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Nonlinear perturbations of systems of partial differential equations with constant coefficients *

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Abstract

In this article, we show the existence of solutions to boundary-value problems, consisting of nonlinear systems of partial differential equations with constant coefficients. For this purpose, we use the right inverse of an associated operator and a fix point argument. As illustrations, we apply this method to Helmholtz equations and to second order systems of elliptic equations.

1 Introduction

Let $G \subset \mathbb{R}^n$ be a bounded region with smooth boundary, and let $(B(G), \|.\|)$ be a Banach space of functions defined on G. For each natural number n, let $B^n(G)$ denote the space of functions f satisfying $D^m f \in B(G)$ for all multi-index m with $|m| \leq n$. Then under the norm $\|f\|_n = \max_{|m|\leq n} \|D^m f\|$, the space $B^n(G)$ becomes a Banach space.

We consider the system

$$D_0\omega = f(\mathbf{x}, \omega, \frac{\partial \omega}{\partial x_1}, \dots, \frac{\partial \omega}{\partial x_n}) \quad \text{in } G\,, \tag{1}$$

where D_0 is a linear differential operator of first order with respect to the real variables $x_1, \dots, x_n$, the vector $\mathbf{x}$ has components $(x_1, \dots, x_n)$, and the unknown ω

and the right-hand side f are vectors of m components, with $m \geq n$. To this system of differential equations, we add the boundary condition

$$Aw = g \quad \text{on } \partial G\,, \tag{2}$$

where g is a given m-dimensional vector-valued function that belongs to the Banach space $B^1(\partial G)$. The operator A is chosen so that (2) leads to a well-posed problem on $B^1(G) \cap \ker D_0$.

For finding a solution to this nonlinear problem, we use a right inverse of the operator D_0 and a fix point argument [9, 8]. First, we construct the right inverse

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for a first order differential operator of constant coefficients. Then using that the operator D_0, in its matrix form, commutes with the elements of the formal adjoint matrix, we obtain the right inverse. In fact, we obtain a formal algebraic inversion through the associated operators determinant and adjoint matrix of D_0. In the last section of this article, we describe a natural generalization to high order systems, and show two applications of this method.

2 The Right Inverse of D_0.

The operator D_0 in (1) is represented in a matrix form as

$$D_0 = \begin{pmatrix} D_{11} & \dots & D_{1m} \\ \vdots & & \\ D_{m1} & \cdots & D_{mm} \end{pmatrix},$$

where D_{ij} is the differential operator of first order with respect to the real variables $x_1 \dots x_n$.

The determinant of D_0 is computed formally, and is a scalar linear differential operator with constant coefficients. Note that $\det D_0$ maps the space $B^m(G)$ into the space $B(G)$. As a general hypothesis, we assume that the differential operator $\det D_0$ possesses a continuous right inverse:

$$T_{\det D_0} : B(G) \to B^m(G) \tag{3}$$

which is an operator that improves the differentiability of functions in $B(G)$ by m orders.

The adjoint matrix associated with D_0, in algebraic sense, is computed formally, resulting a linear matrix differential operator, denotes by $\operatorname{adj} D_0$, with constant coefficients and of order $m-1$ respect to the real variables $x_1 \dots x_n$, i.e., $m-1$ is the order of the highest derivative that appears in the coefficients of the matrix. We observe that $\operatorname{adj} D_0$ maps the space $B^m(G)$ into the space $B^1(G)$. Under the assumptions above, we obtain the following result.

Theorem 2.1 *The differential operator*

$$\operatorname{adj} D_0(T_{\det D_0}) : B(G) \to B^1(G)$$

is a right inverse operator for D_0.

Proof. Note that $D_0 \operatorname{adj} D_0 = \det D_0 I$, which is satisfied due to the fact that D_0 is a differential operator with constant coefficients. From this remark and (3) the proof follows. $\square$

3 First-Order Nonlinear Systems

We define the fitting operator

$$\Omega : B^1(\partial G) \to B^1(G) \cap \ker D_0$$

by the relation

$$A(\Omega\phi) = A(\phi) \quad \text{for each } \phi \in B^1(\partial G). \tag{4}$$

i.e., to each $\phi \in B^1(\partial G)$ we associate the unique $B^1(G)$-solution to (4) in $\ker D_0$.

Theorem 3.1 *The boundary-value problem (1)-(2) is equivalent to the fixed point problem for the operator*

$$T(\omega, h_1, \dots, h_n) = (W, H_1, \dots, H_n)\,, \tag{5}$$

where

$$W = \Omega g + (I - \Omega)\,\mathrm{adj}\,D_0(T_{\det D_0} I) f(\boldsymbol{x}, \omega, h_1, \dots, h_n) \tag{6}$$

$$H_j = \tfrac{\partial}{\partial x_j}(\Omega g + (I - \Omega)\,\mathrm{adj}\,D_0(T_{\det D_0} I)) f(\boldsymbol{x}, \omega, h_1, \dots, h_n), \tag{7}$$

with $j = 1, \dots, n$.

Proof. Let $\omega \in B^1(G)$ be a solution to (1)-(2). To the function

$$\Psi = w - \mathrm{adj}\,D_0(T_{\det D_0} I) f(\mathbf{x}, \omega, \frac{\partial\omega}{\partial x_1}, \dots, \frac{\partial\omega}{\partial x_n}) \tag{8}$$

we apply the operator D_0 to obtain

$$D_0\Psi = D_0\omega - D_0\,\mathrm{adj}\,D_0(T_{\det D_0} I) f(\mathbf{x}, \omega, \frac{\partial\omega}{\partial x_1}, \dots, \frac{\partial\omega}{\partial x_n}) = 0\,.$$

Thus, $\Psi \in \ker D_0$. To Ψ we apply the operator A and obtain

$$\begin{aligned} A\Psi &= A\omega - A\,\mathrm{adj}\,D_0(T_{\det D_0} I) f(\mathbf{x}, \omega, \frac{\partial\omega}{\partial x_1}, \dots, \frac{\partial\omega}{\partial x_n}) \\ &= g - A\,\mathrm{adj}\,D_0(T_{\det D_0} I) f(\mathbf{x}, \omega, \frac{\partial\omega}{\partial x_1}, \dots, \frac{\partial\omega}{\partial x_n})\,. \end{aligned}$$

According to the definition of the operator Ω, we have

$$\Psi = \Omega g - \Omega\,\mathrm{adj}\,D_0(T_{\det D_0} I) f(\mathbf{x}, \omega, \frac{\partial\omega}{\partial x_1}, \dots, \frac{\partial\omega}{\partial x_n})\,.$$

Substituting this expression in (8) and differentiating with respect to x_j, we conclude that $(\omega, \frac{\partial\omega}{\partial x_1}, \dots, \frac{\partial\omega}{\partial x_n})$ is a fixed point of (5).

On the other hand if $(\omega, h_1, \dots, h_n)$ is a fixed point of (5), we can carry out the differentiation of (6) with respect to x_j for each $j = 1, \dots, n$. Because ω is in $B^1(G)$, we obtain

$$\frac{\partial w}{\partial x_j} = \frac{\partial}{\partial x_j}(\Omega g + (I - \Omega)\,\mathrm{adj}\,D_0(T_{\det D_0} I)) f(\mathbf{x}, \omega, h_1, \dots, h_n)\,.$$

Comparing these equations with (7), it follows that $\frac{\partial\omega}{\partial x_j} = h_j$ for $j = 1, \dots, n$. Substituting these equations in (6) and then applying the operator D_0 we obtain $D_0\omega = f(\mathbf{x}, \omega, \frac{\partial\omega}{\partial x_1}, \dots, \frac{\partial\omega}{\partial x_n})$. Applying Ω to (6) we conclude that

$$\Omega\omega = \Omega\Omega g + \Omega(I - \Omega) \operatorname{adj} D_0(T_{\det D_0} I) f(\mathbf{x}, \omega, h_1, \dots, h_n) = \Omega g\,.$$

By the definition of the operator Ω, it follows that $A(\omega) = g$, and hence, ω is a solution of (1)-(2). $\square$

Consider the polycylinder

$$\begin{aligned} M \quad = \quad & \{(\omega, h_1, \dots, h_n) \in \prod_{i=1}^{n+1} B(G) : \|\omega - \omega_0\| \le a_0, \\ & \|h_j - h_{j0}\| \le a_j, j = 1, \dots, n\} \end{aligned}$$

where $\omega_0 \in B^1(G)$ and $h_{j0} \in B(G)$ are taken as the coordinates of the polycylinder mid-point, and $a_0, a_1, \dots, a_n$ are positive real numbers.

From the definition of the operators $T_{\det D_0}, \operatorname{adj} D_0$, and Ω, it follows that the operators

$$(I - \Omega) \operatorname{adj} D_0(T_{\det D_0} I) : B(G) \to B(G) \quad \text{and} \tag{9}$$

$$\tfrac{\partial}{\partial x_j}(I - \Omega) \operatorname{adj} D_0(T_{\det D_0} I) : B(G) \to B(G) \tag{10}$$

are continuous and hence bounded. Therefore, for all $(\omega, h_1, \dots, h_n) \in M$ we have

$$\begin{aligned} \|W - \omega_0\| &= \|\Omega g + (I - \Omega) \operatorname{adj} D_0(T_{\det D_0} I) f(\mathbf{x}, \omega, h_1, \dots, h_n) - \omega_0\| \\ &= \|(I - \Omega) \operatorname{adj} D_0(T_{\det D_0} I)[f(\mathbf{x}, \omega, h_1, \dots, h_n) - D_0\omega_0] \\ &\quad +(I - \Omega) \operatorname{adj} D_0(T_{\det D_0} I) D_0\omega_0 + \Omega g - \omega_0\| \\ &\le \|(I - \Omega) \operatorname{adj} D_0(T_{\det D_0} I)\| \|f(\mathbf{x}, \omega, h_1, \dots, h_n) - D_0\omega_0\| + K_0 \end{aligned}$$

and

$$\begin{aligned} & \|H_j - h_{j0}\| \\ &= \|\frac{\partial}{\partial x_j}[\Omega g + (I - \Omega) \operatorname{adj} D_0(T_{\det D_0} I)] f(\mathbf{x}, \omega, h_1, \dots, h_n) - h_{j0}\| \\ &\le \|\frac{\partial}{\partial x_j}(I - \Omega) \operatorname{adj} D_0(T_{\det D_0} I)\| \|f(\mathbf{x}, \omega, h_1, \dots, h_n) - D_0\omega_0\| + K_j\,, \end{aligned}$$

where

$$\begin{aligned} K_0 &= \|(I - \Omega) \operatorname{adj} D_0(T_{\det D_0} I) D_0\omega_0 + \Omega g - \omega_0\| \\ K_j &= \|\tfrac{\partial}{\partial x_j}(I - \Omega) \operatorname{adj} D_0(T_{\det D_0} I) D_0\omega_0 + \tfrac{\partial}{\partial x_j}\Omega g - h_{j0}\|, \end{aligned}$$

for $j = 1, \dots, n$.

For a positive real number R and $j = 1, 2, \dots n$, we set

$$\begin{aligned} a_0 &= \|(I-\Omega)\operatorname{adj} D_0(T_{\det D_0} I)\| R + K_0 \\ a_j &= \|\frac{\partial}{\partial x_j}(I-\Omega)\operatorname{adj} D_0(T_{\det D_0} I)\| R + K_j\,. \end{aligned}$$

For the rest of this article, we will denote by M_R the polycylinder M with the parameters $a_0, a_1, \dots, a_n$ as defined above.

Theorem 3.2 *Let R be a positive real number such that f maps the polycylinder M_R into $B(G)$ and satisfies the growth condition*

$$\|f(\boldsymbol{x}, \omega, h_1, \dots, h_n) - D_0\omega_0\| \le R, \quad \forall\, (\omega, h_1, \dots, h_n) \in M_R\,.$$

Then the operator T maps continuously the polycylinder M_R into itself.

Proof. Let $(\omega, h_1, \dots, h_n)$ be an element in M_R and $(W, H_1, \dots, H_n)$ its image under T. Since $(\omega, h_1, \dots, h_n) \in M_R$, by the definitions of the operators $T_{\det D_0}, \operatorname{adj} D_0$ and Ω, it follows that $W \in B^1(G) \subset B(G)$. Since $\frac{\partial}{\partial x_j} : B^1(G) \to B(G)$, it follows that $H_j \in B(G)$ for all $j = 1, \dots, n$. Therefore, $T : M_R \to \prod_{i=1}^{n+1} B(G)$. That $(W, H_1, \dots, H_n)$ is in M_R follows from the boundedness of the operators (9)-(10), the hypotheses on f, and the definition of M_R. $\square$

Theorem 3.3 *Suppose f maps the polycylinder M_R into the space $B(G)$, and that f is Lipschitz continuous with constant L satisfying*

$$L < \min\{\|(I-\Omega)\operatorname{adj} D_0(T_{\det D_0} I)\|^{-1},\ \|\frac{\partial}{\partial x_j}(I-\Omega)\operatorname{adj} D_0(T_{\det D_0} I)\|^{-1}\},$$

for $j = 1, \dots, n$. Then T is a contraction.

Proof. Let $(\omega, h_1, \dots, h_n)$, $(\omega', h'_1, \dots, h'_n)$ be elements of M_R, and $(W, H_1, \dots, H_n)$, $(W', H'_1, \dots, H'_n)$ be their images under T. Since the operators (9) and (10) are bounded and f is Lipschitz with constant L, it follows that

$$\begin{aligned} \|W - W'\| &\le \|(I-\Omega)\operatorname{adj} D_0(T_{\det D_0} I)\| L \|(\omega, h_1, \dots, h_n) - (\omega', h'_1, \dots, h'_n)\| \\ &\le \|(\omega, h_1, \dots, h_n) - (\omega', h'_1, \dots, h'_n)\|\,. \end{aligned}$$

Similarly,

$$\|H_j - H'_j\| \le \|(\omega, h_1, \dots, h_n) - (\omega', h'_1, \dots, h'_n)\|$$

for $j = 1, \dots, n$. Therefore, T is a contraction. $\square$

With the aid of Theorems 3.1, 3.2 and 3.3, we obtain existence and uniqueness of a solution for Problem (1)-(2).

Theorem 3.4 *Suppose that f satisfies the hypotheses of Theorems 3.2 and 3.3. Then Problem (1)-(2) possesses exactly one solution in the polycylinder M_R.*

Proof. By definition M_R is a closed subset in the space $B(G)$. Applying Theorems 3.2 and 3.3, we realize that T maps M_R into itself, and it is a contraction; therefore, according to the Fixed Point Theorem there exists a unique fixed point in M_R. As a consequence of Theorem 3.1 this fixed point is a solution to Problem (1)-(2). $\square$

4 High-Order Systems

In this section we apply the method developed in the above section to high-order equations. Consider the system of differential equations

$$D_0\omega = f(\mathbf{x}, D^r\omega) \tag{11}$$

where D^r is a differential operator of order r, and D_0 is a linear differential operator of order r. The unknown ω and the right-hand side f are vector-valued functions of m components, with $m \geq n$.

We will assume that the associated differential operator $\det D_0$ has a continuous right inverse, $T_{\det D_0} : B(G) \to B^{rm}(G)$.

To system (11) we add the boundary condition

$$A\omega = g \quad \text{on } \partial G\,, \tag{12}$$

where g is a vector-valued function with m components in $B^r(\partial G)$. The operator A is chosen so that (12) becomes a well-posed problem on $B^r(G) \cap \ker D_0$.

We define the fitting operator $\Omega : B^r(\partial G) \to B^r(G) \cap \ker D_0$ as follows: For each function $\phi \in B^r(\partial G)$, $\Omega(\phi)$ is the unique $B^r(G)$-solution in $\ker D_0$ to the equation $A(\Omega(\phi)) = A(\phi)$.

The results established in section 3 are also valid for systems of order $r > 1$. However, (6) and (7) need to be increased to include equations corresponding to the higher-order derivatives. We will analyze the case when D_0 is a diagonal operator. Let D_0 be a linear differential operator of order r, which can be represented as $D_0 = PI$, where P is a linear differential operator of order r with a continuous right inverse $T_P : B(G) \to B^r(G)$. Let us assume that the operator T_P satisfies homogeneous boundary condition $A(T_P\phi) = 0$ for all $\phi \in B(G)$; thus the identity $(I - \Omega)\operatorname{adj} D_0(T_{\det D_0} I) = T_P I$ holds. Under these conditions, the equivalent system (6)-(7) can be simplified. Furthermore, we need only the continuity T_P for homogeneous conditions, and an estimate on Ω for non-homogeneous conditions. As a consequence of this we have the following result

Theorem 4.1 *Suppose that*

$$D_0\omega = PI\omega = \tilde{f} \tag{13}$$

$$A(\omega) = 0 \tag{14}$$

is a well-posed problem in the sense of

$$T_P : B(G) \to B^r(G), \tag{15}$$

where $\tilde{f}$ is a vector-valued function of dimension m, depending only on the coordinates $x_1, \dots, x_n$.

If the right-hand side in (11) satisfies a certain growth condition, and is Lipschitz with a constant sufficiently small, then Problem (11)-(12) is well-posed in the sense of (15).

5 Examples.

Example 1: Helmholtz type equations.

Let $G = G_1 \times G_2$ be a bounded simply connected region in $\mathbb{R}^3$ with smooth boundary ∂G. Here G_1 is the region containing the component x_1, and G_2 is the region containing the components x_2 and x_3.

On the domain G, we consider the system

$$D_0\omega = f(\mathbf{x}, \omega, \frac{\partial \omega_1}{\partial x_2}, \frac{\partial \omega_1}{\partial x_3}, \frac{\partial \omega_2}{\partial x_1}, \frac{\partial \omega_2}{\partial x_3}, \frac{\partial \omega_3}{\partial x_1}, \frac{\partial \omega_3}{\partial x_2}), \tag{16}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ is a vector in $\mathbb{R}^3$, $\omega = (\omega_1, \omega_2, \omega_3)$ and $f = (f_1, f_2, f_3)$ are vector-valued functions, and the right-hand side f does not dependent on $\frac{\partial \omega_i}{\partial x_i}$, $i = 1, 2, 3$.

For $\lambda > 0$, let

$$D_0 = \begin{pmatrix} \lambda & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & \lambda & -\frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \lambda \end{pmatrix}.$$

From (16) it follows that for $i \neq j$,

$$\operatorname{curl}\omega + \lambda\omega = \begin{pmatrix} f_1(x, \omega, \frac{\partial \omega_1}{\partial x_2}, \dots, \frac{\partial \omega_i}{\partial x_j}, \dots) \\ f_2(x, \omega, \frac{\partial \omega_1}{\partial x_2}, \dots, \frac{\partial \omega_i}{\partial x_j}, \dots) \\ f_3(x, \omega, \frac{\partial \omega_1}{\partial x_2}, \dots, \frac{\partial \omega_i}{\partial x_j}, \dots) \end{pmatrix}.$$

To the system (16) we add the Dirichlet boundary condition

$$\begin{aligned} \omega_1 &= g_1 \quad \text{on } \partial G \\ \omega_2 &= g_2 \quad \text{on } \partial G_1 \times \partial G_2\,, \end{aligned} \tag{17}$$

where g_1 and g_2 are given real-valued functions in the space of α-Hölder continuous and differentiable functions $C^{1,\alpha}$. We look for solutions to Problem (16)-(17) in the space of α-Hölder continuous functions $C^\alpha(G)$.

After some calculations, we obtain $\det D_0 = \lambda(\lambda^2 + \Delta)$, where Δ denotes the Laplace operator, and λ^2 is not an eigenvalue for the Helmholtz operator $\Delta + \lambda^2$. Therefore, this operator possesses a continuous right inverse $T_{\Delta+\lambda^2} : C^\alpha(G) \to C^{\alpha,2}(G)$.

Similarly, we obtain the associated adjoint matrix

$$\operatorname{adj} D_0 = \begin{pmatrix} \lambda^2 + \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_2 \partial x_1} + \lambda \frac{\partial}{\partial x_3} & \frac{\partial^2}{\partial x_1 \partial x_3} - \lambda \frac{\partial}{\partial x_2} \\ \frac{\partial^2}{\partial x_1 \partial x_2} - \lambda \frac{\partial}{\partial x_3} & \lambda^2 + \frac{\partial^2}{\partial x_2^2} & \frac{\partial^2}{\partial x_2 \partial x_3} + \lambda \frac{\partial}{\partial x_1} \\ \frac{\partial^2}{\partial x_3 \partial x_1} + \lambda \frac{\partial}{\partial x_2} & \frac{\partial^2}{\partial x_3 \partial x_2} - \lambda \frac{\partial}{\partial x_1} & \lambda^2 + \frac{\partial^2}{\partial x_3^2} \end{pmatrix}.$$

Note that the operator $T_{\Delta+\lambda^2}I$ improves the differentiability properties of a function by two, not by three orders. The operator $\operatorname{adj} D_0$ decreases the differentiability properties by two orders only in the ii components with respect to x_i. However, it was assumed that the derivatives $\frac{\partial \omega_i}{\partial x_i}$, $i = 1, 2, 3$ do not appear in the right-hand side f of (16). Therefore, $\operatorname{adj} D_0(T_{\Delta+\lambda^2}I)$ improves the properties of differentiability by one order, and we can consider all the equations except those associated with $\frac{\partial \omega_i}{\partial x_i}$, $i = 1, 2, 3$ in Problem (6)-(7).

Now, we study the kernel of D_0. Let $(\omega_1, \omega_2, \omega_3)$ be a solution of the homogeneous problem

$$D_0 \omega = 0\,. \tag{18}$$

When we apply the operator $\operatorname{adj} D_0$ on the left in the above equation, it follows that $(\Delta + \lambda^2)\omega_i = 0$ for $i = 1, 2, 3$. Due to (18), the three components are linearly dependent. Therefore, we will assume w_1 as an arbitrary given function which satisfies the equation $(\lambda^2 + \Delta)w_1 = 0$ and is also defined on ∂G.

In view of (18), we obtain

$$\begin{aligned} \lambda w_1 - \frac{\partial \omega_2}{\partial x_3} + \frac{\partial \omega_3}{\partial x_2} &= 0 \\ \frac{\partial \omega_1}{\partial x_3} + \lambda w_2 - \frac{\partial \omega_3}{\partial x_1} &= 0 \\ -\frac{\partial \omega_1}{\partial x_2} + \frac{\partial \omega_2}{\partial x_1} + \lambda w_3 &= 0\,. \end{aligned} \tag{19}$$

When we differentiate the first equation respect to x_1, the second respect to x_2, and the third respect to x_3, after summing the results, we have

$$\frac{\partial \omega_1}{\partial x_1} + \frac{\partial \omega_2}{\partial x_2} + \frac{\partial \omega_3}{\partial x_3} = 0\,. \tag{20}$$

Using (19) and (20) we have, in matrix form,

$$D_1 \begin{pmatrix} w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} -\frac{\partial \omega_1}{\partial x_1} \\ -\lambda w_1 \end{pmatrix} \tag{21}$$

and

$$D_2 \begin{pmatrix} w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} -\frac{\partial \omega_1}{\partial x_3} \\ \frac{\partial \omega_1}{\partial x_2} \end{pmatrix} \tag{22}$$

where

$$D_1 = \begin{pmatrix} \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix} \lambda & -\frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_1} & \lambda \end{pmatrix}.$$

Since $\det D_1 = \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ and $\det D_2 = \lambda^2 + \frac{\partial^2}{\partial x_1^2}$, we can assume the existence of right inverse operators for D_1 and D_2. Since $(\lambda^2 + \Delta)w_1 = 0$, the integrability condition

$$D_2 \begin{pmatrix} -\frac{\partial \omega_1}{\partial x_1} \\ -\lambda w_1 \end{pmatrix} = D_1 \begin{pmatrix} -\frac{\partial \omega_1}{\partial x_3} \\ \frac{\partial \omega_1}{\partial x_2} \end{pmatrix}$$

is fulfilled for the system (21)-(22). Put $w = w_2 + iw_3$ and $z = x_2 - ix_3$. Then from (21), we obtain the non-homogeneous Cauchy-Riemann System

$$\frac{\partial \omega}{\partial \bar{z}} = F(\omega_1, \frac{\partial \omega_1}{\partial x_1}), \tag{23}$$

where F is known. Thus w can be uniquely determined up to a holomorphic function in z. Since ω satisfies $D_2\omega = 0$, we apply the operator $\operatorname{adj} D_2$ on the left to this equation, and obtain

$$(\lambda^2 + \frac{\partial^2}{\partial x_1^2})Iw = 0\,. \tag{24}$$

From (24) it follows that $(\lambda^2 + \frac{\partial^2}{\partial x_1^2})w_2 = 0$ and $(\lambda^2 + \frac{\partial^2}{\partial x_1^2})w_3 = 0$. When we prescribe the boundary values on $\partial G_1 \times \partial G_2$, w_2 becomes a uniquely determined function. Finally from the last equation in (19), we obtain $w_3 = \frac{1}{\lambda}(\frac{\partial \omega_1}{\partial x_2} - \frac{\partial \omega_2}{\partial x_1})$, and we cannot require additional values for w_3.

Since this is a well-posed problem, it follows that (17) is well formulated. Therefore, applying the theory developed in section 3, we assure the existence of an unique solution for Problem (16)-(17).

Example 2: A second order elliptic operator.

Let G be a bounded simply connected region in $\mathbb{R}^n$ with boundary sufficiently smooth. Consider the system

$$D_0\omega = f(x, D^2\omega) \quad \text{in } G\,, \tag{25}$$

where D^2 is a second-order differential operator, not necessarily linear, and D_0 is a linear differential operator of second order. The unknown ω and the right-hand side f are vectors of m components.

We assume that D_0 is a diagonal operator of the form $D_0 = PI$, where P is an elliptic differential operator of second order with constant coefficients, $P = \sum_{i,j=1}^n a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}$. In addition to (25) we impose the Dirichlet boundary condition

$$\omega = g \quad \text{on } \partial G, \tag{26}$$

where g is a given vector-valued m-dimensional function belonging to $C^{2,\alpha}(\partial G)$. Then we look for a solution to (25)-(26) in the space $C^\alpha(\bar{G})$.

It is known that the operator P possesses a continuous right inverse [7], $T_P : C^\alpha(\bar{G}) \to C^{2,\alpha}(\bar{G})$, which satisfies $A(T_P\phi) = 0$ for all $\phi \in C^\alpha(\bar{G})$. Since $\det D_0 = P^m$, there is a continuous right inverse operator $T_{\det D_0} = T_{P^m} : B(G) \to B^{2m}(G)$. We conclude by observing that now all the theory developed in sections 3 and 4 can be applied to this problem.

References

[1] Fichera C., *Linear Elliptic Differential Systems and Eigenvalue Problems*, Lecture Notes in Mathematics, Vol. 8, Springer-Verlag, Berlin-Heidelberg-New York (1965).

[2] Folland G.B., *Introduction to Partial Differential Equations*, Princeton University Press, Princeton,New Jersey (1995).

[3] Gilbert D., Trudinger N.S., *Elliptic Partial Differential Equations of Second Order*, Grundlehren der mathematischen Wissenschaften, 224, Springer-Verlag, Berlin-Heidelberg-New York (1977).

[4] Gilbert R.P., *Constructive Methods for Elliptic Equations*, Lecture Notes in Mathematics, 365, Berlin (1974).

[5] Gilbert R.P., Buchanan J.L., *First Order Elliptic Systems: A Function Theoretic Approach*, Academy Press, New York, London (1983).

[6] Hörmander L., *Linear Partial Differential Operators*, third edition, Springer-Verlag (1969).

[7] Miranda C., *Partial Differential Equations of Elliptic Type*, Springer-Verlag, Berlin-Heidelberg-New York (1970).

[8] Tutschke W., *Partielle Differentialgleichungen. Klassische Funktionalanalytische und Komplexe Methoden*, BSB B.G. Teubner-Verlagsgesellschaft, Leipzig (1983).

[9] Vekua I.N., *New Methods for Solving Elliptic Equations*, Vol. 1, North-Holland Publ., Amsterdam (1968).

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