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EXISTENCE RESULTS FOR A SECOND-ORDER ABSTRACT CAUCHY PROBLEM WITH NONLOCAL CONDITIONS

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Abstract. In this paper we study the existence of mild and classical solutions for a second-order abstract Cauchy problem with nonlocal conditions.

1. Introduction

In this paper we study the existence of mild and classical solutions for a class of second-order abstract Cauchy problem with nonlocal conditions described in the form

$$\frac{d}{dt}[x'(t) + g(t, x(t), x'(t))] = Ax(t) + f(t, x(t), x'(t)), \quad t \in I = [0, a], \tag{1.1}$$

$$x(0) = y_0 + p(x, x'), \tag{1.2}$$

$$x'(0) = y_1 + q(x, x'), \tag{1.3}$$

where A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on a Banach space X and $g, f : I \times X^2 \to X$, $p, q : C(I; X) \times C(I; X) \to X$ are appropriate functions.

The system (1.1)-(1.3) is a simultaneous generalization of the classical second order abstract Cauchy problem studied by Travis and Weeb in [20, 21] and of some recent developments for ordinary differential equations by Staněk in [16, 17, 18, 19]. This generalization and their applications to partial second order differential equations are the main motivations of this paper.

Initial value problems with nonlocal conditions arises to deal specially with some situations in physics. Motivated for numerous applications, Byszewski studied in [5] the existence of mild, strong and classical solutions for the semilinear abstract Cauchy problem with nonlocal conditions

$$x'(t) = Ax(t) \ + \ f(t, x(t)), \quad t \in I = [0, a],$$
$$x(0) = x_0 \ + \ q(t_1, t_2, t_3, \dots, t_n, x(\cdot)) \in X.$$

In this system, A denotes the infinitesimal generator of a strongly continuous semigroup of linear operators on X; $0 < t_1 < \dots < t_n \le a$ are prefixed numbers;

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$f : [0,a] \times X \to X$, $q(t_1, t_2, t_3, \dots, t_n, \cdot) : C(I; X) \to X$ are appropriated functions and the symbol $q(t_1, t_2, t_3, \dots, t_n, u(\cdot))$ is used in the sense that $u(\cdot)$ can be evaluated only in the points t_i, for instance $q(t_1, t_2, t_3, \dots, t_n, u(\cdot)) = \sum_{i=1}^{n} \alpha_i u(t_i)$.

The existence of mild solutions for second order abstract Cauchy problems with nonlocal conditions is studied in Ntouyas & Tsamatos [14, 15], Benchohra & Ntouyas [1, 2, 3, 4], Dauer & Mahmudov [8] and Hernndez [11]. The results in the first two paper are only applicable to ordinary differential equations since the compactness assumption assumed on the cosine function is valid if, only if, the underlying space is finite dimensional, see Travis [20, p. 557] for details. On the other hand, the results in [1, 2, 3, 4] are proved using that the cosine function is continuous in the uniform operator topology which implies that their infinitesimal generator is bounded, see [20, p. 565]. We also observe that, in general, the nonlocal conditions considered in these works are described in the form

$$x(0) = h(x) + x_0, \quad x'(0) = p(x) + \eta,$$

where $h, p : C(I : X) \to X$ are appropriate functions and $\eta \in X$ is prefixed. These restrictions are an additional motivation for our paper.

Concluding this introduction, we remark that the results in this paper can be applied in the study of second order partial differential equations, the operator A is assumed unbounded and the system (1.1)-(1.3) can be considered a generalization at those studied in [1, 2, 3, 4, 8, 11, 16, 17, 18, 19, 20, 21].

2. Preliminaries

Throughout this paper, A is the infinitesimal generator of a strongly continuous cosine family, $(C(t))_{t\in\mathbb{R}}$, of bounded linear operators defined on a Banach space X. We denote by $(S(t))_{t\in\mathbb{R}}$ the sine function associated to $(C(t))_{t\in\mathbb{R}}$ which is defined by

$$S(t)x := \int_0^t C(s)x ds, \quad x \in X, \ t \in \mathbb{R}.$$

Moreover, N and $\tilde{N}$ are positive constants such that $\|C(t)\| \le N$ and $\|S(t)\| \le \tilde{N}$ for every $t \in I$.

In this paper, $[D(A)]$ is the space $D(A) = \{x \in X : C(\cdot)x \text{ is of class } C^2 \text{ on } \mathbb{R}\}$, endowed with the norm $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$. The notation E stands for the space formed by the vectors $x \in X$ for which $C(\cdot)x$ is of class C^1 on $\mathbb{R}$. We know from Kisińsky [12], that E endowed with the norm

$$\|x\|_E = \|x\| \ + \sup_{0\le t\le 1} \|AS(t)x\|, \quad x \in E, \tag{2.1}$$

is a Banach space. The operator valued function $g(t) = \begin{bmatrix} lrC(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this, it follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$ as $t \to 0$, for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is locally integrable, then $y(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function which is a consequence of the fact that

$$\int_0^t g(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\, ds \\ \int_0^t C(t-s)x(s)\, ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function.

The existence of solutions of the second-order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \quad t \in [0, a], \tag{2.2}$$

$$x(0) = y_0, \tag{2.3}$$

$$x'(0) = y_1, \tag{2.4}$$

where $h : [0, a] \to X$ is an integrable function has been discussed in [20]. Similarly, the existence of solutions of semilinear second order abstract Cauchy problem has been treated in [21]. We only mention here that the function

$$x(t) = C(t)y_0 + S(t)y_1 + \int_0^t S(t-s)h(s)\, ds, \quad t \in [0, a], \tag{2.5}$$

is called mild solution of (2.2)-(2.4) and that when $y_0 \in E$, $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)y_0 + C(t)y_1 + \int_0^t C(t-s)h(s)\, ds. \tag{2.6}$$

The regularity of mild solutions of (2.2)-(2.4) is studied in Travis & Weeb [21]. In our work, we adopt the next concept of classical solution of (2.2)-(2.4).

Definition 2.1. A function $u \in C(I; X)$ is a classical solution of (2.2)-(2.4), if $u \in C^2(I; X)$ and (2.2)-(2.4) are verified.

Remark 2.2. As usual, we say that $u \in C^1([\sigma, \mu] : X)$ if $u'(\cdot)$ is continuous on (σ, μ) and the right and left lateral derivatives of $u(\cdot)$ are continuous functions on $[\sigma, \mu)$ and $(\sigma, \mu]$ respectively.

For additional details concern to cosine function theory, we refer the reader to Fattorini [9] and Travis & Weeb [20, 21].

The terminology and notation are those generally used in functional analysis. In particular, if $(Z, \|\cdot\|_Z)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, we indicate by $\mathcal{L}(Z; Y)$ the Banach space of bounded linear operators from Z into Y and we abbreviate this notation to $\mathcal{L}(Z)$ whenever $Z = Y$. In this paper, $B_r(x; Z)$ denotes the closed ball with center at x and radius $r > 0$ in Z. Additionally, for a bounded function $\xi : I \to Z$ and $t \in I$, we will employ the notation $\xi_{Z,\,t}$ for

$$\xi_{Z,t} = \sup\{\|\xi(s)\|_Z : s \in [0, t]\},$$

and we will write simply ξ_t in the place of $\xi_{Z,\,t}$ when no confusion arises.

This paper has five sections. In section 3 we discuss the existence of mild solutions for some abstract Cauchy problems similar to (1.1)-(1.3) and in section 4 we study the existence of classical solutions for (1.1)-(1.3). In section 5 some examples are considered.

3. Existence of mild solutions

To begin this section we study the abstract Cauchy problem with nonlocal conditions

$$\frac{d}{dt}[x'(t) + g(t, x(t))] = Ax(t) + f(t, x(t)), \quad t \in I, \tag{3.1}$$

$$x(0) = y_0 + p(x), \tag{3.2}$$

$$x'(0) = y_1 + q(x), \tag{3.3}$$

where $f, g : I \times X \to X$ and $p, q : C(I; X) \to X$ are appropriate functions.

If $u(\cdot)$ is a solution of (3.1)-(3.3) and the mapping $t \to g(t, u(t))$ is enough smooth, from (2.5) and the relation $A\int_r^s S(\theta)x = C(s)x - C(r)x$, $x \in X$, we obtain

$$u(t) = C(t)(y_0 + p(u)) + S(t)[y_1 + q(u) + g(0, u(0))] - \int_0^t C(t-s)g(s, u(s))ds + \int_0^t S(t-s)f(s, u(s))ds, \quad t \in I.$$

This expression is the motivation of the following definition.

Definition 3.1. A function $u \in C(I; X)$ is a mild solution of (3.1)-(3.3), if $u(0) = y_0 + p(u)$ and

$$u(t) = C(t)(y_0 + p(u)) + S(t)(y_1 + q(u) + g(0, u(0))) - \int_0^t C(t-s)g(s, u(s))ds + \int_0^t S(t-s)f(s, u(s))ds, \quad t \in I.$$

Before establishing our first result of existence, we consider the following general lemma.

Lemma 3.2. *Let $(Z_i, \|\cdot\|_i)$, $i = 1, 2, 3$, be Banach spaces, $L : I \times Z_1 \to Z_2$ be a function, $\{R(t) : t \in I\} \subset \mathcal{L}(Z_2, Z_3)$ and assume that the next conditions hold.*

(a) *The function $L(\cdot)$ satisfies the following conditions.*
  (i) *For every $r > 0$, the set $L(I \times B_r(0; Z_1))$ is relatively compact in Z_2.*
  (ii) *The function $L(t, \cdot) : Z_1 \to Z_2$ is continuous a.e. $t \in I$*
  (iii) *For each $z \in Z_1$, the function $L(\cdot, z) : I \to Z_2$ is strongly measurable.*
  (iv) *There exist an integrable function $m_L : I \to [0, \infty)$ and a continuous function $W_L : [0, \infty) \to [0, \infty)$ such that*
$$\|L(t, z)\|_2 \le m_L(t)W_L(\|z\|_1) \quad (t, z) \in I \times Z_1.$$
(b) *The operator family $(R(t))_{t\in I}$ is strongly continuous, this means that $t \to R(t)z$ is continuous on I for every $z \in Z_2$.*

Then mapping $\Gamma : C(I; Z_1) \to C(I; Z_3)$ defined by
$$\Gamma u(t) = \int_0^t R(t-s)L(s, u(s)),$$
is completely continuous.

Proof. It is clear that $\Gamma(\cdot)$ is well defined and continuous. From conditions (a) and (b), it follows that the set $\{\, R(s)L(\theta, z) : s, \theta \in I, z \in B_r(0; Z_1) \,\}$ is relatively compact in Z_3. If $u \in B_r(0; C(I; Z_1))$, from the mean value Theorem for the Bochner integral, see [13, Lemma 2.1.3], we get
$$\Gamma u(t) \in t\,\overline{\mathrm{co}(\{\, R(s)L(\theta, z) : s, \theta \in I,\ z \in B_r(0; Z_1)\})}^{Z_3} \tag{3.4}$$
where co$(\cdot)$ denote the convex hull. Thus, $\{\Gamma u(t) : u \in B_r(0; C(I; Z_1))\}$ is relatively compact in Z_3 for every $t \in I$.

Next, we prove that $\Gamma(B_r(0; C(I; Z_1)) = \{\Gamma u : u \in B_r(0; C(I; Z_1))\}$ is equicontinuous on I. Let $\varepsilon > 0$ and $r > 0$. From the strong continuity of $(R(t))_{t\in I}$ and the compactness of $L(I \times B_r(0; Z_1))$, we can choose $\delta > 0$ such that
$$\|R(t)L(s, z) - R(t')L(s, z)\|_3 \le \varepsilon, \quad t', t, s \in I,\ z \in B_r(0; Z_1),$$

when $|t-t'| \le \delta$. Consequently, for $u \in B_r(0; C(I; Z_1))$, $t \in I$ and $|h| \le \delta$ such that $t+h \in I$, we get

$$\begin{aligned}\|\Gamma u(t+h) - \Gamma u(t)\|_3 &\le \int_0^t \|(R(t+h-s) - R(t-s))L(s,u(s))\|_3 ds \\ &\quad + \sup_{\theta \in I} \|R(\theta)\|_{\mathcal{L}(Z_2;Z_3)} \int_t^{t+h} \|L(s,u(s))\|_2 ds \\ &\le \varepsilon a + \sup_{\theta \in I} \|R(\theta)\|_{\mathcal{L}(Z_2;Z_3)} W_L(r) \int_t^{t+h} m_L(s) ds,\end{aligned}$$

which shows the equicontinuity at $t \in I$ and so that $\Gamma(B_r(0; C(I; Z_1))$ is equicontinuous on I. The assertion is now consequence of the Azcoli-Arzela criterion. The proof is complete. □

For the rest of this article we use the following hypotheses:

(H1) The functions $f, g : I \times X \to X$ satisfies the following conditions.
   (i) The functions $f(t,\cdot) : X \to X$, $g(t,\cdot) : X \to X$ are continuous *a.e.* $t \in I$;
   (ii) For each $x \in X$, the functions $f(\cdot, x) : I \to X$, $g(\cdot, x) : I \to X$ are strongly measurable.

(H2) The functions $p, q : C(I; X) \to X$ are continuous and there are positive constants l_p, l_q such that

$$\begin{aligned}\|p(u) - p(v)\| &\le l_p \|u - v\|_a, \qquad u, v \in C(I; X), \\ \|q(u) - q(v)\| &\le l_q \|u - v\|_a, \qquad u, v \in C(I; X).\end{aligned}$$

Now, we establish our first result of existence.

Theorem 3.3. *Assume (H1), (H2), and the following conditions:*

(a) *For every $r > 0$, the set $g(I \times B_r(0; X))$ is relatively compact in X and there exists a constant α_r^g such that $\|g(t,x)\| \le \alpha_r^g$ for every $(t,x) \in I \times B_r(0; X)$.*

(b) *For every $0 < t' < t \le a$ and every $r > 0$, the set*

$$U(t, t', r) = \{S(t') f(s,x) : s \in [0,t], x \in B_r(0; X)\}$$

is relatively compact in X and there exists a positive constant α_r^f such that $\|f(t,x)\| \le \alpha_r^f$ for every $(t,x) \in I \times B_r(0; X)$.

If

$$(N l_p + \tilde{N} l_q) + \liminf_{r \to +\infty} \frac{\tilde{N}\alpha_r^g + (N\alpha_r^g + \tilde{N}\alpha_r^f)a}{r} < 1,$$

then there exists a mild solution of (3.1)-(3.3).

Proof. On the space $Y = C(I; X)$ endowed with the norm of the uniform convergence, we define the operator $\Gamma : Y \to Y$ by

$$\begin{aligned}\Gamma u(t) &= C(t)(y_0 + p(u)) + S(t)(y_1 + q(u) + g(0, u(0))) \\ &\quad - \int_0^t C(t-s) g(s, u(s)) ds + \int_0^t S(t-s) f(s, u(s)) ds.\end{aligned}$$

We claim that there exists $r^* > 0$ such that $\Gamma(B_{r^*}(0, Y)) \subset B_{r^*}(0, Y)$. Assuming that the claim is false, then for every $r > 0$ there exists $x^r \in B_r(0; Y)$ and $t^r \in I$

such that $\|\Gamma x^r(t^r)\| > r$. This yields

$$r < \|x^r(t^r)\| \leq N(\|y_0\| + l_p r + \|p(0)\|) + \tilde{N}(\|y_1\| + l_q r + \|q(0)\| + \alpha_r^g) + N\int_0^a \alpha_r^g ds + \tilde{N}\int_0^a \alpha_r^f ds,$$

and then

$$1 \leq (Nl_p + \tilde{N}l_q) + \liminf_{r\to+\infty} \frac{\tilde{N}\alpha_r^g + (N\alpha_r^g + \tilde{N}\alpha_r^f)a}{r},$$

which contradicts our assumptions.

Now, we prove that $\Gamma(\cdot)$ is a condensing operator on $B_{r^*}(0,Y)$. For this purpose, we introduce the decomposition $\Gamma = \sum_{i=1}^3 \Gamma_i$, where

$$\Gamma_1 u(t) = C(t)(y_0 + p(u)) + S(t)(y_1 + q(u)),$$
$$\Gamma_2 u(t) = S(t)g(0,u(0)) - \int_0^t C(t-s)g(s,u(s))ds,$$
$$\Gamma_3 u(t) = \int_0^t S(t-s)f(s,u(s))ds.$$

From Lemma 3.2, condition (a) and the Lipschitz continuity of $t \to S(t)$ we infer that $\Gamma_2(\cdot)$ is completely continuous on Y and from the estimate

$$\|\Gamma_1 u - \Gamma_1 v\|_a \leq \Big(Nl_p + \tilde{N}l_q\Big)\|u-v\|_a, \quad u,v \in C(I;X),$$

that $\Gamma_1(\cdot)$ is a contraction on Y.

Next, by using the Ascoli-Arzela criterion, we prove that $\Gamma_3(\cdot)$ is completely continuous on Y. In the next steps r is a positive number.

Step 1 The set $\Gamma_3(B_r(0;Y))(t) = \{\Gamma_3 u(t) : u \in B_r(0;Y)\}$ is relatively compact in X for every $t \in I$. Let $t \in I, \varepsilon > 0$ and $0 = s_1 < s_2 < \dots < s_k = t$ be numbers such that $|s_i - s_{i+1}| \leq \varepsilon$ for every $i = 1,2,\dots k-1$. If $u \in B_r(0;Y)$, from the mean value Theorem for Bochner integral, see [13, Lemma 2.1.3], we find that

$$\begin{aligned}\Gamma_3 u(t) &= \sum_{i=1}^{k-1}\int_{s_i}^{s_{i+1}} S(s_i)f(t-s,u(t-s))ds \\ &\quad + \sum_{i=1}^{k-1}\int_{s_i}^{s_{i+1}} (S(s)-S(s_i))f(t-s,u(t-s))ds \\ &\in \sum_{i=1}^{k-1}(s_{i+1}-s_i)\overline{\mathrm{co}(U(t,s_i,r))} + \epsilon N\alpha_r^f aB_1(0,X),\end{aligned}$$

where $co(\cdot)$ denote the convex hull. Thus, $\Gamma_3(B_r(0;Y))(t)$ is relatively compact in X.

Step 2. The set $\Gamma_3(B_r(0;Y))$ is uniformly equicontinuous on I. For $u \in B_r(0;Y)$, $t \in I$ and $h \in \mathbb{R}$ such that $t+h \in I$, we get

$$\begin{aligned}&\|\Gamma_3 u(t+h) - \Gamma_3 u(t)\| \\ &\leq \int_0^t \|(S(t+h-s) - S(t-s))f(s,u(s))\|ds + \tilde{N}\int_t^{t+h}\|f(s,u(s))\|ds \\ &\leq N\alpha_r^f ah + \tilde{N}\alpha_r^f h,\end{aligned}$$

which implies that $\Gamma_3(B_r(0;Y))$ is uniformly equicontinuous on I.

It follows from steps 1 and 2 that $\Gamma_3(\cdot)$ is completely continuous on Y. The previous remarks show that $\Gamma(\cdot)$ is condensing from $B_{r^*}(0,Y)$ into $B_{r^*}(0,Y)$. The existence of a mild solution of system (3.1)-(3.3) is now a consequence of [13, Corollary 4.3.2]. The proof is completed. $\square$

Using arguments similar to the ones above, we can prove the next result.

Proposition 3.4. *Let assumptions (H1), (H2) be satisfied. Suppose, furthermore, that condition* (a) *of Theorem 3.3 holds and that there exists* $l_g \geq 0$ *such that*

$$\|g(t,x)-g(t,y)\| \leq l_g\|x-y\|, \quad t \in I, x,y \in X.$$

If

$$(Nl_p+\tilde{N}l_q)+(\tilde{N}+Na)l_g+\tilde{N}a \liminf_{r\to+\infty}\frac{\alpha_r^f}{r}<1,$$

then there exists a mild solution of (3.1)-(3.3).

Using the classical principle of contraction, we can prove the following result.

Theorem 3.5. *Let (H1), (H2) be satisfied and assume that there exist constants* l_f, l_g *such that*

$$\begin{gathered}
\|g(t,x)-g(t,y)\| \leq l_g\|x-y\|, \quad t \in I, x,y \in X, \\
\|f(t,x)-f(t,y)\| \leq l_f\|x-y\|, \quad t \in I, x,y \in X.
\end{gathered}$$

If $[N(l_p+al_g)+\tilde{N}(l_q+l_g+al_f)]<1$, *then there exists a unique mild solution of* (3.1)-(3.3).

Next, we study the abstract Cauchy problem (1.1)-(1.3).

Definition 3.6. A function $u \in C(I;X)$ is called a mild solution of (1.1)-(1.3) if $u \in C^1(I;X)$, conditions (1.2) and (1.3) are satisfied and

$$\begin{aligned}
u(t) &= C(t)(y_0+p(u,u'))+S(t)(y_1+q(u,u')+g(0,u(0),u'(0))) \\
&\quad -\int_0^t C(t-s)g(s,u(s),u'(s))ds+\int_0^t S(t-s)f(s,u(s),u'(s))ds, \quad t \in I.
\end{aligned}$$

To study the system (1.1)-(1.3) we introduce the following conditions.

- (H3) The function $f,g : I\times X\times X \to X$ satisfies the following conditions;
  - (i) The function $f(t,\cdot) : X\times X \to X$ is continuous $a.e.$ $t \in I$;
  - (ii) The function $f(\cdot,x,y) : I \to X$ is strongly measurable for each $(x,y) \in X\times X$.
  - (iii) The function $g(\cdot)$ is E-valued and $g : I\times X\times X \to E$ is continuous.
- (H4) The function $p,q : C(I;X)\times C(I;X) \to X$ are continuous, $p(\cdot)$ is E-valued and there exist positive constants l_p, l_q such that

$$\begin{gathered}
\|p(u_1,v_1)-p(u_2,v_2)\|_E \leq l_p(\|u_1-u_2\|_a+\|v_1-v_2\|_a), \\
\|q(u_1,v_1)-q(u_2,v_2)\| \leq l_q(\|u_1-u_2\|_a+\|v_1-v_2\|_a).
\end{gathered}$$

  for every $u_i, v_i \in C(I;X)$.

Remark 3.7. In the rest of this paper, $\rho=\sup_{\theta\in I}\|AS(\theta)\|_{\mathcal{L}(E;X)}$.

Theorem 3.8. *Let* $(y_0,y_1) \in E\times X$ *and assume (H3), (H4) be satisfied. Suppose in addition that the following conditions hold:*

(**a**) *For every $r > 0$, the set $f(I \times B_r(0;X) \times B_r(0;X))$ is relatively compact in X and there exists a constant α_r^f such that $\|f(t,x,y)\| \le \alpha_r^f$ for every $(t,x,y) \in I \times B_r(0;X) \times B_r(0;X)$.*

(**b**) *The function $g(\cdot) : I \times X \times X \to E$ is completely continuous and for every $r > 0$ there exists a constant α_r^g such that $\|g(t,x,y)\|_E \le \alpha_r^g$ for every $(t,x,y) \in I \times B_r(0;X) \times B_r(0;X)$.*

(**c**) *For every $r > 0$, the set $\{t \to g(t,u(t),v(t)) : u,v \in B_r(0;C(I;X))\}$ is a equicontinuous subset of $C(I;X)$.*

If

$$(N+\rho)l_p + (N+\tilde{N})l_q + \liminf_{r\to\infty} \frac{(N+\tilde{N})(\alpha_r^g + a\alpha_r^f) + \alpha_r^g(1+a(N+\rho))}{r} < 1,$$

then there exists a mild solution of (1.1)-(1.3).

Proof. On the space $Y = C(I;X)\times C(I;X)$ endowed with the norm of the uniform convergence, $\|(u,v)\|_a = \|u\|_a + \|v\|_a$, we define the operator $\Gamma : Y \to Y$ by $\Gamma(u,v) = (\Gamma_1(u,v), \Gamma_2(u,v))$ where

$$\begin{aligned}
\Gamma_1(u,v)(t) &= C(t)(y_0 + p(u,v)) + S(t)(y_1 + q(u,v) + g(0,u(0),v(0))) \\
&\quad - \int_0^t C(t-s)g(s,u(s),v(s))ds + \int_0^t S(t-s)f(s,u(s),v(s))ds, \\
\Gamma_2(u,v)(t) &= AS(t)(y_0 + p(u,v)) + C(t)(y_1 + q(u,v) + g(0,u(0),v(0))) \\
&\quad - g(t,u(t),v(t)) - \int_0^t AS(t-s)g(s,u(s),v(s))ds \\
&\quad + \int_0^t C(t-s)f(s,u(s),v(s))ds.
\end{aligned}$$

Using that $g(\cdot)$ and $p(\cdot)$ are E-valued continuous, it's easy to prove that $\Gamma(\cdot)$ is well defined and continuous.

Now, we show that there exists $r^* > 0$ such that $\Gamma(B_{r^*}(0,Y)) \subset B_{r^*}(0,Y)$. Assume that this property is false. Then for every $r > 0$ there exists $(u^r, v^r) \in B_r(0;Y)$ such that $r < \|\Gamma(u^r,v^r)\|_a$. This yields

$$\begin{aligned}
& r < \|\Gamma^1(u,v)\|_a + \|\Gamma^2(u,v)\|_a \\
&\le N\left(\|y_0\| + l_p r + \|p(0,0)\|\right) + \tilde{N}(\|y_1\| + l_q r + \|q(0,0)\| + \alpha_r^g) \\
&\quad + a(N\alpha_r^g + \tilde{N}\alpha_r^f) + \sup_{\theta\in I}\|AS(\theta)\|_{\mathcal{L}(E;X)}\left(\|y_0\|_E + l_p r + \|p(0,0)\|_E\right) \\
&\quad + N(\|y_1\| + l_q r + \|q(0,0)\| + \alpha_r^g) + \alpha_r^g \\
&\quad + \int_0^a \sup_{\theta\in I}\|AS(\theta)\|_{\mathcal{L}(E;X)}\|g(s,u(s),v(s))\|_E ds + N\alpha_r^f a \\
&\le (N+\rho)\left(\|y_0\|_E + l_p r + \|p(0,0)\|_E\right) + \alpha_r^g \\
&\quad + (N+\tilde{N})\left(\|y_1\| + l_q r + \|q(0,0)\| + \alpha_r^g\right) + a\left(\alpha_r^g(N+\rho) + \alpha_r^f(N+\tilde{N})\right)
\end{aligned}$$

and hence

$$1 \le (N+\rho)l_p + (N+\tilde{N})l_q + \liminf_{r\to\infty}\frac{(N+\tilde{N})(\alpha_r^g + a\alpha_r^f) + \alpha_r^g(1+a(N+\rho))}{r},$$

which is contrary to the hypotheses.

Next, we prove that $\Gamma(\cdot)$ is condensing from $B_{r^*}(0,Y)$ into $B_{r^*}(0,Y)$. Consider the decomposition $\Gamma=\bar{\Gamma}_1+\bar{\Gamma}_2$ where $\bar{\Gamma}_2(u,v)=(\bar{\Gamma}_2^1(u,v),\bar{\Gamma}_2^2(u,v))$ and

$$\begin{aligned}
\bar{\Gamma}_2^1(u,v)(t) &= S(t)g(0,u(0),v(0))-\int_0^t C(t-s)g(s,u(s),v(s))ds\\
&\quad+\int_0^t S(t-s)f(s,u(s),v(s))ds,\\
\bar{\Gamma}_2^2(u,v)(t) &= C(t)g(0,u(0),v(0))-g(t,u(t),v(t))\\
&\quad-\int_0^t AS(t-s)g(s,u(s),v(s))ds+\int_0^t C(t-s)f(s,u(s),v(s))ds.
\end{aligned}$$

Simple calculus using the properties of $p(\cdot)$ and $q(\cdot)$ proves that

$$\|\bar{\Gamma}_1(u,v)-\bar{\Gamma}_1(w,z)\|_a \leq \Big((N+\rho)l_p+(N+\tilde{N})l_q\Big)\,\|(u,v)-(w,z)\|_a, \tag{3.5}$$

and so that $\bar{\Gamma}_1(\cdot)$ is a contraction on Y.

On the other hand, from Lemma 3.2 and the properties of $f(\cdot)$ and $g(\cdot)$, it's easy to infer that $\bar{\Gamma}_2(\cdot)$ is completely continuous on Y. From the previous remark, it follows that $\Gamma(\cdot)$ is a condensing operator from $B_{r^*}(0,Y)$ into $B_{r^*}(0,Y)$. The assertion is now a consequence of [13, Corollary 4.3.2]. $\square$

Proceeding as in the proof of Theorem 3.8 we can prove the next existence result.

Proposition 3.9. *Let $(y_0,y_1)\in E\times X$ and conditions (H3), (H4) be satisfied. Suppose that $f(\cdot)$ satisfies condition* (**a**) *of Theorem 3.8 and that there exists a constant $l_g\geq 0$ such that*

$$\|g(t,x_1,z_1)-g(t,x_2,z_2)\|_E \leq l_g(\|x_1-x_2\|+\|z_1-z_2\|), \tag{3.6}$$

for every $t\in I$ and every $x_i,z_i\in X$. If

$$(N+\rho)l_p+(N+\tilde{N})l_q+l_g((N+\rho)a+\tilde{N}+N+1)+(N+\tilde{N})\liminf_{r\to\infty}\frac{\alpha_r^f}{r})<1,$$

then there exists a mild solution of (1.1)-(1.3).

Theorem 3.10. *Assume (H3), (H4), $(y_0,y_1)\in E\times X$ and that there exist constants l_f,l_g such that*

$$\begin{gathered}
\|f(t,x_1,z_1)-f(t,x_2,z_2)\|\leq l_f(\|x_1-x_2\|+\|z_1-z_2\|),\\
\|g(t,x_1,z_1)-g(t,x_2,z_2)\|_E\leq l_g(\|x_1-x_2\|+\|z_1-z_2\|),
\end{gathered}$$

for every $x_i,z_i\in X$.

If $\max\{N(l_p+al_g)+\tilde{N}(l_q+l_g+al_f),\ N(l_q+l_g+al_f)+\rho(l_p+al_g)+l_g\}<1$, *then there exists a unique mild solution of* (1.1)-(1.3).

Proof. Let $\Gamma(\cdot)$ be the map defined in the proof of Theorem 3.8. It's clear that $\Gamma(\cdot)$ is well defined and continuous. Moreover, for $u_i,v_i\in C(I;X)$

$$\|\Gamma_1(u_1,v_1)-\Gamma_1(u_2,v_2)\|_a\leq[N(l_p+al_g)+\tilde{N}(l_q+l_g+al_f)]\|(u_1,v_1)-(u_2,v_2)\|_a$$

and

$$\begin{aligned}
&\|\Gamma_2(u_1,v_1)-\Gamma_2(u_2,v_2)\|_a \\
&\le \|AS(t)\|_{\mathcal{L}(E;X)}\|p(u_1,v_1)-p(u_2,v_2)\|_E \\
&\quad + (N(l_q+l_g)+l_g+aNl_f)\|(u_1,v_1)-(u_2,v_2)\|_a \\
&\quad + \int_0^t \|AS(t-s)\|_{\mathcal{L}(E;X)}\|g(s,u_1(s),v_1(s))-g(s,u_2(s),v_2(s)))\|_E ds \\
&\le (\rho l_p + N(l_q+l_g)+l_g+aNl_f+a\rho l_g)\,\|(u_1,v_1)-(u_2,v_2)\|_a \\
&\le (N(l_q+l_g+al_f)+\rho(l_p+al_g)+l_g)\|(u_1,v_1)-(u_2,v_2)\|_a,
\end{aligned}$$

which implies that Γ is a contraction. The statement of the theorem is now a consequence of the contraction mapping principle. □

4. Classical Solutions

In this section we establish the existence of classical solutions for (1.1)-(1.3). First, we introduce some definitions, notation and preliminary results.

Definition 4.1. A function $u \in C^2(I;X)$ is a classical solution of (1.1)-(1.3), if the mapping $t \to u(t) + \int_0^t g(s,u(s),u'(s))ds$ is in $C^2(I:X)$, $u(t) \in D(A)$ for every $t \in I$, and (1.1)-(1.3) are satisfied.

In the next pages, we use the assumption

(H5) The function $g(\cdot)$ is $[D(A)]$-valued and $g : I\times X\times X \to [D(A)]$ is continuous.

The remark below is a consequence of our preliminary results.

Remark 4.2. If $u(\cdot)$ is a mild solution of (1.1)-(1.3), $\varphi(0) \in E$ and the function $s \to g(s,u(s),u'(s))$ is continuous from I into E, then $u \in C^1$ and

$$\begin{aligned}
u'(t) &= AS(t)(y_0+p(u,u')) + C(t)(y_1+q(u,u')+g(0,u(0),u'(0))) \\
&\quad - g(t,u(t),u'(t)) - \int_0^t AS(t-s)g(s,u(s),u'(s))ds \\
&\quad + \int_0^t C(t-s)f(s,u(s),u'(s))ds.
\end{aligned}$$

Lemma 4.3. *Let $u(\cdot)$ be a mild solution of* (1.1)-(1.3) *and assume that (H5) holds. If $y_0+p(u,u') \in D(A)$, $y_1+q(u,u') \in E$, $f(\cdot)$ is Lipschitz continuous on bounded subsets of $I\times X\times X$ and there exist constants $l_g^1 > 0$, $0 < l_g^2 < 1$ such that*

$$\|g(t,x_1,y_1)-g(s,x_2,y_2)\|_E \le l_g^1(|t-s|+\|x_1-x_2\|)+l_g^2\|y_1-y_2\|\,,$$

for every $x_i, y_i \in X$ and every $t, s \in I$, then $u'(\cdot)$ is Lipschitz continuous on I.

Proof. Let $t \in I$ and $h \in \mathbb{R}$ be such that $t+h \in I$. Using Remark 4.2 and the Lipschitz continuity of $u(\cdot)$ on I, we obtain

$$\begin{aligned}
&\|u'(t+h) - u'(t)\| \\
&\le C_1 h + l_g^2 \|u'(t+h) - u'(t)\| + \int_t^{t+h} \|S(t+h-s)Ag(s,u(s),u'(s))\| ds \\
&\quad + \int_0^t \|(S(t+h-s) - S(t-s))Ag(s,u(s),u'(s))\| ds \\
&\quad + \int_0^h \|C(t+h-s)f(s,u(s),u'(s))\| ds \\
&\quad + N \int_0^t C_2 [\, h + \|u(s+h) - u(s)\| + \|u'(s+h) - u'(s)\| \,] ds \\
&\le C_3 h + l_g^2 \|u'(t+h) - u'(t)\| + N C_2 \int_0^t \int_0^t \|u'(s+h) - u'(s)\| ds,
\end{aligned}$$

where the constants C_i are independent of t and h. Since $l_g^2 < 1$, we can rewrite the last inequality in the form

$$\|u'(t+h) - u'(t)\| \le C_4 h + C_5 \int_0^t \|u'(s+h) - u'(s)\| ds,$$

where C_4, C_5 are independent of t and h. This proves that $u'(\cdot)$ is Lipschitz on I. The proof is complete □

Let $(Z_i, \|\cdot\|_i)$, $i = 1,2,3$, be Banach spaces and $j(\cdot) : I \times Z_1 \times Z_2 \to Z_3$ be a differentiable function. We will use the decomposition

$$\begin{aligned}
&j(s, \bar{z}_1, \bar{z}_2) - j(t, z_1, z_2) \\
&= (D_1 j(t,z_1,z_2), D_2 j(t,z_1,z_2), D_3 j(t,z_1,z_2))(s-t, \bar{z}_1 - z_1, \bar{z}_2 - z_2) \\
&\quad + \|(s-t, \bar{z}_1 - z_1, \bar{z}_2 - z_2)\|_{Z_1,Z_2} R^{Z_3}_{Z_1,Z_2}(j(t,z_1,z_2), s-t, \bar{z}_1 - z_1, \bar{z}_2 - z_2),
\end{aligned}$$

where

$$\|R^{Z_3}_{Z_1,Z_2}(j(t,z_1,z_2), h, w_1, w_2)\|_{Z_3} \to 0,$$

when $\|(h,w_1,w_2)\|_{Z_1,Z_2} = |h| + \|w_1\|_{Z_1} + \|w_2\|_{Z_2} \to 0$. Moreover, we will write simply $R^{Z_3}_{Z_1}$ and $\|(s,y,w)\|_{Z_1}$ when $Z_1 = Z_2$.

The proof of the next Lemma will be omitted.

Lemma 4.4. *Let $(Z_i, \|\cdot\|_{Z_i})$, $i = 1,2,3$, be Banach spaces, $\Omega_1 \times \Omega_2 \subset Z_1 \times Z_2$ open, $K \subset \Omega_1 \times \Omega_2$ compact and $j : I \times \Omega_1 \times \Omega_2 \to Z_3$ be a continuously differentiable function. Then, for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\|R^{Z_3}_{Z_1,Z_2}(j(t,z_1,z_2), s-t, \bar{z}_1 - z_1, \bar{z}_2 - z_2)\|_{Z_3} < \varepsilon, \quad t,s \in I,\ (z_1,z_2), (\bar{z}_1,\bar{z}_2) \in K$$

when $\|(s-t, \bar{z}_1 - z_1, \bar{z}_2 - z_2)\|_{Z_1,Z_2} \le \delta$.

Theorem 4.5. *Let condition (H5) be satisfied and $u(\cdot)$ be a mild solution of* (1.1)-(1.3)*. Assume that the functions $f : I \times X^2 \to X$, $g : I \times X^2 \to E$ are continuously differentiable, $(y_0 + p(u,u'), y_1 + q(u,u')) \in D(A) \times E$ and that there exist constants $l_g^1 > 0$, $0 < l_g^2 < 1$ such that*

$$\|g(t,x_1,y_1) - g(s,x_2,y_2)\|_E \le l_g^1 (|t-s| + \|x_1 - x_2\|) + l_g^2 \|y_1 - y_2\|,$$

for every $x_i, y_i \in X$ and every $t, s \in I$. If

$$\|D_3 g(w)\|_{\mathcal{L}(X),\,a} + \int_0^a [\rho\|D_3 g(w(s))\|_{\mathcal{L}(X;E)} + \|D_3 f(w(s))\|_{\mathcal{L}(X)}]ds < 1, \tag{4.1}$$

where $w(t) = (t, u(t), u'(t))$, then $u(\cdot)$ is a classical solution.

Proof. First, we prove that $u(\cdot)$ is of class C^2 on I and for this purpose we introduce the integral equation

$$\begin{aligned} v(t) = P(t) - D_3 g(w(t))(v(t)) - \int_0^t AS(t-s) D_3 g(w(s))(v(s))ds \\ + \int_0^t C(t-s) D_3 f(w(s))(v(s))ds, \quad t \in I, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} P(t) &= C(t)Au(0) + AS(t)u'(0) - D_1 g(w(t)) - D_2 g(w(t))(u'(t)) \\ &\quad - \int_0^t AS(t-s)[D_1 g(w(s)) + D_2 g(w(s))(u'(s))]ds + C(t)\tilde{f}(0) \\ &\quad + \int_0^t C(t-s)\left(D_1 f(w(s)) + D_2 f(w(s))(u'(s))\right) ds. \end{aligned}$$

The existence and uniqueness of solutions of the integral equation (4.2) is consequence of the contraction mapping principle and (4.1), we omit additional details. Let $v(\cdot)$ be the solution (4.2) and let $t \in I$, $h \in \mathbb{R}$ be such that $t + h \in I$. By using the relation $A\int_r^s S(\theta)x = C(s)x - C(r)x$, the notation $\zeta_h(t) = \partial_h u'(t) - v(t)$, $\tilde{f} = f(w(t))$, $\tilde{g} = g(w(t))$ and

$$\begin{aligned} \Lambda g(t) &= D_1 g(w(t)) + D_2 g(w(t))(u'(t)) + D_3 g(w(t))(v(t)), \\ \Lambda f(t) &= D_1 f(w(t)) + D_2 f(w(t))(u'(t)) + D_3 f(w(t))(v(t)), \end{aligned}$$

we find that

$$\begin{aligned} &\|\zeta_h(t)\| \\ &\le \xi_1(h,t) + \|[\partial_h C(t)]\tilde{g}(0) - \frac{1}{h}\int_0^h AS(t+h-s)\tilde{g}(s)ds\| + \|\Lambda g(t) - \partial_h \tilde{g}(t)\| \\ &\quad + \rho\int_0^t \|\Lambda g(s) - \partial_h \tilde{g}(s)\|_E ds + \|\frac{1}{h}\int_0^h C(t+h-s)\tilde{f}(s)ds - C(t)\tilde{f}(0)\| \\ &\quad + N\int_0^t \|\partial_h \tilde{f}(s) - \Lambda f(s)\|ds \\ &\le \xi_2(h,t) + \frac{1}{h}\int_0^h \|S(t+h-s)(A\tilde{g}(0) - A\tilde{g}(s)\|ds + \|D_3 g(w(t))\|_{\mathcal{L}(X)}\|\zeta_h(t)\| \\ &\quad + \|(1, \partial_h u(t), \partial_h u'(t))\|_X \|R_X^X(\tilde{g}(t), h, h\partial_h u(t), h\partial_h u'(t))\| \\ &\quad + \int_0^t \left[\rho\|D_3 g(w(s))\|_{\mathcal{L}(X;E)} + N\|D_3 f(w(s))\|_{\mathcal{L}(X)}\right]\|\zeta_h(s)\|ds \\ &\quad + \rho\int_0^t \|(1, \partial_h u(s), \partial_h u'(s))\|_X \|R_X^E(\tilde{g}(s), h, h\partial_h u(s), h\partial_h u'(s))\|_E ds \\ &\quad + N\int_0^t \|(1, \partial_h u(s), \partial_h u'(s))\|_X \|R_X^X(\tilde{f}(s), h, h\partial_h u(s), h\partial_h u'(s))\|ds, \end{aligned}$$

where $\xi_i(h,t) \to 0$, $i = 1,2$, as $h \to 0$. Since $\mu = 1 - \|D_3 g(w(\cdot))\|_{\mathcal{L}(X),a} > 0$, we obtain

$$
\begin{aligned}
\|\zeta_h(t)\| \le \xi_3(h,t) &+ \frac{1}{\mu}\|(1,\partial_h u(t),\partial_h u'(t))\|_X \|R_X^X(\tilde g(t),h,h\partial_h u(t),h\partial_h u'(t))\| \\
&+ \frac{1}{\mu}\int_0^t \big[\rho\|D_3 g(w(s))\|_{\mathcal{L}(X;E)} + N\|D_3 f(w(s))\|_{\mathcal{L}(X)}\big]\,\|\zeta_h(s)\|ds \\
&+ \frac{\rho}{\mu}\int_0^t \|(1,\partial_h u(s),\partial_h u'(s))\|_X \|R_X^E(\tilde g(s),h,h\partial_h u(s),h\partial_h u'(s))\|_E ds \\
&+ \frac{N}{\mu}\int_0^t \|(1,\partial_h u(s),\partial_h u'(s))\|_X \|R_X^X(\tilde f(s),h,h\partial_h u(s),h\partial_h u'(s))\| ds
\end{aligned}
$$

where $\xi_3(h,t) \to 0$ as $h \to 0$. This inequality, jointly with the Lipschitz continuity of $u(\cdot)$ and $u'(\cdot)$, see Lemma 4.3, the Gronwall Bellman inequality and Lemma 4.4, permit to conclude that $u''(\cdot)$ exists and that $u''(\cdot) = v(\cdot)$ on I.

From [21, Proposition 2.4], we know that the mild solution, $y(\cdot)$, of the abstract Cauchy problem

$$
\begin{gathered}
x''(t) = Ax(t) + f(t,u(t),u'(t)) - A\int_0^t g(s,u(s),u'(s))ds, \quad t \in I, \\
x(0) = y_0 + p(u,u') \quad x'(0) = y_1 + q(u,u') + g(0,u(0),u'(0)),
\end{gathered}
\tag{4.3}
$$

is a classical solution (see Definition 2.1). The uniqueness of solution of (4.3) and Remark 4.2, permit to conclude that $y(t) = u(t) + \int_0^t g(s,u(s),u'(s))ds$ is a function of class C^2 on I and that $u(t) \in D(A)$ for every $t \in I$ since $g(\cdot)$ is $[D(A)]$-valued continuous. This completes the proof that $u(\cdot)$ is a classical solution. $\square$

5. Applications

In this section we apply some of the results established in this paper. First, we introduce the required technical framework. On the space $X = L^2([0,\pi])$ we consider the operator $Af(\xi) = f''(\xi)$ with domain $D(A) = \{f(\cdot) \in H^2(0,\pi) : f(0) = f(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a strongly continuous cosine function, $(C(t))_{t\in\mathbb{R}}$, on X. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding normalized eigenvectors $z_n(\xi) := (\frac{2}{\pi})^{1/2}\sin(n\xi)$ and

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.
(b) If $\varphi \in D(A)$ then $A\varphi = -\sum_{n=1}^\infty n^2\langle\varphi,z_n\rangle z_n$.
(c) For $\varphi \in X$, $C(t)\varphi = \sum_{n=1}^\infty \cos(nt)\langle\varphi,z_n\rangle z_n$. It follows from this expression that $S(t)\varphi = \sum_{n=1}^\infty \frac{\sin(nt)}{n}\langle\varphi,z_n\rangle z_n$ for every $\varphi \in \mathcal{B}$. Moreover, $S(t)$ is a compact operator and $\|C(t)\| = \|S(t)\| = 1$ for every $t \in \mathbb{R}$.
(d) If Φ is the group of translations on X defined by $\Phi(t)x(\xi) = \tilde x(\xi+t)$, where $\tilde x(\cdot)$ is the extension of $x(\cdot)$ with period 2π, then $C(t) = \frac{1}{2}(\Phi(t)+\Phi(-t))$ and $A = B^2$, where B is the infinitesimal generator of Φ and $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$, see [9] for details.

First, we consider the partial second-order differential equation with nonlocal conditions

$$\frac{\partial}{\partial t}[\frac{\partial u(t,\xi)}{\partial t}+G(t,\xi,u(t,\xi))]=\frac{\partial^2 u(t,\xi)}{\partial \xi^2}+F(t,\xi,u(t,\xi)),\quad \xi\in J=[0,\pi],\ t\in I=[0,a], \tag{5.1}$$

$$u(t,0)=u(t,\pi)=0,\quad t\in I, \tag{5.2}$$

$$u(0,\xi)=y_0(\xi)+\sum_{i=1}^{n}\alpha_i u(t_i,\xi),\quad \xi\in J, \tag{5.3}$$

$$\frac{\partial u(0,\xi)}{\partial t}=y_1(\xi)+\sum_{i=1}^{k}\beta_i u(s_i,\xi),\quad \xi\in J, \tag{5.4}$$

where $0<t_i,s_j<a$, $\alpha_i,\beta_j\in\mathbb{R}$ are fixed numbers, $y_0,y_1\in X$ and the functions $G,F:I\times J\times\mathbb{R}\to\mathbb{R}$ satisfy the following conditions:

(i) $F(\cdot)$ is continuous and there exist functions $\eta_1^F,\eta_2^F\in C(I\times J:\mathbb{R}^+)$ such that

$$|F(t,\xi,w)|\le \eta_1^F(t,\xi)+\eta_2^F(t,\xi)|w|,\quad t\in I,\xi\in J,w\in\mathbb{R}.$$

(ii) $G(\cdot)$ is continuous and there exists $\eta^G\in C(I\times J;\mathbb{R}^+)$ such that

$$\mid G(t,\xi,x_1)-G(t,\xi,x_2)\mid\le\eta^G(t,\xi)\mid x_1-x_2\mid,$$

for every $(t,\xi)\in I\times J$ and every $x_1,x_2\in\mathbb{R}$.

By defining the functions $f,g:I\times X\to X$ and $p,q:C(I;X)\to X$ by $g(t,x)(\xi)=G(t,\xi,x(\xi))$, $f(t,x)(\xi)=F(t,\xi,x(\xi))$, $p(u)(\xi)=\sum_{i=1}^{n}\alpha_i u(t_i,\xi)$ and $q(u)(\xi)=\sum_{i=1}^{k}\beta_i u(s_i,\xi)$, the system (5.1)-(5.4) can be described as the abstract Cauchy problem with nonlocal conditions (3.1)-(3.3). It is easy to see that $f(\cdot)$, $g(\cdot)$, $p(\cdot)$, $q(\cdot)$ satisfies the assumption of Proposition 3.4 and that $l_g=\sup_{(s,\xi)\in I\times J}\eta^G(s,\xi)$, $l_p=\sum_{i=1}^{n}|\alpha_i|$, $l_q=\sum_{i=1}^{k}|\beta_i|$ and

$$\alpha_r^f=\sup\Big\{\Big(\int_0^\pi \eta_1^F(t,\xi)^2 d\xi\Big)^{1/2}+r\eta_2^F(t,\cdot)_\pi\ :\ t\in I\Big\}.$$

The next result is a consequence of Proposition 3.4.

Theorem 5.1. *Assume that (i) and (ii) are satisfied. If*

$$\sum_{i=1}^{n}|\alpha_i|+\sum_{i=1}^{k}|\beta_i|+(1+a)\sup_{(s,\xi)\in I\times J}\eta^G(s,\xi)+a\sup_{s\in I}\eta_2^F(s,\cdot)_\pi ds<1,$$

then there exists a mild solution of (3.1)-(3.3).

Now, we consider briefly the partial differential equation

$$\frac{\partial}{\partial t}[\frac{\partial u(t,\xi)}{\partial t}+\int_0^\pi b(t,\eta,\xi)u(t,\eta)d\eta]=\frac{\partial^2 u(t,\xi)}{\partial \xi^2}+F(t,\xi,u(t,\xi)), \tag{5.5}$$

for $\xi\in J$, $t\in I$, submitted to the conditions (5.2)-(5.4). To study this system we introduce the next condition.

(iii) The functions $b(s,\eta,\xi)$, $\dfrac{\partial^i b(s,\eta,\xi)}{\partial\xi^i}$, $i=1,2$, are continuous on $\mathbb{R}^3$ and $b(\cdot,\pi)=b(\cdot,0)=0$ on $I\times J$.

Let $f(\cdot), p(\cdot), q(\cdot)$ defined as before and $g(\cdot) : I \times X \to X$ be the function defined by $g(t,x)(\xi) = \int_0^\pi b(t,\eta,\xi)x(\eta)d\eta$. From the properties of $b(\cdot)$, we infer that $g(t,\cdot)$ is a $D(A)$-valued linear operator and that

$$\sup\{\|g(t,\cdot)\|, \|g(t,\cdot)\|_E, \|Ag(t,\cdot)\|_{\mathcal{L}(X)} : t \in I\} \leq \alpha^{1/2},$$

where

$$\alpha := \sup_{t\in[0,a]} \Big\{ \int_0^\pi \int_0^\pi b(t,\eta,\xi)^2 d\eta d\xi, \int_0^\pi \int_0^\pi \big(\frac{\partial^j b(t,\eta,\xi)}{\partial \xi^j}\big)^2 d\eta d\xi : j = 1,2 \Big\}.$$

Moreover, $g(\cdot)$ is completely continuous since the inclusion $i_c : [D(A)] \to X$ is compact.

In the next result, the existence of a mild solution can be deduced from Theorem 3.3 or from Proposition 3.4.

Theorem 5.2. *Assume (i) and (iii) be satisfied and that*

$$\sum_{i=1}^n |\alpha_i| + \sum_{i=1}^k |\beta_i| + (1+a)\alpha^{\frac{1}{2}} + a \sup_{s\in I} \eta_2^F(s,\cdot)_\pi ds < 1.$$

Then the partial differential equation (5.5) submitted to the conditions (5.2)-(5.4) has a mild solution.

To finish this section, we consider the differential system

$$\frac{\partial}{\partial t}\Big[\frac{\partial u(t,\xi)}{\partial t} + \int_0^\pi b(t,\eta,\xi)\frac{\partial u(t,\eta)}{\partial t}d\eta\Big] = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + F(t,u(t,\xi),\frac{\partial u}{\partial t}(t,\xi)), \quad (5.6)$$

for $\xi \in J$, $t \in I$, subject to the conditions:

$$u(t,0) = u(t,\pi) = 0, \quad t \in I, \quad (5.7)$$

$$u(0,\xi) = y_0(\xi) + \int_0^a P(u(s),\frac{\partial u}{\partial t}(s))(\xi)d\mu(s), \quad (5.8)$$

$$\frac{\partial u}{\partial t}(0,\xi) = y_1(\xi) + \sum_{i=1}^n \alpha_i u(t_i,\xi) + \sum_{i=1}^k \beta_i \frac{\partial u}{\partial t}(s_i,\xi), \quad (5.9)$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $0 < t_i, s_j < a$ are prefixed numbers, $\mu(\cdot)$ is a real function of bounded variation on I and $F : I \times J \times \mathbb{R}^2 \to \mathbb{R}$, $P : X \times X \to X$ satisfies the next conditions.

(iv) $F(\cdot)$ is continuous and there exists a constant L_F such that

$$|F(t,x_1,w_1) - F(s,x_2,w_2)| \leq L_F\left(|t-s| + |x_1-x_2| + |w_1-w_2|\right),$$

for every $t,s \in I$ and every $x_i, w_i \in \mathbb{R}$;

(v) P is E-valued and there exist l_P such that

$$\|P(x_1,w_1) - P(x_2,w_2)\|_E \leq l_P(\|x_1-x_2\| + \|w_1-w_2\|), \quad x_i, w_i \in X.$$

(for examples of operators satisfying (v), see [13]).

By defining the operators $f, g : I \times X \times X \to X$ and $p, q : C(I; X) \times C(I; X) \to X$ by

$$f(t, x, y)(\xi) = F(t, x(\xi), y(\xi)),$$
$$g(t, x, y)(\xi) = \int_0^{\pi} b(t, \eta, \xi) y(\eta) d\eta, \quad x, y \in X,$$
$$p(u, v)(\xi) = \int_0^{\pi} P(u(s), v(s))(\xi) d\mu(s), \quad u, v \in C(I; X),$$
$$q(u, v)(\xi) = \sum_{i=1}^{n} \alpha_i u(t_i, \xi) + \sum_{i=1}^{k} \beta_i v(s_i, \xi), \quad u, v \in C(I; X),$$

we can model (5.6)-(5.9) as the abstract Cauchy problem (1.1)-(1.3). As in the previous example, $g(\cdot)$ is $[D(A)]$-valued continuous and $\|Ag(t, \cdot)\|_{\mathcal{L}(X)} \le \alpha^{\frac{1}{2}}$ for every $t \in I$. Moreover, the assumptions of Theorem 3.10 are satisfied with, $l_p = l_P V(\mu)$, where $V(\mu)$ is the variation of μ, $l_q = \sum_{i=1}^{n} |\alpha_i| + \sum_{i=1}^{k} |\beta_i|$, $l_f = L_F$, $l_g = \alpha^{\frac{1}{2}}$ and $\rho = 1$. The next result is a consequence of Theorems 3.10.

Theorem 5.3. *Assume conditions (iii)-(v) are satisfied and*

$$l_P V(\mu) + \sum_{i=1}^{n} |\alpha_i| + \sum_{i=1}^{k} |\beta_i| + 3\alpha^{\frac{1}{2}} + L_F < 1.$$

Then there exists a unique mild solution, $u(\cdot)$, of (5.6)-(5.9).

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References

[1] Benchohra, M.; Ntouyas, S. K.; *Existence of mild solutions of second order initial value problems for delay integrodifferential inclusions with nonlocal conditions.* Math. Bohem. 127 (2002), no. 4, 613–622.

[2] Benchohra, M.; Ntouyas, S. K.; *Existence of mild solutions of second order initial value problems for differential inclusions with nonlocal conditions.* Atti Sem. Mat. Fis. Univ. Modena 49 (2001), no. 2, 351–361.

[3] Benchohra, M.; Ntouyas, S. K.; *Controllability of second-order differential inclusions in Banach spaces with nonlocal conditions.* J. Optim. Theory Appl. 107 (2000), no. 3, 559–571.

[4] Benchohra, M.; Ntouyas, S. K.; *Existence of mild solutions on noncompact intervals to second-order initial value problems for a class of differential inclusions with nonlocal conditions.* Comput. Math. Appl. 39 (2000), no. 12, 11–18.

[5] L. Byszewski, V. Lakshmikantham; *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space.* Appl. Anal. **40** (1)(1991) 11-19.

[6] L. Byszewski; *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem.* J. Math. Anal. Appl. **162** (2) (1991) 494–505.

[7] L. Byszewski, H. Akca; *Existence of solutions of a semilinear functional-differential evolution nonlocal problem.* Nonlinear Analysis TMA. **34** (1) (1998) 65-72.

[8] Dauer, J. P.; Mahmudov, N. I.; *Remark on existence result for second order evolution equations in Banach spaces.* Int. J. Pure Appl. Math. 12 (2004), no. 4, 471–482.

[9] H. O. Fattorini; *Second Order Linear Differential Equations in Banach Spaces,* North-Holland Mathematics Studies, Vol. 108, North-Holland, Amsterdam, 1985.

[10] Granas, A. and Dugundji, J., Fixed Point Theory. Springer-Verlag, New York, 2003.

[11] Hernndez, Eduardo; *Existence of solutions to a second order partial differential equation with nonlocal conditions,* Electron. J. Diff. Eqns., Vol. 2003(2003), No. 51, pp. 1-10.

[12] J. Kisyński; *On cosine operator functions and one parameter group of operators,* Studia Math. **49** (1972), 93-105.

[13] R. H. Martin; *Nonlinear Operators and Differential Equations in Banach Spaces,* Robert E. Krieger Publ. Co., Florida, 1987.

[14] S. K. Ntouyas; *Global existence results for certain second order delay integrodifferential equations with nonlocal conditions.* Dynam. Systems Appl. **7** (3) (1998), 415-425.

[15] S. K. Ntouyas, P. Ch. Tsamatos; *Global existence for second order semilinear ordinary and delay integrodifferential equations with nonlocal conditions.* Appl. Anal. **67** (3-4) (1997) 245-257.

[16] Staněk, S.; *On solvability of nonlinear boundary value problems for the equation* $(x' + g(t,x,x'))' = f(t,x,x')$ *with one-sided growth restrictions on* f. Arch. Math. (Brno) **38** (2) (2002), 129–148.

[17] Staněk, S.; *The degree method for condensing operators in periodic boundary value problems.* Nonlinear Anal. **48** (4)(2002), 535–550.

[18] Staněk, S.; *Functional boundary value problems for second order functional differential equations of the neutral type.* Glas. Mat. Ser. III **36** (56)(1) (2001), 73–84.

[19] Staněk, S.; *Boundary value problems for systems of second-order functional differential equations.* Proceedings of the 6th Colloquium on the Qualitative Theory of Differential Equations (Szeged, 1999), No. 28, 14 pp. (electronic), Proc. Colloq. Qual. Theory Differ. Equ., Electron. J. Qual. Theory Differ. Equ., Szeged, 2000.

[20] C. C. Travis, G. F. Webb; *Compactness, regularity, and uniform continuity properties of strongly continuous cosine families.* Houston J. Math. **3** (4) (1977) 555-567.

[21] C. C. Travis, G. F. Webb; *Cosine families and abstract nonlinear second order differential equations.* Acta Math. Acad. Sci. Hungaricae, **32** (1978) 76-96.

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