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Boundary-value problems for the one-dimensional p-Laplacian with even superlinearity *

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Abstract

This paper is concerned with a study of the quasilinear problem

$$-(|u'|^{p-2}u')' = |u|^p - \lambda, \quad \text{in } (0,1)\,,$$
$$u(0) = u(1) = 0\,,$$

where $p > 1$ and $\lambda \in \mathbb{R}$ are parameters. For $\lambda > 0$, we determine a lower bound for the number of solutions and establish their nodal properties. For $\lambda \leq 0$, we determine the exact number of solutions. In both cases we use a quadrature method.

1 Introduction

This paper is devoted to a study of existence and multiplicity of solutions to the quasilinear two-point boundary-value problem

$$\begin{gathered} -(\varphi_p(u'))' = f(\lambda, u), \quad \text{in } (0,1)\,, \\ u(0) = u(1) = 0\,, \end{gathered} \tag{1}$$

where $\varphi_p(s) = |s|^{p-2}s$ and $f(\lambda, u) = |u|^p - \lambda$. Here $(\varphi_p(u'))'$ is the one-dimensional p-Laplacian, and $p > 1$.

When the differential operator is linear, i.e., $p = 2$, several existence and multiplicity results, related to superlinear boundary value problems with Dirichlet boundary data, are available in the literature. Let us recall some of them for the one-dimensional case.

Lupo et al [14] have studied the non-autonomous case

$$\begin{gathered} -u''(x) = u^2(x) - t\sin x, \quad \text{in } (0,\pi)\,, \\ u(0) = u(\pi) = 0\,. \end{gathered} \tag{2}$$

Using a combination of shooting and topological arguments, they show that for any $k \in \mathbb{N}$ there exists $t_k > 0$ such that for all $t \geq t_k$, problem (2) admits at least k solutions.

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Castro and Shivaji [8], using phase-plane analysis, consider the problem

$$\begin{gathered} -u''(x) = g(u(x)) - \rho(x) - t, \quad \text{in } (0,1)\,, \\ u(0) = u(1) = 0\,, \end{gathered} \tag{3}$$

where ρ a continuous function on $[0,1]$, $g \in C^1(\mathbb{R})$,

$$\lim_{s \to -\infty} \frac{g(s)}{s} = M \in \mathbb{R}, \quad \text{and} \quad \lim_{s \to +\infty} \frac{g(s)}{s^{1+\sigma}} = +\infty \quad \text{with} \quad \sigma > 0\,.$$

They show that for $k \in \mathbb{N}$ there exists $t_k(M)$ such that $\lim_{k \to +\infty} t_k(M) = +\infty$, and for all $t > t_k$, problem (3) has at least two solutions with k nodes in $(0,1)$.

The autonomous case has been studied by many authors. Let us mention some of them. Independently of Castro and Shivaji, Ruf and Solimini [16] consider the problem

$$\begin{gathered} -u''(x) = g(u(x)) - t, \quad \text{in } (0,\pi)\,, \\ u(0) = u(\pi) = 0\,, \end{gathered} \tag{4}$$

where

$$g \in C^1(\mathbb{R}), \quad \limsup_{s \to -\infty} g'(s) < +\infty, \quad \text{and} \quad \lim_{s \to +\infty} g'(s) = +\infty\,.$$

Using variational methods, they show that for any $k \in \mathbb{N}$ there exists $t_k \in \mathbb{R}$ such that for $t > t_k$ problem (4) has at least k distinct solutions.

Prior to the papers mentioned above, Scovel [17] obtained the same result as Ruf and Solimini [16] in the special case where $g(u) = 6u^2$. He has shown that for any integer $k \geq 1$, there exist values $t_1 < \cdots < t_k$ such that for $t > t_k$ problem (4) (with $g(u) = 6u^2$) admits at least k distinct solutions.

Independently and prior to Scovel, in 1983, Ammar Khodja [7] obtained a complete description of the solution set of the problem

$$\begin{gathered} -u''(x) = u^2(x) - \lambda, \quad \text{in } (0,1)\,, \\ u(0) = u(1) = 0\,. \end{gathered} \tag{5}$$

He detects all the solutions to (5) for any value of $\lambda \in \mathbb{R}$, and thus obtains the exact number of solutions to (5) for all λ. To state his result, for any integer $k \geq 1$, denote

$$\begin{gathered} S_k^+ = \left\{ u \in C_0^2[0,1] : u'(0) > 0, u \text{ admits } k-1 \text{ nodes in } (0,1) \right\}, \\ S_k^- = -S_k^+ \quad \text{and} \quad S_k = S_k^+ \cup S_k^-\,. \end{gathered}$$

Theorem 1 *[7] There exists a sequence $(\lambda_k)_{k \geq 0}$ such that*

$$-\infty < \lambda_0 < 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$$

and:

(i) *If $\lambda < \lambda_0$, problem (5) admits no solution.*
If $\lambda_0 < \lambda < 0$, problem (5) admits exactly two solutions and they are positive.
If $\lambda = \lambda_0$ or $0 \leq \lambda < \lambda_1$, there exists a unique positive solution.
If $\lambda > \lambda_1$, there is no positive solution.

(ii) *If $\lambda > 0$, there exists a unique solution in S_1^-.*

(iii) *If (and only if) $\lambda > \lambda_k$:*

- *there exist exactly 2 solutions in S_{2k}*
- *there exists exactly one solution in S_{2k+1}^-*

(iv) *There exists a sequence $(\mu_k)_{k\geq 1}$ such that*

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_k < \lambda_{k+1} < \cdots$$

and such that:
if (and only if) $\mu_k < \lambda < \lambda_{k+1}$, there exist exactly two solutions in S_{2k+1}^+,
if (and only if) $\lambda = \mu_k$ or $\lambda > \lambda_{k+1}$, there exists a unique solution in S_{2k+1}^+.

The objective of this paper is to extend Ammar Khodja's result to the general quasilinear case $p > 1$. In particular, we will show that if $\lambda \leq 0$ the same result holds for all $p > 1$, but if $\lambda > 0$ and $p > 2$ the situation is different from that obtained in [7]. So, the behavior of the solution set of problem (1) depends not only on the values of λ (as was shown in [7]) but also on those of the parameter p.

These changes in the behavior of the solution set when the parameter p varies is not new in the literature. Guedda and Veron [11] consider the problem

$$\begin{gathered} -(\varphi_p(u'))' = \lambda\varphi_p(u) - f(u), \quad \text{in } (0,1)\,, \\ u(0) = u(1) = 0\,, \end{gathered} \tag{6}$$

where f is a C^1 odd function such that the function $s \mapsto f(s)/s^{p-1}$ is strictly increasing on $(0, +\infty)$ with limit 0 at 0 and $\lim_{s\to+\infty} f(s)/s^{p-1} = +\infty$. They denote by E_λ the solution set of problem (6) and show, under some technical assumptions, that when $1 < p \leq 2$ the structure of E_λ is exactly the same as in the case $p = 2$, and strictly different in the cases $p > 2$.

This paper is organized as follows. In Section 2 we introduce notation and state the main results (Theorems 2 and 3). Section 3 is devoted to explain of the method used in proving our results. In Section 4 we prove Theorem 2 and finally, in Section 5, we prove Theorem 3.

2 Notation and main results

In order to state the main results, for any $k \in \mathbb{N}^*$, let

$$S_k^+ = \left\{ \begin{array}{r} u \in C^1([\alpha,\beta]) : \ u \text{ admits exactly } (k-1) \text{ zeros in } (\alpha,\beta) \\ \text{all simple, } u(\alpha) = u(\beta) = 0 \text{ and } u'(\alpha) > 0 \end{array} \right\},$$

$S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$.

Definition Let $u \in C([\alpha,\beta])$ be a function with two consecutive zeros $x_1 < x_2$. We call the *I-hump of* u the restriction of u to the open interval $I = (x_1, x_2)$. When there is no confusion we refer to a hump of u.

With this definition in mind, each function in S_k^+ has exactly k humps such that the first one is positive, the second is negative, and so on with alternations. Let A_k^+ $(k \geq 1)$ be the subset of S_k^+ consisting of the functions u satisfying:

- Every hump of u is symmetrical about the center of the interval of its definition.
- Every positive (resp. negative) hump of u can be obtained by translating the first positive (resp. negative) hump.
- The derivative of each hump of u vanishes once and only once.

Let $A_k^- = -A_k^+$ and $A_k = A_k^+ \cup A_k^-$. Let B_k^+ $(k \geq 1)$ be the subset of $C^1([\alpha,\beta])$ consisting of the functions u satisfying:

- $u(x) \geq 0, \forall x \in [\alpha,\beta]$, and $u(\alpha) = u(\beta) = u'(\alpha) = 0$.
- u admits exactly $(k-1)$ zero(s), all double, in the open interval (α,β).
- If $k > 1$, u is $((\beta-\alpha)/k)-$ periodic.
- Every hump of u (necessarily positive) is symmetrical about the center of the interval of its definition.
- The derivative of each hump of u vanishes once and only once.

Let $B_k^- = -B_k^+$ and $B_k = B_k^+ \cup B_k^-$.

The first result concerns the case $\lambda \leq 0$ and gives the *exact* number of solutions to (1).

Theorem 2 (Case $\lambda \leq 0$) *There exists a number $\lambda_* < 0$ such that:*

(i) *If $\lambda < \lambda_*$, problem (1) admits no solution.*

(ii) *If $\lambda = \lambda_*$, problem (1) admits a unique solution and it belongs to A_1^+.*

(iii) *If $\lambda_* < \lambda < 0$, problem (1) admits exactly two solutions and they belong to A_1^+.*

(iv) *If $\lambda = 0$, beside the trivial solution, problem (1) admits a unique solution and it belongs to A_1^+.*

The second result concerns the case $\lambda > 0$.

Theorem 3 (Case $\lambda > 0$) *For any $p > 1$ there exist two real numbers $J(p) > J_+(p) > 0$ and for all $p > 2$ there exists a positive real number $J_-(p) < J(p)$ such that for all integer $n \geq 1$:*

(i) *Problem (1) admits a solution in B_n^+ if and only if $\lambda = (2nJ(p))^{p^2}$, and in this case, the solution is unique.*

(ii) *Problem (1) admits no solution in $\bigcup_{n\geq 1} B_n^-$.*

(iii) *Problem (1) (with $\lambda > 0$) admits a solution in A_1^+ if and only if $0 < \lambda < (2J(p))^{p^2}$, and in this case, the solution is unique.*

(iv) *Problem (1) admits a solution in A_1^- if and only if ($1 < p \leq 2$ and $\lambda > 0$)* **or** *($p > 2$ and $0 < \lambda < (2J_-(p))^{p^2}$), and in this case, the solution is unique.*

(v) *Problem (1) admits a solution $u_{2n}^{\pm}$ in $A_{2n}^{\pm}$ provided $1 < p \leq 2$ and $\lambda > (2nJ(p))^{p^2}$* **or** *$p > 2$ and*

$$\begin{aligned}&\inf\left\{(2nJ(p))^{p^2}, (2n(J_-(p) + J_+(p)))^{p^2}\right\}\\ <\quad \lambda \quad &< \sup\left\{(2nJ(p))^{p^2}, (2n(J_-(p) + J_+(p)))^{p^2}\right\}\end{aligned}$$

(vi) *Problem (1) admits a solution in A_{2n+1}^+ provided $1 < p \leq 2$ and $\lambda > (2(n+1)J(p))^{p^2}$* **or** *$p > 2$ and*

$$\begin{aligned}&\inf\left\{(2(n+1)J(p))^{p^2}, (2((n+1)J_+(p) + nJ_-(p)))^{p^2}\right\}\\ <\quad \lambda \quad &< \sup\left\{(2(n+1)J(p))^{p^2}, (2((n+1)J_+(p) + nJ_-(p)))^{p^2}\right\}\end{aligned}$$

(vii) *Problem (1) admits a solution in A_{2n+1}^- provided $1 < p \leq 2$ and $\lambda > (2nJ(p))^{p^2}$* **or** *$p > 2$ and*

$$\begin{aligned}&\inf\left\{(2nJ(p))^{p^2}, (2((n+1)J_-(p) + nJ_+(p)))^{p^2}\right\}\\ <\quad \lambda \quad &< \sup\left\{(2nJ(p))^{p^2}, (2((n+1)J_-(p) + nJ_+(p)))^{p^2}\right\}\end{aligned}$$

Remark According to Proposition 5 below, if $\lambda > 0$ and $p \in (1, 2]$ then $\tilde{S} \subset (\bigcup_{k\geq 1} A_k) \cup (\bigcup_{k\geq 1} B_k)$, where $\tilde{S}$ denotes the solution set of problem (1).

Remark The results obtained in [7], for $p = 2$, concerning solutions in A_{2n}, A^-_{2n+1}, and A^+_{2n+1}, are more precise than those stated in Theorem 3, assertions **(v), (vi)** and **(vii)** for $p \neq 2$. In fact, these assertions do not provide the exact number of solutions in A_{2n}, A^-_{2n+1}, and A^+_{2n+1}. The proof given in [7] uses strongly the fact that the nonlinearity $u \mapsto u^2 - \lambda$ is a second degree polynomial function. We were not able to obtain the same degree of precision.

3 The method used

To obtain our results, we make use of the well known time mapping approach. See, for instance, Laetsch [12], de Mottoni & Tesei [9], [10], Smoller & Wasserman [19], Ammar Khodja [7], Shivaji [18], Guedda & Veron [11], Ubilla [20], Manásevich et al [15], Addou & Ammar Khodja [1], Addou et al [4], Addou & BenmezaÏ [2]. To describe this method we denote by g a nonlinearity and by p a real parameter, and we assume one of the following conditions:

$$g \in C(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad 1 < p < +\infty \tag{7}$$

$$g \in C(\mathbb{R}, \mathbb{R}), \quad 1 < p < +\infty, \quad \text{and} \quad xg(x) > 0, \forall x \in \mathbb{R}^* \tag{8}$$

$$g \text{ is locally Lipschitzian and } 1 < p \leq 2. \tag{9}$$

One may observe that (8) or (9) implies (7), hence (8) and (9) are more restrictive than (7), but they furnish better results as we will see later (see Proposition 5).

We denote by $S(p)$ the solution set of problem

$$\begin{gathered} -(|u'|^{p-2}u')' = g(u), \quad \text{in } (\alpha, \beta) \\ u(\alpha) = u(\beta) = 0\,. \end{gathered} \tag{10}$$

When there is no confusion we write S instead of $S(p)$.

Denote by $p' = p/(p-1)$ the conjugate exponent of p. Define $G(s) := \int_0^s g(t)dt$. For any $E \geq 0$ and $\kappa = +, -$, let

$$X_\kappa(E) = \{s \in \mathbb{R} : \kappa s > 0 \quad \text{and} \quad E^p - p'G(\xi) > 0, \forall\, \xi, 0 < \kappa\xi < \kappa s\}$$

and

$$r_\kappa(E) = \begin{cases} 0 & \text{if } X_\kappa(E) = \emptyset\,, \\ \kappa \sup(\kappa X_\kappa(E)) & \text{otherwise.} \end{cases}$$

Note that r_κ may be infinite. We shall also make use of the following sets:

$$D_\kappa = \{E \geq 0 : 0 < |r_\kappa(E)| < +\infty \text{ and } \kappa g(r_\kappa(E)) > 0\}$$

and $D = D_+ \cap D_-$. Define the following time-maps:

$$\begin{aligned} T_\kappa(E) &= \kappa \textstyle\int_0^{r_\kappa(E)} (E^p - p'G(t))^{-1/p} dt, \quad E \in D_\kappa\,. \\ T_{2n}(E) &= n(T_+(E) + T_-(E)), \quad n \in \mathbb{N}, \quad E \in D\,, \\ T^\kappa_{2n+1}(E) &= T_{2n}(E) + T_\kappa(E), \quad n \in \mathbb{N}, \quad E \in D\,. \end{aligned}$$

Theorem 4 (Quadrature method) *Assume that (7) holds. Let $E \geq 0$, $\kappa = +, -$. Then*

1. *Problem (10) admits a solution $u_\kappa \in A_1^\kappa$ satisfying $u'_\kappa(\alpha) = \kappa E$ if and only if $E \in D_\kappa \cap (0, +\infty)$ and $T_\kappa(E) = (\beta - \alpha)/2$, and in this case the solution is unique.*

2. *Problem (10) admits a solution $u_\kappa \in A_{2n}^\kappa$ $(n \neq 0)$ satisfying $u'_\kappa(\alpha) = \kappa E$ if and only if $E \in D \cap (0, +\infty)$ and $T_{2n}(E) = (\beta - \alpha)/2$, and in this case the solution is unique.*

3. *Problem (10) admits a solution $u_\kappa \in A_{2n+1}^\kappa$ $(n \neq 0)$ satisfying $u'_\kappa(\alpha) = \kappa E$ if and only if $E \in D \cap (0, +\infty)$ and $T_{2n+1}^\kappa(E) = (\beta - \alpha)/2$, and in this case the solution is unique.*

4. *Problem (10) admits a solution $u_\kappa \in B_n^\kappa$ $(n \neq 0)$ if and only if $0 \in D_\kappa$ and $nT_\kappa(0) = (\beta - \alpha)/2$, and in this case the solution is unique.*

One may observe that this result does not give information about solutions to (10) outside $\bigcup\limits_{k \geq 1} (A_k \cup B_k)$. The following proposition gives some useful information.

Proposition 5 *If (8) holds then $S \subset \{0\} \cup (\bigcup\limits_{k \geq 1} A_k)$. If (9) holds then*

(i) $g(0) = 0$ *implies* $S \subset \{0\} \cup (\bigcup\limits_{k \geq 1} A_k)$,

(ii) $g(0) \neq 0$ *implies* $S \subset (\bigcup\limits_{k \geq 1} A_k) \cup (\bigcup\limits_{k \geq 1} B_k)$.

Theorem 4 and Proposition 5 are certainly well known, but we did not find a convenient reference to the precise statements used later.

4 Proof of Theorem 2

Since $\lambda \leq 0$, any solution to (1) is positive. In fact, if u is a solution to (1) then

$$u'(x) = \varphi_{p'}(\varphi_p(u'(x))), \; \forall x \in (0, 1) \, .$$

Since $x \mapsto \varphi_p(u'(x))$ is decreasing (from $(\varphi_p(u'))'(x) = -|u(x)|^p + \lambda$, for all $x \in (0, 1)$ and $\lambda \leq 0$) and $\varphi_{p'}$ is increasing, it follows that u' is decreasing. This shows that u is concave, and since $u(0) = u(1) = 0$ it follows that u is positive.

Moreover, the nonlinear term $f(\lambda, u) = |u|^p - \lambda$ satisfies (8) so, from Proposition 5, it follows that any nontrivial solution is necessarily in A_1^+. Hence, we have only to define the time map T_+. In order to do this, we need the following technical lemma.

Lemma 6 *Consider the equation in $s \in \mathbb{R}$:*

$$E^p - p' F(\lambda, s) = 0\,, \tag{11}$$

where $p > 1$, $\lambda \leq 0$ and $E \geq 0$ are real parameters and $F(\lambda, s) = \int_0^s f(\lambda, t)dt$. Then for any $E > 0$, (resp. $E = 0$) equation (11) admits a unique positive zero $s_+ = s_+(p, \lambda, E)$ (resp. a unique zero $s_+ = s_+(p, \lambda, 0) = 0$). Moreover:

(a) *The function $E \longmapsto s_+(p, \lambda, E)$ is C^1 in $(0, +\infty)$ and*

$$\frac{\partial s_+}{\partial E}(p, \lambda, E) = \frac{(p-1)E^{p-1}}{f(\lambda, s_+(p, \lambda, E))} > 0$$

for all $p > 1$, all $\lambda \leq 0$, and all $E > 0$.

(b) $\lim_{E \to 0^+} s_+(p, \lambda, E) = 0.$

(c) $\lim_{E \to +\infty} s_+(p, \lambda, E) = +\infty.$

Proof. For a fixed $p > 1$, $\lambda \leq 0$ and $E \geq 0$, consider the function

$$s \longmapsto M(p, \lambda, E, s) := E^p - p' F(\lambda, s) = E^p - p' s\Big(\frac{|s|^p}{p+1} - \lambda\Big)\,,$$

defined in $\mathbb{R}$, which is strictly decreasing and such that

$$M(p, \lambda, E, 0) = E^p \geq 0\,, \quad \text{and} \quad \lim_{s \to +\infty} M(s) = -\infty\,.$$

It is clear that (11) admits, for any $E > 0$, a unique positive zero, $s_+ = s_+(p, \lambda, E)$; and if $E = 0$, it admits a unique zero $s_+ = 0$.

Now, for any $p > 1$ and $\lambda \leq 0$, consider the real-valued function

$$(E, s) \longmapsto M_+(E, s) := E^p - p' s\Big(\frac{s^p}{p+1} - \lambda\Big)$$

defined on $\Omega_+ = (0, +\infty)^2$. One has $M_+ \in C^1(\Omega_+)$ and

$$\frac{\partial M_+}{\partial s}(E, s) = -p' f(\lambda, s) = -p'(|s|^p - \lambda) \quad \text{in } \Omega_+,$$

hence

$$\frac{\partial M_+}{\partial s}(E, s) < 0 \quad \text{in } \Omega_+$$

and one may observe that $s_+(p, \lambda, E)$ belongs to the open interval $(0, +\infty)$ and satisfies from its definition

$$M_+(E, s_+(p, \lambda, E)) = 0\,. \tag{12}$$

So, one can make use of the implicit function theorem to show that the function $E \longmapsto s_+(p, \lambda, E)$ is $C^1((0, +\infty), \mathbb{R})$ and to obtain the expression for

$\frac{\partial s_+}{\partial E}(p,\lambda,E)$ given in **(a)**. Hence, for any fixed $p>1$ and $\lambda \le 0$, the function defined in $(0,+\infty)$ by $E \longmapsto s_+(p,\lambda,E)$ is strictly increasing and bounded from below by 0 and from above by $+\infty$. Thus the limit $\lim_{E\to 0^+} s_+(p,\lambda,E) = l_0^+$ exists as a real number and the limit $\lim_{E\to+\infty} s_+(p,\lambda,E) = l_{+\infty}$ exists and belongs to $(0,+\infty]$. Moreover

$$0 \le l_0^+ < l_{+\infty} \le +\infty\,.$$

One may observe that, for any fixed $p>1$ and $\lambda \le 0$, the function $(E,s) \longmapsto M_+(E,s)$ is continuous in $[0,+\infty)^2$ and the function $E \longmapsto s_+(p,\lambda,E)$ is continuous in $(0,+\infty)$ and satisfies (12). So, by passing to the limit in (12) as E tends to 0^+ one gets:

$$0 = \lim_{E\to 0^+} M_+(E, s_+(p,\lambda,E)) = M_+(0,l_0^+).$$

Hence, l_0^+ is a zero, belonging to $[0,+\infty)$, of the equation in s: $M_+(0,s)=0$. By resolving this equation in $[0,+\infty)$ one gets: $l_0^+=0$. The assertion **(b)** is proved.

Assume that $l_{+\infty} < +\infty$ then by passing to the limit in (12) as E tends to $+\infty$ one gets:

$$+\infty = p' l_{+\infty}\Big(\frac{(l_{+\infty})^p}{p+1} - \lambda\Big) < +\infty,$$

which is impossible. So, $l_{+\infty} = +\infty$. Therefore, Lemma 6 is proved. ♢

Now we are ready to compute $X_+(p,\lambda,E)$ as defined in Section 3, for any $p>1, \lambda \le 0$ and $E \ge 0$. In fact, $X_+(p,\lambda,E) = (0, s_+(p,\lambda,E))$ if $E>0$ and $X_+(p,\lambda,0)=\emptyset$. Thus

$$r_+(p,\lambda,E) := \sup X_+(p,\lambda,E) = s_+(p,\lambda,E) \text{ if } E>0 \quad\text{and}\quad r_+(p,\lambda,0)=0\,,$$

and since $f(\lambda,s) = |s|^p - \lambda > 0$, $\forall(\lambda,s)\in(-\infty,0]\times\mathbb{R}, (\lambda,s)\neq(0,0)$, it follows that

$$\begin{aligned} D_+ &:= \{E\ge 0 : 0 < r_+(p,\lambda,E) < +\infty \text{ and } f(\lambda, r_+(p,\lambda,E)) > 0\} \\ &= (0,+\infty)\,. \end{aligned}$$

Before going further in the investigation, we deduce from Lemma 6 the following:

$$\lim_{E\to 0^+} r_+(p,\lambda,E) = 0 \quad\text{and}\quad \lim_{E\to+\infty} r_+(p,\lambda,E) = +\infty\,, \tag{13}$$

$$\frac{\partial r_+}{\partial E}(p,\lambda,E) = \frac{(p-1)E^{p-1}}{f(\lambda, r_+(p,\lambda,E))} > 0, \forall E \in D_+ = (0,+\infty) \quad \forall\lambda\le 0\,. \tag{14}$$

We define, for any $p>1, \lambda\le 0$, and $E\in D_+ = (0,+\infty)$ the time map

$$T_+(p,\lambda,E) := \int_0^{r_+(p,\lambda,E)} \{E^p - p'F(\lambda,\xi)\}^{-1/p}\, d\xi,\ E\in D_+ \tag{15}$$

and a simple change of variables shows that

$$T_+(p,\lambda,E) = r_+(p,\lambda,E)\int_0^1 \{E^p - p'F(\lambda, r_+(p,\lambda,E)\xi)\}^{-1/p}\, d\xi. \tag{16}$$

Observe that from the definition of $s_+(p,\lambda,E)$ one has

$$E^p - p' F(\lambda, s_+(p,\lambda,E)) = 0$$

and so, from the definition of $r_+(p,\lambda,E)$, one has $E^p = p'F(\lambda, r_+(p,\lambda,E))$. So, (16) may be written as

$$\begin{aligned} &T_+(p,\lambda,E) \\ &= r_+(p,\lambda,E)(p')^{-1/p} \int_0^1 \{F(\lambda, r_+(p,\lambda,E)) - F(\lambda, r_+(p,\lambda,E)\xi)\}^{-1/p}\, d\xi\,. \end{aligned} \tag{17}$$

After some rearrangements one has

$$\begin{aligned} &T_+(p,\lambda,E) \\ &= r_+^{1-\frac{1}{p}}(p,\lambda,E)(p')^{-1/p} \int_0^1 \left\{ \frac{r_+^p(p,\lambda,E)(1-\xi^{p+1})}{p+1} - \lambda(1-\xi) \right\}^{-1/p} d\xi\,. \end{aligned} \tag{18}$$

Lemma 7 *If $\lambda \le 0$ then one has*

(i) $\lim_{E\to 0^+} T_+(p,\lambda,E) = 0$, *if* $\lambda < 0$ *and* $\lim_{E\to 0^+} T_+(p,0,E) = +\infty$,

(ii) $\lim_{E\to +\infty} T_+(p,\lambda,E) = 0, \forall \lambda \le 0$,

(iii) *If $\lambda < 0$, $T_+(p,\lambda,\cdot)$ admits a unique critical point, $E^*(\lambda)$, at which it attains its global maximum value. Moreover,*

- **(a)** *The function $\lambda \mapsto T_+(p,\lambda,E^*(\lambda))$ is strictly increasing in $(-\infty,0)$.*
- **(b)** $\lim_{\lambda\to -\infty} T_+(p,\lambda,E^*(\lambda)) = 0$.
- **(c)** $\lim_{\lambda\to 0^-} T_+(p,\lambda,E^*(\lambda)) = +\infty$.

(iv) *If $\lambda = 0, (\partial T_+/\partial E)(p,0,\cdot) < 0$ in $(0,+\infty)$.*

Proof. **(i)** If $\lambda < 0$, from (18) one has

$$0 \le T_+(p,\lambda,E) \le r_+^{1-\frac{1}{p}}(p,\lambda,E)(p')^{-\frac{1}{p}} \int_0^1 \{-\lambda(1-\xi)\}^{-1/p}\, d\xi\,.$$

So, by passing to the limit as E tends to 0, one gets

$$0 \le \lim_{E\to 0} T_+(p,\lambda,E) \le \lim_{E\to 0} r_+^{1-\frac{1}{p}}(p,\lambda,E)(p')^{-1/p} \int_0^1 \{-\lambda(1-\xi)\}^{-1/p}\, d\xi = 0\,.$$

If $\lambda = 0$, then from (18) one gets

$$T_+(p,0,E) = (p')^{-1/p} r_+^{-1/p}(p,0,E) \int_0^1 (\frac{1-\xi^{p+1}}{p+1})^{-1/p}\, d\xi,$$

and from (13) one gets $\lim_{E\to 0^+} T_+(p,0,E) = +\infty$.

(ii) From (18) one has for any $\lambda \le 0$,

$$0 \le T_+(p,\lambda,E) \le r_+^{-1/p}(p,\lambda,E)(p')^{-\frac{1}{p}} \int_0^1 \left\{\frac{1-\xi^{p+1}}{p+1}\right\}^{-1/p} d\xi.$$

So, by passing to the limit as E tends to $+\infty$, one gets

$$\begin{aligned} 0 &\le \lim_{E\to+\infty} T_+(p,\lambda,E) \\ &\le \lim_{E\to+\infty} r_+^{-1/p}(p,\lambda,E)(p')^{-1/p} \int_0^1 \left\{\frac{1-\xi^{p+1}}{p+1}\right\}^{-1/p} d\xi = 0\,. \end{aligned}$$

(iii) If $\lambda < 0$, then from **(i)** and **(ii)** one deduces that $T_+(p,\lambda,\cdot)$ admits at least one critical point. Here, we are going to prove its uniqueness. From (17), one may observe that

$$T_+(p,\lambda,E) = (p')^{-\frac{1}{p}} S(p,\lambda,\rho(p,\lambda,E))$$

where $\rho(p,\lambda,E) = r_+(p,\lambda,E)$ and

$$S(p,\lambda,\rho) = \int_0^\rho \{F(p,\lambda,\rho) - F(p,\lambda,\xi)\}^{-\frac{1}{p}} d\xi\,.$$

On the other hand, observe that for each fixed $\lambda < 0$ the function $E \mapsto \rho(p,\lambda,E)$ is an increasing C^1-diffeomorphism from $(0,+\infty)$ onto itself (Lemma 6, assertions (a), (b) and (c)), and

$$\frac{\partial T_+}{\partial E}(p,\lambda,E) = (p')^{-1/p} \times \frac{\partial S}{\partial \rho}(p,\lambda,\rho(p,\lambda,E)) \times \frac{\partial \rho}{\partial E}(p,\lambda,E). \tag{19}$$

So, to study the variations of $E \mapsto T_+(p,\lambda,E)$ it suffices to study those of $\rho \mapsto S(p,\lambda,\rho)$. That is, $S(p,\lambda,\cdot)$ attains a local maximum (resp. minimum) value at ρ_* iff $T_+(p,\lambda,\cdot)$ does so at $\rho_{p,\lambda}^{-1}(\rho_*)$, where $\rho_{p,\lambda}^{-1}$ is the function inverse to $\rho(p,\lambda,\cdot)$. From **(i)** and **(ii)**, it follows that $\lim_{\rho\to 0} S(\rho) = \lim_{\rho\to+\infty} S(\rho) = 0$, that is, S admits at least a maximum value. To prove uniqueness, we first find a priori estimates on the critical points of $S(p,\lambda,\cdot)$. That is, for each $\lambda < 0$, we look for a compact interval $J(\lambda)$ which contains all possible critical points of $S(p,\lambda,\cdot)$. Next, we prove that $S(p,\lambda,\cdot)$ is concave in $J(\lambda)$.

One has

$$\frac{\partial S}{\partial \rho}(p,\lambda,\rho) = \int_0^\rho \frac{H(p,\lambda,\rho) - H(p,\lambda,u)}{p\rho(F(p,\lambda,\rho) - F(p,\lambda,u))^{\frac{p+1}{p}}} du \tag{20}$$

where $H(p,\lambda,u) = pF(p,\lambda,u) - uf(p,\lambda,u) = \frac{-u^{p+1}}{p+1} - \lambda(p-1)u$, $\forall u > 0$. The variations of $u \mapsto H(p,\lambda,u)$ can be described as follows. $H(p,\lambda,\cdot)$ is strictly increasing in $(0,\rho_1(p,\lambda))$ and strictly decreasing in $(\rho_1(p,\lambda),+\infty)$ where

$\rho_1(p,\lambda) = (-\lambda(p-1))^{1/p}$. Moreover, $H(p,\lambda,0) = H(p,\lambda,\rho_2(p,\lambda)) = 0$ where $\rho_2(p,\lambda) = (-\lambda(p^2-1))^{1/p} > \rho_1(p,\lambda)$. So, it follows that:

$$\frac{\partial S}{\partial \rho}(p,\lambda,\rho) > 0,\ \forall \rho \in (0,\rho_1(p,\lambda))$$

and

$$\frac{\partial S}{\partial \rho}(p,\lambda,\rho) < 0,\ \forall \rho \in (\rho_2(p,\lambda),+\infty).$$

That is, we get the a priori estimates as follows : $\forall p > 1, \forall \lambda < 0, \forall \rho_* > 0$,

$$\frac{\partial S}{\partial \rho}(p,\lambda,\rho_*) = 0 \Longrightarrow \rho_* \in J(\lambda) := [\rho_1(p,\lambda),\rho_2(p,\lambda)].$$

Easy computations show that for any $\rho > 0$ and $\lambda < 0$, one has

$$\begin{aligned}\frac{\partial^2 S}{\partial \rho^2}(p,\lambda,\rho) \quad &= \quad \int_0^1 \frac{(p+1)(H(p,\lambda,\rho)-H(p,\lambda,\rho\xi))^2}{p^2\rho(F(p,\lambda,\rho)-F(p,\lambda,\rho\xi))^{\frac{2p+1}{p}}}d\xi \\ &\quad + \int_0^1 \frac{p(\Psi(p,\lambda,\rho)-\Psi(p,\lambda,\rho\xi))(F(p,\lambda,\rho)-F(p,\lambda,\rho\xi))}{p^2\rho(F(p,\lambda,\rho)-F(p,\lambda,\rho\xi))^{\frac{2p+1}{p}}}\,d\xi\,,\end{aligned}$$

where

$$\begin{aligned}\Psi(p,\lambda,u) \quad &= \quad -p(p+1)F(p,\lambda,u) + 2puf(p,\lambda,u) - u^2 f'_u(p,\lambda,u) \\ &= \quad \lambda p(p-1)u, \forall u > 0\,.\end{aligned}$$

After some substitutions one gets

$$\frac{\partial^2 S}{\partial \rho^2}(p,\lambda,\rho) = \int_0^1 \frac{\rho(1-\xi)^2 P(X(\xi))}{p^2(F(p,\lambda,\rho)-F(p,\lambda,\rho\xi))^{\frac{2p+1}{p}}}\,d\xi\,,$$

where

$$X(\xi) = \begin{cases} p+1 & \text{if } \xi = 1 \\ \frac{1-\xi^{p+1}}{1-\xi} & \text{if } \xi \in [0,1) \end{cases}$$

and P is the polynomial function

$$P(X) = (\frac{\rho^{2p}}{p+1})X^2 + \frac{(p-1)(p^2+2p+2)}{(p+1)}\lambda\rho^p X - (p-1)\lambda^2.$$

An easily checked fact is that $X(\xi) \in [1,p+1]$, for all $\xi \in [0,1]$. In fact, the function $\xi \mapsto h(\xi) := \xi^{p+1}$ is convex in $(0,+\infty)$, and

$$X(\xi) = \frac{h(1)-h(\xi)}{1-\xi} \le h'(1) = p+1\,, \forall \xi \in (0,1)\,.$$

So, we are interested in the sign of $P(X)$ when $X \in [1,p+1]$. First, its discriminant is $\Delta = (\mu(p)/(p+1)^2)\lambda^2\rho^{2p} > 0$, where

$$\mu(p) = (p-1)^2(p^2+2p+2)^2 + 4(p^2-1)\,,$$

and its roots are, for each $\lambda < 0$ and $\rho > 0$,

$$X_1(p,\lambda,\rho) = \frac{\lambda}{2\rho^p}(\sqrt{\mu(p)} - (p-1)(p^2+2p+2)) < 0\,,$$
$$X_2(p,\lambda,\rho) = \frac{-\lambda}{2\rho^p}(\sqrt{\mu(p)} + (p-1)(p^2+2p+2)) > 0\,.$$

It can be verified that $\rho \mapsto X_2(p,\lambda,\rho)$ is decreasing in $(0,+\infty)$ and one can deduce, from $H(p,\lambda,\rho_2(p,\lambda)) = 0$, that

$$X_2(p,\lambda,\rho_2(p,\lambda)) = \frac{\sqrt{\mu(p)} + (p-1)(p^2+2p+2)}{2(p^2-1)},\ \forall\lambda < 0\,.$$

Hence, one can deduce that $X_2(p,\lambda,\rho_2(p,\lambda)) > p+1$. (In fact, to prove this it suffices to show that

$$\sqrt{\mu(p)} + (p-1)(p^2+2p+2) > 2(p+1)(p^2-1)$$

which is equivalent to proving that $\mu(p) > (p(p-1)(p+2))^2$, and this is (after some simple computations) equivalent to $4(p+1)p^2 > 0$ which is true since $p > 1$). Then

$$[1,p+1] \subset (X_1(p,\lambda,\rho), X_2(p,\lambda,\rho)) : \forall\lambda < 0, \forall\rho \in [\rho_1(p,\lambda), \rho_2(p,\lambda)],$$

hence, $P(X(\xi)) < 0$, for all $\xi \in [0,1]$, so,

$$\frac{\partial^2 S}{\partial\rho^2}(p,\lambda,\rho) < 0 \quad \forall\lambda < 0,\ \forall\rho \in J(\lambda) := [\rho_1(p,\lambda), \rho_2(p,\lambda)]\,,$$

which proves the uniqueness of the critical point of $S(p,\lambda,\cdot)$ and of $T_+(p,\lambda,\cdot)$.

(a) Some easy computations show that

$$\begin{aligned} &\frac{\partial T_+}{\partial E}(p,\lambda,E) \qquad (21)\\ &= (p')^{-1/p}\frac{\partial r_+}{\partial E}(p,\lambda,E)\int_0^{r_+(p,\lambda,E)} \frac{H(p,\lambda,r_+) - H(p,\lambda,\xi)}{pr_+(F(p,\lambda,r_+) - F(p,\lambda,\xi))^{\frac{p+1}{p}}}\,d\xi\,,\end{aligned}$$

and that

$$\begin{aligned} &\frac{\partial T_+}{\partial \lambda}(p,\lambda,E) \qquad (22)\\ &= (p')^{-1/p}\frac{\partial r_+}{\partial \lambda}(p,\lambda,E)\int_0^{r_+(p,\lambda,E)} \frac{H(p,\lambda,r_+) - H(p,\lambda,\xi)}{pr_+(F(p,\lambda,r_+) - F(p,\lambda,\xi))^{\frac{p+1}{p}}}d\xi\\ &\quad +(p')^{-\frac{1}{p}}\int_0^{r_+(p,\lambda,E)} \frac{r_+(p,\lambda,E) - \xi}{p(F(p,\lambda,r_+) - F(p,\lambda,\xi))^{\frac{p+1}{p}}}\,d\xi\,,\end{aligned}$$

and then combining (21) and (22) one gets

$$\begin{aligned} & -\frac{\partial r_+}{\partial \lambda}\frac{\partial T_+}{\partial E}+\frac{\partial r_+}{\partial E}\frac{\partial T_+}{\partial \lambda} \\ = \;& (p')^{-1/p}\frac{\partial r_+}{\partial E}(p,\lambda,E)\int_0^{r_+(p,\lambda,E)}\frac{r_+(p,\lambda,E)-\xi}{p(F(p,\lambda,r_+)-F(p,\lambda,\xi))^{\frac{p+1}{p}}}\,d\xi\,, \end{aligned}$$

so,

$$-\frac{\partial r_+}{\partial \lambda}\frac{\partial T_+}{\partial E}+\frac{\partial r_+}{\partial E}\frac{\partial T_+}{\partial \lambda}>0,\quad \forall E>0\,,\ \lambda<0\,. \tag{23}$$

Since, $(\partial T_+/\partial E)(p,\lambda,E^*(\lambda))=0$, using (23) and (14) one gets:

$$\frac{\partial T_+}{\partial \lambda}(p,\lambda,E^*(\lambda))>0,\ \forall \lambda<0\,.$$

(b) Since $H(p,\lambda,\cdot)$ is strictly increasing on $(0,\rho_1(p,\lambda))$,

$$\frac{\partial T_+}{\partial E}(p,\lambda,E)>0,\forall E\in(0,E_1(p,\lambda))$$

where $E_1(p,\lambda):=(p'F(p,\lambda,\rho_1(p,\lambda)))^{1/p}$. Since $(\partial T_+/\partial E)(p,\lambda,E^*(\lambda))=0$, it follows that $E^*(\lambda)\geq E_1(\lambda)$, and since

$$F(r_+(E^*(\lambda)))=\frac{(E^*)^p(\lambda)}{p'}\geq\frac{E_1^p(p,\lambda)}{p'}=F(\rho_1(p,\lambda))$$

and F is continuous and strictly increasing ($\lambda<0$), it follows that

$$r_+(p,\lambda,E^*(\lambda))\geq\rho_1(p,\lambda)=r_+(p,\lambda,E_1(p,\lambda)).$$

One has from (18),

$$\begin{aligned} T_+(p,\lambda,E^*(\lambda)) &\leq (p')^{-1/p}r_+^{-1/p}(p,\lambda,E^*(\lambda))\int_0^1(\frac{1-\xi^{p+1}}{p+1})^{-1/p}d\xi \\ &\leq (p')^{-1/p}\left\{\rho_1(p,\lambda)\right\}^{-1/p}\int_0^1(\frac{1-\xi^{p+1}}{p+1})^{-1/p}\,d\xi \end{aligned}$$

and by passing to the limit as λ tends to $-\infty$, one gets

$$\begin{aligned} 0 &\leq \lim_{\lambda\to-\infty}T_+(p,\lambda,E^*(\lambda)) \\ &\leq (p')^{-1/p}\int_0^1(\frac{1-\xi^{p+1}}{p+1})^{-1/p}d\xi\lim_{\lambda\to-\infty}\left\{-\lambda(p-1)\right\}^{-1/p^2}=0\,. \end{aligned}$$

(c) For each $\lambda<0$, one has

$$T_+(p,\lambda,E^*(\lambda))=\sup_{E>0}T_+(p,\lambda,E)\geq T_+(p,\lambda,E_1(\lambda))$$

and from (18) and the fact $\rho_1(p,\lambda) = r_+(p,\lambda,E_1(p,\lambda))$ one has

$$\begin{aligned} & T_+(p,\lambda,E_1(\lambda)) \\ & = \quad (-\lambda)^{-1/p^2}(p')^{-1/p}(p-1)^{\frac{p-1}{p^2}} \int_0^1 \left\{ \frac{p-1}{p+1}(1-\xi^{p+1}) + (1-\xi) \right\}^{-1/p} d\xi \,. \end{aligned}$$

So,

$$\lim_{\lambda\to 0^-} T_+(p,\lambda,E^*(\lambda)) \geq \lim_{\lambda\to 0^-} T_+(p,\lambda,E_1(\lambda)) = +\infty \,.$$

(iv) If $\lambda = 0$, the function $\rho \mapsto S(p,\lambda,\rho)$ decreases strictly on $(0,+\infty)$, since the function $u \mapsto H(p,0,u) := -u^{p+1}/(p+1)$ does so in $(0,+\infty)$ (see (20)). Then, from (19) and (14) it follows that $(\partial T_+/\partial E)(p,0,\cdot) < 0$ in $(0,+\infty)$. Therefore, Lemma 7 is proved. $\diamondsuit$

Completion of the proof of Theorem 2. The proof is an easy consequence of the previous lemmas. In fact, there exists a unique $\lambda^* < 0$ which satisfies $T_+(p,\lambda^*,E^*(\lambda^*)) = \frac{1}{2}$, and the function $\lambda \mapsto T_+(p,\lambda,E^*(\lambda))$ is strictly increasing in $(-\infty,0)$. So, if $\lambda < \lambda^*$, for any $E > 0$ and $\lambda < 0$,

$$T_+(p,\lambda,E) \leq \sup_{E>0} T_+(p,\lambda,E) = T_+(p,\lambda,E^*(\lambda)) < T_+(p,\lambda^*,E^*(\lambda^*)) = \frac{1}{2} \,.$$

Thus equation $T_+(p,\lambda,E) = \frac{1}{2}$ admits no solution. If $\lambda = \lambda^*$, $E^*(\lambda^*)$ is the unique solution of the equation $T_+(p,\lambda^*,E) = \frac{1}{2}$. So, problem (1) admits a unique positive solution and this one is in A_1^+. Finally, if $0 > \lambda > \lambda^*$, then $T_+(p,\lambda,E^*(\lambda)) > T_+(p,\lambda^*,E^*(\lambda^*)) = \frac{1}{2}$. So, equation $T_+(p,\lambda,E) = \frac{1}{2}$ admits exactly two solutions and then problem (1) admits exactly two positive solutions in A_1^+. If $\lambda = 0$, $T_+(p,0,\cdot)$ is strictly decreasing in $(0,+\infty)$ and $\lim_{E\to 0^+} T_+(p,0,E) = +\infty$ and $\lim_{E\to+\infty} T_+(p,0,E) = 0$. So, equation $T_+(p,0,E) = (1/2)$ admits a unique solution in $(0,+\infty)$. Thus, Theorem 2 is proved. $\diamondsuit$

5 Proof of Theorem 3

As for the proof of Theorem 2, we begin this section by some preliminary lemmas. In order to define the time-maps we need as usual the following technical lemma.

Lemma 8 *Consider the equation in the variable $s \in \mathbb{R}^*$,*

$$E^p - p'F(\lambda,s) = 0 \tag{24}$$

where $p > 1$, $\lambda > 0$ and $E \geq 0$ are real parameters. First, if $E = 0$, equation (24) admits a unique positive zero $s_+ = s_+(p,\lambda,0)$ and a unique negative zero $s_- = s_-(p,\lambda,0)$ such that $|s_\pm| = (\lambda(p+1))^{1/p}$. Moreover, for any $E > 0$, equation (24) admits a unique positive zero $s_+ = s_+(p,\lambda,E)$ and this zero belongs to the open interval $((\lambda(p+1))^{1/p},+\infty)$. On the other hand,

(i) *If* $E > E_*(p,\lambda) := ((\frac{pp\prime}{p+1})\lambda^{1+\frac{1}{p}})^{1/p}$*, equation (24) admits no negative zero.*

(ii) *If* $E = E_*(p,\lambda)$*, equation (24) admits a unique negative zero* $s_- = s_-(p,\lambda) = -\lambda^{1/p}$.

(iii) *If* $0 < E < E_*(p,\lambda)$*, equation (24) admits, in the open interval* $(-\lambda^{1/p}, 0)$*, a unique zero* $s_- = s_-(p,\lambda,E)$.

Moreover,

(a) *The function* $E \longmapsto s_\pm(p,\lambda,E)$ *is* C^1 *in* $(0,+\infty)$ *(resp.* $(0, E_*(p,\lambda))$*) and*

$$\pm\frac{\partial s_\pm}{\partial E}(p,\lambda,E) = \frac{\pm(p-1)E^{p-1}}{f(\lambda, s_\pm(p,\lambda,E))} > 0\,,$$

for all $p > 1$*, for all* $\lambda > 0$*, and for all* $E > 0$*. (resp. for all* $E \in (0, E_*(p,\lambda))$*).*

(b) $\lim\limits_{E\to 0^+} s_+(p,\lambda,E) = ((p+1)\lambda)^{1/p}$ *and* $\lim\limits_{E\to 0^+} s_-(p,\lambda,E) = 0$.

(c) $\lim\limits_{E\to +\infty} s_+(p,\lambda,E) = +\infty$ *and* $\lim\limits_{E\to E_*} s_-(p,\lambda,E) = -\lambda^{1/p}$.

Proof. For a fixed $p > 1$, $\lambda > 0$ and $E \geq 0$, consider the function

$$s \longmapsto N(p,\lambda,E,s) := E^p - p'F(\lambda,s) = E^p - p's(\frac{|s|^p}{p+1} - \lambda),$$

defined in $\mathbb{R}$. From a study of its variations, it is clear that equation (24) admits, if $E = 0$, a unique positive zero s_+ and a unique negative zero s_-. Their values are obtained by simple resolution of equation (24). Moreover, for any $E > 0$, equation (24) admits a unique positive zero, $s_+ = s_+(p,\lambda,E)$, and this zero belongs to the open interval $((\lambda(p+1))^{1/p}, +\infty)$ (since

$$N(p,\lambda,E,(\lambda(p+1))^{\frac{1}{p}}) = N(p,\lambda,E,0) = E^p > 0).$$

Also, the assertions **(i)** **(ii)** and **(iii)** follow readily from the variations of $N(p,\lambda,E,\cdot)$.

Now, for any $p > 1$ and $\lambda > 0$, consider the real-valued function

$$(E,s) \longmapsto N_\pm(E,s) = E^p - p's(\frac{(\pm s)^p}{p+1} - \lambda)$$

defined on $\Omega_+ = (0,+\infty) \times ((\lambda(p+1))^{1/p}, +\infty)$ (resp. $\Omega_- = (0, E_*(p,\lambda)) \times (-\lambda^{1/p}, 0)$). One has $N_\pm \in C^1(\Omega_\pm)$ and

$$\frac{\partial N_\pm}{\partial s}(E,s) = -p'f(\lambda,s) = -p'(|s|^p - \lambda) \quad \text{in } \Omega_\pm,$$

hence

$$\mp\frac{\partial N_\pm}{\partial s}(E,s) > 0 \quad \text{in } \Omega_\pm$$

and one may observe that $s_\pm(p,\lambda,E)$ belongs to the open interval $((\lambda(p+1))^{1/p},+\infty)$ (resp. $(-\lambda^{1/p},0)$) and satisfies (from its definition)

$$N_\pm(E,s_\pm(p,\lambda,E))=0. \tag{25}$$

So, one can make use of the implicit function theorem to show that the function $E\longmapsto s_\pm(p,\lambda,E)$ is $C^1((0,+\infty),\mathbb{R})$ (resp. $C^1((0,E_*(p,\lambda)),\mathbb{R})$) and to obtain the expression of $\frac{\partial s_\pm}{\partial E}(p,\lambda,E)$ given in **(a)**. Hence, for any fixed $p>1$ and $\lambda>0$, the function defined in $(0,+\infty)$ (resp. $(0,E_*(p,\lambda))$) by $E\longmapsto s_\pm(p,\lambda,E)$ is strictly increasing (resp. decreasing) and bounded from below by $(\lambda(p+1))^{1/p}$ (resp. $-\lambda^{1/p}$) and from above by $+\infty$ (resp. by 0). Then, the limit $\lim\limits_{E\to 0^+} s_\pm(p,\lambda,E)=l_0^\pm$ exists as a real number and the limit $\lim\limits_{E\to+\infty} s_+(p,\lambda,E)=l_{+\infty}$ (resp. $\lim\limits_{E\to E_*} s_-(p,\lambda,E)=l_*$) exists and belongs to $((\lambda(p+1))^{1/p},+\infty]$ (resp. $[-\lambda^{1/p},0]$). Moreover

$$-\infty<-\lambda^{1/p}\le l_*<l_0^-\le 0<(\lambda(p+1))^{1/p}\le l_0^+<l_{+\infty}\le+\infty\,.$$

One may observe that, for any fixed $p>1$ and $\lambda>0$, the function

$$(E,s)\longmapsto N_\pm(E,s)$$

is continuous in $[0,+\infty)\times[(\lambda(p+1))^{1/p},+\infty)$ (resp. $[0,E_*(p,\lambda)]\times(-\infty,0]$) and the function $E\longmapsto s_\pm(p,\lambda,E)$ is continuous in $(0,+\infty)$ (resp. $(0,E_*(p,\lambda))$) and satisfies $(25)_\pm$. So, by passing to the limit in $(25)_\pm$ as E tends to 0^+ one gets

$$0=\lim_{E\to 0^+} N_\pm(E,s_\pm(p,\lambda,E))=N_\pm(0,l_0^\pm).$$

Hence, $l_0^\pm$ is a zero, belonging to $[(\lambda(p+1))^{1/p},+\infty)$ (resp. $[-\lambda^{1/p},0]$), to the equation in the variable s:

$$N_\pm(0,s)=0.$$

By resolving this equation in the indicated interval one gets : $l_0^+=((p+1)\lambda)^{1/p}$ (resp. $l_0^-=0$). The assertion **(b)** is proved.

Assume that $l_{+\infty}<+\infty$. Then by passing to the limit in $(25)_+$ as E tends to $+\infty$ one gets

$$+\infty=p'l_{+\infty}(\frac{(l_{+\infty})^p}{p+1}-\lambda)<+\infty,$$

which is impossible. So, $l_{+\infty}=+\infty$.

To prove that $l_*=-\lambda^{1/p}$, it suffices to pass to the limit in $(25)_-$ as E tends to $E_*(p,\lambda)$ to get

$$N_-(E_*(p,\lambda),l_*)=0$$

and to resolve this equation in $[-\lambda^{1/p},0]$. (To this end, one may observe that the function $s\longmapsto N_-(E_*(p,\lambda),s)$ is strictly increasing in $[-\lambda^{1/p},0]$ and

$$N_-(E_*(p,\lambda),-\lambda^{1/p})=0).$$

Therefore, Lemma 8 is proved. $\diamondsuit$

Now we are ready to compute $X_\pm(p,\lambda,E)$ as defined in Section 3, for any $p>1$, $\lambda>0$ and $E\geq 0$. In fact, $X_+(p,\lambda,E)=(0,s_+(p,\lambda,E))$ and

$$X_-(p,\lambda,E)=\begin{cases}(-\infty,0) & \text{if } E>E_*(p,\lambda)\\ (s_-(p,\lambda,E),0) & \text{if } 0\leq E\leq E_*(p,\lambda)\,,\end{cases}$$

where $s_\pm(p,\lambda,E)$ is defined in Lemma 8. Then

$$r_+(p,\lambda,E):=\sup X_+(p,\lambda,E)=s_+(p,\lambda,E)$$

and

$$r_-(p,\lambda,E):=\inf X_-(p,\lambda,E)=\begin{cases}-\infty & \text{if } E>E_*(p,\lambda)\\ s_-(p,\lambda,E) & \text{if } 0\leq E\leq E_*(p,\lambda).\end{cases}$$

Recall that for any $E\geq 0$, $s_+(p,\lambda,E)$ belongs to $[(\lambda(p+1))^{\frac{1}{p}},+\infty)$. Thus

$$0<r_+(p,\lambda,E)<+\infty \quad \text{if and only if} \quad E>0\,.$$

Also recall that, for any $0<E\leq E_*(p,\lambda)$, $s_-(p,\lambda,E)$ belongs to $[-\lambda^{1/p},0)$ and $s_-(p,\lambda,0)=-((p+1)\lambda)^{1/p}$, so

$$-\infty<r_-(p,\lambda,E)<0 \quad \text{if and only if} \quad 0\leq E\leq E_*(p,\lambda)\,.$$

One may observe that $f(\lambda,r_+(p,\lambda,E))=r_+^p(p,\lambda,E)-\lambda>0,\forall E\geq 0$ and

$$f(\lambda,r_-(p,\lambda,E))=(-r_-(p,\lambda,E))^p-\lambda<0\leftrightarrow E\in(0,E_*(p,\lambda)),$$

so that

$$D_+:=\{E\geq 0\mid 0<r_+(p,\lambda,E)<+\infty \text{ and } f(\lambda,r_+(p,\lambda,E))>0\}=[0,+\infty).$$

and

$$D_-:=\{E\geq 0\mid -\infty<r_-(p,\lambda,E)<0 \text{ and } f(\lambda,r_-(p,\lambda,E))<0\}=(0,E_*(p,\lambda)).$$

So, $D:=D_+\cap D_-=(0,E_*(p,\lambda))$.

Before going further in the investigation, we deduce from Lemma 8 that

$$\lim_{E\to 0^+} r_+(p,\lambda,E)=((p+1)\lambda)^{1/p} \quad \text{and} \quad \lim_{E\to 0^+} r_-(p,\lambda,E)=0, \tag{26}$$

$$\lim_{E\to +\infty} r_+(p,\lambda,E)=+\infty \quad \text{and} \quad \lim_{E\to E_*} r_-(p,\lambda,E)=-\lambda^{1/p}, \tag{27}$$

$$\frac{\partial r_\pm}{\partial E}(p,\lambda,E)=\frac{(p-1)E^{p-1}}{f(\lambda,r_\pm(p,\lambda,E))}\,,\forall E\in \operatorname{int}(D_\pm), \tag{28}$$

$$\pm\frac{\partial r_\pm}{\partial E}(p,\lambda,E)>0,\,\forall E\in\operatorname{int}(D_\pm)\,. \tag{29}$$

At present, we define, for any $p > 1, \lambda > 0$, and $E \in D_\pm$, the time map

$$T_\pm(p,\lambda,E) := \pm \int_0^{r_\pm(p,\lambda,E)} \{E^p - p'F(\lambda,\xi)\}^{-1/p}\, d\xi,\ E \in D_\pm, \tag{30}$$

and a simple change of variables shows that

$$T_\pm(p,\lambda,E) = \pm r_\pm(p,\lambda,E) \int_0^1 \{E^p - p'F(\lambda, r_\pm(p,\lambda,E)\xi)\}^{-1/p}\, d\xi\,. \tag{31}$$

Observe that from the definition of $s_\pm(p,\lambda,E)$ one has $E^p - p'F(\lambda, s_\pm(p,\lambda,E)) = 0$, and so, from the definition of $r_\pm(p,\lambda,E)$, one has $E^p = p'F(\lambda, r_\pm(p,\lambda,E))$. So, (31) may be written as

$$\begin{aligned} T_\pm(p,\lambda,E) \;&=\; \pm r_\pm(p,\lambda,E)(p')^{-1/p} \times \\ &\quad \int_0^1 \{F(\lambda, r_\pm(p,\lambda,E)) - F(\lambda, r_\pm(p,\lambda,E)\xi)\}^{-1/p}\, d\xi\,. \end{aligned} \tag{32}$$

After some substitutions one has

$$\begin{aligned} T_+(p,\lambda,E) \;&=\; (r_+(p,\lambda,E))^{-1/p}(p')^{-1/p} \times \\ &\quad \int_0^1 \left\{\frac{1-\xi^{p+1}}{p+1} - \lambda\frac{1-\xi}{(r_+(p,\lambda,E))^p}\right\}^{-1/p} d\xi,\ E \in D_+ \end{aligned} \tag{33}$$

and

$$\begin{aligned} T_-(p,\lambda,E) \;&=\; (-r_-(p,\lambda,E))^{1-\frac{1}{p}}(p')^{-1/p} \times \\ &\quad \int_0^1 \left\{\lambda(1-\xi) - \frac{(-r_-(p,\lambda,E))^p}{p+1}(1-\xi^{p+1})\right\}^{-1/p} d\xi,\ E \in D_-\,. \end{aligned} \tag{34}$$

Also, we define for any $E \in D = D_+ \cap D_-$ and $n \in \mathbb{N}$ the time maps:

$$T_{2n}(p,\lambda,E) := n(T_+(p,\lambda,E) + T_-(p,\lambda,E)), E \in D, \tag{35}$$

$$T^\pm_{2n+1}(p,\lambda,E) := T_{2n}(p,\lambda,E) + T_\pm(p,\lambda,E), E \in D\,. \tag{36}$$

The limits of these time maps are the aim of the following lemmas.

Lemma 9 *For any $p > 1$ and $\lambda > 0$, one has $T_+(p,\lambda,E_*(p,\lambda)) = \lambda^{-1/p^2} \times J_+(p)$, where*

$$J_+(p) := (p')^{-1/p}(p+1)^{1/p}\theta(p) \times \int_0^1 \left\{p - (\theta(p)\xi)^{p+1} + (p+1)\theta(p)\xi\right\}^{-1/p} d\xi$$

and $\theta(p) > (p+1)^{1/p}$ is the unique positive zero of the equation

$$\theta^{p+1} - (p+1)\theta - p = 0\,. \tag{37}$$

Lemma 10 *For any $p > 1$ and $\lambda > 0$, let*

$$J(p) := \frac{1}{p}(p')^{-1/p}(p+1)^{\frac{p-1}{p^2}} \cdot \frac{\Gamma(\frac{p-1}{p^2})\Gamma(\frac{p-1}{p})}{\Gamma(\frac{(p-1)(p+1)}{p^2})}$$

and

$$J_-(p) := (p')^{-1/p}(p+1)^{1/p} \int_0^1 \left\{p - (p+1)\xi + \xi^{p+1}\right\}^{-1/p} d\xi.$$

Then one has:

$$J_-(p) < +\infty \leftrightarrow p > 2, \tag{38}$$

(i) $\lim_{E\to 0^+} T_+(p,\lambda,E) = J(p)\lambda^{-1/p^2}$, **(ii)** $\lim_{E\to 0^+} T_-(p,\lambda,E) = 0$,
(iii) $\lim_{E\to +\infty} T_+(p,\lambda,E) = 0$, **(iv)** $\lim_{E\to E_*} T_-(p,\lambda,E) = J_-(p)\lambda^{-1/p^2}$.

Lemma 11 *For any $p > 1$ and $\lambda > 0$, one has*

(a) $\lim_{E\to 0^+} T_{2n}(p,\lambda,E) = nJ(p)\lambda^{-1/p^2}$

(b) $\lim_{E\to 0^+} T^+_{2n+1}(p,\lambda,E) = (n+1)J(p)\lambda^{-1/p^2}$,

(c) $\lim_{E\to 0^+} T^-_{2n+1}(p,\lambda,E) = nJ(p)\lambda^{-1/p^2}$,

(d) $\lim_{E\to E_*} T_{2n}(p,\lambda,E) = n(J_+(p) + J_-(p)) \times \lambda^{-1/p^2}$,

(e) $\lim_{E\to E_*} T^+_{2n+1}(p,\lambda,E) = ((n+1)J_+(p) + nJ_-(p)) \times \lambda^{-1/p^2}$,

(f) $\lim_{E\to E_*} T^-_{2n+1}(p,\lambda,E) = (nJ_+(p) + (n+1)J_-(p)) \times \lambda^{-1/p^2}$.

Proof of Lemma 9. For any $p > 1$, let us consider the function Θ defined in $(0,+\infty)$ by $\Theta(\theta) := \theta^{p+1} - (p+1)\theta - p$. A study of its variations implies that equation (37) admits a unique zero in $(0,+\infty)$, denoted by $\theta(p)$, and this zero belongs to $((p+1)^{1/p}, +\infty)$ (Note that $\Theta((p+1)^{1/p}) = -p$). Furthermore, recall (Lemma 8) that, for any $\lambda > 0$ and $E > 0$, $r_+(p,\lambda,E)$ is the unique positive solution of equation (24). In particular, if $E = E_*(p,\lambda) := ((\frac{pp'}{p+1})\lambda^{1+\frac{1}{p}})^{1/p}$ then $r_+(p,\lambda,E_*(p,\lambda))$ is the unique positive solution of the following equation in the variable s :

$$s^{p+1} - \lambda(p+1)s - p\lambda^{1+\frac{1}{p}} = 0. \tag{39}$$

Some easy computations show that $\theta(p)\lambda^{1/p}$ is also a positive solution of (39), and since (39) admits a unique positive solution (which is $r_+(p,\lambda,E_*(p,\lambda))$) it follows that

$$r_+(p,\lambda,E_*(p,\lambda)) = \theta(p)\lambda^{1/p},\ \forall p > 1, \forall\lambda > 0.$$

Now, from (33), some simple computations show that $T_+(p,\lambda,E_*(p,\lambda)) = \lambda^{-1/p^2}J_+(p)$ where

$$J_+(p) := (p')^{-1/p}(p+1)^{1/p}\theta(p)\int_0^1 (p-(\theta(p)\xi)^{p+1}+(p+1)\theta(p)\xi)^{-1/p}d\xi\,.$$

Therefore, Lemma 9 is proved. $\diamondsuit$

Proof of Lemma 10. In order to prove the first assertion we first claim that there exists $\varepsilon_0 > 0$ (sufficiently small) such that for any $\xi \in (1-\varepsilon_0, 1)$,

$$\frac{p(p+1)}{4}(1-\xi)^2 \leq p-(p+1)\xi+\xi^{p+1} \leq p(p+1)(1-\xi)^2.$$

To proof this claim, for any $x > 0$, let

$$h_x(\xi) := p-(p+1)\xi+\xi^{p+1}-x(1-\xi)^2, \xi \in (0,1]\,.$$

Simple computations lead to

$$\frac{dh_x}{d\xi}(\xi) = 2(1-\xi)(x-(\frac{p+1}{2})\frac{1-\xi^p}{1-\xi}), \xi \in (0,1)\,.$$

Using l'Hôpital's rule one gets

$$\lim_{\xi\to 1^-}(x-(\frac{p+1}{2})\frac{1-\xi^p}{1-\xi}) = (x-\frac{p(p+1)}{2}).$$

So, because of continuity properties, there exists $\varepsilon_1 > 0$ (resp. $\varepsilon_2 > 0$) sufficiently small such that

$$\frac{dh_{p(p+1)}}{d\xi}(\xi) > 0, \forall \xi \in (1-\varepsilon_1, 1)$$

(resp. $\frac{dh_{p(p+1)/4}}{d\xi}(\xi) < 0, \forall \xi \in (1-\varepsilon_2, 1)$).

Notice that $h_x(1) = 0, \forall x > 0$, so that

$$h_{p(p+1)}(\xi) < 0, \forall \xi \in (1-\varepsilon_0, 1)$$

(resp. $h_{p(p+1)/4}(\xi) > 0, \forall \xi \in (1-\varepsilon_0, 1)$) where $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$. Then the claim is proved.

With this claim we are able to prove easily the first assertion of this lemma. In fact, the integral which appears in the definition of $J_-(p)$ may be written as

$$\int_0^{1-\varepsilon_0}(p-(p+1)\xi+\xi^{p+1})^{-1/p}d\xi + \int_{1-\varepsilon_0}^1 (p-(p+1)\xi+\xi^{p+1})^{-1/p}d\xi\,.$$

The first integral converges because the integrand function is continuous on the compact interval $[0, 1-\varepsilon_0]$. For the second integral, one has from the claim

$$A(p)\int_{1-\varepsilon_0}^{1}\frac{d\xi}{(1-\xi)^{\frac{2}{p}}} \le \int_{1-\varepsilon_0}^{1}(p-(p+1)\xi+\xi^{p+1})^{-1/p}d\xi \le B(p)\int_{1-\varepsilon_0}^{1}\frac{d\xi}{(1-\xi)^{\frac{2}{p}}}$$

where $A(p) = (p(p+1))^{-1/p}$ and $B(p) = (p(p+1)/4)^{-1/p}$. So, from the well-known fact

$$\int_{1-\varepsilon_0}^{1}\frac{d\xi}{(1-\xi)^{\frac{2}{p}}} < +\infty \leftrightarrow p > 2$$

the first assertion follows.

Proof of (i). One has from (31)

$$T_+(p,\lambda,E) = r_+(p,\lambda,E)\int_0^1 (E^p - p'F(\lambda, r_+(p,\lambda,E)\xi))^{-1/p}d\xi.$$

Using (26) one gets:

$$\begin{aligned}\lim_{E\to 0^+} E^p - p'F(\lambda, r_+(p,\lambda,E)\xi) &= -p'F(\lambda, ((p+1)\lambda)^{1/p}\xi)\\ &= p'((p+1)\lambda)^{1/p}\lambda\xi(1-\xi^p),\end{aligned}$$

so, some simple computations yield

$$\lim_{E\to 0^+} T_+(p,\lambda,E) = (p+1)^{\frac{p-1}{p^2}}(p')^{-1/p}\lambda^{-1/p^2}\int_0^1 \xi^{-1/p}(1-\xi^p)^{-1/p}d\xi\,.$$

To compute this integral, one can make use of the change of variables $x = \xi^p$ and then make use of the relationship between the Euler beta and gamma functions, see for instance [13, Chap. VII, no 90, example 2, pp. 595-596], to obtain:

$$\int_0^1 \xi^{-1/p}(1-\xi^p)^{-1/p}d\xi = \frac{1}{p}\frac{\Gamma(\frac{p-1}{p^2})\Gamma(\frac{p-1}{p})}{\Gamma(\frac{(p-1)(p+1)}{p^2})}.$$

This completes the proof of **(i)**.

Proof of (ii). Consider the expression for $T_-(p,\lambda,E)$ given by (34). From (26) one gets

$$\begin{aligned}&\lim_{E\to 0^+}\int_0^1 (\lambda(1-\xi) - \frac{(-r_-(p,\lambda,E))^p}{p+1}(1-\xi^{p+1}))^{-1/p}d\xi\\ &= \int_0^1 (\lambda(1-\xi))^{-1/p}d\xi = \lambda^{-1/p}p'\,.\end{aligned}$$

So, from (26) and (34) one gets

$$\lim_{E\to 0^+} T_-(p,\lambda,E) = (p')^{-1/p}\times(0)^{1-\frac{1}{p}}\times\lambda^{-1/p}\times p' = 0.$$

This completes the proof of **(ii)**.

Proof of (iii). Consider the expression for $T_+(p,\lambda,E)$ given by (33). From (26) one gets

$$\lim_{E\to+\infty}\int_0^1 (\frac{1-\xi^{p+1}}{p+1}-\lambda\frac{1-\xi}{r_+^p(p,\lambda,E)})^{-\frac{1}{p}}d\xi = \frac{1}{(p+1)^{-1/p}}\int_0^1 (1-\xi^{p+1})^{-1/p}d\xi,$$

and this integral may be computed by making use of the change of variables $x=\xi^{p+1}$ to get

$$\int_0^1 (1-\xi^{p+1})^{-1/p}d\xi = \frac{1}{p+1}\frac{\Gamma(\frac{1}{p+1})\Gamma(\frac{p-1}{p})}{\Gamma(\frac{p-1}{p(p+1)})}\,.$$

So, from (26) and (33) one gets

$$\lim_{E\to+\infty} T_+(p,\lambda,E) = (p')^{-1/p}\times 0\times\frac{1}{(p+1)^{1-\frac{1}{p}}}\times\frac{\Gamma(\frac{1}{p+1})\Gamma(\frac{p-1}{p})}{\Gamma(\frac{p-1}{p(p+1)})}=0\,.$$

This completes the proof of **(iii)**.

Proof of (iv). Consider the expression for $T_-(p,\lambda,E)$ given by (32). One has

$$\begin{aligned}\lim_{E\to E_*}(F(\lambda,r_-(p,\lambda,E))-F(\lambda,r_-(p,\lambda,E)\xi)) &= \lim_{x\to-\lambda^{1/p}}(F(\lambda,x)-F(\lambda,x\xi))\\ &= \frac{\lambda^{1+\frac{1}{p}}}{p+1}(p-(p+1)\xi+\xi^{p+1})\end{aligned}$$

so that

$$\lim_{E\to E_*} T_-(p,\lambda,E) = \lambda^{1/p}\times(p')^{-1/p}\times(\frac{\lambda^{1+\frac{1}{p}}}{p+1})^{-1/p}\times\int_0^1 (p-(p+1)\xi+\xi^{p+1})^{-1/p}d\xi$$

which is the same as

$$\lim_{E\to E_*} T_-(p,\lambda,E) = \lambda^{-1/p^2}\times(p')^{-1/p}\times(p+1)^{1/p}\times\int_0^1 (p-(p+1)\xi+\xi^{p+1})^{-1/p}d\xi.$$

This completes the proof of **(iv)** and of Lemma 10. $\diamondsuit$

Proof of Lemma 11. This proof is an immediate consequence of the two preceding lemmas and the definitions (35) and (36) of the time maps $T_{2n}, T_{2n+1}^{\pm}$.

Lemma 12 *For any $p>1, \lambda>0$, one has:*

$$\pm\frac{\partial T_\pm}{\partial E}(p,\lambda,E)<0,\ \forall E\in D_\pm.$$

Proof. From (32) one has

$$
\begin{aligned}
&\pm\frac{\partial T_\pm}{\partial E}(p,\lambda,E)\\
&= (p')^{-1/p}\Big\{\frac{\partial r_\pm}{\partial E}(p,\lambda,E)\int_0^1 (F(\lambda,r_\pm(p,\lambda,E))-F(\lambda,r_\pm(p,\lambda,E)\xi))^{-1/p}d\xi\\
&\qquad +r_\pm(p,\lambda,E)\int_0^1 \frac{\partial}{\partial E}(F(\lambda,r_\pm(p,\lambda,E))-F(\lambda,r_\pm(p,\lambda,E)\xi))^{-1/p}d\xi\Big\}\\
&= (p')^{-1/p}\Big\{\frac{\partial r_\pm}{\partial E}(p,\lambda,E)\int_0^1 \frac{(F(\lambda,r_\pm(p,\lambda,E))-F(\lambda,r_\pm(p,\lambda,E)\xi))}{(F(\lambda,r_\pm(p,\lambda,E))-F(\lambda,r_\pm(p,\lambda,E)\xi))^{1+\frac{1}{p}}}d\xi\\
&\qquad -\frac{1}{p}r_\pm(p,\lambda,E)\frac{\partial r_\pm}{\partial E}(p,\lambda,E)\times\\
&\qquad \int_0^1 \frac{f(\lambda,r_\pm(p,\lambda,E))-f(\lambda,r_\pm(p,\lambda,E)\xi)\xi}{(F(\lambda,r_\pm(p,\lambda,E))-F(\lambda,r_\pm(p,\lambda,E)\xi))^{1+\frac{1}{p}}}d\xi\Big\}
\end{aligned}
$$

so that

$$
\begin{aligned}
\pm\frac{\partial T_\pm}{\partial E}(p,\lambda,E) &= \frac{1}{p}(p')^{-1/p}(\frac{\pm\partial r_\pm}{\partial E}(p,\lambda,E))\times \\
&\quad \int_0^1 \frac{\pm(H(\lambda,r_\pm(p,\lambda,E))-H(\lambda,r_\pm(p,\lambda,E)\xi))}{(F(\lambda,r_\pm(p,\lambda,E))-F(\lambda,r_\pm(p,\lambda,E)\xi))^{1+\frac{1}{p}}}d\xi
\end{aligned}
\tag{40}
$$

where $H(\lambda,x)=pF(\lambda,x)-xf(\lambda,x)=\frac{-1}{p+1}|x|^px-(p-1)\lambda x$. Because the function $x\mapsto H(\lambda,x)$ is decreasing for each fixed $\lambda>0$ (in fact, $\frac{\partial H}{\partial x}(\lambda,x)<0$), it follows that

$$\pm(H(\lambda,r_\pm(p,\lambda,E))-H(\lambda,r_\pm(p,\lambda,E)\xi))<0,\forall\lambda>0,\forall\xi\in(0,1).$$

Hence, the integral in (40) is negative. So, because of (29), the proof of Lemma 12 is achieved. $\diamondsuit$

Completion of the proof of Theorem 3. The proof is carried out by making use of the quadrature method (Theorem 4). We have to resolve equations of the type $T(E)=\frac{1}{2}$, where T designates, in each case, the appropriate time map.

Solution in B_n^+. Recall that $r_+(p,\lambda,0)=((p+1)\lambda)^{1/p}$. Furthermore

$$T_+(p,\lambda,0)=\int_0^{r_+(p,\lambda,0)}(-p'F(p,\lambda,\xi))^{-1/p}d\xi=J(p)\lambda^{-1/p^2},$$

where $J(p)$ is defined in Lemma 10. Then problem (1) admits a solution in B_n^+ if and only if $nJ(p)\lambda^{-1/p^2}=(1/2)$, that is, if and only if $\lambda=(2nJ(p))^{p^2}$.

Solution in B_n^-. Since $0 \notin D_- = (0, E_*(p,\lambda))$, problem (1) admits no solution in $\bigcup_{n\geq 1} B_n^-$.

Solution in A_1^+. Recall that for any $p > 1$ and $\lambda > 0$ the function $E \mapsto T_+(p,\lambda,E)$ is defined in $[0,+\infty)$, is strictly decreasing (Lemma 12), and by Lemma 10,

$$\lim_{E\to 0^+} T_+(p,\lambda,E) = J(p)\lambda^{-1/p^2}, \quad \lim_{E\to +\infty} T_+(p,\lambda,E) = 0\,.$$

Then, the equation $T_+(p,\lambda,E) = (1/2)$ in the variable $E \in (0,+\infty)$ admits a solution in $[0,+\infty)$ if and only if $J(p)\lambda^{-1/p^2} > 1/2$, that is, if and only if $\lambda < (2J(p))^{p^2}$, and in this case, the solution is unique since the function $T_+(p,\lambda,\cdot)$ is strictly decreasing.

Solution in A_1^-. **Case** $1 < p \leq 2$**.** In this case, for each $\lambda > 0$, the function $E \mapsto T_-(p,\lambda,E)$ is defined in $D_- = (0, E_*(p,\lambda))$, is strictly increasing (Lemma 12), and

$$\lim_{E\to 0^+} T_-(p,\lambda,E) = 0, \quad \lim_{E\to E_*} T_-(p,\lambda,E) = +\infty$$

(Lemma 10, **(ii)** and assertion (38)). So, the equation $T_-(p,\lambda,E) = (1/2)$ in the variable $E \in D_-$ admits a unique solution in D_- for any $\lambda > 0$.

Case $p > 2$. In this case, for each $\lambda > 0$, the function $E \mapsto T_-(p,\lambda,E)$ is defined in $D_- = (0, E_*(p,\lambda))$, is strictly increasing (Lemma 12), and

$$\lim_{E\to 0^+} T_-(p,\lambda,E) = 0, \quad \lim_{E\to E_*} T_-(p,\lambda,E) = J_-(p)\lambda^{-1/p^2} < +\infty$$

(Lemma 10). So, the equation $T_-(p,\lambda,E) = (1/2)$ in the variable $E \in D_-$ admits a solution in D_- if and only if $(1/2) < J_-(p)\lambda^{-1/p^2}$, that is, if and only if $\lambda < (2J_-(p))^{p^2}$, and in this case the solution is unique since $T_-(p,\lambda,\cdot)$ is strictly increasing.

Solution in $A_{2n}^\pm$. **Case** $1 < p \leq 2$. In this case, for each $\lambda > 0$, the function $E \mapsto T_{2n}(p,\lambda,E)$ is defined in $D = (0, E_*(p,\lambda))$, and

$$\lim_{E\to 0^+} T_{2n}(p,\lambda,E) = nJ(p)\lambda^{-1/p^2}, \quad \lim_{E\to E_*} T_{2n}(p,\lambda,E) = +\infty$$

(Lemma 11 and Lemma 10, assertion (38)). So, the equation $T_{2n}(p,\lambda,E) = (1/2)$ in the variable $E \in D$ admits a solution in D provided that $(1/2) > nJ(p)\lambda^{-1/p^2}$, that is, provided that $\lambda > (2nJ(p))^{p^2}$.

Case $p > 2$. In this case, for each $\lambda > 0$, the function $E \mapsto T_{2n}(p,\lambda,E)$ is defined in $D = (0, E_*(p,\lambda))$, and

$$\lim_{E\to 0^+} T_{2n}(p,\lambda,E) = nJ(p)\lambda^{-1/p^2},$$
$$\lim_{E\to E_*} T_{2n}(p,\lambda,E) = n\lambda^{-1/p^2}(J_-(p) + J_+(p)) < +\infty$$

(Lemma 11 and Lemma 10, assertion (38)). So, the equation $T_{2n}(p,\lambda,E)=(1/2)$ in the variable $E\in D$ admits a solution in D provided that

$$n\lambda^{-1/p^2}\inf(J(p),J_-(p)+J_+(p))<\frac{1}{2}<n\lambda^{-\frac{1}{p^2}}\sup(J(p),J_-(p)+J_+(p)),$$

that is, provided that

$$\{2n\inf(J(p),J_-(p)+J_+(p))\}^{p^2}<\lambda<\{2n\sup(J(p),J_-(p)+J_+(p))\}^{p^2}.$$

Solution in A^+_{2n+1}. **Case** $1<p\le 2$. In this case, for each $\lambda>0$, the function $E\mapsto T^+_{2n+1}(p,\lambda,E)$ is defined in $D=(0,E_*(p,\lambda))$, and

$$\lim_{E\to 0^+}T^+_{2n+1}(p,\lambda,E)=(n+1)J(p)\lambda^{-1/p^2},\quad \lim_{E\to E_*}T^+_{2n+1}(p,\lambda,E)=+\infty$$

(Lemma 11 and Lemma 10, assertion (38)). So, the equation $T^+_{2n+1}(p,\lambda,E)=(1/2)$ in the variable $E\in D$ admits a solution in D provided that $(n+1)J(p)\lambda^{-1/p^2}<(1/2)$, that is, provided that $\lambda>(2(n+1)J(p))^{p^2}$.

Case $p>2$. In this case, for each $\lambda>0$, the function $E\mapsto T^+_{2n+1}(p,\lambda,E)$ is defined in $D=(0,E_*(p,\lambda))$, and

$$\begin{aligned}\lim_{E\to 0^+}T^+_{2n+1}(p,\lambda,E)&=(n+1)J(p)\lambda^{-1/p^2},\\ \lim_{E\to E_*}T^+_{2n+1}(p,\lambda,E)&=\lambda^{-1/p^2}((n+1)J_+(p)+nJ_-(p))<+\infty\end{aligned}$$

(Lemma 11 and Lemma 10, assertion (38)). So, the equation $T^+_{2n+1}(p,\lambda,E)=(1/2)$ in the variable $E\in D$ admits a solution in D provided that

$$\begin{aligned}&\lambda^{-1/p^2}\inf((n+1)J(p),(n+1)J_+(p)+nJ_-(p))\\ <\ &\tfrac{1}{2}\ <\lambda^{-\frac{1}{p^2}}\sup((n+1)J(p),(n+1)J_+(p)+nJ_-(p)),\end{aligned}$$

that is, provided that

$$\begin{aligned}&\{2\inf((n+1)J(p),(n+1)J_+(p)+nJ_-(p))\}^{p^2}\\ <\ &\lambda\ <\{2\sup((n+1)J(p),(n+1)J_+(p)+nJ_-(p))\}^{p^2}.\end{aligned}$$

Solution in A^-_{2n+1}. **Case** $1<p\le 2$. In this case, for each $\lambda>0$, the function $E\mapsto T^-_{2n+1}(p,\lambda,E)$ is defined in $D=(0,E_*(p,\lambda))$, and

$$\lim_{E\to 0^+}T^-_{2n+1}(p,\lambda,E)=nJ(p)\lambda^{-1/p^2},\quad \lim_{E\to E_*}T^-_{2n+1}(p,\lambda,E)=+\infty$$

(Lemma 11 and Lemma 10, assertion (38)). So, the equation $T^-_{2n+1}(p,\lambda,E)=(1/2)$ in the variable $E\in D$ admits a solution in D provided that $nJ(p)\lambda^{-1/p^2}<(1/2)$, that is, provided that $\lambda>(2nJ(p))^{p^2}$.

Case $p > 2$. In this case, for each $\lambda > 0$, the function $E \mapsto T^-_{2n+1}(p, \lambda, E)$ is defined in $D = (0, E_*(p, \lambda))$, and

$$\lim_{E \to 0^+} T^-_{2n+1}(p, \lambda, E) = nJ(p)\lambda^{-1/p^2},$$

$$\lim_{E \to E_*} T^-_{2n+1}(p, \lambda, E) = \lambda^{-1/p^2}(nJ_+(p) + (n+1)J_-(p)) < +\infty$$

(Lemma 11 and Lemma 10, assertion (38)). So, the equation $T^-_{2n+1}(p, \lambda, E) = (1/2)$ in the variable $E \in D$ admits a solution in D provided that

$$\begin{aligned} & \lambda^{-1/p^2} \inf(nJ(p), nJ_+(p) + (n+1)J_-(p)) \\ < \ & \tfrac{1}{2} \ < \lambda^{-1/p^2} \sup(nJ(p), nJ_+(p) + (n+1)J_-(p)), \end{aligned}$$

that is, provided that

$$\begin{aligned} & \{2\inf((nJ(p), nJ_+(p) + (n+1)J_-(p)))\}^{p^2} \\ < \ & \lambda \ < \{2\sup((nJ(p), nJ_+(p) + (n+1)J_-(p)))\}^{p^2}\ . \end{aligned}$$

Then the proof of Theorem 3 is complete.

Remark. Theorem 3 shows that for $1 < p \leq 2$ (resp. $p > 2$) solutions to (1) with $k \geq 1$ interior nodes exist for all λ belonging to an interval unbounded from above (resp. a bounded interval). Hence, for $1 < p \leq 2$, if problem (1) admits a solution with a prescribed number $k_0 \geq 1$ of nodes for a certain value λ_0 of λ, it still admits solutions with k_0 nodes for all λ greater than λ_0. In [5] it was shown that this is not the case for $p > 2$, and these changes in the behavior of the solution set as p varies depend strongly on the nonlinearity of the problem.

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