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POSITIVE PERIODIC SOLUTIONS FOR THE KORTEWEG-DE VRIES EQUATION

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Abstract. In this paper we prove that the Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0$$

has unique positive solution $u(t,x)$ which is ω-periodic with respect to the time variable t and $u(0,x) \in \dot{B}^{\gamma}_{p,q}([a,b])$, $\gamma > 0$, $\gamma \notin \{1,2,\dots\}$, $p > 1$, $q \geq 1$, $a < b$ are fixed constants, $x \in [a,b]$. The period $\omega > 0$ is arbitrary chosen and fixed.

1. Introduction

In this paper we consider the initial-value problem for the Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad t \in \mathbb{R}, \quad x \in [a,b], \tag{1.1}$$

$$u \text{ is periodic in } t, \tag{1.2}$$

$$u(0,x) = u_0, \quad u_0 \in \dot{B}^{\gamma}_{p,q}([a,b]), \tag{1.3}$$

where $q \geq 1$, $1 < p < \infty$, $\gamma > 0$, $\gamma \notin \{1,2,\dots\}$. We prove that the (1.1)–(1.3) has unique positive solution in the form $u(t,x) = v(t)q(x)$, which is continuous ω-periodic with respect to the time variable t. When we say that the solution $u(t,x)$ of the (1.1) is positive we understand: $u(t,x) > 0$ for $t \in \mathbb{R}$, $x \in [a,b]$. Here the period $\omega > 0$ is arbitrary chosen and fixed.

Bourgain [1] consider the initial-value problem

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0,$$
$$u \text{ is periodic in } x,$$
$$u(0,x) = u_0.$$

He proved that the above problem is globally well-posed for H^s-data ($s \geq 0$, integer). Bourgain [1] used the Fourier restriction space method, which he introduced.

Here we use the theory of completely continuous vector field presented by Krasnosel'skii and Zabrejko and we prove that the Korteweg-de Vries (1.1) has unique positive solution $u(t,x) = v(t)q(x)$, which is continuous ω-periodic with respect to

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the time variable t and infinitely differentiable with respect to the space variable $x \in [a,b]$ and $u(0,x) \in \dot{B}^{\gamma}_{p,q}([a,b])$, $p > 1$, $q \geq 1$, $\gamma > 0$, $\gamma \notin \{1,2,\dots\}$.

To state our main result we use the following hypotheses:

(H1) $q \in \mathcal{C}^{\infty}([a,b])$, $q(x) > 0$ for all $x \in [a,b]$;
(H2) $q'(x) < 0$, $q'''(x) > 0$ for all $x \in [a,b]$.

Theorem 1.1. *Let $q \geq 1$, $1 < p < \infty$, $\gamma > 0$, $\gamma \notin \{1,2,\dots\}$ be fixed. Then the initial-value problem* (1.1)–(1.3) *has unique positive solution $u(t,x) = v(t)q(x)$, which is continuous ω-periodic with respect to the time variable t and infinitely differentiable with respect to the space variable $x \in [a,b]$, where $q(x)$ is a fixed function satisfying* (H1)–(H2).

This paper is organized as follows: In section 2 we prove that the (1.1)–(1.3) has positive solution $u(t,x) = v(t)q(x)$ which is continuous ω-periodic with respect to the time variable t and infinitely differentiable with respect to the space variable $x \in [a,b]$, where $q(x)$ is fixed function satisfying (H1)–(H2). In section 3 we prove that the solution obtained in section 2, is unique.

2. Existence of positive periodic solutions

Here and bellow we will suppose that $q(x)$ is fixed function satisfying (H1)–(H2). As an example of such function, we have $q(x) = 2 + \sin x$ with $[a,b] = [2\pi/3, 5\pi/6]$.

Proposition 2.1. *If for every fixed $x \in [a,b]$, $u(t,x) = v(t)q(x)$ satisfies*

$$u(t,x) = -\int_0^{\omega} \frac{e^{-\frac{q'''(x)}{q(x)}s}}{1 - e^{-\frac{q'''(x)}{q(x)}\omega}} u^2(t-s,x) \frac{q'(x)}{q(x)} ds, \tag{2.1}$$

then $u(t,x) = v(t)q(x)$ satisfies the (1.1) *for every fixed $x \in [a,b]$. Here $v(t)$ is a positive continuous ω-periodic function.*

Proof. For every fixed $x \in [a,b]$ if $u(t,x) = v(t)q(x)$ is a solution to (2.1), we have

$$\begin{aligned} v(t)q(x) &= -\int_0^{\omega} \frac{e^{-\frac{q'''(x)}{q(x)}s}}{1 - e^{-\frac{q'''(x)}{q(x)}\omega}} v^2(t-s) q^2(x) \frac{q'(x)}{q(x)} ds \\ &= -\int_0^{\omega} \frac{e^{-\frac{q'''(x)}{q(x)}s}}{1 - e^{-\frac{q'''(x)}{q(x)}\omega}} v^2(t-s) q(x) q'(x) ds. \end{aligned}$$

From here,

$$v(t) = -\int_0^{\omega} \frac{e^{-\frac{q'''(x)}{q(x)}s}}{1 - e^{-\frac{q'''(x)}{q(x)}\omega}} v^2(t-s) q'(x) ds;$$

i.e., for every fixed $x \in [a,b]$, if $u(t,x) = v(t)q(x)$ is a solution to (2.1) we have

$$v(t) = -\frac{q'(x)}{1 - e^{-\frac{q'''(x)}{q(x)}\omega}} \int_0^{\omega} e^{-\frac{q'''(x)}{q(x)}s} v^2(t-s) ds. \tag{2.2}$$

Let us consider the integral

$$\int_0^{\omega} e^{-\frac{q'''(x)}{q(x)}s} v^2(t-s) ds.$$

We make the change of variable $s = t - z$, from where $ds = -dz$ and

$$\begin{aligned}\int_0^\omega e^{-\frac{q'''(x)}{q(x)}s} v^2(t-s)ds &= -\int_t^{t-\omega} e^{-\frac{q'''(x)}{q(x)}(t-z)} v^2(z)dz\\ &= e^{-\frac{q'''(x)}{q(x)}t}\Big(\int_0^t e^{\frac{q'''(x)}{q(x)}z} v^2(z)dz - \int_0^{t-\omega} e^{\frac{q'''(x)}{q(x)}z} v^2(z)dz\Big).\end{aligned}$$

Then the equality (2.2) takes the form

$$v(t) = -\frac{q'(x)}{1-e^{-\frac{q'''(x)}{q(x)}\omega}} e^{-\frac{q'''(x)}{q(x)}t}\Big(\int_0^t e^{\frac{q'''(x)}{q(x)}z} v^2(z)dz - \int_0^{t-\omega} e^{\frac{q'''(x)}{q(x)}z} v^2(z)dz\Big).$$

From the above equality, for every fixed $x \in [a, b]$, we get

$$\begin{aligned}v'(t) &= -\frac{q'(x)}{1-e^{-\frac{q'''(x)}{q(x)}\omega}} e^{-\frac{q'''(x)}{q(x)}t}\Big[-\frac{q'''(x)}{q(x)}\Big(\int_0^t e^{\frac{q'''(x)}{q(x)}z} v^2(z)dz\\ &\quad -\int_0^{t-\omega} e^{\frac{q'''(x)}{q(x)}z} v^2(z)dz\Big) + e^{\frac{q'''(x)}{q(x)}t} v^2(t) - e^{\frac{q'''(x)}{q(x)}(t-\omega)} v^2(t-\omega)\Big]\\ &= \frac{q'''(x)}{q(x)}\frac{q'(x)}{1-e^{-\frac{q'''(x)}{q(x)}\omega}} e^{-\frac{q'''(x)}{q(x)}t}\Big(\int_0^t e^{\frac{q'''(x)}{q(x)}z} v^2(z)dz - \int_0^{t-\omega} e^{\frac{q'''(x)}{q(x)}z} v^2(z)dz\Big)\\ &\quad - \frac{q'(x)}{1-e^{-\frac{q'''(x)}{q(x)}\omega}}\Big(1-e^{-\frac{q'''(x)}{q(x)}\omega}\Big)v^2(t)\\ &= -\frac{q'''(x)}{q(x)}v(t) - q'(x)v^2(t);\end{aligned}$$

i.e., for every fixed $x \in [a, b]$ we have

$$v'(t) = -\frac{q'''(x)}{q(x)}v(t) - q'(x)v^2(t).$$

Then

$$q(x)v'(t) = -q'''(x)v(t) - q'(x)q(x)v^2(t) \tag{2.3}$$

for every fixed $x \in [a, b]$. Since for every fixed $x \in [a, b]$ we have

$$\begin{gathered}u_t = v'(t)q(x),\\ \partial_x^3 u = q'''(x)v(t),\\ u\partial_x u = q'(x)q(x)v^2(t).\end{gathered}$$

From the equality (2.3) we take

$$u_t = -\partial_x^3 u - u\partial_x u;$$

i.e., for every fixed $x \in [a, b]$, if $u(t, x) = v(t)q(x)$ is a solution to the (2.1), then $u(t, x)$ satisfies the Korteweg-de Vries equation (1.1). □

Proposition 2.2. *If for every fixed $x \in [a, b]$, $u(t, x) = v(t)q(x)$ satisfies the Korteweg-de Vries equation* (1.1) *then $u(t, x) = v(t)q(x)$ satisfies the integral* (2.1)*. Here $v(t)$ is positive continuous ω-periodic function.*

Proof. Let $x \in [a, b]$ is fixed and $u(t, x) = v(t)q(x)$ is a solution to the Korteweg-de Vries (1.1), where $v(t)$ is positive continuous ω-periodic function. Then

$$v'(t)q(x) = -q'''(x)v(t) - v^2(t)q'(x)q(x).$$

After we use the definition of the function $q(x)$ (see (H1), (H2)) from the last equation we get

$$v'(t) = -\frac{q'''(x)}{q(x)}v(t) - q'(x)v^2(t).$$

Since $x \in [a,b]$ is fixed, the last equation we may consider as ordinary differential equation with respect to the variable t. Therefore

$$\begin{aligned} v(t) &= e^{-\int_0^t \frac{q'''(x)}{q(x)}ds}\Big(v(0) - \int_0^t q'(x)v^2(s)e^{\int_0^s \frac{q'''(x)}{q(x)}d\tau}ds\Big) \\ &= e^{-\frac{q'''(x)}{q(x)}t}\Big(v(0) - \int_0^t q'(x)v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds\Big). \end{aligned}$$

For $q'''(x) > 0$, $q(x) > 0$ for $x \in [a,b]$ we have $\lim_{t\to-\infty} e^{-\frac{q'''(x)}{q(x)}t} = \infty$. Therefore,

$$v(0) = q'(x)\int_0^{-\infty} v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds = -q'(x)\int_{-\infty}^0 v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds$$

or

$$v(t) = -q'(x)e^{-\frac{q'''(x)}{q(x)}t}\int_{-\infty}^t v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds. \tag{2.4}$$

Now we consider the integral

$$\int_{-\infty}^t v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds.$$

We have

$$\int_{-\infty}^t v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds = \int_{t-\omega}^t v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds + \int_{t-2\omega}^{t-\omega} v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds + \dots. \tag{2.5}$$

Let

$$J = \int_{t-\omega}^t v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds.$$

Let us consider the integral

$$\int_{t-2\omega}^{t-\omega} v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds.$$

After the change of variable $s + \omega = \tau$, we obtain

$$\int_{t-2\omega}^{t-\omega} v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds = e^{-\frac{q'''(x)}{q(x)}\omega}\int_{t-\omega}^t v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds = e^{-\frac{q'''(x)}{q(x)}\omega}J.$$

In the same way,

$$\int_{t-3\omega}^{t-2\omega} v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds = e^{-\frac{q'''(x)}{q(x)}\omega}\int_{t-2\omega}^{t-\omega} v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds = e^{-2\frac{q'''(x)}{q(x)}\omega}J$$

and so on and so forth. Then the equality (2.5) takes the form

$$\int_{-\infty}^t v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds = J\Big(1 + e^{-\frac{q'''(x)}{q(x)}\omega} + e^{-2\frac{q'''(x)}{q(x)}\omega} + \dots\Big) = J\frac{1}{1 - e^{-\frac{q'''(x)}{q(x)}\omega}},$$

because $\frac{q'''(x)}{q(x)} > 0$ for every fixed $x \in [a,b]$, $e^{-\frac{q'''(x)}{q(x)}\omega} < 1$ for every fixed $x \in [a,b]$. Therefore, from (2.4), for every fixed $x \in [a,b]$ we get

$$v(t) = -q'(x)e^{-\frac{q'''(x)}{q(x)}t}\frac{1}{1-e^{-\frac{q'''(x)}{q(x)}\omega}}\int_{t-\omega}^{t} v^2(s)e^{\frac{q'''(x)}{q(x)}s}ds.$$

Now we make the change of variable $s - t = \tau$. Then

$$\begin{aligned} v(t) &= -q'(x)e^{-\frac{q'''(x)}{q(x)}t}\frac{1}{1-e^{-\frac{q'''(x)}{q(x)}\omega}}\int_{-\omega}^{0} v^2(t+\tau)e^{\frac{q'''(x)}{q(x)}\tau}e^{\frac{q'''(x)}{q(x)}t}d\tau \\ &= -q'(x)\frac{1}{1-e^{-\frac{q'''(x)}{q(x)}\omega}}\int_{-\omega}^{0} v^2(t+\tau)e^{\frac{q'''(x)}{q(x)}\tau}d\tau. \end{aligned}$$

Let $\tau = -z$. Then

$$v(t) = -q'(x)\frac{1}{1-e^{-\frac{q'''(x)}{q(x)}\omega}}\int_{0}^{\omega} v^2(t-z)e^{-\frac{q'''(x)}{q(x)}z}dz.$$

From where for every fixed $x \in [a,b]$,

$$u(t,x) = -\frac{q'(x)}{q(x)}\frac{1}{1-e^{-\frac{q'''(x)}{q(x)}\omega}}\int_{0}^{\omega} u^2(t-z,x)e^{-\frac{q'''(x)}{q(x)}z}dz;$$

i. e., for every fixed $x \in [a,b]$, $u(t,x)$ satisfies (2.1). $\square$

Let $\mathcal{C}(\omega)$ be the space of the real continuous ω-periodic functions defined on the whole axis. With $\mathcal{C}_+(\omega)$ we denote the space of the positive continuous ω-periodic functions defined on the whole axis. Let

$$D_q^+ = \max_{0\le s\le\omega,\, x\in[a,b]} e^{-\frac{q'''(x)}{q(x)}s}, \quad D_q^- = \min_{0\le s\le\omega,\, x\in[a,b]} e^{-\frac{q'''(x)}{q(x)}s}.$$

With $\mathcal{C}_+^\circ(\omega) \subset \mathcal{C}_+(\omega)$ we denote the cone

$$\mathcal{C}_+^\circ(\omega) = \big\{x \in \mathcal{C}_+(\omega) : \min_t x(t) \ge \frac{D_q^-}{D_q^+}\max_t x(t)\big\}.$$

For every fixed $x \in [a,b]$ we define the operator

$$\chi(u) = -\frac{q'(x)}{q(x)}\int_0^\omega u^2(t-s,x)\frac{e^{-\frac{q'''(x)}{q(x)}s}}{1-e^{-\frac{q'''(x)}{q(x)}\omega}}ds,$$

where $u(t,x) = v(t)q(x)$, $v(t)$ is a positive continuous ω-periodic function, $q(x)$ is a function satisfying (H1), (H2).

Proposition 2.3. *For every fixed $x \in [a,b]$ we have $\chi : \mathcal{C}_+(\omega) \to \mathcal{C}_+^\circ(\omega)$.*

Proof. Let $x \in [a,b]$ is fixed. Let also $u(t,x) \in \mathcal{C}_+(\omega)$. $u(t,x)$ is continuous ω-periodic with respect to the time variable t. Then

$$\begin{aligned}\chi(u) &= -\frac{q'(x)}{q(x)} \int_0^\omega u^2(t-s,x) \frac{e^{-\frac{q'''(x)}{q(x)}s}}{1-e^{-\frac{q'''(x)}{q(x)}\omega}} ds \\ &\geq D_q^- \frac{1}{1-e^{-\frac{q'''(x)}{q(x)}\omega}} \Big(-\frac{q'(x)}{q(x)} \int_0^\omega u^2(t-s,x)ds\Big) \\ &= D_q^- \frac{1}{1-e^{-\frac{q'''(x)}{q(x)}\omega}} \Big(-\frac{q'(x)}{q(x)} \int_0^\omega u^2(s,x)ds\Big);\end{aligned}$$

i.e., for every fixed $x \in [a,b]$ we have

$$\chi(u) \geq D_q^- \frac{1}{1-e^{-\frac{q'''(x)}{q(x)}\omega}} \Big(-\frac{q'(x)}{q(x)} \int_0^\omega u^2(s,x)ds\Big).$$

From where, for every fixed $x \in [a,b]$, we have

$$\min_t \chi(u) \geq D_q^- \frac{1}{1-e^{-\frac{q'''(x)}{q(x)}\omega}} \Big(-\frac{q'(x)}{q(x)} \int_0^\omega u^2(s,x)ds\Big). \tag{2.6}$$

On the other hand, for every fixed $x \in [a,b]$, we have

$$\chi(u) \leq D_q^+ \frac{1}{1-e^{-\frac{q'''(x)}{q(x)}\omega}} \Big(-\frac{q'(x)}{q(x)} \int_0^\omega u^2(s,x)ds\Big).$$

Therefore, for every fixed $x \in [a,b]$, we have

$$\max_t \chi(u) \leq D_q^+ \frac{1}{1-e^{-\frac{q'''(x)}{q(x)}\omega}} \Big(-\frac{q'(x)}{q(x)} \int_0^\omega u^2(s,x)ds\Big).$$

From this inequality and (2.6),

$$\min_t \chi(u) \geq \frac{D_q^-}{D_q^+} \max_t \chi(u)$$

for every fixed $x \in [a,b]$. Consequently for every fixed $x \in [a,b]$ we have

$$\chi : \mathcal{C}_+(\omega) \to \mathcal{C}_+^\circ(\omega).$$

□

From proposition 2.3, we have that $\chi : \mathcal{C}_+^\circ(\omega) \to \mathcal{C}_+^\circ(\omega)$, i.e. the operator χ is positive with respect to the cone $\mathcal{C}_+^\circ(\omega)$ for every fixed $x \in [a,b]$.

Proposition 2.4. *The operator χ is completely continuous in the space $\mathcal{C}(\omega)$ for every fixed $x \in [a,b]$.*

Proof. Let $x \in [a,b]$ be fixed. Let also $u(t,x) \in \mathcal{C}(\omega)$, $\max_{t\in[0,\omega]} |u(t,x)| = r$, $r > 0$. $u(t,x)$ is continuous ω- periodic with respect to the time variable t. From the definition of the operator χ we have

$$|\chi(u)|(t) \leq \max_{x\in[a,b]} \Big(-\frac{q'(x)}{q(x)}\Big)\omega r^2 \frac{1}{1-e^{\max_{x\in[a,b]}\left(-\frac{q'''(x)}{q(x)}\right)\omega}}.$$

Consequently the functions $\chi(u)(t)$ are uniformly bounded in the space $\mathcal{C}(\omega)$ for every fixed $x \in [a,b]$.

Let $\epsilon > 0$. Then there exists $\delta > 0$ such that
$$-\frac{q'(x)}{q(x)} \frac{e^{-\frac{q'''(x)}{q(x)}s}}{1-e^{-\frac{q'''(x)}{q(x)}\omega}} |u^2(t_1 - s, x) - u^2(t_2 - s, x)| < \frac{\epsilon}{\omega}$$
for $|t_1 - t_2| < \delta$ and for every $s \in [0, \omega]$, for every fixed $x \in [a, b]$. Therefore
$$|\chi(u)(t_1) - \chi(u)(t_2)| < \epsilon$$
for $|t_1 - t_2| < \delta$, for every fixed $x \in [a, b]$. Then $\chi(u)$ is equicontinuous for every fixed $x \in [a, b]$. From the Arzela-Ascoli theorem follows that the set $\{\chi(u)(t)\}$ is compact subset in the space $\mathcal{C}(\omega)$ for every fixed $x \in [a, b]$. From here and from uniformly bounded of the functions $\chi(u)(t)$ follows that the operator χ is completely continuous in the space $\mathcal{C}(\omega)$ for every fixed $x \in [a, b]$. □

Proposition 2.5. *Let $v(t)$ is continuous ω-periodic function and $q(x) \in \mathcal{C}^\infty([a, b])$. Then for every $\gamma > 0$, $\gamma \notin \{1, 2, \dots\}$, $p > 1$, $q \geq 1$ we have $u(t, x) = v(t)q(x) \in \dot{B}^\gamma_{p,q}([a, b])$ for every $t \in [0, \omega]$.*

Proof. Here we use the following definition of the $\dot{B}^\gamma_{p,q}([a, b])$-norm (see [3]).
$$\|u\|^q_{\dot{B}^\gamma_{p,q}([a,b])} = \int_0^1 h^{-1-(\gamma-k)q} \Big\|\Delta_h \frac{\partial^k}{\partial x^k} u\Big\|^q_{L^p([a,b])} dh,$$
where
$$\Delta_h u(t, x) = u(t, x + h) - u(t, x),$$
$k \in \{0, 1, 2, \dots\}$, $\gamma - k = \{\gamma\}$, $\{\gamma\}$ is the fractional part of γ, $0 < \{\gamma\} < 1$. Then, after we use the middle point theorem we have
$$\begin{aligned}
\|u\|^q_{\dot{B}^\gamma_{p,q}([a,b])} &= \int_0^1 h^{-1-(\gamma-k)q} \big\|\Delta_h \frac{\partial^k}{\partial x^k} u\big\|^q_{L^p([a,b])} dh \\
&\leq C_1 \int_0^1 h^{-(\gamma-k)q+q-1} \big\| \frac{\partial^{k+1}}{\partial x^{k+1}} u\big\|^q_{L^p[a,b]} dh \\
&\leq C_2 \int_0^1 h^{-(\gamma-k)q+q-1} dh < \infty,
\end{aligned}$$
because $q - (\gamma - k)q > 0$. Here C_1 and C_2 are positive constants. □

The proof for existence of nontrivial solution to the Korteweg-de Vries equation, which is positive continuous ω-periodic with respect to the variable t and positive continuous with respect to the variable x is based on the theory of completely continuous vector field presented by Krasnosel'skii and Zabrejko in [2]. More precisely we will prove that the (1.1) has nontrivial solution, which is positive continuous ω-periodic with respect to the variable t and positive continuous with respect to the variable x after we use the following theorem which is extracted from [2].

Theorem 2.6 ([2])**.** *Let Y be a real Banach space with a cone Q and $L : Y \to Y$ be a completely continuous and positive with respect to Q operator. Then the following propositions are valid.*

- (i) *Let $L(0) = 0$. Let also for every sufficiently small $r > 0$ there is no $y \in Q$, $\|y\|_Y = r$, with $y \overset{\circ}{\leq} L(y)$. Then there exists* $\operatorname{ind}(0, L; Q) = 1$.
- (ii) *Let for every sufficiently large R there is no $y \in Q$ with $\|y\|_Y = R$ and $L(y) \overset{\circ}{\leq} y$. Then there exists* $\operatorname{ind}(\infty, L; Q) = 0$.

(iii) *Let* $L(0) = 0$ *and let there exist* $\operatorname{ind}(0, L; Q) \neq \operatorname{ind}(\infty, L; Q)$. *Then* L *has nontrivial fixed point in* Q.

Here $\operatorname{ind}(\cdot, L; Q)$ denotes an index of a point with respect to L and Q. The symbol $\overset{\circ}{\leq}$ denotes the semiordering generated by Q.

Theorem 2.7. *Let* $\gamma > 0$, $\gamma \notin \{1, 2, \dots\}$, $p > 1$, $q \geq 1$. *Let also* $q(x)$ *is a function which satisfies the hypothesis (H1) and (H2). Then the Korteweg- de Vries* (1.1) *has a positive solution in the form* $u(t,x) = v(t)q(x)$, *which is* ω*-periodic with respect to the time variable* t *and* $u(0,x) \in \dot{B}^{\gamma}_{p,q}([a,b])$.

Proof. First we note that $\chi(0) = 0$. Also, from Propositions 2.3 and 2.4, we have that the operator χ is positive and completely continuous with respect to the cone $\mathcal{C}^{\circ}_{+}(\omega)$ for every fixed $x \in [a,b]$. Let $x \in [a,b]$ is fixed.
(1) Let $r > 0$ satisfy the inequality

$$r < \frac{D_q^-}{{D_q^+}^2 \max_{x\in[a,b]}\Big(-\frac{q'(x)}{q(x)}\Big)\omega}\Big(1 - e^{\max_{x\in[a,b]}\big(-\frac{q'(x)}{q(x)}\big)\omega}\Big). \tag{2.7}$$

We suppose that there exists $u(t,x) \in \mathcal{C}^{\circ}_{+}(\omega)$ for which

$$\max_t u(t,x) = r, \quad u \leq \chi(u), \quad t \in [0,\omega],$$

for every fixed $x \in [a,b]$. Then

$$u(t,x) \leq D_q^+ \max_{x\in[a,b]}\Big(-\frac{q'(x)}{q(x)}\Big)\frac{1}{1 - e^{\max_{x\in[a,b]}\big(-\frac{q'(x)}{q(x)}\big)\omega}}\int_0^\omega u^2(t-s,x)ds. \tag{2.8}$$

From the definition of the cone $\mathcal{C}^{\circ}_{+}(\omega)$ we have for every fixed $x \in [a,b]$,

$$u(t,x) \leq \max_t u(t,x) \leq \frac{D_q^+}{D_q^-}\min_t u(t,x) \leq \frac{D_q^+}{D_q^-}\max_t u(t,x) = r\frac{D_q^+}{D_q^-}.$$

From this and (2.8), we have

$$u(t,x) \leq r\frac{{D_q^+}^2}{D_q^-}\max_{x\in[a,b]}\Big(-\frac{q'(x)}{q(x)}\Big)\frac{1}{1 - e^{\max_{x\in[a,b]}\big(-\frac{q'(x)}{q(x)}\big)}}\int_0^\omega u(t-s,x)ds.$$

Now we integrate the last inequality from 0 to ω with respect to the time variable t and we get

$$\int_0^\omega u(s,x)ds \leq \omega r\frac{{D_q^+}^2}{D_q^-}\max_{x\in[a,b]}\Big(-\frac{q'(x)}{q(x)}\Big)\frac{1}{1 - e^{\max_{x\in[a,b]}\big(-\frac{q'(x)}{q(x)}\big)}}\int_0^\omega u(s,x)ds.$$

From the last inequality we have

$$1 \leq \omega r\frac{{D_q^+}^2}{D_q^-}\max_{x\in[a,b]}\Big(-\frac{q'(x)}{q(x)}\Big)\frac{1}{1 - e^{\max_{x\in[a,b]}\big(-\frac{q'(x)}{q(x)}\big)}}$$

or

$$r \geq \frac{D_q^-}{{D_q^+}^2 \max_{x\in[a,b]}\Big(-\frac{q'(x)}{q(x)}\Big)\omega}\Big(1 - e^{\max_{x\in[a,b]}\big(-\frac{q'(x)}{q(x)}\big)\omega}\Big)$$

which is a contradiction with (2.7). Consequently for every enough small $r > 0$ there is no $u(t,x) \in \mathcal{C}_+^\circ(\omega)$ such that $\max_t u(t,x) = r$ for every fixed $x \in [a,b]$, $u(t,x) \leq \chi(u)$ for every fixed $x \in [a,b]$ and $t \in [0,\omega]$. From here and from Theorem 2.6(i) we get that there exists $\mathrm{ind}(0,\chi;\mathcal{C}_+^\circ(\omega)) = 1$.

(2) Let $R > 0$ be large enough so that

$$R > \frac{D_q^+}{{D_q^-}^2 \min_{x\in[a,b]}\Big(-\frac{q'(x)}{q(x)}\Big)\omega}\Big(1 - e^{\min_{x\in[a,b]}\big(-\frac{q'(x)}{q(x)}\big)\omega}\Big). \tag{2.9}$$

We suppose that there exists $u(t,x) \in \mathcal{C}_+^\circ(\omega)$ for which

$$\max_t u(t,x) = R, \quad u \geq \chi(u)$$

for every fixed $x \in [a,b]$ and for every $t \in [0,\omega]$. Then

$$u(t,x) \geq D_q^- \min_{x\in[a,b]}\Big(-\frac{q'(x)}{q(x)}\Big)\frac{1}{1 - e^{\min_{x\in[a,b]}\big(-\frac{q'(x)}{q(x)}\big)}}\int_0^\omega u^2(t-s,x)ds. \tag{2.10}$$

From the definition of the cone $\mathcal{C}_+^\circ(\omega)$ we have for every fixed $x \in [a,b]$

$$u(t,x) \geq \min_t u(t,x) \geq \frac{D_q^-}{D_q^+}\max_t u(t,x) = R\frac{D_q^-}{D_q^+}.$$

Therefore, from (2.10), we have

$$u(t,x) \geq R\frac{{D_q^-}^2}{D_q^+}\min_{x\in[a,b]}\Big(-\frac{q'(x)}{q(x)}\Big)\frac{1}{1 - e^{\min_{x\in[a,b]}\big(-\frac{q'(x)}{q(x)}\big)}}\int_0^\omega u(t-s,x)ds.$$

Now we integrate the above inequality from 0 to ω with respect to t and obtain

$$\int_0^\omega u(s,x)ds \geq \omega R\frac{{D_q^-}^2}{D_q^+}\min_{x\in[a,b]}\Big(-\frac{q'(x)}{q(x)}\Big)\frac{1}{1 - e^{\min_{x\in[a,b]}\big(-\frac{q'(x)}{q(x)}\big)}}\int_0^\omega u(s,x)ds.$$

From the above inequality we have

$$1 \geq \omega R\frac{{D_q^-}^2}{D_q^+}\min_{x\in[a,b]}\Big(-\frac{q'(x)}{q(x)}\Big)\frac{1}{1 - e^{\min_{x\in[a,b]}\big(-\frac{q'(x)}{q(x)}\big)}}$$

or

$$R \leq \frac{D_q^+}{{D_q^-}^2 \min_{x\in[a,b]}\Big(-\frac{q'(x)}{q(x)}\Big)\omega}\Big(1 - e^{\min_{x\in[a,b]}\big(-\frac{q'(x)}{q(x)}\big)\omega}\Big)$$

which is a contradiction with (2.9). Consequently for every enough large $R > 0$ there is no $u(t,x) \in \mathcal{C}_+^\circ(\omega)$ such that $\max_t u(t,x) = R$ for every fixed $x \in [a,b]$, $u(t,x) \geq \chi(u)$ for every fixed $x \in [a,b]$ and $t \in [0,\omega]$. From here and from Theorem 2.6(ii) we get that there exists $\mathrm{ind}(\infty,\chi;\mathcal{C}_+^\circ(\omega)) = 0$.

From (1) and (2) follows that there exist

$$\mathrm{ind}(\infty,\chi;\mathcal{C}_+^\circ(\omega)) \neq \mathrm{ind}(0,\chi;\mathcal{C}_+^\circ(\omega)).$$

Consequently, from Theorem 2.6 (iii), we conclude that the operator χ has a non-trivial fixed point in the cone $\mathcal{C}_+^\circ(\omega)$ for every fixed $x \in [a,b]$. Therefore the Korteweg - de Vries equation (1.1) has positive solution $u(t,x) = v(t)q(x)$, which is continuous ω-periodic with respect to the time variable t and from Proposition 2.5 we have $u(0,x) \in \dot{B}^{\gamma}_{p,q}([a,b])$ for every $x \in [a,b]$. $\square$

3. Uniqueness of the positive periodic solutions

Here we use the following theorem.

Theorem 3.1 ([2]). *Let Q is a physical cone in the Banach space Y and the operator $A : Y \to Q$ is monotonous u_0-convex operator ($u_0 \in Q$). Let also for every two solutions x_1 and x_2 to the equation $x = Ax$ one of the differences $x_1 - x_2$, $x_2 - x_1$ is equal to zero or is inside element for the cone Q. Then the equation $x = Ax$ has in the cone Q no more than one nontrivial solution.*

We say that the operator $A : Y \to Y$, where Y is a Banach space with a cone Q, is *monotonous* if: $y_1 \in Y$, $y_2 \in Y$, with $y_1 \overset{\circ}{\leq} y_2$ then $Ay_1 \overset{\circ}{\leq} Ay_2$. Here $\overset{\circ}{\leq}$ denotes the semiordering generating by Q.

We say that the operator $A : Y \to Y$, Y is a Banach space with a cone Q, $A : Q \to Q$, is a u_0-*convex* operator ($u_0 \in Q$) if for every $x \in Q$, $x \neq 0$, then

$$\alpha(x)u_0 \leq Ax \leq \beta(x)u_0,$$

where $\alpha(x) > 0$, $\beta(x) > 0$; and for every $x \in Q$ for which

$$\alpha_1(x)u_0 \leq Ax \leq \beta_1(x)u_0$$

($\alpha_1(x) > 0$, $\beta_1(x) > 0$) we have

$$A(\lambda x) \leq [1 - \eta(x,\lambda)]\lambda Ax, \quad 0 < \lambda < 1,$$

where $\eta(x,\lambda) > 0$.

Here and bellow we suppose that $q(x)$ is the function satisfying the conditions in Theorem 2.7. Let

$$K(x,s) = -\frac{q'(x)}{q(x)} \frac{e^{-\frac{q'''(x)}{q(x)}s}}{1 - e^{-\frac{q'''(x)}{q(x)}\omega}}, \quad x \in [a,b], s \in [0,\omega].$$

From the above assumptions follows that there exist constants $m > 0$, $M > 0$ such that

$$m \leq K(x,s) \leq M, \quad \forall x \in [a,b], \quad \forall s \in [0,\omega].$$

For instance

$$m = \min_{x\in[a,b]} \Big(-\frac{q'(x)}{q(x)}\Big) \frac{e^{-\max_{x\in[a,b]} \frac{q'''(x)}{q(x)}\omega}}{1 - e^{-\max_{x\in[a,b]} \frac{q'''(x)}{q(x)}\omega}},$$

$$M = \max_{x\in[a,b]} \Big(-\frac{q'(x)}{q(x)}\Big) \frac{1}{1 - e^{-\min_{x\in[a,b]} \frac{q'''(x)}{q(x)}\omega}}.$$

Now we consider the integral equation (for a fixed $x \in [a,b]$)

$$u(t,x) = \int_0^{\omega} K(x,s)u^2(t-s,x)ds, \quad t \in [0,\omega]. \tag{3.1}$$

The operator χ (see section 2) we may rewriten in the form

$$\chi(u) = \int_0^{\omega} K(x,s)u^2(t-s,x)ds. \tag{3.2}$$

Theorem 3.2. *Let $\gamma > 0$, $\gamma \notin \{1, 2, \dots\}$, $p > 1$, $q \geq 1$. Let also*
$$\frac{M^2}{m^2} - \frac{m^2}{M^2} < \frac{1}{2}.$$
Then (1.1) has a unique positive solution $u(t,x) = v(t)q(x)$ which is continuous ω-periodic with respect to the time variable t and $u(0,x) \in \dot{B}^{\gamma}_{p,q}([a,b])$.

Proof. From Theorem 2.7 follows that the problem (1.1)–(1.3) has positive solution $u(t,x) = v(t)q(x)$. Let $x \in [a,b]$ is fixed. Let also $T \subset \mathcal{C}^{\circ}_{+}(\omega)$ is the set
$$T = \big\{u(t,x) \in \mathcal{C}^{\circ}_{+}(\omega), \quad \frac{m}{M^2\omega} \leq u(t,x) \leq \frac{M}{m^2\omega}, \forall t \in [0,\omega]\big\}.$$
If $u(t,x)$ is positive solution to (1.1), which is ω-periodic with respect to the time variable t then $u(t,x) \in T$. Indeed, for every fixed $x \in [a,b]$ we have
$$u(t,x) = \chi(u) \leq \Big(\max_{t\in[0,\omega]} u(t,x)\Big)^2 M\omega$$
for every $t \in [0,\omega]$. From where,
$$\max_{t\in[0,\omega]} u(t,x) \leq \Big(\max_{t\in[0,\omega]} u(t,x)\Big)^2 M\omega$$
or $\max_{t\in[0,\omega]} u(t,x) \geq 1/M\omega$ for every fixed $x \in [a,b]$. On the other hand from proposition 2.3, we have
$$u(t,x) \geq \frac{m}{M} \max_{t\in[0,\omega]} u(t,x) \geq \frac{m}{M^2\omega} \quad \forall t \in [0,\omega],$$
for every fixed $x \in [a,b]$. Also, for every fixed $x \in [a,b]$
$$u(t,x) = \chi(u) \geq m\omega\Big(\min_{t\in[0,\omega]} u(t,x)\Big)^2, \quad \forall t \in [0,\omega].$$
From the above inequality,
$$\min_{t\in[0,\omega]} u(t,x) \leq \frac{1}{m\omega}. \tag{3.3}$$
Since $u(t,x) \in \mathcal{C}^{\circ}_{+}(\omega)$, we have
$$\min_{t\in[0,\omega]} u(t,x) \geq \frac{m}{M} \max_{t\in[0,\omega]} u(t,x)$$
for every fixed $x \in [a,b]$. From the above inequality and (3.3),
$$\max_{t\in[0,\omega]} u(t,x) \leq \frac{M}{m^2\omega} \tag{3.4}$$
for every fixed $x \in [a,b]$. From (3) and (3.4) it follows that $u(t,x) \in T$ for every $t \in [0,\omega]$ and for every fixed $x \in [a,b]$.

Let u_1 and u_2 be two solutions to the integral equation (3.1). Let $y = u_1 - u_2$. We suppose that y changes its sign. Then for every positive constants c we have
$$\|y - c\| \geq \frac{1}{2}\|y\|.$$
(because y changes your sign) We note that in our case $\|y\| = \max_{t\in[0,\omega]} |y|$ for every fixed $x \in [a,b]$, $y \in \mathcal{C}(\omega)$. Let
$$b_1 = 2\frac{m^2}{M^2\omega}, \quad b_2 = 2\frac{M^2}{m^2\omega}.$$

In particular we have

$$\Big\|y - \frac{b_1+b_2}{2}\int_0^\omega y(s)ds\Big\| \geq \frac{1}{2}\|y\|$$

for every fixed $x \in [a,b]$. Also, we have

$$y(t,x) = \int_0^\omega K(x,s)(u_1^2(t-s,x) - u_2^2(t-s,x))ds = 2\int_0^\omega K(x,s)z(s)y(s)ds$$

for every fixed $x \in [a,b]$. In the last equality we use the middle point theorem. Here

$$\min\{u_1,u_2\} \leq z \leq \max\{u_1,u_2\}.$$

From where it follows that $z \in T$ for every fixed $x \in [a,b]$. Then

$$\begin{aligned}
2K(x,s)z(s) &\geq 2m\frac{m}{M^2\omega} = b_1,\\
2K(x,s)z(s) &\leq 2M\frac{M}{m^2\omega} = b_2.
\end{aligned}$$

Consequently

$$\Big|2K(x,s)z(s) - \frac{b_1+b_2}{2}\Big| \leq \frac{b_2-b_1}{2}$$

for every fixed $x \in [a,b]$. On the other hand

$$\begin{aligned}
\Big|y(t) - \frac{b_1+b_2}{2}\int_0^\omega y(s)ds\Big| &= \Big|2\int_0^\omega K(x,s)z(s)y(s)ds - \frac{b_1+b_2}{2}\int_0^\omega y(s)ds\Big|\\
&= \Big|\int_0^\omega \Big(2K(x,s)z(s) - \frac{b_1+b_2}{2}\Big)y(s)ds\Big|\\
&\leq \int_0^\omega \Big|2K(x,s)z(s) - \frac{b_1+b_2}{2}\Big||y(s)|ds\\
&\leq \frac{b_2-b_1}{2}\int_0^\omega |y(s)|ds \leq \frac{b_2-b_1}{2}\|y\|\omega
\end{aligned}$$

for every fixed $x \in [a,b]$. From where,

$$\Big\|y - \frac{b_1+b_2}{2}\int_0^\omega y(s)ds\Big\| \leq \frac{b_2-b_1}{2}\|y\|\omega$$

for every fixed $x \in [a,b]$. Now we use the inequality (3) and we get

$$\frac{1}{2}\|y\| \leq \frac{b_2-b_1}{2}\|y\|\omega$$

or

$$1 \leq (b_2-b_1)\omega = 2\Big(\frac{M^2}{m^2\omega} - \frac{m^2}{M^2\omega}\Big)\omega,$$

from where,

$$\frac{1}{2} \leq \frac{M^2}{m^2} - \frac{m^2}{M^2},$$

which is a contradiction with the conditions of the theorem 3.2. Consequently, if u_1 and u_2 are two solutions to the integral equation $u = \chi(u)$ we have $u_1 \equiv u_2$ or $u_1 - u_2$ or $u_2 - u_1$ is inside element for the cone $\mathcal{C}_+^\circ(\omega)$. Now we will show that the operator χ is 1-convex operator with respect to the cone $\mathcal{C}_+^\circ(\omega)$. First we note that $1 \in \mathcal{C}_+^\circ(\omega)$. Let $\eta(x,\lambda) = 1-\lambda$, $\lambda \in (0,1)$. Then we have

$$\chi(\lambda u) = \lambda^2\chi(u) = (1-\eta(x,\lambda))\lambda\chi(u).$$

Consequently the operator χ is 1-convex operator with respect to the cone $\mathcal{C}_+^\circ(\omega)$.

From here and from Theorems 2.7, 3.1, it follows that the Korteveg-de Vries (1.1) has unique positive solution $u(t,x) = v(t)q(x)$, which is ω-periodic with respect to the time variable t and $u(0,x) \in \dot{B}^{\gamma}_{p,q}([a,b])$. □

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