

Analysis of the mushy region in conduction- convection problems with change of phase *

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Abstract

A conduction-convection problem with change of phase is studied, where convective motion of the liquid affects the change of phase. The mushy region is the portion of the system to which temperature and enthalpy do not assign a phase, solid or liquid. In this paper we show that the enthalpy density remains constant in time almost everywhere in the mushy region.

1 Introduction

The conduction–convection problem with change of phase is the problem of determining the temperature and motion of a system with liquid and solid components where the evolution and change of phase depend on both conduction and the convective motion of the liquid phase. The phase of the material is determined by the temperature $u(x, t)$ and enthalpy density, or thermal energy density, $w(x, t)$ of the material, normalized so that $w = 0$ for solid at the phase change temperature, which for convenience we select to be zero. Then if $u(x, t) < 0$, or more generally if $w(x, t) \leq 0$, the material is solid, while if $u(x, t) > 0$, or more generally if $w(x, t) \geq L$, where L is the latent heat of fusion per unit mass, the material is liquid. The mushy region is that portion of the material for which $u(x, t) = 0$ and $0 < w(x, t) < L$. (For additional details concerning the physics of the problem, see [1].) In this note we characterize the mushy region for weak solutions of conduction–convection systems; roughly speaking, we show that if some portion of the material is mushy at a given instant of time, then the enthalpy density of the material at almost every point of such a region has the same value as it did at almost every prior time; the precise statement of the result is Theorem 1.

The general weak form of the conduction–convection system is

$$w_t - \Delta K(u) + \mathbf{v} \cdot \nabla u = 0, \tag{1}$$

$$w \subseteq \beta(u), \tag{2}$$

* 1991 Mathematics Subject Classifications: 35R35, 76D99, 76R99, 80A22. .

Key words and phrases: conduction, convection, free boundary, Stefan problem, mushy region

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Submitted: August 12, 1996. Published January 30, 1997.

coupled with equations governing the motion of the material in the liquid phase, like the Stokes or Navier-Stokes equations with the Boussinesq approximation. Here $\beta(s)$ describes the relationship between the temperature and the enthalpy, and $K(s)$ describes the ratio of the thermal conductivity to the specific heat at constant volume of the material. We assume $\beta(s)$ is a strictly monotone increasing graph with a single jump discontinuity at the origin, so that

$$\beta(0) = [0, L] \tag{3}$$

and

$$0 < \beta_o \leq \beta'(s) \leq \beta_1 \quad \text{if } s \neq 0. \tag{4}$$

We also assume $K(s)$ is a continuous monotone function, differentiable away from the origin, so that

$$K(0) = 0 \tag{5}$$

and

$$0 < K_o \leq K'(s) \leq K_1 \quad \text{if } s \neq 0. \tag{6}$$

Weak systems of this type have been studied in [2, 3, 4, 5, 6, 14, 15], primarily with a view to obtaining existence results with various systems governing the flow of the fluid; in our analysis, we shall merely assume that \mathbf{v} is of class L_2 and weakly solenoidal. The Stefan problem is this system without the convective term $\mathbf{v} \cdot \nabla u$; it has been extensively studied, in particular in the monographs [12, 16] and the references contained therein.

An analysis of the mushy region is important for a number of reasons. First the usual physical model from which the equations (1)-(2) are derived begins with the assumption that the phase change occurs on a sharp interface; i.e. the mushy region is a smooth surface (see [1, §1.2.B] and [6]). It is important to see to what extent this condition holds for weak solutions. Secondly the general question of the behavior and modeling of mushy regions and partially solidified systems is still active, see for example [9, 13]. Thirdly, there is the interest in the construction of accurate numerical methods to solve conduction-convection systems, especially in and near mushy regions, see for example [17, 18]. Lastly, the proof of existence of solutions to the conduction-convection system in three dimensions with the Stokes system and the Boussinesq approximation [5] allowed for the possibility of a "singular set" on the boundary of the mushy region where the velocity of the fluid might become unbounded. It is unclear if this is an artifact of the techniques used in the proof, or if such sets might actually be realized.

2 Results

Theorem 1. *Let $\Omega \subseteq \mathbb{R}^N$ for $N \geq 2$ be an arbitrary domain, let $T > 0$, and let $\Omega_T = \Omega \times (0, T)$. Let $\mathbf{v} \in L_2(\Omega_T)$ be weakly solenoidal, let $u \in L_\infty(\Omega_T) \cap$*

$L_2(0, T; W_2^1(\Omega))$ and let $w \subseteq \beta(u)$ be any selection from the graph. Suppose that

$$w_t - \Delta K(u) + \mathbf{v} \cdot \nabla u = 0 \tag{7}$$

weakly in Ω_T . If

$$M(t) = \{x \in \Omega : 0 < w(x, t) < L\}, \tag{8}$$

then there exists a set $\Sigma \subset [0, T]$ of measure zero so that if $t_1 < t_2$ and $t_1, t_2 \in [0, T] \setminus \Sigma$ then for almost every $x \in M(t_2)$,

$$w(x, t_2) = w(x, t_1). \tag{9}$$

The general idea for the proof of this result follows the techniques of Meirmanov [12, Chapter 1, Section 3]. In particular, taking the velocity \mathbf{v} as given, we shall prove an existence result for solutions of the temperature equations (7) with appropriate Dirichlet boundary conditions. This shall be done by solving approximating problems that are discretized in time; this discretization allows us to prove the analogue of (9) for the approximations; then we pass to the limit. A simple uniqueness result shall yield the result.

Remark. The set of times $\{t_i\}$ for which (9) holds contains all times t for which

$$\lim_{\tau \rightarrow t} w(x, \tau) = w(x, t) \quad \text{for a.e. } x. \tag{10}$$

Moreover, if

$$\lim_{\tau \downarrow 0} w(x, \tau) = w(x, 0) \quad \text{for a.e. } x, \tag{11}$$

then we can set $t_1 = 0$ in the result.

Indeed, suppose $\lim_{\tau \downarrow 0} w(x, \tau) = w(x, 0)$ for almost every $x \in \Omega$. Choose $t_2 \in (0, T] \setminus \Sigma$, let $\tau_i \in [0, t_2] \setminus \Sigma$, with $\tau_i \downarrow 0$, and let $S_o \subset \Omega$ be a set of measure zero so that $w(x, \tau_i) \rightarrow w(x, 0)$ if $x \in \Omega \setminus S_o$. For each i , there exists a set $S_i \subset \Omega$ of measure zero so that if $x \in M(t_2) \setminus S_i$, then $w(x, t_2) = w(x, \tau_i)$. Set $S = \cup_{i=0}^\infty S_i$; then S has measure zero and for every $x \in M(t_2) \setminus S$ and every i we have $w(x, t_2) = w(x, \tau_i) \rightarrow w(x, 0)$. The statement (10) is proven similarly.

3 Proof of the main Theorem

Proposition 2. Let $\Omega \subset \mathbb{R}^N$ for $N \geq 2$ be a smooth bounded domain and let $T > 0$. Let $\mathbf{v} \in L_2(0, T; W_2^1(\Omega)) \cap L_\infty(0, T; L_2(\Omega))$ be weakly solenoidal, let $g \in L_\infty(\partial\Omega \times (0, T)) \cap L_2(0, T; W_2^1(\partial\Omega))$ and suppose that $w_o \in C^2(\bar{\Omega})$ and $u_o = \beta^{-1}(w_o)$. Then there exists a function $u \in L_\infty(\Omega_T) \cap L_2(0, T; W_2^1(\Omega))$, and a function $w \subseteq \beta(u)$ so that

$$w_t - \Delta K(u) + \mathbf{v} \cdot \nabla u = 0 \quad \text{weakly in } \Omega_T, \tag{12}$$

$$u|_{\partial\Omega} = g \quad \text{as traces}, \tag{13}$$

$$w(\cdot, t) \xrightarrow{t \downarrow 0} w_o \quad \text{weakly in } L_2(\Omega). \tag{14}$$

Further, if

$$M(t) = \{x \in \Omega : 0 < w(x, t) < L\} \quad (15)$$

then there exists a set $\Sigma \subset [0, T]$ of measure zero so that if $t_1 < t_2$ and $t_1, t_2 \in [0, T] \setminus \Sigma$ then for almost every $x \in M(t_2)$,

$$w(x, t_2) = w(x, t_1). \quad (16)$$

PROOF: Let $\epsilon > 0$ be a regularizing parameter, and let β_ϵ and K_ϵ be smooth approximations of β and K so that

$$\beta_\epsilon(0) = 0, \quad 0 < \beta_o \leq \beta'_\epsilon(s) \leq \frac{1}{\epsilon}, \quad (17)$$

$$K_\epsilon(0) = 0, \quad 0 < K_o \leq K'_\epsilon(s) \leq K_1. \quad (18)$$

We also suppose that $|\beta'_\epsilon(s)| \leq \beta_1$ for $|s| \geq \epsilon$ and that $|\beta_\epsilon(s)| \leq \beta_1|s| + L$ for each s . Finally, let \mathbf{v}_ϵ be smooth solenoidal approximations of \mathbf{v} and g_ϵ smooth approximations of g .

Let $h = T/n$, and consider the family of problems

$$\frac{1}{h}(w_\epsilon^i - w_\epsilon^{i-1}) - \Delta K_\epsilon(u_\epsilon^i) + \mathbf{v}_\epsilon^i \cdot \nabla u_\epsilon^i = 0, \quad i = 1, 2, \dots, n-1 \quad (19)$$

$$\mathbf{v}_\epsilon^i(x) = \frac{1}{h} \int_{ih}^{(i+1)h} \mathbf{v}_\epsilon(x, s) ds, \quad (20)$$

$$u_\epsilon^i|_{\partial\Omega} = \frac{1}{h} \int_{ih}^{(i+1)h} g_\epsilon(x, s) ds, \quad (21)$$

$$w_\epsilon^i = \beta_\epsilon(u_\epsilon^i), \quad i = 1, 2, \dots, n-1 \quad (22)$$

$$w_\epsilon^o(x) = w_o \subseteq \beta(u_o). \quad (23)$$

For ϵ , n and i fixed, this is a quasi-linear elliptic equation for $w_\epsilon^i = \beta_\epsilon(u_\epsilon^i)$. Monotonicity of β_ϵ and K_ϵ imply the classical maximum principle (c.f. [11, Chapter 3, §1])

$$\sup_{\bar{\Omega}} |w_\epsilon^i(x)| \leq \max\left\{ \sup_{\partial\Omega \times (0, T)} |\beta_\epsilon \circ g_\epsilon|, \sup_{\bar{\Omega}} |w_\epsilon^{i-1}| \right\} \quad (24)$$

for $1 \leq i \leq N-1$. Standard theory [7, Theorem 15.11, Theorem 6.19] implies we have a classical solution $w_\epsilon^i \in C^3(\bar{\Omega})$ for each ϵ and i ; boundedness of the data and the maximum principle imply that $\|w_\epsilon^i\|_{L^\infty(\Omega)}$ and $\|u_\epsilon^i\|_{L^\infty(\Omega)}$ are bounded uniformly in ϵ and i .

For each $h = T/n$, define the functions

$$u_{\epsilon, h}(x, t) = u_\epsilon^i(x) \text{ if } ih \leq t < (i+1)h, \quad (25)$$

$$w_{\epsilon, h}(x, t) = w_\epsilon^i(x) \text{ if } ih \leq t < (i+1)h, \quad (26)$$

$$\mathbf{v}_{\epsilon, h}(x, t) = \mathbf{v}_\epsilon^i(x) \text{ if } ih \leq t < (i+1)h. \quad (27)$$

Since (19) holds classically, multiply by hu_ϵ^i , integrate over Ω and sum to obtain

$$\begin{aligned} \sum_{i=1}^{n-1} K_o h \int_{\Omega} |\nabla u_\epsilon^i|^2 dx &\leq \sum_{i=1}^{n-1} h \int_{\partial\Omega} u_\epsilon^i \nabla K_\epsilon(u_\epsilon^i) \cdot \nu d\sigma + \int_{\Omega} w^o u_\epsilon^1 dx \\ &\quad - \int_{\Omega} w_\epsilon^{n-1} u_\epsilon^{n-1} dx + \sum_{i=2}^{n-1} \int_{\Omega} w_\epsilon^{i-1} (u_\epsilon^i - u_\epsilon^{i-1}) dx \\ &\quad + \sum_{i=1}^{n-1} h \int_{\Omega} u_\epsilon^i \mathbf{v}_\epsilon^i \cdot \nabla u_\epsilon^i dx, \end{aligned} \tag{28}$$

where ν is the outward unit normal to Ω . For each $2 \leq i \leq n - 1$, the continuity of β_ϵ and the mean value theorem for integrals implies the existence of a number $a_i(x)$ between $u_\epsilon^{i-1}(x)$ and $u_\epsilon^i(x)$ so that

$$\beta_\epsilon(a_i)(u_\epsilon^i - u_\epsilon^{i-1}) = \int_{u_\epsilon^{i-1}}^{u_\epsilon^i} \beta_\epsilon(s) ds; \tag{29}$$

thus

$$\begin{aligned} \sum_{i=2}^{n-1} \int_{\Omega} w_\epsilon^{i-1} (u_\epsilon^i - u_\epsilon^{i-1}) dx &= \\ \int_{\Omega} \int_{u_\epsilon^1}^{u_\epsilon^{n-1}} \beta_\epsilon(s) ds dx &+ \sum_{i=2}^{n-1} \int_{\Omega} (\beta_\epsilon(u_\epsilon^{i-1}) - \beta_\epsilon(a_i))(u_\epsilon^i - u_\epsilon^{i-1}) dx. \end{aligned} \tag{30}$$

Monotonicity of β_ϵ implies that the last term above is nonpositive. Young’s inequality applied to the first term on the right side of (28) together with (30) imply

$$\begin{aligned} \int_h^T \int_{\Omega} |\nabla u_{\epsilon,h}|^2 dx dt &\leq C \int_0^T \int_{\partial\Omega} \{ |g_\epsilon|^2 + |\nabla g_\epsilon|^2 \} d\sigma(x) dt + C \int_{\Omega} w^o u_\epsilon^1 dx \\ &\quad + C \int_{\Omega} \int_{u_\epsilon^1}^{u_\epsilon^{n-1}} \beta_\epsilon(s) ds dx + C \int_h^T \int_{\Omega} u_{\epsilon,h} \mathbf{v}_{\epsilon,h} \cdot \nabla u_{\epsilon,h} dx dt \leq C, \end{aligned} \tag{31}$$

as boundedness of the data, the maximum principle, and Young’s inequality applied to the last term above imply $\|\nabla u_{\epsilon,h}\|_{L_2(\Omega \times (h,T))}$ is bounded uniformly in ϵ and h .

Next, we wish to show the pre-compactness of $\{w_{\epsilon,h}\}$ in $L_1(\Omega_T)$; to that end we begin with an estimate of $\nabla w_{\epsilon,h}$.

Let $\Omega' \Subset \Omega$ and let $\zeta \in C_0^\infty(\Omega)$ be a cutoff function so that $\zeta(x) = 1$ if $x \in \Omega'$. Let $\delta > 0$ and define

$$\phi^i(x) = -\frac{\partial}{\partial x_k} \left\{ \zeta(x) \frac{w_{\epsilon,x_k}^i}{\sqrt{(w_{\epsilon,x_k}^i)^2 + \delta}} \right\} = -\frac{\partial}{\partial x_k} (\zeta \operatorname{sgn}^\delta w_{\epsilon,x_k}^i) \tag{32}$$

for some $k = 1, 2, \dots, N$. Multiply (19) by $h\phi_i$, integrate, and sum from $i = 1$ to m , for some arbitrary $1 \leq m \leq n - 1$ to obtain

$$\begin{aligned} \sum_{i=1}^m \int_{\Omega} (w_{\epsilon, x_k}^i - w_{\epsilon, x_k}^{i-1}) \zeta \operatorname{sgn}^{\delta} w_{\epsilon, x_k}^i dx + h \sum_{i=1}^m \int_{\Omega} \Delta K_{\epsilon}(u_{\epsilon}^i) \frac{\partial}{\partial x_k} (\zeta \operatorname{sgn}^{\delta} w_{\epsilon, x_k}^i) dx \\ + h \sum_{i=1}^m \int_{\Omega} \frac{\partial}{\partial x_k} (\mathbf{v}_{\epsilon}^i \cdot \nabla u_{\epsilon}^i) \zeta \operatorname{sgn}^{\delta} w_{\epsilon, x_k}^i dx = 0. \end{aligned} \quad (33)$$

To estimate the first term, note that

$$\begin{aligned} \lim_{\delta \downarrow 0} \sum_{i=1}^m \int_{\Omega} (w_{\epsilon, x_k}^i - w_{\epsilon, x_k}^{i-1}) \zeta \operatorname{sgn}^{\delta} w_{\epsilon, x_k}^i dx \\ = \int_{\Omega} |w_{x_k}^m| \zeta dx - \int_{\Omega} |w_{x_k}^o| \zeta dx - \sum_{i=1}^m \int_{\Omega} |w_{\epsilon, x_k}^{i-1}| \zeta (\operatorname{sgn} w_{\epsilon, x_k}^i \operatorname{sgn} w_{\epsilon, x_k}^{i-1} - 1) dx \\ \geq \int_{\Omega} |w_{x_k}^m| \zeta dx - \int_{\Omega} |w_{x_k}^o| \zeta dx. \end{aligned} \quad (34)$$

To handle the second term in (33), begin with a pair of integrations by parts yielding

$$\begin{aligned} h \sum_{i=1}^m \int_{\Omega} \Delta K_{\epsilon}(u_{\epsilon}^i) \frac{\partial}{\partial x_k} (\zeta \operatorname{sgn}^{\delta} w_{\epsilon, x_k}^i) dx \\ = h \sum_{i=1}^m \int_{\Omega} \nabla \{ (K_{\epsilon} \circ \beta_{\epsilon}^{-1})'(w_{\epsilon}^i) w_{\epsilon, x_k}^i \} (\zeta \nabla \operatorname{sgn}^{\delta} w_{\epsilon, x_k}^i + \nabla \zeta \operatorname{sgn}^{\delta} w_{\epsilon, x_k}^i) dx, \end{aligned} \quad (35)$$

and integrating by parts again in the last term we obtain

$$\begin{aligned} h \sum_{i=1}^m \int_{\Omega} \Delta K_{\epsilon}(u_{\epsilon}^i) \frac{\partial}{\partial x_k} (\zeta \operatorname{sgn}^{\delta} w_{\epsilon, x_k}^i) dx \\ = h \sum_{i=1}^m \int_{\Omega} [\nabla (K_{\epsilon} \circ \beta_{\epsilon}^{-1})'(w_{\epsilon}^i)] \zeta w_{\epsilon, x_k}^i \nabla \operatorname{sgn}^{\delta} w_{\epsilon, x_k}^i dx \\ + h \sum_{i=1}^m \int_{\Omega} (K_{\epsilon} \circ \beta_{\epsilon}^{-1})'(w_{\epsilon}^i) \zeta \nabla w_{\epsilon, x_k}^i \nabla \operatorname{sgn}^{\delta} w_{\epsilon, x_k}^i dx \\ - h \sum_{i=1}^m \int_{\Omega} (K_{\epsilon} \circ \beta_{\epsilon}^{-1})'(w_{\epsilon}^i) w_{\epsilon, x_k}^i \operatorname{div}(\nabla \zeta \operatorname{sgn}^{\delta} w_{\epsilon, x_k}^i) dx \\ = I_1 + I_2 + I_3. \end{aligned} \quad (36)$$

By computation, we see that

$$I_1 = h \sum_{i=1}^m \int_{\Omega} [\nabla (K_{\epsilon} \circ \beta_{\epsilon}^{-1})'(w_{\epsilon}^i)] \zeta w_{\epsilon, x_k}^i \frac{\delta \nabla w_{\epsilon, x_k}^i}{[(w_{\epsilon, x_k}^i)^2 + \delta]^{3/2}} dx. \quad (37)$$

Now for almost every $x \in \Omega$

$$\frac{\delta w_{\epsilon, x_k}^i(x)}{[(w_{\epsilon, x_k}^i(x))^2 + \delta]^{3/2}} \leq 1 \quad \text{and} \quad \lim_{\delta \downarrow 0} \frac{\delta w_{\epsilon, x_k}^i(x)}{[(w_{\epsilon, x_k}^i(x))^2 + \delta]^{3/2}} = 0. \quad (38)$$

Since $w_\epsilon^i \in C^3(\bar{\Omega})$, we can apply Lebesgue's dominated convergence theorem to conclude

$$\lim_{\delta \downarrow 0} I_1 = \lim_{\delta \downarrow 0} h \sum_{i=1}^m \int_{\Omega} [\nabla(K_\epsilon \circ \beta_\epsilon^{-1})'(w_\epsilon^i)] \zeta w_{\epsilon, x_k}^i \frac{\delta \nabla w_{\epsilon, x_k}^i}{[(w_{\epsilon, x_k}^i)^2 + \delta]^{3/2}} dx = 0. \quad (39)$$

To estimate I_2 , note that monotonicity of K_ϵ and β_ϵ^{-1} implies

$$\begin{aligned} I_2 &= h \sum_{i=1}^m \int_{\Omega} (K_\epsilon \circ \beta_\epsilon^{-1})'(w_\epsilon^i) \zeta \nabla w_{\epsilon, x_k}^i \nabla \operatorname{sgn}^\delta w_{\epsilon, x_k}^i dx \\ &= h \sum_{i=1}^m \int_{\Omega} (K_\epsilon \circ \beta_\epsilon^{-1})'(w_\epsilon^i) \zeta \frac{\delta |\nabla w_{\epsilon, x_k}^i|^2}{[(w_{\epsilon, x_k}^i)^2 + \delta]^{3/2}} dx \geq 0. \end{aligned} \quad (40)$$

Finally, decompose I_3 as

$$\begin{aligned} I_3 &= h \sum_{i=1}^m \int_{\Omega} (K_\epsilon \circ \beta_\epsilon^{-1})'(w_\epsilon^i) w_{\epsilon, x_k}^i \operatorname{div}(\nabla \zeta \operatorname{sgn}^\delta w_{\epsilon, x_k}^i) dx \\ &= h \sum_{i=1}^m \int_{\Omega} (K_\epsilon \circ \beta_\epsilon^{-1})'(w_\epsilon^i) \nabla \zeta w_{\epsilon, x_k}^i \nabla \operatorname{sgn}^\delta w_{\epsilon, x_k}^i dx \\ &\quad + h \sum_{i=1}^m \int_{\Omega} (K_\epsilon \circ \beta_\epsilon^{-1})'(w_\epsilon^i) w_{\epsilon, x_k}^i \operatorname{sgn}^\delta w_{\epsilon, x_k}^i \Delta \zeta dx. \end{aligned} \quad (41)$$

The first of these terms tends to zero as $\delta \downarrow 0$ for the same reasons that I_1 does, while the second can be written as

$$\begin{aligned} \lim_{\delta \downarrow 0} h \sum_{i=1}^m \int_{\Omega} (K_\epsilon \circ \beta_\epsilon^{-1})'(w_\epsilon^i) w_{\epsilon, x_k}^i \operatorname{sgn}^\delta w_{\epsilon, x_k}^i \Delta \zeta dx \\ = h \sum_{i=1}^m \int_{\Omega} (K_\epsilon \circ \beta_\epsilon^{-1})'(w_\epsilon^i) \Delta \zeta |w_{\epsilon, x_k}^i| dx \\ = h \sum_{i=1}^m \int_{\Omega} \left| \frac{\partial}{\partial x_k} K_\epsilon(w_\epsilon^i) \right| \Delta \zeta dx \end{aligned} \quad (42)$$

where we have again used the monotonicity of K_ϵ and β_ϵ^{-1} .

To estimate the last term of (33), integrate by parts and use the fact that

\mathbf{v}_ϵ is solenoidal to obtain

$$\begin{aligned} h \sum_{i=1}^m \int_{\Omega} \frac{\partial}{\partial x_k} (\mathbf{v}_\epsilon^i \cdot \nabla u_\epsilon^i) \zeta \operatorname{sgn}^\delta w_{\epsilon, x_k}^i dx &= h \sum_{i=1}^m \int_{\Omega} \frac{\partial \mathbf{v}_\epsilon^i}{\partial x_k} \cdot \nabla u_\epsilon^i \zeta \operatorname{sgn}^\delta w_{\epsilon, x_k}^i dx \\ &\quad - h \sum_{i=1}^m \int_{\Omega} u_{\epsilon, x_k}^i \mathbf{v}_\epsilon^i \cdot \nabla \zeta \operatorname{sgn}^\delta w_{\epsilon, x_k}^i dx \\ &\quad - h \sum_{i=1}^m \int_{\Omega} (\beta_\epsilon^{-1})' (w_\epsilon^i) w_{\epsilon, x_k}^i \zeta \mathbf{v}_\epsilon^i \cdot \nabla \operatorname{sgn}^\delta w_{\epsilon, x_k}^i dx. \end{aligned} \quad (43)$$

The last term above vanishes as $\delta \downarrow 0$ for the same reasons that I_1 did, thus

$$\begin{aligned} \lim_{\delta \downarrow 0} h \sum_{i=1}^m \int_{\Omega} \frac{\partial}{\partial x_k} (\mathbf{v}_\epsilon^i \cdot \nabla u_\epsilon^i) \zeta \operatorname{sgn}^\delta w_{\epsilon, x_k}^i dx \\ \geq -h \sum_{i=1}^m \int_{\Omega} \left| \frac{\partial \mathbf{v}_\epsilon^i}{\partial x_k} \right| |\nabla u_\epsilon^i| \zeta dx - h \sum_{i=1}^m \int_{\Omega} |\mathbf{v}_\epsilon^i| |u_{\epsilon, x_k}^i| |\nabla \zeta| dx. \end{aligned} \quad (44)$$

Combining these results, we obtain for any $\Omega' \Subset \Omega$ and for any $0 \leq t \leq T$

$$\begin{aligned} \int_{\Omega'} |\nabla w_{\epsilon, h}(x, t)| dx &\leq \int_{\Omega} |\nabla w_o| dx \\ &\quad + C \int_h^T \int_{\Omega} \{ |\nabla K_\epsilon(u_{\epsilon, h})| + |\nabla \mathbf{v}_{\epsilon, h}| |\nabla u_{\epsilon, h}| + |\mathbf{v}_{\epsilon, h}| |\nabla u_{\epsilon, h}| \} dx \leq C \end{aligned} \quad (45)$$

for a constant C depending only upon the data, including $\|w_o\|_{C^1(\overline{\Omega})}$, Ω , and Ω' , but independent of ϵ and h .

To obtain the pre-compactness of $\{w_{\epsilon, h}\}$, we require an additional estimate. Let $0 < t < t + \tau < T$, and choose m_1 and m_2 so that $w^{m_1}(x) = w_{\epsilon, h}(x, t)$ and $w^{m_2}(x) = w_{\epsilon, h}(x, t + \tau)$. Let $\psi \in C_0^2(\Omega)$, multiply (19) by ψ , integrate and sum from $m_1 + 1$ to m_2 to obtain

$$\begin{aligned} \sum_{i=m_1+1}^{m_2} \int_{\Omega} (w_{\epsilon, h}^i - w_{\epsilon, h}^{i-1}) \psi dx \\ = h \sum_{i=m_1+1}^{m_2} \int_{\Omega} K(u_{\epsilon, h}^i) \Delta \psi dx + h \sum_{i=m_1+1}^{m_2} \int_{\Omega} u_{\epsilon, h}^i \mathbf{v}_{\epsilon, h}^i \cdot \nabla \psi dx. \end{aligned} \quad (46)$$

Thus

$$\begin{aligned} \left| \int_{\Omega} \{w_{\epsilon, h}(x, t + \tau) - w_{\epsilon, h}(x, t)\} \psi dx \right| \\ \leq \int_t^{t+\tau} \int_{\Omega} |K(u_{\epsilon, h}) \Delta \psi + u_{\epsilon, h} \mathbf{v}_{\epsilon, h} \cdot \nabla \psi| dx ds \\ \leq C \tau \sup_{\Omega} \{|\Delta \psi| + |\nabla \psi|\} \end{aligned} \quad (47)$$

where C depends only upon $K_1, |\Omega|, \|u_{\epsilon,h}\|_{L^\infty(\Omega_T)}$ and $\sup_{0 < t < T} \|\mathbf{v}_\epsilon(\cdot, t)\|_{L_2(\Omega)}$, and so can be bounded independently of ϵ and h .

The following result, proven in [8], suffices to show that $\{w_{\epsilon,h}\}$ is precompact in $L_1(\Omega_T)$.

Lemma 3. *Let $B_R \subseteq \mathbb{R}^N$ be an open ball, and let $T > 0$. Let $r < R$, suppose that $w \in L^\infty(B_R \times [0, T])$ and that*

$$\operatorname{ess\,sup}_{0 < t < T} \int_{B_R} |\nabla w(x, t)| \, dx \leq M. \tag{48}$$

If there is a constant γ so that for any $\psi \in C_0^2(B_r)$ and almost every $0 < t < t + \tau < T$

$$\left| \int_{B_r} \{w(x, t + \tau) - w(x, t)\} \psi(x) \, dx \right| \leq \gamma \tau \sup_{B_r} \{|\psi| + |\nabla \psi| + |\Delta \psi|\}, \tag{49}$$

then for almost every $0 < t < t + \tau < T$,

$$\int_{B_r} |w(x, t + \tau) - w(x, t)| \, dx \leq C \min_{0 < \sigma < R-r} \left\{ \tau + \sigma + \frac{\tau}{\sigma} + \frac{\tau}{\sigma^2} \right\} \tag{50}$$

where C depends only on $\|w\|_{L^\infty(B_R)}, M, \gamma, r$, and N .

The pre-compactness implied by (24), (31), (45), (47) and (50) allow us to select a subsequence along which w_ϵ^i and u_ϵ^i will converge almost everywhere in Ω and ∇u_ϵ^i will converge weakly in $L_2(\Omega)$; thus for each h we obtain $w^i \subseteq \beta(u^i)$ so that

$$\frac{1}{h}(w^i - w^{i-1}) - \Delta K(u^i) + \mathbf{v}^i \cdot \nabla u^i = 0. \tag{51}$$

Because $\mathbf{v} \in L_2(0, T; W_2^1(\Omega))$, the Sobolev embedding theorem implies $\mathbf{v}^i \cdot \nabla u^i \in L_{(N+2)/(N+1)}(\Omega)$ for each $i \geq 1$; the usual estimates for elliptic equations [7, Theorem 9.11] imply that $D_x^2 u^i \in L_{(N+2)/(N+1), \text{loc}}(\Omega)$ and the equation holds almost everywhere.

If $0 < w^i(x) < L$, then $u^i(x) = 0$. Moreover, for almost every x with $u^i(x) = 0$, we know $\nabla u^i(x) = 0$ and $\Delta K(u^i(x)) = 0$ [7, Lemma 7.7]. Thus, for almost every x with $0 < w^i(x) < L$, the equation (51) implies

$$w^i(x) = w^{i-1}(x). \tag{52}$$

The compactness described above allows us to let $h \downarrow 0$ along a subsequence; standard arguments show that the limit satisfies (12)-(14). To see that (16) follows from (52), let $\{h_j\}$ be a subsequence so that $w_{h_j}(x, t) \rightarrow w(x, t)$ for almost every $(x, t) \in \Omega_T$. We begin with the claim that there exists a set $\Sigma \subset [0, T]$ with $\operatorname{meas} \Sigma = 0$ so that $w_{h_j}(x, t) \rightarrow w(x, t)$ for almost every $x \in \Omega$, for each $t \in [0, T] \setminus \Sigma$. Indeed, if $S \subset \Omega_T$ is the set upon which w_{h_j} does not

converge to w , then the sets $\{x : (x, t) \in S\} \subseteq \Omega$ are measurable, and the set $\Sigma = \{t : \text{meas}\{x : (x, t) \in S\} > 0\}$ is measurable. If $\text{meas } \Sigma > 0$, then

$$\text{meas } S = \iint_{\Omega_T} \chi_S dx dt \geq \int_{\Sigma} \text{meas}\{x : (x, t) \in S\} dt > 0 \quad (53)$$

so that $\text{meas } \Sigma = 0$.

Let $0 \leq t_1 \leq t_2 \leq T$ with $t_1, t_2 \in [0, T] \setminus \Sigma$. Recall the definition

$$M(t_2) = \{x \in \Omega : 0 < w(x, t_2) < L\}; \quad (54)$$

let $\delta > 0$ and define

$$M_\delta(t_2) = \{x \in \Omega : \delta < w(x, t_2) < L - \delta\}. \quad (55)$$

Let $\eta > 0$; by Egoroff's theorem there is a set $E_\eta \subseteq \Omega$ so that $w_{h_j}(x, t_2) \rightarrow w(x, t_2)$ uniformly in E_η and so that $\text{meas}(\Omega \setminus E_\eta) \leq \eta$. Thus, there exists an integer J so that if $j > J$ and $x \in M_\delta(t_2) \cap E_\eta$, then $0 < w_{h_j}(x, t_2) < L$.

From (52), we see that $w_{h_j}(x, t_1) = w_{h_j}(x, t_2)$ for almost every $x \in M_\delta(t_2) \cap E_\eta$; since there are only countably many functions w_{h_j} this holds independently of the choice of j . Pass to the limit in j to conclude that, for almost every $x \in M_\delta(t_2) \cap E_\eta$ we have $w(x, t_1) = w(x, t_2)$. Since δ and η are arbitrary, we can send $\eta \downarrow 0$ then $\delta \downarrow 0$ to conclude (52). \square

Proposition 4. *Let $\Omega \subset \mathbb{R}^N$ for $N \geq 2$ be a smooth domain and let $T > 0$. Suppose that $\mathbf{v}_1, \mathbf{v}_2 \in L_2(\Omega_T)$ are weakly solenoidal. If $w_i \subseteq \beta(u_i)$ and $u_i \in L_\infty(\Omega_T) \cap L_2(0, T, W_2^1(\Omega))$ satisfy*

$$\frac{\partial}{\partial t} w_i - \Delta K(u_i) + \mathbf{v}_i \cdot \nabla u_i = 0 \quad (56)$$

$$u_i|_{\partial\Omega} = g \quad (57)$$

$$w_i|_{t=0} = w_{o,i} \quad (58)$$

for $i = 1, 2$, then

$$\text{ess sup}_{0 < \tau < T} \|w_2(\cdot, \tau) - w_1(\cdot, \tau)\|_{L_2(\Omega)}^2 \leq C \|w_{o,2} - w_{o,1}\|_{L_1(\Omega)} + C \|\mathbf{v}_2 - \mathbf{v}_1\|_{L_2(\Omega_T)} \quad (59)$$

where C depends on $\|u_i\|_{L_\infty(\Omega_T)}$ and $\|\nabla u_i\|_{L_2(\Omega_T)}$.

PROOF: Set $u = u_2 - u_1$, $w = w_2 - w_1$, and $w_o = w_{o,2} - w_{o,1}$. For any $\phi \in C^\infty(\Omega_T)$ with $\phi|_{\partial\Omega \times (0, T)} = 0$ and for almost every $0 < \tau < T$,

$$\begin{aligned} & \int_{\Omega} w_i(x, \tau) \phi(x, \tau) dx - \int_0^\tau \int_{\Omega} \{w_i \phi_t + K(u_i) \Delta \phi\} dx dt \\ & + \int_0^\tau \int_{\partial\Omega} K(g) \frac{\partial \phi}{\partial \nu} d\sigma(x) dt + \int_0^\tau \int_{\Omega} u_i \mathbf{v}_i \cdot \nabla \phi dx dt = \int_{\Omega} w_{o,i} \phi(x, 0) dx \quad (60) \end{aligned}$$

where $\frac{\partial \phi}{\partial \nu}$ is the derivative in the direction of the outward normal. Subtract these identities for $i = 1, 2$ to obtain

$$\begin{aligned} \int_{\Omega} w(x, \tau) \phi(x, \tau) \, dx - \int_0^{\tau} \int_{\Omega} w \{ \phi_t + \kappa \mu \Delta \phi + \mu \mathbf{v}_2 \cdot \nabla \phi \} \, dx \, dt \\ = \int_0^{\tau} \int_{\Omega} u_1 \mathbf{v} \cdot \nabla \phi \, dx \, dt + \int_0^{\tau} \int_{\Omega} w_o \phi(x, 0) \, dx \end{aligned} \tag{61}$$

where

$$\kappa = \frac{K(u_2) - K(u_1)}{u_2 - u_1} \quad \text{and} \quad \mu = \frac{u_2 - u_1}{w_2 - w_1}. \tag{62}$$

Note that $0 < K_1 \leq \kappa \leq K_2$ and $0 \leq \mu \leq 1/\beta_0$.

To construct the test function that we shall use above, let $\epsilon > 0$, $\delta > 0$, and $\sigma > 0$, and let $\mathbf{v}_{2,\delta}$, μ_{σ} , and κ_{σ} be smooth approximations of \mathbf{v}_2 , μ and κ , with $\mathbf{v}_{2,\delta}$ solenoidal. Let $\phi_o \in C_0^3(\bar{\Omega})$ and consider

$$\frac{\partial \phi_{\delta,\epsilon,\sigma}}{\partial t} + (\kappa_{\sigma} \mu_{\sigma} + \epsilon) \Delta \phi_{\delta,\epsilon,\sigma} + \mu_{\sigma} \mathbf{v}_{2,\delta} \cdot \nabla \phi_{\delta,\epsilon,\sigma} = 0, \tag{63}$$

$$\phi_{\delta,\epsilon,\sigma} |_{\partial \Omega \times (0,\tau)} = 0, \tag{64}$$

$$\phi_{\delta,\epsilon,\sigma}(x, \tau) = \phi_o(x). \tag{65}$$

The usual parabolic theory [10, Chapter 4, Theorem 5.2] implies that this problem has a solution $\phi_{\delta,\epsilon,\sigma} \in C_{x,t}^{2,1}(\bar{\Omega}_{\tau})$. To obtain uniform estimates, multiply (63) by $\Delta \phi_{\delta,\epsilon,\sigma}$ and integrate over Ω to obtain

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \phi_{\delta,\epsilon,\sigma}|^2 \, dx \\ + \int_{\Omega} (\kappa_{\sigma} \mu_{\sigma} + \epsilon) |\Delta \phi_{\delta,\epsilon,\sigma}|^2 \, dx + \int_{\Omega} \mu_{\sigma} \mathbf{v}_{2,\delta} \cdot \nabla \phi_{\delta,\epsilon,\sigma} \Delta \phi_{\delta,\epsilon,\sigma} \, dx = 0. \end{aligned} \tag{66}$$

Then Young’s inequality implies the existence of a constant depending on K_1 and β_o so that

$$-\frac{d}{dt} \int_{\Omega} |\nabla \phi_{\delta,\epsilon,\sigma}|^2 \, dx + \int_{\Omega} (\kappa_{\sigma} \mu_{\sigma} + \epsilon) |\Delta \phi_{\delta,\epsilon,\sigma}|^2 \, dx \leq C \int_{\Omega} |\mathbf{v}_{2,\delta}|^2 |\nabla \phi_{\delta,\epsilon,\sigma}|^2 \, dx. \tag{67}$$

Thus if we allow C to depend upon δ through $\|\mathbf{v}_{2,\delta}\|_{L_{\infty}(\Omega_T)}$, we can apply Gronwall’s inequality to obtain the estimate

$$\sup_{0 < \tau < T} \int_{\Omega} |\nabla \phi_{\delta,\epsilon,\sigma}|^2 \, dx + \int_0^{\tau} \int_{\Omega} (\kappa_{\sigma} \mu_{\sigma} + \epsilon) |\Delta \phi_{\delta,\epsilon,\sigma}|^2 \, dx \, dt \leq C_{\delta} \int_{\Omega} |\nabla \phi_o|^2 \, dx. \tag{68}$$

This, together with the maximum principle, allows us to send $\sigma \downarrow 0$ and implies the existence of a function $\phi_{\delta,\epsilon} \in W_2^{2,1}(\Omega_{\tau})$ so that

$$\frac{\partial \phi_{\delta,\epsilon}}{\partial t} + (\kappa \mu + \epsilon) \Delta \phi_{\delta,\epsilon} + \mu \mathbf{v}_{2,\delta} \cdot \nabla \phi_{\delta,\epsilon} = 0, \tag{69}$$

$$\phi_{\delta,\epsilon} |_{\partial \Omega \times (0,\tau)} = 0, \tag{70}$$

$$\phi_{\delta,\epsilon}(x, \tau) = \phi_o(x), \tag{71}$$

and so that

$$\sup_{\overline{\Omega} \times [0, \tau]} |\phi_{\delta, \epsilon}| \leq \sup_{\overline{\Omega}} |\phi_o|, \quad (72)$$

$$\int_0^\tau \int_{\Omega} \left\{ \epsilon |\Delta \phi_{\delta, \epsilon}|^2 + |\nabla \phi_{\delta, \epsilon}|^2 + \left| \frac{\partial \phi_{\delta, \epsilon}}{\partial t} \right|^2 \right\} dx dt \leq C_\delta \int_{\Omega} |\nabla \phi_o|^2 dx. \quad (73)$$

Substitute this function $\phi_{\delta, \epsilon}$ into (61) to obtain

$$\begin{aligned} \int_{\Omega} w(x, \tau) \phi_o(x) dx &= - \int_0^\tau \int_{\Omega} \{ \epsilon w \Delta \phi_{\delta, \epsilon} + \phi_{\delta, \epsilon} (\mathbf{v}_2 - \mathbf{v}_{2, \delta}) \cdot \nabla u \} dx dt \\ &\quad - \int_0^\tau \int_{\Omega} \phi_{\delta, \epsilon} \mathbf{v} \cdot \nabla u_1 dx dt + \int_{\Omega} w_o \phi_{\delta, \epsilon}(x, 0) dx \end{aligned} \quad (74)$$

where we have integrated by parts. The estimate (73) implies

$$\begin{aligned} &\left| \int_0^\tau \int_{\Omega} \epsilon w \Delta \phi_{\delta, \epsilon} dx dt \right| \\ &\leq \|w\|_{L^\infty(\Omega_T)} (\epsilon |\Omega| \tau)^{1/2} \left(\int_0^\tau \int_{\Omega} \epsilon |\Delta \phi_{\delta, \epsilon}|^2 dx dt \right)^{1/2} \leq C_\delta \sqrt{\epsilon} \end{aligned} \quad (75)$$

so that we may pass to the limit as $\epsilon \downarrow 0$ to obtain

$$\begin{aligned} \left| \int_{\Omega} w(x, \tau) \phi_o(x) dx \right| &\leq \|\phi_o\|_{L^\infty(\Omega_T)} \int_0^\tau \int_{\Omega} \{ |\mathbf{v} - \mathbf{v}_\delta| |\nabla u| + |\mathbf{v}| |\nabla u_2| \} dx dt \\ &\quad + \|\phi_o\|_{L^\infty(\Omega)} \int_{\Omega} |w_o| dx \end{aligned} \quad (76)$$

which, by completeness, holds for every $\phi_o \in L^\infty(\Omega)$. Let $\delta \downarrow 0$ and set $\phi_o = w(x, \tau)$ to obtain the result. \square

PROOF OF THEOREM 1.

Let $Q_R(x_o, t_o) = B_R(x_o) \times (t_o - R, t_o) \subseteq \Omega_T$. Then for almost every $R/2 < r < R$,

- (i.) $u \in L^\infty(\partial B_r(x_o) \times (t_o - R, t_o))$,
- (ii.) $\nabla u \in L_2(\partial B_r(x_o) \times (t_o - R, t_o))$,
- (iii.) $w(\cdot, t_o - r) \in L^\infty(B_R(x_o))$,
- (iv.) $w(\cdot, t)$ converges weakly in $L_2(\Omega)$ to $w(\cdot, t_o - r)$ as $t \downarrow t_o - r$.

Let \mathbf{v}_ϵ be a smooth solenoidal approximation of \mathbf{v} , and let $w_{o, \epsilon}$ be a smooth approximation of $w(\cdot, t_o - r)$. Proposition 2 implies the existence of functions

$w_\epsilon \subseteq \beta(u_\epsilon)$ satisfying

$$\frac{\partial}{\partial t} w_\epsilon - \Delta K(u_\epsilon) + \mathbf{v}_\epsilon \cdot \nabla u_\epsilon = 0 \quad \text{in } Q_r(x_o), \quad (77)$$

$$u_\epsilon|_{\partial B_r(x_o) \times (t_o-r, t_o)} = w, \quad (78)$$

$$w_\epsilon|_{t=t_o-r} = w_{o,\epsilon}. \quad (79)$$

Proposition 3 implies that $w_\epsilon \rightarrow w$ almost everywhere in $Q_r(x_o, t_o)$, and hence for almost every $t_o - r \leq t_1 \leq t_2 \leq t_o$ and almost every $x \in B_r(x_o)$ with $0 < w(x, t_2) < L$,

$$w(x, t_2) = w(x, t_1). \quad (80)$$

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