired estimates follow. □

Proposition 3.11. *Let $f\in L^1(\partial D_1)$ and $g\in L^1(\partial D_2)$. If $u\in L^p_{\rm loc}([0,\infty)\times D)$ is a global weak solution to* (1.1)*, then for all $\varphi\in\Phi$,*

$$\begin{aligned}
&-\int_{\Gamma_1}\frac{\partial_x\varphi}{\partial n_1}f(x)\,d\sigma_x\,dy\,dt-\int_{\Gamma_2}|x|^{2\ell}\frac{\partial_y\varphi}{\partial n_2}g(y)\,dx\,d\sigma_y\,dt\\
&\leq C\Big(\int_\Omega \varphi^{\frac{-1}{p-1}}\,|\varphi_{tt}|^{\frac{p}{p-1}}\,dy\,dx\,dt+\int_\Omega\varphi^{\frac{-1}{p-1}}|\Delta_x\varphi|^{\frac{p}{p-1}}\,dy\,dx\,dt\\
&\quad+\int_\Omega|x|^{\frac{2\ell p}{p-1}}\varphi^{\frac{-1}{p-1}}|\Delta_y\varphi|^{\frac{p}{p-1}}\,dy\,dx\,dt\Big).
\end{aligned}$$

Proof. Let $u\in L^p_{\rm loc}([0,\infty)\times D)$ be a global weak solution to (1.1). Then by (2.2), for all $\varphi\in\Phi$, we have

$$\begin{aligned}
&\int_\Omega|u|^p\varphi\,dy\,dx\,dt-\int_{\Gamma_1}\frac{\partial_x\varphi}{\partial n_1}f(x)\,d\sigma_x\,dy\,dt-\int_{\Gamma_2}|x|^{2\ell}\frac{\partial_y\varphi}{\partial n_2}g(y)\,dx\,d\sigma_y\,dt\\
&\leq\int_\Omega u\left(\varphi_{tt}-\Delta_x\varphi-|x|^{2\ell}\Delta_y\varphi\right)\,dy\,dx\,dt\\
&\leq\int_\Omega|u||\varphi_{tt}|\,dy\,dx\,dt+\int_\Omega|u||\Delta_x\varphi|\,dy\,dx\,dt+\int_\Omega|x|^{2\ell}|u||\Delta_y\varphi|\,dy\,dx\,dt.
\end{aligned}\tag{3.9}$$

On the other hand, by Young's inequality, we obtain

$$\int_\Omega |u||\varphi_{tt}|\,dy\,dx\,dt \le \frac{1}{3}\int_\Omega |u|^p\varphi\,dy\,dx\,dt + C\int_\Omega \varphi^{\frac{-1}{p-1}}\,|\varphi_{tt}|^{\frac{p}{p-1}}\,dy\,dx\,dt. \tag{3.10}$$

Similarly, we have

$$\int_\Omega |u||\Delta_x\varphi|\,dy\,dx\,dt \le \frac{1}{3}\int_\Omega |u|^p\varphi\,dy\,dx\,dt + C\int_\Omega \varphi^{\frac{-1}{p-1}}|\Delta_x\varphi|^{\frac{p}{p-1}}\,dy\,dx\,dt \tag{3.11}$$

and

$$\begin{aligned}&\int_\Omega |x|^{\frac{2\ell p}{p-1}}\varphi^{\frac{-1}{p-1}}|\Delta_y\varphi|^{\frac{p}{p-1}}\,dy\,dx\,dt\\ &\le \frac{1}{3}\int_\Omega |u|^p\varphi\,dy\,dx\,dt + C\int_\Omega |x|^{\frac{2\ell p}{p-1}}\varphi^{\frac{-1}{p-1}}|\Delta_y\varphi|^{\frac{p}{p-1}}\,dy\,dx\,dt.\end{aligned} \tag{3.12}$$

The desired estimate follows from (3.9), (3.10), (3.11), and (3.12). $\square$

4. Proofs of main results

Lemma 4.1. *Let $N_1 = N_2 = 2$, $f \in L^1(\partial D_1)$, and $g \in L^1(\partial D_2)$. Suppose that $u \in L^p_{\rm loc}([0,\infty)\times D)$ is a global weak solution to* (1.1)*. Then, for sufficiently large R, we have*

$$\begin{aligned}&R^{2\sigma}\ln R\int_{\partial D_1} f(x)d\sigma_x + G(R)\int_{\partial D_2} g(y)d\sigma_y\\ &\le C\Big(R^{-\frac{2p}{p-1}+2\theta+2\sigma}(\ln R)^2 + R^{-\frac{2\theta}{p-1}+2\sigma}(\ln R)^2 + R^{-\frac{2\sigma}{p-1}}(\ln R)A(R)\Big),\end{aligned}$$

where $A(R)$ is given by (3.8) *and*

$$G(R) = \begin{cases} 1 & \text{if } \ell < -1,\\ (\ln R)^2 & \text{if } \ell = -1,\\ R^{2\theta(\ell+1)}\ln R & \text{if } \ell > -1.\end{cases} \tag{4.1}$$

Proof. Let $u \in L^p_{\rm loc}([0,\infty)\times D)$ be a global weak solution to (1.1). By Propositions 3.1 and 3.11, for sufficiently large R, we have

$$\begin{aligned}&-\int_{\Gamma_1}\frac{\partial_x\varphi_R}{\partial n_1}f(x)\,d\sigma_x\,dy\,dt - \int_{\Gamma_2}|x|^{2\ell}\frac{\partial_y\varphi_R}{\partial n_2}g(y)\,dx\,d\sigma_y\,dt\\ &\le C\Big(\int_\Omega \varphi_R^{\frac{-1}{p-1}}\,|(\varphi_R)_{tt}|^{\frac{p}{p-1}}\,dy\,dx\,dt + \int_\Omega \varphi_R^{\frac{-1}{p-1}}|\Delta_x\varphi_R|^{\frac{p}{p-1}}\,dy\,dx\,dt\\ &\quad + \int_\Omega |x|^{\frac{2\ell p}{p-1}}\varphi_R^{\frac{-1}{p-1}}|\Delta_y\varphi_R|^{\frac{p}{p-1}}\,dy\,dx\,dt\Big).\end{aligned} \tag{4.2}$$

On the other hand, by (3.1), (3.2), (3.3), and Lemma 3.4-(i), we obtain

$$\begin{aligned}&-\int_{\Gamma_1}\frac{\partial_x\varphi_r}{\partial n_1}f(x)\,d\sigma_x\,dy\,dt - \int_{\Gamma_2}|x|^{2\ell}\frac{\partial\varphi}{\partial n_2}g(y)\,dx\,d\sigma_y\,dt\\ &= \Big(\int_0^R \eta^\lambda\Big(\frac{t}{R}\Big)\,dt\Big)\Big(\int_{1<|y|<2R^\sigma}\ln|y|\xi^\lambda\big(\frac{|y|}{R^\sigma}\big)\,dy\Big)\Big(\int_{\partial D_1} f(x)d\sigma_x\Big)\\ &\quad + \Big(\int_0^R \eta^\lambda\Big(\frac{t}{R}\Big)\,dt\Big)\Big(\int_{1<|x|<2R^\theta}|x|^{2\ell}\ln|x|\xi^\lambda\big(\frac{|x|}{R^\theta}\big)\,dx\Big)\Big(\int_{\partial D_2} g(y)d\sigma_y\Big)\\ &\ge CR\Big(\int_{1<|y|<2R^\sigma}\ln|y|\xi^\lambda\big(\frac{|y|}{R^\sigma}\big)\,dy\Big)\Big(\int_{\partial D_1} f(x)d\sigma_x\Big)\end{aligned}$$

$$+ CR\Big(\int_{1<|x|<2R^\theta} |x|^{2\ell} \ln|x| \xi^\lambda\big(\frac{|x|}{R^\theta}\big)\,dx\Big)\Big(\int_{\partial D_2} g(y)d\sigma_y\Big).$$

Since

$$\int_{1<|y|<R^\sigma} \ln|y|\,dy \le \int_{1<|y|<2R^\sigma} \ln|y|\xi^\lambda\big(\frac{|y|}{R^\sigma}\big)\,dy \le \int_{1<|y|<2R^\sigma} \ln|y|\,dy,$$

by Lemma 3.2, as $R\to\infty$, it follows that

$$\int_{1<|y|<2R^\sigma} \ln|y|\xi^\lambda\big(\frac{|y|}{R^\sigma}\big)\,dy \sim R^{2\sigma}\ln R.$$

Similarly, since

$$\begin{aligned}\int_{1<|x|<R^\theta} |x|^{2\ell}\ln|x|\,dx &\le \int_{1<|x|<2R^\theta} |x|^{2\ell}\ln|x|\xi^\lambda\big(\frac{|x|}{R^\theta}\big)\,dx\\ &\le \int_{1<|x|<2R^\theta} |x|^{2\ell}\ln|x|\,dx,\end{aligned}$$

by Lemma 3.2, as $R\to\infty$, it follows that

$$\int_{1<|x|<2R^\theta} |x|^{2\ell}\ln|x|\xi^\lambda\big(\frac{|x|}{R^\theta}\big)\,dx \sim G(R). \tag{4.3}$$

Hence, for sufficiently large R, we deduce that

$$\begin{aligned}&-\int_{\Gamma_1} \frac{\partial_x\varphi_r}{\partial n_1} f(x)\,d\sigma_x\,dy\,dt - \int_{\Gamma_2} |x|^{2\ell}\frac{\partial\varphi}{\partial n_2} g(y)\,dx\,d\sigma_y\,dt\\ &\ge CR\Big(R^{2\sigma}\ln R\int_{\partial D_1} f(x)d\sigma_x + G(R)\int_{\partial D_2} g(y)d\sigma_y\Big).\end{aligned} \tag{4.4}$$

Finally, using (4.2), (4.4), and Lemmas 3.8, 3.9, and 3.10, the desired estimate follows. □

Proof of Theorem 2.2. Suppose that $u\in L^p_{\rm loc}([0,\infty)\times D)$ is a global weak solution to (1.1). Let

$$\ell\le -1 \quad\text{and}\quad \int_{\partial D_1} f(x)d\sigma_x > 0.$$

By Lemma 4.1 and (3.8), for sufficiently large R, we obtain

$$\int_{\partial D_1} f(x)d\sigma_x \le C\Big(R^{-\frac{2p}{p-1}+2\theta}\ln R + R^{-\frac{2\theta}{p-1}}\ln R + R^{-\frac{2\sigma}{p-1}-2\sigma}\Big).$$

In particular, for $\theta=1$, we have

$$\int_{\partial D_1} f(x)d\sigma_x \le C\Big(R^{-\frac{2}{p-1}}\ln R + R^{-\frac{2\sigma}{p-1}-2\sigma}\Big).$$

Passing to the limit as $R\to\infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_1} f(x)d\sigma_x>0$. This shows that (1.1) admits no global weak solution for all $p>1$.

Let

$$\ell\le -1,\quad \int_{\partial D_1} f(x)d\sigma_x = 0,\quad\text{and}\quad \int_{\partial D_2} g(y)d\sigma_y > 0.$$

By Lemma 4.1 and (3.8), for sufficiently large R, we obtain

$$\int_{\partial D_2} g(y)d\sigma_y \le C\Big(R^{-\frac{2p}{p-1}+2\theta+2\sigma}(\ln R)^2 + R^{-\frac{2\theta}{p-1}+2\sigma}(\ln R)^2 + R^{-\frac{2\sigma}{p-1}}(\ln R)\Big).$$

Taking $\theta = 1$, $0 < \sigma < \frac{1}{p-1}$, and passing to the limit as $R \to \infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_2} g(y) d\sigma_y > 0$. This shows that (1.1) admits no global weak solution for all $p > 1$. Therefore, part (I) of Theorem 2.2 is proved.

Let

$$\ell > -1 \quad \text{and} \quad \int_{\partial D_1} f(x) d\sigma_x > 0.$$

Using Lemma 4.1 with $\theta = 1$ and $\sigma > 2(\ell+1)$, for sufficiently large R, we obtain

$$\int_{\partial D_1} f(x) d\sigma_x \le C \left(R^{-\frac{2}{p-1}} \ln R + R^{-\frac{2\sigma}{p-1} - 2\sigma} A(R) \right).$$

If $p(\ell+1) \le 1$, by (3.8), we have $A(R) \le (\ln R)^2$. Then

$$\int_{\partial D_1} f(x) d\sigma_x \le C \left(R^{-\frac{2}{p-1}} \ln R + R^{-\frac{2\sigma}{p-1} - 2\sigma} (\ln R)^2 \right).$$

Passing to the limit as $R \to \infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_1} f(x) d\sigma_x > 0$. If $p(\ell+1) > 1$, by (3.8), we have $A(R) = R^{2\left(\frac{\ell p}{p-1}+1\right)} \ln R$. Then

$$\int_{\partial D_1} f(x) d\sigma_x \le C \Big(R^{-\frac{2}{p-1}} \ln R + R^{-\frac{2\sigma}{p-1} - 2\sigma + 2\left(\frac{\ell p}{p-1}+1\right)} \ln R \Big). \tag{4.5}$$

On the other hand, for $\sigma > 2(\ell+1)$, we have

$$-\frac{2\sigma}{p-1} - 2\sigma + 2\Big(\frac{\ell p}{p-1} + 1\Big) < 0.$$

Hence, Passing to the limit as $R \to \infty$ in (4.5), we obtain a contradiction with $\int_{\partial D_1} f(x) d\sigma_x > 0$. Then, we deduce that (1.1) admits no global weak solution for all $p > 1$.

Let

$$\ell > -1 \quad \text{and} \quad \int_{\partial D_2} g(y) d\sigma_y > 0.$$

Using Lemma 4.1 with $\theta = 1$ and $0 < \sigma < \ell + 1$, for sufficiently large R, we obtain

$$R^{2(\ell+1)} \ln R \int_{\partial D_2} g(y) d\sigma_y \le C \Big(R^{-\frac{2}{p-1} + 2\sigma} (\ln R)^2 + R^{-\frac{2\sigma}{p-1}} (\ln R) A(R) \Big),$$

that is,

$$\int_{\partial D_2} g(y) d\sigma_y \le C \Big(R^{-\frac{2}{p-1} + 2\sigma - 2(\ell+1)} \ln R + R^{-\frac{2\sigma}{p-1} - 2(\ell+1)} A(R) \Big).$$

If $p(\ell+1) \le 1$, by (3.8), we have $A(R) \le (\ln R)^2$. Then

$$\int_{\partial D_2} g(y) d\sigma_y \le C \Big(R^{-\frac{2}{p-1} + 2\sigma - 2(\ell+1)} \ln R + R^{-\frac{2\sigma}{p-1} - 2(\ell+1)} (\ln R)^2 \Big).$$

Hence, passing to the limit as $R \to \infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_2} g(y) d\sigma_y > 0$. If $p(\ell+1) > 1$, by (3.8), we have $A(R) = R^{2\left(\frac{\ell p}{p-1}+1\right)} \ln R$. Then

$$\int_{\partial D_2} g(y) d\sigma_y \le C \Big(R^{-\frac{2}{p-1} + 2\sigma - 2(\ell+1)} \ln R + R^{-\frac{2\sigma}{p-1} - 2(\ell+1) + 2\left(\frac{\ell p}{p-1}+1\right)} \ln R \Big). \tag{4.6}$$

Observe that for $\sigma > \ell$,

$$-\frac{2\sigma}{p-1} - 2(\ell+1) + 2\Big(\frac{\ell p}{p-1} + 1\Big) < 0.$$

Hence, for $\max\{0,\ell\} < \sigma < \ell+1$, passing to the limit as $R \to \infty$ in (4.6), we obtain a contradiction with $\int_{\partial D_2} g(y)d\sigma_y > 0$. Therefore, we deduce that (1.1) admits no global weak solution for all $p > 1$. Then part (II) of Theorem 2.2 is proved. $\square$

Lemma 4.2. *Let $N_1 = 2$, $N_2 \geq 3$, $f \in L^1(\partial D_1)$, and $g \in L^1(\partial D_2)$. Suppose that $u \in L^p_{\rm loc}([0,\infty) \times D)$ is a global weak solution to* (1.1)*. Then, for sufficiently large R,*

$$\begin{aligned}
&R^{\sigma N_2} \int_{\partial D_1} f(x)d\sigma_x + G(R) \int_{\partial D_2} g(y)d\sigma_y \\
&\leq C\Big(R^{-\frac{2p}{p-1}+2\theta+\sigma N_2} \ln R + R^{-\frac{2\theta}{p-1}+\sigma N_2} \ln R + R^{-\frac{2\sigma p}{p-1}+\sigma N_2} A(R)\Big),
\end{aligned}$$

where $A(R)$ and $G(R)$ are given respectively by (3.8) *and* (4.1)*.*

Proof. Let $u \in L^p_{\rm loc}([0,\infty) \times D)$ be a global weak solution to (1.1). By (3.1), (3.2), (3.3), (4.3), and Lemma 3.4-(i), for sufficiently large R, we obtain

$$\begin{aligned}
&-\int_{\Gamma_1} \frac{\partial_x \varphi_r}{\partial n_1} f(x)\, d\sigma_x\, dy\, dt - \int_{\Gamma_2} |x|^{2\ell} \frac{\partial \varphi}{\partial n_2} g(y)\, dx\, d\sigma_y\, dt \\
&= \Big(\int_0^R \eta^\lambda\Big(\frac{t}{R}\Big)\, dt\Big)\Big(\int_{1<|y|<2R^\sigma} \big(1-|y|^{2-N_2}\big)\, \xi^\lambda\big(\frac{|y|}{R^\sigma}\big)\, dy\Big)\Big(\int_{\partial D_1} f(x)d\sigma_x\Big) \\
&\quad + \Big(\int_0^R \eta^\lambda\Big(\frac{t}{R}\Big)\, dt\Big)\Big(\int_{1<|x|<2R^\theta} |x|^{2\ell} \ln|x| \xi^\lambda\big(\frac{|x|}{R^\theta}\big)\, dx\Big)\Big(\int_{\partial D_2} g(y)d\sigma_y\Big) \\
&\geq CR\Big(\int_{1<|y|<2R^\sigma} \big(1-|y|^{2-N_2}\big)\, \xi^\lambda\big(\frac{|y|}{R^\sigma}\big)\, dy\Big)\Big(\int_{\partial D_1} f(x)d\sigma_x\Big) \\
&\quad + CRG(R)\int_{\partial D_2} g(y)d\sigma_y.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{1<|y|<R^\sigma} \big(1-|y|^{2-N_2}\big)\, dy &\leq \int_{1<|y|<2R^\sigma} \big(1-|y|^{2-N_2}\big)\, \xi^\lambda\big(\frac{|y|}{R^\sigma}\big)\, dy \\
&\leq \int_{1<|y|<2R^\sigma} \big(1-|y|^{2-N_2}\big)\, dy,
\end{aligned}$$

by Lemma 3.3, as $R \to \infty$, we have

$$\int_{1<|y|<2R^\sigma} \big(1-|y|^{2-N_2}\big)\, \xi^\lambda\big(\frac{|y|}{R^\sigma}\big)\, dy \sim R^{\sigma N_2}.$$

Hence, we deduce that

$$\begin{aligned}
&-\int_{\Gamma_1} \frac{\partial_x \varphi_r}{\partial n_1} f(x)\, d\sigma_x\, dy\, dt - \int_{\Gamma_2} |x|^{2\ell} \frac{\partial \varphi}{\partial n_2} g(y)\, dx\, d\sigma_y\, dt \\
&\geq CR\Big(R^{\sigma N_2} \int_{\partial D_1} f(x)d\sigma_x + G(R)\int_{\partial D_2} g(y)d\sigma_y\Big).
\end{aligned}$$

Finally, using (4.2), Lemmas 3.8, 3.9, and 3.10, the desired estimate follows. $\square$

Proof of Theorem 2.6. Suppose that $u \in L^p_{\rm loc}([0,\infty)\times D)$ is a global weak solution to (1.1). Let

$$\ell \le -1 \quad \text{and} \quad \int_{\partial D_1} f(x)d\sigma_x > 0.$$

By Lemma 4.2, (3.8), and (4.1), for sufficiently large R, we obtain

$$R^{\sigma N_2}\int_{\partial D_1} f(x)d\sigma_x \le C\Big(R^{-\frac{2p}{p-1}+2\theta+\sigma N_2}\ln R + R^{-\frac{2\theta}{p-1}+\sigma N_2}\ln R + R^{-\frac{2\sigma p}{p-1}+\sigma N_2}\Big),$$

that is,

$$\int_{\partial D_1} f(x)d\sigma_x \le C\Big(R^{-\frac{2p}{p-1}+2\theta}\ln R + R^{-\frac{2\theta}{p-1}}\ln R + R^{-\frac{2\sigma p}{p-1}}\Big).$$

Taking $\theta = 1$, we obtain

$$\int_{\partial D_1} f(x)d\sigma_x \le C\Big(R^{-\frac{2\theta}{p-1}}\ln R + R^{-\frac{2\sigma p}{p-1}}\Big).$$

Passing to the limit as $R\to\infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_1} f(x)d\sigma_x > 0$. Hence, for all $p>1$, (1.1) admits no global weak solution. This proves parts (I)-(i) and (II)-(i) of Theorem 2.6. Let

$$\ell > -1 \quad \text{and} \quad \int_{\partial D_1} f(x)d\sigma_x > 0.$$

Using (4.1) and Lemma 4.2 with $\theta = 1$ and $\sigma > \ell+1$ (so $\sigma N_2 > 2(\ell+1)$), for sufficiently large R, we obtain

$$R^{\sigma N_2}\int_{\partial D_1} f(x)d\sigma_x \le C\Big(R^{-\frac{2}{p-1}+\sigma N_2}\ln R + R^{-\frac{2\sigma p}{p-1}+\sigma N_2}A(R)\Big),$$

that is,

$$\int_{\partial D_1} f(x)d\sigma_x \le C\left(R^{-\frac{2}{p-1}}\ln R + R^{-\frac{2\sigma p}{p-1}}A(R)\right).$$

If $p(\ell+1)\le 1$, then by (3.8), we have $A(R)\le (\ln R)^2$. Then

$$\int_{\partial D_1} f(x)d\sigma_x \le C\left(R^{-\frac{2}{p-1}}\ln R + R^{-\frac{2\sigma p}{p-1}}(\ln R)^2\right).$$

Passing to the limit as $R\to\infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_1} f(x)d\sigma_x > 0$. If $p(\ell+1)>1$, then by (3.8), we have $A(R) = R^{2\left(\frac{\ell p}{p-1}+1\right)}\ln R$. Then

$$\int_{\partial D_1} f(x)d\sigma_x \le C\left(R^{-\frac{2}{p-1}}\ln R + R^{-\frac{2\sigma p}{p-1}+2\left(\frac{\ell p}{p-1}+1\right)}\right). \tag{4.7}$$

Notice that for $\sigma > \ell+1$,

$$-\frac{2\sigma p}{p-1} + 2\Big(\frac{\ell p}{p-1}+1\Big) < 0.$$

Hence, passing to the limit as $R\to\infty$ in (4.7), we obtain a contradiction with $\int_{\partial D_1} f(x)d\sigma_x > 0$. Then, we deduce that for all $p>1$, (1.1) admits no global weak solution. This proves part (III) when $\int_{\partial D_1} f(x)d\sigma_x > 0$, and part (IV)-(i).

Let

$$\ell < -1 \quad \text{and} \quad \int_{\partial D_1} f(x)d\sigma_x = 0,\ \int_{\partial D_2} g(y)d\sigma_y > 0.$$

In this case, by Lemma 4.2, (3.8), and (4.1), for sufficiently large R, we obtain

$$\int_{\partial D_2} g(y)d\sigma_y \le C\Big(R^{-\frac{2p}{p-1}+2\theta+\sigma N_2}\ln R + R^{-\frac{2\theta}{p-1}+\sigma N_2}\ln R + R^{-\frac{2\sigma p}{p-1}+\sigma N_2}\Big).$$

In particular, for $\theta = 1$, we have

$$\int_{\partial D_2} g(y)d\sigma_y \le C\Big(R^{-\frac{2}{p-1}+\sigma N_2}\ln R + R^{-\sigma\big(\frac{2p}{p-1}-N_2\big)}\Big).$$

Let $1 < p < \frac{N_2}{N_2-2}$. Taking $0 < \sigma N_2 < \frac{2}{p-1}$, and passing to the limit as $R \to \infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_2} g(y)d\sigma_y > 0$. Hence, for all $1 < p < \frac{N_2}{N_2-2}$, (1.1) admits no global weak solution. This proves part (I)-(ii) of Theorem 2.6.

Let

$$\ell = -1 \quad \text{and} \quad \int_{\partial D_1} f(x)d\sigma_x = 0,\ \int_{\partial D_2} g(y)d\sigma_y > 0.$$

In this case, by Lemma 4.2, (3.8), and (4.1), for sufficiently large R, we obtain

$$(\ln R)^2\int_{\partial D_2} g(y)d\sigma_y \le C\Big(R^{-\frac{2p}{p-1}+2\theta+\sigma N_2}\ln R + R^{-\frac{2\theta}{p-1}+\sigma N_2}\ln R + R^{-\frac{2\sigma p}{p-1}+\sigma N_2}\Big),$$

that is,

$$\begin{aligned}&\int_{\partial D_2} g(y)d\sigma_y\\ &\le C\Big(R^{-\frac{2p}{p-1}+2\theta+\sigma N_2}(\ln R)^{-1} + R^{-\frac{2\theta}{p-1}+\sigma N_2}(\ln R)^{-1} + R^{-\frac{2\sigma p}{p-1}+\sigma N_2}(\ln R)^{-2}\Big).\end{aligned}$$

In particular, for $\theta = 1$, we obtain

$$\int_{\partial D_2} g(y)d\sigma_y \le C\left(R^{-\frac{2}{p-1}+\sigma N_2}(\ln R)^{-1} + R^{\sigma\big(N_2-\frac{2p}{p-1}\big)}(\ln R)^{-2}\right).$$

Let $1 < p \le \frac{N_2}{N_2-2}$. Taking $0 < \sigma N_2 < \frac{2}{p-1}$, and passing to the limit as $R \to \infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_2} g(y)d\sigma_y > 0$. Hence, for all $1 < p \le \frac{N_2}{N_2-2}$, (1.1) admits no global weak solution. This proves part (II)-(ii).

Let

$$-1 < \ell < 0 \quad \text{and} \quad \int_{\partial D_2} g(y)d\sigma_y > 0.$$

By (4.1) and using Lemma 4.2 with $\theta = 1$ and $0 < \sigma N_2 < 2(\ell+1)$, for sufficiently large R, we obtain

$$R^{2(\ell+1)}\ln R\int_{\partial D_2} g(y)d\sigma_y \le C\Big(R^{-\frac{2}{p-1}+\sigma N_2}\ln R + R^{\sigma\big(N_2-\frac{2p}{p-1}\big)}A(R)\Big),$$

that is,

$$\int_{\partial D_2} g(y)d\sigma_y \le C\Big(R^{-\frac{2}{p-1}+\sigma N_2-2(\ell+1)}\ln R + R^{\sigma\big(N_2-\frac{2p}{p-1}\big)-2(\ell+1)}A(R)\Big). \tag{4.8}$$

If $p(\ell+1) \le 1$, by (3.8) we have $A(R) \le (\ln R)^2$. Then

$$\int_{\partial D_2} g(y)d\sigma_y \le C\Big(R^{-\frac{2}{p-1}+\sigma N_2-2(\ell+1)}\ln R + R^{\sigma\big(N_2-\frac{2p}{p-1}\big)-2(\ell+1)}(\ln R)^2\Big).$$

Passing to the limit as $R \to \infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_2} g(y) d\sigma_y > 0$. If $p(\ell+1) > 1$, by (3.8) we have $A(R) = R^{2\left(\frac{\ell p}{p-1}+1\right)} \ln R$. Then

$$\begin{aligned} &\int_{\partial D_2} g(y) d\sigma_y \\ &\le C\Big(R^{-\frac{2}{p-1}+\sigma N_2 - 2(\ell+1)} \ln R + R^{\sigma\left(N_2 - \frac{2p}{p-1}\right) - 2(\ell+1) + 2\left(\frac{\ell p}{p-1}+1\right)} \ln R\Big). \end{aligned} \tag{4.9}$$

Observe that for $0 < \sigma N_2 < \frac{-2\ell}{p-1}$ (so $0 < \sigma N_2 < \min\{2(\ell+1), \frac{-2\ell}{p-1}\}$), we have

$$\sigma\Big(N_2 - \frac{2p}{p-1}\Big) - 2(\ell+1) + 2\Big(\frac{\ell p}{p-1}+1\Big) < \sigma N_2 - 2(\ell+1) + 2\Big(\frac{\ell p}{p-1}+1\Big) < 0.$$

Hence, passing to the limit as $R \to \infty$ in (4.9), we obtain a contradiction with $\int_{\partial D_2} g(y) d\sigma_y > 0$. Consequently, (1.1) admits no global weak solution for all $p > 1$. This proves part (III) in the case $\int_{\partial D_2} g(y) d\sigma_y > 0$.

Let

$$\ell \ge 0 \quad \text{and} \quad \int_{\partial D_2} g(y) d\sigma_y > 0.$$

As previously, by (4.1) and using Lemma 4.2 with $\theta = 1$ and $0 < \sigma N_2 < 2(\ell+1)$, for sufficiently large R, we obtain (4.8). Moreover, since $\ell \ge 0$ and $p(\ell+1) \ge p > 1$, by (3.8) we have $A(R) = R^{2\left(\frac{\ell p}{p-1}+1\right)} \ln R$, and (4.9) holds. Observe that for all $1 < p < \frac{N_2}{N_2-2}$,

$$\sigma\Big(N_2 - \frac{2p}{p-1}\Big) - 2(\ell+1) + 2\Big(\frac{\ell p}{p-1}+1\Big) < 0.$$

Hence, passing to the limit as $R \to \infty$ in (4.9), we obtain a contradiction with $\int_{\partial D_2} g(y) d\sigma_y > 0$. Consequently, for all $1 < p < \frac{N_2}{N_2-2}$, (1.1) admits no global weak solution. This proves part (IV)-(ii). The proof of Theorem 2.6 is complete. □

Case $N_1 \ge 3$ **and** $N_2 = 2$**.** Proceeding as in the proofs of Lemmas 4.1 and 4.2, we obtain the following estimate.

Lemma 4.3. *Let $N_1 \ge 3$, $N_2 = 2$, $f \in L^1(\partial D_1)$, and $g \in L^1(\partial D_2)$. Suppose that $u \in L^p_{\rm loc}([0,\infty) \times D)$ is a global weak solution to* (1.1)*. Then, for sufficiently large R,*

$$\begin{aligned} &R^{2\sigma} \ln R \int_{\partial D_1} f(x) d\sigma_x + \mathcal{G}(R) \int_{\partial D_2} g(y) d\sigma_y \\ &\le C\Big(R^{-\frac{2p}{p-1}+\theta N_1 + 2\sigma} \ln R + R^{2\sigma - \frac{2\theta p}{p-1} + \theta N_1} \ln R + R^{-\frac{2\sigma p}{p-1}} (\ln R) B(R)\Big), \end{aligned}$$

where $B(R)$ is given by (3.8) *and*

$$\mathcal{G}(R) = \begin{cases} 1 & \text{if } \ell < -\frac{N_1}{2}, \\ \ln R & \text{if } \ell = -\frac{N_1}{2}, \\ R^{\theta(2\ell+N_1)} & \text{if } \ell > -\frac{N_1}{2}. \end{cases} \tag{4.10}$$

Proof of Theorem 2.11. Suppose that $u \in L^p_{\rm loc}([0,\infty) \times D)$ is a global weak solution to (1.1). Let

$$\ell \le -\frac{N_1}{2} \quad \text{and} \quad \int_{\partial D_1} f(x) d\sigma_x > 0.$$

Then, by Lemma 4.3, for sufficiently large R, we obtain

$$\begin{aligned}&R^{2\sigma}\ln R\int_{\partial D_1} f(x)d\sigma_x\\&\le C\Big(R^{-\frac{2p}{p-1}+\theta N_1+2\sigma}\ln R+R^{2\sigma-\frac{2\theta p}{p-1}+\theta N_1}\ln R+R^{-\frac{2\sigma p}{p-1}}(\ln R)B(R)\Big),\end{aligned}$$

that is,

$$\int_{\partial D_1} f(x)d\sigma_x\le C\left(R^{-\frac{2p}{p-1}+\theta N_1}+R^{-\frac{2\theta p}{p-1}+\theta N_1}+R^{-\frac{2\sigma p}{p-1}-2\sigma}B(R)\right).$$

In particular, for $\theta=1$, we obtain

$$\int_{\partial D_1} f(x)d\sigma_x\le C\left(R^{-\frac{2p}{p-1}+N_1}+R^{-\frac{2\sigma p}{p-1}-2\sigma}B(R)\right). \tag{4.11}$$

Notice that in this case, $p(2\ell+N_1)\le 0<N_1$. Hence, by (3.8) we obtain

$$\int_{\partial D_1} f(x)d\sigma_x\le C\left(R^{-\frac{2p}{p-1}+N_1}+R^{-\frac{2\sigma p}{p-1}-2\sigma}\right).$$

For $1<p<\frac{N_1}{N_1-2}$, passing to the limit as $R\to\infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_1} f(x)d\sigma_x>0$. Consequently, (1.1) admits no global weak solution for all $1<p<\frac{N_1}{N_1-2}$. This proves part (I) of Theorem 2.11 when $\int_{\partial D_1} f(x)d\sigma_x>0$.

Let

$$\ell>-\frac{N_1}{2}\quad\text{and}\quad\int_{\partial D_1} f(x)d\sigma_x>0.$$

In this case, using Lemma 4.3 with $\theta=1$ and $2\sigma>2\ell+N_1$, for sufficiently large R, we obtain (4.11). Let $1<p<\frac{N_1}{N_1-2}$. If $p(2\ell+N_1)\le N_1$, by (3.8) and (4.11) we have $B(R)\le\ln R$ and

$$\int_{\partial D_1} f(x)d\sigma_x\le C\left(R^{-\frac{2p}{p-1}+N_1}+R^{-\frac{2\sigma p}{p-1}-2\sigma}\ln R\right).$$

Then, passing to the limit as $R\to\infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_1} f(x)d\sigma_x>0$. If $p(2\ell+N_1)>N_1$, by (3.8) we have $B(R)=R^{\frac{2\ell p}{p-1}+N_1}$. Then

$$\int_{\partial D_1} f(x)d\sigma_x\le C\Big(R^{-\frac{2p}{p-1}+N_1}+R^{-\frac{2\sigma p}{p-1}-2\sigma+\frac{2\ell p}{p-1}+N_1}\Big).$$

Taking $2\sigma>\frac{2\ell p}{p-1}+N_1$ (so $2\sigma>\max\{2\ell+N_1,\frac{2\ell p}{p-1}+N_1\}$) and passing to the limit as $R\to\infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_1} f(x)d\sigma_x>0$. Consequently, (1.1) admits no global weak solution for all $1<p<\frac{N_1}{N_1-2}$. This proves parts (II)-(i) and (III)-(i).

Let

$$\ell\le-\frac{N_1}{2}\quad\text{and}\quad\int_{\partial D_1} f(x)d\sigma_x=0,\ \int_{\partial D_2} g(y)d\sigma_y>0.$$

By Lemma 4.3 and (3.8), for sufficiently large R, we obtain

$$\int_{\partial D_2} g(y)d\sigma_y\le C\left(R^{-\frac{2p}{p-1}+\theta N_1+2\sigma}\ln R+R^{2\sigma-\frac{2\theta p}{p-1}+\theta N_1}\ln R+R^{-\frac{2\sigma p}{p-1}}\ln R\right).$$

In particular, for $\theta = 1$, we have

$$\int_{\partial D_2} g(y)d\sigma_y \leq C\Big(R^{-\frac{2p}{p-1}+N_1+2\sigma}\ln R + R^{-\frac{2\sigma p}{p-1}}\ln R\Big).$$

Hence, for $1 < p < \frac{N_1}{N_1-2}$, taking $0 < 2\sigma < \frac{2p}{p-1} - N_1$ and passing to the limit as $R \to \infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_1} g(y)d\sigma_y > 0$. Consequently, (1.1) admits no global weak solution for all $1 < p < \frac{N_1}{N_1-2}$. This proves part (I) of Theorem 2.11 when $\int_{\partial D_1} f(x)d\sigma_x = 0$ and $\int_{\partial D_2} g(y)d\sigma_y > 0$.

Let

$$-\frac{N_1}{2} < \ell < -1 \quad \text{and} \quad \int_{\partial D_2} g(y)d\sigma_y > 0.$$

Using Lemma 4.3 with $\theta = 1$ and $0 < 2\sigma < 2\ell + N_1$, for sufficiently large R, we obtain

$$R^{2\ell+N_1}\int_{\partial D_2} g(y)d\sigma_y \leq C\left(R^{-\frac{2p}{p-1}+N_1+2\sigma}\ln R + R^{-\frac{2\sigma p}{p-1}}(\ln R)B(R)\right),$$

that is

$$\int_{\partial D_2} g(y)d\sigma_y \leq C\left(R^{-\frac{2p}{p-1}+2\sigma-2\ell}\ln R + R^{-\frac{2\sigma p}{p-1}-2\ell-N_1}(\ln R)B(R)\right). \tag{4.12}$$

Let $1 < p < \frac{\ell}{\ell+1}$. If $p(2\ell + N_1) \leq N_1$, by (3.8) we have $B(R) \leq \ln R$. Then

$$\int_{\partial D_2} g(y)d\sigma_y \leq C\left(R^{-\frac{2p}{p-1}+2\sigma-2\ell}\ln R + R^{-\frac{2\sigma p}{p-1}-2\ell-N_1}(\ln R)^2\right). \tag{4.13}$$

Taking $0 < \sigma < \ell + \frac{p}{p-1}$ (so $0 < 2\sigma < \min\{2\ell + N_1, 2(\ell + \frac{p}{p-1})\}$) and passing to the limit as $R \to \infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_2} g(y)d\sigma_y > 0$. If $p(2\ell + N_1) > N_1$, by (3.8) we have $B(R) = R^{\frac{2\ell p}{p-1}+N_1}$. Then

$$\int_{\partial D_2} g(y)d\sigma_y \leq C\left(R^{-\frac{2p}{p-1}+2\sigma-2\ell}\ln R + R^{-\frac{2\sigma p}{p-1}-2\ell+\frac{2\ell p}{p-1}}(\ln R)\right). \tag{4.14}$$

Taking $0 < \sigma < \ell + \frac{p}{p-1}$ and passing to the limit as $R \to \infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_2} g(y)d\sigma_y > 0$. Hence, we deduce that (1.1) admits no global weak solution for all $1 < p < \frac{\ell}{\ell+1}$. This proves part (II)-(ii) of Theorem 2.11.

Let

$$\ell \geq -1 \quad \text{and} \quad \int_{\partial D_2} g(y)d\sigma_y > 0.$$

As in the previous case, using Lemma 4.3 with $\theta = 1$ and $0 < 2\sigma < 2\ell + N_1$, for sufficiently large R, we obtain (4.12). If $p(2\ell+N_1) \leq N_1$, by (3.8) we obtain (4.13). Notice that in this case, $\ell + \frac{p}{p-1} > 0$. So, taking $0 < \sigma < \ell + \frac{p}{p-1}$ and passing to the limit as $R \to \infty$ in (4.13), we obtain a contradiction with $\int_{\partial D_2} g(y)d\sigma_y > 0$. If $p(2\ell + N_1) > N_1$, we obtain (4.14), and the same conclusion as above follows. Consequently, (1.1) admits no global weak solution for all $p > 1$. This proves part (III)-(ii). The proof of Theorem 2.11 is complete. □

Case $N_1, N_2 \geq 3$. Proceeding as in the proofs of Lemmas 4.1 and 4.2, we obtain the following estimate.

Lemma 4.4. *Let $N_1, N_2 \geq 3$, $f \in L^1(\partial D_1)$, and $g \in L^1(\partial D_2)$. Suppose that $u \in L^p_{\rm loc}([0,\infty)\times D)$ is a global weak solution to* (1.1). *Then, for sufficiently large R,*

$$\begin{aligned} &R^{\sigma N_2}\Big(\int_{\partial D_1} f(x)d\sigma_x\Big)+\mathcal{G}(R)\Big(\int_{\partial D_2} g(y)d\sigma_y\Big)\\ &\leq C\left(R^{-\frac{2p}{p-1}+\theta N_1+\sigma N_2}+R^{-\frac{2\theta p}{p-1}+\theta N_1+\sigma N_2}+R^{-\frac{2\sigma p}{p-1}+\sigma N_2}B(R)\right), \end{aligned}$$

where $B(R)$ and $\mathcal{G}(R)$ are given respectively by (3.8) *and* (4.10).

Proof of Theorem 2.15. Suppose that $u \in L^p_{\rm loc}([0,\infty)\times D)$ is a global weak solution to (1.1). Let

$$\ell \leq -\frac{N_1}{2} \quad \text{and} \quad \int_{\partial D_1} f(x)d\sigma_x > 0.$$

Then by Lemma 4.4, (3.8), and (4.10), for sufficiently large R, we obtain

$$\int_{\partial D_1} f(x)d\sigma_x \leq C\left(R^{-\frac{2p}{p-1}+\theta N_1}+R^{-\frac{2\theta p}{p-1}+\theta N_1}+R^{-\frac{2\sigma p}{p-1}}\right).$$

In particular, for $\theta=1$, we have

$$\int_{\partial D_1} f(x)d\sigma_x \leq C\left(R^{-\frac{2p}{p-1}+N_1}+R^{-\frac{2\sigma p}{p-1}}\right).$$

Hence, for $1<p<\frac{N_1}{N_1-2}$, passing to the limit as $R\to\infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_1} f(x)d\sigma_x>0$. Therefore, (1.1) admits no global weak solution for all $1<p<\frac{N_1}{N_1-2}$. This proves part (I)-(i) of Theorem 2.15.

Let

$$\ell > -\frac{N_1}{2} \quad \text{and} \quad \int_{\partial D_1} f(x)d\sigma_x > 0.$$

In this case, using Lemma 4.4 with $\theta=1$ and $\sigma N_2 > 2\ell+N_1$, by (4.10), for sufficiently large R, we obtain

$$\int_{\partial D_1} f(x)d\sigma_x \leq C\left(R^{-\frac{2p}{p-1}+N_1}+R^{-\frac{2\sigma p}{p-1}}B(R)\right).$$

Let $1<p<\frac{N_1}{N_1-2}$. If $p(2\ell+N_1)\leq N_1$, by (3.8) we have $B(R)\leq \ln R$. Then

$$\int_{\partial D_1} f(x)d\sigma_x \leq C\left(R^{-\frac{2p}{p-1}+N_1}+R^{-\frac{2\sigma p}{p-1}}\ln R\right).$$

Passing to the limit as $R\to\infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_1} f(x)d\sigma_x>0$. If $p(2\ell+N_1)>N_1$, by (3.8) we have $B(R)=R^{\frac{2\ell p}{p-1}+N_1}$. Then

$$\int_{\partial D_1} f(x)d\sigma_x \leq C\Big(R^{-\frac{2p}{p-1}+N_1}+R^{-\frac{2\sigma p}{p-1}+\frac{2\ell p}{p-1}+N_1}\Big).$$

Taking $\sigma > \ell+\frac{N_1(p-1)}{2p}$ (so $\sigma N_2 > \max\{2\ell+N_1, N_2(\ell+\frac{N_1(p-1)}{2p})\}$ and passing to the limit as $R\to\infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_1} f(x)d\sigma_x>0$. Then, we deduce that (1.1) admits no global weak solution for all $1<p<\frac{N_1}{N_1-2}$. This proves parts (II)-(i), (III)-(i), and (IV)-(i) of Theorem 2.15.

Let

$$\ell < -\frac{N_1}{2} \quad \text{and} \quad \int_{\partial D_1} f(x)d\sigma_x = 0, \int_{\partial D_2} g(y)d\sigma_y > 0.$$

In this case, using Lemma 4.4, (3.8), and (4.10), for sufficiently large R, we obtain

$$\int_{\partial D_2} g(y)d\sigma_y \le C\Big(R^{-\frac{2p}{p-1}+\theta N_1+\sigma N_2} + R^{-\frac{2\theta p}{p-1}+\theta N_1+\sigma N_2} + R^{-\frac{2\sigma p}{p-1}+\sigma N_2}\Big).$$

In particular, for $\theta = 1$ we have

$$\int_{\partial D_2} g(y)d\sigma_y \le C\Big(R^{-\frac{2p}{p-1}+N_1+\sigma N_2} + R^{-\frac{2\sigma p}{p-1}+\sigma N_2}\Big).$$

Hence, for $1 < p < \min\{\frac{N_1}{N_1-2}, \frac{N_2}{N_2-2}\}$, taking $0 < \sigma N_2 < \frac{2p}{p-1} - N_1$ and passing to the limit as $R \to \infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_2} g(y)d\sigma_y > 0$. Consequently, (1.1) admits no global weak solution for all $1 < p < \min\{\frac{N_1}{N_1-2}, \frac{N_2}{N_2-2}\}$. This proves part (I)-(ii) of Theorem 2.15.

Let

$$\ell = -\frac{N_1}{2} \quad \text{and} \quad \int_{\partial D_1} f(x)d\sigma_x = 0, \int_{\partial D_2} g(y)d\sigma_y > 0.$$

Using Lemma 4.4 with $\theta = 1$, (3.8), and (4.10), for sufficiently large R, we obtain

$$\ln R \int_{\partial D_2} g(y)d\sigma_y \le C\Big(R^{-\frac{2p}{p-1}+N_1+\sigma N_2} + R^{-\frac{2\sigma p}{p-1}+\sigma N_2}\Big),$$

that is,

$$\int_{\partial D_2} g(y)d\sigma_y \le C\Big(R^{-\frac{2p}{p-1}+N_1+\sigma N_2}(\ln R)^{-1} + R^{-\frac{2\sigma p}{p-1}+\sigma N_2}(\ln R)^{-1}\Big).$$

Hence, for $1 < p < \min\{\frac{N_1}{N_1-2}, \frac{N_2}{N_2-2}\}$ or $p = \frac{N_2}{N_2-2} < \frac{N_1}{N_1-2}$, taking $0 < \sigma N_2 \le \frac{2p}{p-1} - N_1$ and passing to the limit as $R \to \infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_2} g(y)d\sigma_y > 0$. Therefore, (1.1) admits no global weak solution for all $1 < p < \min\{\frac{N_1}{N_1-2}, \frac{N_2}{N_2-2}\}$ or $p = \frac{N_2}{N_2-2} < \frac{N_1}{N_1-2}$. This proves part (I)-(iii).

Let

$$-\frac{N_1}{2} < \ell < -1 \quad \text{and} \quad \int_{\partial D_2} g(y)d\sigma_y > 0.$$

In this case, using (4.10) and Lemma 4.4 with $\theta = 1$ and $0 < \sigma N_2 < 2\ell + N_1$, for sufficiently large R, we obtain

$$R^{2\ell+N_1} \int_{\partial D_2} g(y)d\sigma_y \le C\left(R^{-\frac{2p}{p-1}+N_1+\sigma N_2} + R^{-\frac{2\sigma p}{p-1}+\sigma N_2}B(R)\right),$$

that is,

$$\int_{\partial D_2} g(y)d\sigma_y \le C\Big(R^{-\frac{2p}{p-1}+\sigma N_2-2\ell} + R^{-\frac{2\sigma p}{p-1}+\sigma N_2-2\ell-N_1}B(R)\Big).$$

Let $1 < p < \frac{\ell}{\ell+1}$. If $p(2\ell + N_1) \le N_1$, by (3.8) we have $B(R) \le \ln R$. Then

$$\int_{\partial D_2} g(y)d\sigma_y \le C\Big(R^{-\frac{2p}{p-1}+\sigma N_2-2\ell} + R^{-\frac{2\sigma p}{p-1}+\sigma N_2-2\ell-N_1}\ln R\Big). \tag{4.15}$$

Taking $0<\sigma N_2<2(\ell+\frac{p}{p-1})$ (so $0<\sigma N_2<\min\{2\ell+N_1,2(\ell+\frac{p}{p-1})\}$) and passing to the limit as $R\to\infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_2} g(y)d\sigma_y>0$. If $p(2\ell+N_1)>N_1$, by (3.8) we have $B(R)=R^{\frac{2\ell p}{p-1}+N_1}$. Then

$$\int_{\partial D_2} g(y)d\sigma_y\le C\left(R^{-\frac{2p}{p-1}+\sigma N_2-2\ell}+R^{-\frac{2\sigma p}{p-1}+\sigma N_2-2\ell+\frac{2\ell p}{p-1}}\right). \tag{4.16}$$

Similarly, taking $0<\sigma N_2<2(\ell+\frac{p}{p-1})$ (so $0<\sigma N_2<\min\{2\ell+N_1,2(\ell+\frac{p}{p-1})\}$) and passing to the limit as $R\to\infty$ in the above inequality, we obtain a contradiction with $\int_{\partial D_2} g(y)d\sigma_y>0$. Hence, (1.1) admits no global weak solution for all $1<p<\frac{\ell}{\ell+1}$. This proves part (II)-(ii).

Let

$$-1\le\ell<0\quad\text{and}\quad\int_{\partial D_2} g(y)d\sigma_y>0.$$

We use Lemma 4.4 with $\theta=1$ and $0<\sigma N_2<2\ell+N_1$. Proceeding as in the previous case, if $p(2\ell+N_1)\le N_1$, for sufficiently large R, we obtain (4.15). Notice that since $\ell\ge-1$, one has $\ell+\frac{p}{p-1}>0$. Hence, taking $0<\sigma N_2<2(\ell+\frac{p}{p-1})$ (so $0<\sigma N_2<\min\{2\ell+N_1,2(\ell+\frac{p}{p-1})\}$) and passing to the limit as $R\to\infty$ in (4.15), we obtain a contradiction with $\int_{\partial D_2} g(y)d\sigma_y>0$. If $p(2\ell+N_1)>N_1$, then for sufficiently large R, (4.16) holds. Taking $0<\sigma N_2<-\frac{2\ell}{p-1}=\min\{-\frac{2\ell}{p-1},2(\ell+\frac{p}{p-1}),2\ell+N_1\}$ and passing to the limit as $R\to\infty$ in (4.16), the same conclusion follows. Consequently, (1.1) admits no global weak solution for all $p>1$. This proves part (III)-(ii).

Let

$$\ell\ge0\quad\text{and}\quad\int_{\partial D_2} g(y)d\sigma_y>0.$$

Using (3.8), (4.10), and Lemma 4.4 with $\theta=1$ and $0<\sigma N_2<2\ell+N_1$, for sufficiently large R, we obtain (4.16). For $1<p<\frac{N_2}{N_2-2}$, taking $0<\sigma N_2<2(\ell+\frac{p}{p-1})$ (so $0<\sigma N_2<\min\{2\ell+N_1,2(\ell+\frac{p}{p-1})\}$) and passing to the limit as $R\to\infty$ in (4.16), we obtain a contradiction with $\int_{\partial D_2} g(y)d\sigma_y>0$. Hence, (1.1) admits no global weak solution for all $1<p<\frac{N_2}{N_2-2}$. This proves part (IV)-(ii). The proof of Theorem 2.15 is complete. □

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