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INFINITELY MANY RADIAL SOLUTIONS FOR A SUB-SUPER CRITICAL DIRICHLET BOUNDARY VALUE PROBLEM IN A BALL

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Abstract. We prove the existence of infinitely many solutions to a semilinear Dirichlet boundary value problem in a ball for a nonlinearity $g(u)$ that grows subcritically for u positive and supercritically for u negative.

1. Introduction

In this paper we consider the *sub-super critical* boundary-value problem

$$\begin{gathered}\Delta u + g(u(x)) = 0, \quad x \in \mathbb{R}^N,\ \|x\| \le 1 \\ u(x) = 0 \quad \text{for } \|x\| = 1,\end{gathered} \tag{1.1}$$

where

$$g(u) = \begin{cases} u^p, & u \ge 0 \\ |u|^{q-1}u, & u < 0, \end{cases} \tag{1.2}$$

with

$$1 < p < \frac{N+2}{N-2} < q < \infty, \tag{1.3}$$

that is, g has subcritical growth for $u > 0$ and supercritical growth for $u < 0$. Our results hold for more general nonlinearities. For example, it is easy to see that (1.2) may be replaced by $\lim_{u\to+\infty} g(r,u)/u^p \in (0,\infty)$ and $\lim_{u\to-\infty} g(r,u)/(|u|^{q-1}u) \in (0,\infty)$, uniformly for $r \in [0,1]$.

Our main result is as follows.

Theorem 1.1. *Problem* (1.1) *has infinitely many radial solutions.*

This theorem extends the results of [4] where it was established that if $1 < p < (N+1)/(N-1)$ and $q > 1$, or $p, q \in (1, (N+2)/(N-2))$, or $p \in (1, (N+2)/(N-2))$ and $q = (N+2)/(N-2)$, then (1.1) has infinitely many radial solutions. This result is optimal in the sense that if $p, q \in [(N+2)/(N-2), \infty)$ then $u = 0$ is the only solution to (1.1) (see [12]). For related results for quasilinear equations the reader is referred to [8] and [10]. Studies on positive solutions for sub-super critical problems may be found in [9]. For other studies on the critical case, $p = q = (N+2)/(N-2)$

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and $\lim_{|u|\to\infty}/(u|u|^{p-1}) \in \mathbb{R}$, see [1, 3, 4, 6, 7]. In [2] the reader can find a complete classification of the radial solutions to (1.1) for $1 < p = q < (N+2)/(N-2)$. For a recent survey of radial solutions for elliptic boundary-value problems that includes the case where the Laplacian operator is replaced by the more general *k-Hessian* operator, see [11].

Radial solutions to (1.1) are the solutions to the singular ordinary differential equation

$$\begin{gathered} u'' + \frac{n}{t}u' + g(u(t)) = 0 \\ u'(0) = u(1) = 0, \end{gathered} \tag{1.4}$$

where, and henceforth, $n = N - 1$.

For $d > 0$ let $u(t, d)$ be the solution to the initial-value problem

$$\begin{gathered} u'' + \frac{n}{t}u' + g(u(t)) = 0 \\ u(0) = d,\ u'(0) = 0. \end{gathered} \tag{1.5}$$

We define the *energy* function

$$E(t, d) \equiv \frac{(u'(t, d))^2}{2} + G(u(t, d)), \tag{1.6}$$

where $G(u) = \int_0^u g(s)ds$. For future reference we note that

$$\frac{dE}{dt}(t) = -\frac{n}{t}(u'(t))^2 \leq 0. \tag{1.7}$$

The proof of Theorem 1.1 is based on the properties of the energy and the *argument* function defined below (see (1.9)).

Theorem 1.2. *There exists $D > 0$ such that if $d \geq D$, then*

$$t^{N-1}\Big(tE(t) + \frac{N-2}{2}u(t)u'(t)\Big) \geq cd^{\xi} \quad \text{for all } t \geq \sqrt{N}d^{(1-p)/2}, \tag{1.8}$$

where $\xi = \frac{N+2-p(N-2)}{2}$. Also $u(t) \geq d/2$ for $t \in [0, \sqrt{N}d^{(1-p)/2}]$.

As a consequence of Theorem 1.2 we see that, for $d \geq D$, $\rho(t, d) \equiv u^2(t, d) + (u'(t, d))^2 > 0$ for all $t \in [0, 1]$. Hence there exists a continuous function $\theta : [0, 1] \times [D, \infty) \to \mathbb{R}$ such that

$$u(t, d) = \rho(t, d)\cos(\theta(t, d)) \quad \text{and} \quad u'(t, d) = -\rho(t, d)\sin(\theta(t, d)). \tag{1.9}$$

In section 7 we prove that

$$\lim_{d\to\infty} \theta(1, d) = +\infty, \tag{1.10}$$

see (7.7) below.

2. First zero

Let $d > 0$ and $t_0 > 0$ be such that $u(t_0) = d/2$, and $u(t) > d/2$ for $t \in (0, t_0)$. Following the arguments in [4], based on

$$-u'(t) = t^{-n}\int_0^t s^n g(u(s))ds, \tag{2.1}$$

it is easily seen that

$$\sqrt{N}d^{(1-p)/2} \leq t_0 \leq \sqrt{2^p N}d^{(1-p)/2}. \tag{2.2}$$

Multiplying (1.5) by $r^{N-1}u$ and integrating on $[s,t]$, then multiplying the same equation by $r^N u'$ and integrating also on $[s,t]$ one has the following identity, known as Pohozaev's identity,

$$t^n H(t) = s^n H(s) + \int_s^t r^n \Big(NG(u(r)) - \frac{N-2}{2} u(r) g(u(r)) \Big)\, dr, \tag{2.3}$$

where $H(x) \equiv xE(x) + \frac{N-2}{2}\, u'(x)\, u(x)$. In particular, taking $s=0$ and $t=t_0$ we have

$$t_0^n H(t_0) \geq \frac{t_0^N \gamma d^{p+1}}{2^{p+1}N} \quad \geq \frac{N^{(N-2)/2}\gamma}{2^{p+1}} d^{\xi} \equiv c_1 d^{\xi}, \tag{2.4}$$

where $\gamma = N/(p+1) - (N-2)/2$ and ξ is as in (1.8). Also, from (2.3), if $u(r) \geq 0$ for all $r \in [0,t]$ we have

$$t^N (u'(t))^2 = -(N-2)u \cdot t^n u' - 2t^N \frac{u^{p+1}}{p+1} + 2\int_0^t \gamma r^n u^{p+1}\, dr. \tag{2.5}$$

Thus from (2.5) and the fact that $t^{-n}\int_0^t s^n u^{p+1}\, ds \geq -u(t)u'(t)$ we have

$$\begin{aligned}
\Big(\frac{-tu'}{u}\Big)' &= \frac{(-tu'' - u')u + t(u')^2}{u^2} \\
&= \frac{-t(-\frac{n}{t}u' - u^p)u - uu' + t(u')^2}{u^2} \\
&= \frac{\overbrace{(n-1)}^{N-2} uu' + tu^{p+1} + t(u')^2}{u^2} \\
&= \frac{2t^{-n}\int_0^t s^n \gamma u^{p+1}\, ds - 2\frac{tu^{p+1}}{p+1} + tu^{p+1}}{u^2} \\
&= \frac{2t^{-n}\int_0^t s^n \gamma u^{p+1}\, ds + t\big(\frac{p-1}{p+1}\big)u^{p+1}}{u^2} \\
&\geq \frac{2\gamma}{t}\Big(\frac{-tu'(t)}{u(t)}\Big),
\end{aligned} \tag{2.6}$$

provided $u(s) > 0$ for $s \in (0,t)$. Integrating (2.6) on $[t_0, t)$ we have

$$\ln\left(\frac{-tu'(t)/u(t)}{-t_0u'(t_0)/u(t_0)}\right) \geq \ln\big(\frac{t}{t_0}\big)^{\gamma}.$$

Letting $\Gamma = -t_0 u'(t_0)/u(t_0)$ we conclude that

$$\frac{-tu'(t)}{u(t)} \geq \Gamma\big(\frac{t}{t_0}\big)^{\gamma}.$$

For future reference we note that

$$\Gamma \geq 2^{1-p}, \tag{2.7}$$

where we have used (2.2), and $-u'(t_0) \geq t_0 d^p/(2^p N)$ (see (2.1)). Integrating again in $[t_0, t]$ yields

$$\ln\big(\frac{u(t_0)}{u(t)}\big) \geq \frac{\Gamma}{\gamma t_0^{\gamma}}[t^{\gamma} - t_0^{\gamma}].$$

Assuming that $u(t) \geq 0$ for all $t \in [t_0, t_0 \ln^{1/\gamma}(d) \equiv T]$ we have

$$u(T) \leq u(t_0)(ed^{-1})^{\Gamma/\gamma} = \frac{e^{\Gamma/\gamma}}{2} d^{1-\Gamma/\gamma}.$$

Now we estimate E for $t \ge t_0$ with $u(s) \ge 0$ for $s \in (t_0, t]$. Since $E'(t) \ge -2nE(t)/t$,

$$E(t) \ge E(s)(s/t)^{2n} \quad \text{for any} \quad 0 \le s \le t \le 1. \tag{2.8}$$

Thus

$$\begin{aligned} \frac{(u'(T))^2}{2} &\ge E(t_0)\big(\frac{t_0}{T}\big)^{2n} - \frac{u^{p+1}(T)}{p+1} \\ &\ge \frac{d^{p+1}}{(p+1)2^{p+1}\ln^{2n/\gamma}(d)} - \frac{1}{p+1}\Big(\frac{e^{\Gamma/\gamma}}{2}d^{1-\Gamma/\gamma}\Big)^{p+1} \\ &\ge \frac{d^{p+1}}{(p+1)2^{p+2}\ln^{2n/\gamma}(d)}, \end{aligned} \tag{2.9}$$

for d sufficiently large.

Let us suppose now that that $u(t) > 0$, for any $t \in [T, 2T]$. Arguing as in (2.9) we have

$$\begin{aligned} \frac{(u'(t))^2}{2} &\ge E(T)\left(\frac{T}{t}\right)^{2n} - \frac{u^{p+1}(T)}{p+1} \\ &\ge \frac{d^{p+1}}{(p+1)2^{p+1+2n}\ln^{2n/\gamma}(d)} - \frac{1}{p+1}\left(\frac{e^{\Gamma/\gamma}}{2}d^{1-\Gamma/\gamma}\right)^{p+1} \\ &\ge \frac{d^{p+1}}{(p+1)2^{p+2+2n}\ln^{2n/\gamma}(d)}, \end{aligned} \tag{2.10}$$

for d large. Integrating on $[T, t]$ we have

$$\begin{aligned} 0 \le u(t) &= u(T) + \int_T^t u'(s)ds \\ &\le \frac{e^{\Gamma/\gamma}}{2}d^{1-\Gamma/\gamma} - (t-T)\frac{\sqrt{2}d^{(p+1)/2}}{2^{1+n+(p/2)}\ln^{n/\gamma}(d)\sqrt{p+1}}. \end{aligned} \tag{2.11}$$

Hence u has a zero in $[d^{(1-p)/2}, T + e^{\Gamma/\gamma}d^{(1-p)/2-\Gamma/\gamma}2^{n+(p/2)}\ln^{n/\gamma}(d)\sqrt{p+1}]$. We summarize the above in the following lemma.

Lemma 2.1. *For $d > 0$ sufficiently large, there exists*

$$t_1 \in (\sqrt{N}d^{(1-p)/2}, 2\sqrt{N}d^{(1-p)/2}\ln^{1/\gamma}(d)) \tag{2.12}$$

such that $u(t_1) = 0$, $u(s) > 0$ for $s \in [0, t_1)$, and

$$\frac{d^{p+1}}{(p+1)2^{p+2+2n}\ln^{2n}(d)} \le E(t_1) \le \frac{d^{p+1}}{p+1} \tag{2.13}$$

3. First local minimum

Let $t \in (t_1, t_1 + (1/2)(2/(q+1))^{q/(q+1)}|u'(t_1)|^{(1-q)/(1+q)} \equiv t_1 + \tau)$. From (2.12) and (2.13),

$$\begin{aligned}
\frac{t_1}{t} &\geq 1 - \frac{\tau}{t_1+\tau} \geq 1 - \frac{\tau}{t_1} \\
&\geq 1 - \frac{(1/2)(2/(q+1))^{q/(q+1)}|u'(t_1)|^{(1-q)/(1+q)}}{\sqrt{N} d^{(1-p)/2}} \\
&\geq 1 - \frac{(1/2)(2/(q+1))^{q/(q+1)}(d^{(p+1)/2}/\sqrt{p+1})^{(1-q)/(1+q)}}{\sqrt{N} d^{(1-p)/2}} \\
&\equiv 1 - md^{(p-q)/(1+q)} \geq 0.9^{1/n}.
\end{aligned} \tag{3.1}$$

for d large. Hence for d positive and large

$$\begin{aligned}
u'(t) &= t^{-n}\Big[t_1^n u'(t_1) - \int_{t_1}^t s^n |u(s)|^{q-1}u(s)ds\Big] \\
&\leq 0.9u'(t_1) + (t-t_1)\big(\frac{q+1}{2}\big)^{q/(q+1)}|u'(t_1)|^{(2q)/(q+1)} \\
&\leq 0.4u'(t_1),
\end{aligned} \tag{3.2}$$

where we have used that, since $E' \leq 0$, $|u(t)|^{q+1} \leq (q+1)(u'(t_1))^2/2$ for $t \geq t_1$ with $u(t) \leq 0$. This and (3.2) yield

$$u(t_1+\tau) \leq 0.4u'(t_1)\tau \leq -0.2(2/(q+1))^{q/(q+1)}|u'(t_1)|^{2/(1+q)}. \tag{3.3}$$

Now for $t \geq t_1 + \tau$ with $u(s) \leq -0.2(2/(q+1))^{q/(q+1)}|u'(t_1)|^{2/(1+q)}$ for all $s \in (t_1+\tau, t)$ we have

$$\begin{aligned}
&u'(t) \\
&= t^{-n}\Big[t_1^n u'(t_1) - \int_{t_1}^t s^n |u(s)|^{q-1}u(s)ds\Big] \\
&\geq u'(t_1) + t^{-n}(0.2(2/(q+1))^{q/(q+1)})^q |u'(t_1)|^{2q/(1+q)} \int_{t_1+\tau}^t s^n ds \\
&\geq -u'(t_1)\Big[-1 + t^{-n}(0.2(2/(q+1))^{q/(q+1)})^q |u'(t_1)|^{(q-1)/(1+q)} \frac{t^N - (t_1+\tau)^N}{N}\Big] \\
&\geq -u'(t_1)\Big[-1 + \frac{(0.2(2/(q+1))^{q/(q+1)})^q}{N}|u'(t_1)|^{(q-1)/(1+q)}(t-(t_1+\tau))\Big].
\end{aligned} \tag{3.4}$$

This and the definition of τ imply the following lemma.

Lemma 3.1. *There exists τ_1 in*

$$\begin{aligned}
&\Big(t_1, t_1 + \big\{(1/2)(2/(q+1))^{q/(q+1)} + \frac{N}{(.2(2/(q+1))^{q/(q+1)})^q}\big\}|u'(t_1)|^{(1-q)/(1+q)}\Big] \\
&\equiv \big(t_1, t_1 + \kappa_1|u'(t_1)|^{(1-q)/(1+q)}\big]
\end{aligned}$$

such that $u'(\tau_1) = 0$.

4. Second zero

Let $\tau_0 > \tau_1$ be such that $u(s) \leq 0.5u(\tau_1)$ for all $s \in [\tau_1, \tau_0]$. Imitating the arguments leading to (2.2) we see that

$$\tau_1 + |u(\tau_1)|^{(1-q)/2} \leq \tau_0 \leq \tau_1 + \sqrt{2^q N}|u(\tau_1)|^{(1-q)/2}. \tag{4.1}$$

Hence

$$\begin{aligned} u'(\tau_0) &= \tau_0^{-n} \int_{\tau_1}^{\tau_0} s^n |u(s)|^q ds \\ &\geq \frac{|u(\tau_1)|^q(\tau_0^N - \tau_1^N)}{N2^q\tau_0^n} \\ &\geq \frac{|u(\tau_1)|^q(\tau_0 - \tau_1)}{2^q N} \\ &\geq \frac{|u(\tau_1)|^{(1+q)/2}}{2^q N}, \end{aligned} \tag{4.2}$$

and

$$\tau_0^n \geq .9s^n \quad \text{for any } s \in (\tau_0, \tau_0 + 2^{q+2}N|u(\tau_1)|^{(1-q)/2}], \tag{4.3}$$

for $d > 0$ sufficiently large.

Suppose now that for all $s \in [\tau_0, r \equiv \tau_0 + 2^{q+1}N|u(\tau_1)|^{(1-q)/2}]$ we have $u(s) \leq 0$. Then

$$u'(s) \geq 0.9u'(\tau_0) \quad \text{for all } s \in [\tau_0, r]. \tag{4.4}$$

This and and the definition of r give

$$\begin{aligned} 0 \geq u(r) &\geq \frac{u(\tau_1)}{2} + .9(r - \tau_0)u'(r) \\ &\geq \frac{u(\tau_1)}{2} + .9(2^{q+1})N|u(\tau_1)|^{(1-q)/2}\frac{|u(\tau_1)|^{(1+q)/2}}{2^q N} \\ &= 1.3|u(\tau_1)|, \end{aligned}$$

which is a contradiction. From (3.3), $|u(\tau_0)| \geq 0.2(2/(q+1))^{q/(q+1)}|u'(t_1)|^{2/(1+q)}$. Since also $\tau_0 \leq t_1 + (\kappa_1 + 0.2\sqrt{2^q N}(2/(q+1))^{q/(q+1)})|u'(t_1)|^{(1-q)/(1+q)}$ (see Lemma 3.1 and (4.2)). Thus

$$\begin{aligned} &\tau_0 + 2^{q+1}N|u(\tau_1)|^{(1-q)/2} \\ &\leq t_1 + (\kappa_1 + .2(2^{q+2}N)(2/(q+1))^{q/(q+1)})|u'(t_1)|^{(1-q)/(1+q)} \\ &\equiv t_1 + k_2|u'(t_1)|^{(1-q)/(1+q)}. \end{aligned}$$

Thus we have proven the following lemma.

Lemma 4.1. *There exists $t_2 \in [t_1, t_1 + k_2|u'(t_1)|^{(1-q)/(1+q)}]$ such that $u(t_2) = 0$ and $u(s) < 0$ in (t_1, t_2).*

5. First positive maximum

Let $t > t_2$ be such that $u'(s) > 0$ on $[t_2, t]$. Thus $u'' \leq 0$ in $[t_2, t]$. Hence $u(s) \leq u'(t_2)(s - t_2)$ for all $s \in [t_2, t]$. Integrating (1.5) on $[t_2, s]$, we have

$$\begin{aligned} s^n u'(s) &= t_2^n u'(t_2) - \int_{t_2}^{s} r^n |u(r)|^{p-1} u(r) dr \\ &\geq t_2^n u'(t_2) - s^n \frac{|u'(t_2)|^p (s - t_2)^{p+1}}{p+1} \\ &\geq u'(t_2)\big(t_2^n - \frac{s^n}{p+1}\big), \end{aligned} \tag{5.1}$$

for $s \leq t_2 + u'(t_2)^{(1-p)/(1+p)}$. Since $t_2^N |u'(t_2)|^2 \geq 2c_1 d^{\xi}$ (see (2.4)) and $(u'(t_2))^2 \leq 2d^{p+1}/(p+1)$, we have

$$t_2^N \geq 2c_1 \big(\frac{p+1}{2}\big)^{\xi/(p+1)} |u'(t_2)|^{N(1-p)/(1+p)}. \tag{5.2}$$

Now for

$$s \in [t_2, \min\big\{2^{1/n}, 1 + (2c_1)^{-1/N} \big(\frac{2}{p+1}\big)^{\frac{\xi}{N(p+1)}}\big\} t_2 \equiv \alpha t_2],$$

from (5.1) and (5.2), we have

$$u'(s) \geq u'(t_2)\big(\frac{t_2^n}{s^n} - \frac{1}{p+1}\big) \geq u'(t_2)\frac{p-1}{p+1}. \tag{5.3}$$

Integration on $[t_2, \alpha t_2]$ yields

$$u(\alpha t_2) \geq \frac{p-1}{p+1} \alpha t_2 u'(t_2). \tag{5.4}$$

Therefore, assuming again that $u' > 0$ on $[t_2, t]$, we have

$$\begin{aligned} t^n u'(t) &\leq t_2^n u'(t_2) - \int_{\alpha t_2}^{t} r^n |u(r)|^{p-1} u(r) dr \\ &\leq t_2^n u'(t_2) - t_2^n (t - \alpha t_2) \Big(\frac{p-1}{p+1} \alpha t_2 u'(t_2)\Big)^p. \end{aligned} \tag{5.5}$$

This and (5.2) imply

$$\begin{aligned} t - \alpha t_2 &\leq \big(\frac{p-1}{p+1}\alpha\big)^{-p} t_2^{-p} |u'(t_2)|^{1-p} \\ &\leq \big(\frac{p-1}{p+1}\alpha\big)^{-p} (2c_1)^{-p/N} \big(\frac{p+1}{2}\big)^{-p\xi/(N(p+1))} |u'(t_2)|^{(1-p)/(p+1)} \\ &\equiv \kappa_2 |u'(t_2)|^{(1-p)/(p+1)}. \end{aligned} \tag{5.6}$$

This proves the following result.

Lemma 5.1. *There exists $\tau_2 \in [t_2, \alpha t_2 + \kappa_2 |u'(t_2)|^{(1-p)/(p+1)}]$ such that $u'(\tau_2) = 0$ and $u'(s) > 0$ on $[t_2, \tau_2)$.*

6. Energy on the interval $[t_0, \tau_2]$

Now we estimate the energy on $[t_0, \tau_2]$.

Lemma 6.1. *For $t \in [t_0, \tau_2]$,*

$$t^n H(t) \geq c_1 d^{\xi}. \tag{6.1}$$

Proof. Let us prove first that

$$\int_{t_0}^{t_1} t^n \gamma u^{p+1}(t)dt \geq \int_{t_1}^{t_2} t^n \gamma_1 |u(t)^{q+1}|dt, \tag{6.2}$$

where $\gamma_1 = ((q+1)(N-2)-2N)/(2(q+1))$. Let $\hat{t}_0 \in [t_0, t_1]$ be such that $u(\hat{t}_0) = d/4$. Then, for $t \in [t_0, \hat{t}_0]$, we have

$$-u'(t) = t^{-n} \int_0^t s^n u^p(s)\, ds \leq \frac{td^p}{N}. \tag{6.3}$$

Integrating on $[t_0, \hat{t}_0]$ we have $(d/4) \leq (\hat{t}_0^2 - t_0^2)d^p/(2N)$. This and (2.2) yield

$$\hat{t}_0 \geq \sqrt{\frac{Nd^{1-p}}{2} + t_0^2} = t_0\sqrt{1 + \frac{Nd^{1-p}}{2t_0^2}} \geq t_0\sqrt{1 + \frac{1}{2^{p+1}}}, \tag{6.4}$$

which combined with (2.2) gives

$$\begin{aligned}
\int_{t_0}^{t_1} t^n \gamma u^{p+1}(t)dt &\geq \int_{t_0}^{\hat{t}_0} t^n \gamma u^{p+1}(t)dt \\
&\geq \gamma (d/4)^{p+1} \frac{\hat{t}_0^N - t_0^N}{N} \\
&\geq \frac{\gamma}{4^{p+1}N} t_0^N \Big(\big(1 + \frac{1}{2^{p+1}}\big)^{N/2} - 1\Big) d^{p+1} \\
&\geq \frac{\gamma}{4^{p+1}N} \Big(\big(1 + \frac{1}{2^{p+1}}\big)^{N/2} - 1\Big) N^{N/2} d^{\xi}.
\end{aligned} \tag{6.5}$$

Using (1.7), we have $|u(t)|^{q+1} \leq (q+1)d^{p+1}/(p+1)$. Also from (2.3) and (2.4), we have $t_1^N |u'(t_1)|^2/2 = t_1 H(t_1) \geq c_1 d^{\xi}$. This implies that $k_2|u'(t_1)|^{(1-q)/(1+q)} < t_1$ for $d > 0$ large. These inequalities and Lemma 4.1 imply

$$\begin{aligned}
\int_{t_1}^{t_2} t^n |u(t)|^{q+1} dt &\leq \big(\frac{q+1}{p+1} d^{p+1}\big) \frac{t_2^N - t_1^N}{N} \\
&\leq \big(\frac{q+1}{p+1} d^{p+1}\big) \frac{(t_1 + k_2 |u'(t_1)|^{(1-q)/(1+q)})^N - t_1^N}{N} \\
&\leq \big(\frac{q+1}{p+1} d^{p+1}\big) t_1^n \frac{(2^N - 1)k_2 |u'(t_1)|^{(1-q)/(1+q)}}{N} \\
&\leq \big(\frac{q+1}{p+1} d^{p+1}\big) \frac{(2^N - 1)k_2}{N}\, t_1^n \Big(d^{\xi/2} t_1^{-N/2}\Big)^{(1-q)/(1+q)} \\
&= \big(\frac{q+1}{p+1}\big) \frac{(2^N - 1)k_2}{N}\, d^{p+1+(\xi(1-q)/(2(1+q))} t_1^{N-1-N(1-q)/(2(1+q))} \\
&\leq \big(\frac{q+1}{p+1}\big) \frac{(2^N - 1)k_2}{N} \ln^{M/\gamma}(d)\, d^{\eta},
\end{aligned} \tag{6.6}$$

where

$$\begin{gathered}
\eta = p + 1 + \frac{\xi(1-q) + (1-p)(2(N-1)(1+q) - N(1-q))}{2(1+q)}, \\
M = (2(N-1)(1+q) - N(1-q))/(2(1+q)).
\end{gathered}$$

An elementary calculation shows that $\xi > \eta$. Thus from (6.5) and (6.6), (6.2) follows. Thus for $t \in [t_1, \tau_2]$,

$$\begin{aligned} t^n H(t) &= t_0^n H(t_0) + \int_{t_0}^{t} s^n \left(NG(u(s)) - \frac{N-2}{2} u(s) g(u(s)) \right) ds \\ &\geq t_0^n H(t_0) + \int_{t_0}^{t_2} s^n \left(NG(u(s)) - \frac{N-2}{2} u(s) g(u(s)) \right) ds \\ &\geq t_0^n H(t_0) \\ &\geq c_1 d^{\xi}, \end{aligned} \tag{6.7}$$

which proves the lemma. □

7. Proof of Theorem 1.1

Arguing as in Lemmas 2.1 and 4.1, we see that for $d > 0$ sufficiently large there exist numbers

$$t_3 < \cdots < t_k \leq 1 \tag{7.1}$$

such that

$$u(t) < 0 \quad \text{in } (t_{2i-1}, t_{2i}), \quad \text{and} \quad u(t) > 0 \quad \text{in } (t_{2i}, t_{2i+1}),\ i = 1, \dots \min\{\frac{k}{2}, \frac{k+1}{2}\}. \tag{7.2}$$

Imitating the arguments leading to (6.2), one sees that

$$\int_{t_{2i}}^{t_{2i+1}} t^n \gamma u^{p+1}(t) dt \geq \int_{t_{2i+1}}^{t_{2i+2}} t^n \gamma_1 |u(t)^{q+1}| dt. \tag{7.3}$$

This in turn (see (6.7)) leads to

$$t^n H(t) \geq c_1 d^{\xi} \quad \text{for all } t \in [t_0, 1]. \tag{7.4}$$

This, together with Lemma 2.1, proves Theorem 1.2. From (7.4) we see that

$$\rho^2(t) \equiv u^2(t) + (u'(t))^2 \to \infty \quad \text{as } d \to +\infty, \tag{7.5}$$

uniformly for $t \in [0, 1]$. Therefore, there exists a continuous *argument* function $\theta(t, d) \equiv \theta(t)$ such that

$$u(t) = \rho(t) \cos(\theta(t)) \quad \text{and} \quad u'(t) = -\rho(t) \sin(\theta(t)). \tag{7.6}$$

From this we see that $\theta'(t) = \{((n/t)u'(t) + g(u(t)))u(t) + (u'(t))^2\}/\rho^2(t)$. Thus $\theta'(t) > 0$ for $\theta(t) = j\pi/2$ with $j = 1, \dots,$ which implies that if $\theta(t) = j\pi/2$ then $\theta(s) > j\pi/2$ for all $s \in (t, 1]$.

Imitating the arguments of Lemmas 2.1 and 4.1, we see that $t_{2i} - t_{2(i-1)} \leq c_3 \ln^{1/\gamma}(d) d^{(1-p)/2}$. Thus $k \geq c_4 \ln^{-1/\gamma}(d) d^{(p-1)/2}$ (see (7.1)), which implies that

$$\lim_{d \to +\infty} \theta(1, d) = +\infty. \tag{7.7}$$

By the continuity of θ and the intermediate value theorem we see that there exists a sequence $d_1 < \cdots < d_j < \cdots \to \infty$ such that $\theta(1, d_j) = j\pi + (\pi/2)$. Hence $u(t, d_j)$ is a solution to (1.1) having exactly j zeroes in $(0, 1)$, which proves Theorem 1.1.

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