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THE CAUCHY PROBLEM AND STEADY STATE SOLUTIONS FOR A NONLOCAL CAHN-HILLIARD EQUATION

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Abstract. We study the existence, uniqueness, and continuous dependence on initial data of the solution to the Cauchy problem and steady state solutions of a nonlocal Cahn-Hilliard equation on a bounded domain.

1. Introduction

We are concerned with two different problems, the first being the Cauchy problem for a nonlocal Cahn-Hilliard equation

$$\begin{gathered} \frac{\partial u}{\partial t} = \triangle(\varphi(u) - J * u) \quad \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x), \end{gathered} \tag{1.1}$$

where $\varphi(u) = u + f(u)$, f is bistable (e.g. $f(u) = au(u^2-1)$ for some $a > 0$), $*$ is convolution, and $\int_{\mathbb{R}^n} J = 1$.

The second problem is for the steady state equation

$$\begin{gathered} \int_\Omega J(x-y)dyu(x) - \int_\Omega J(x-y)u(y)dy + f(u) = C \quad \text{in } \Omega, \\ \int_\Omega u(x)dx = 0, \end{gathered} \tag{1.2}$$

where Ω is a bounded domain, C is a constant. The case when $\Omega = \mathbb{R}$ or $\mathbb{R}^n$ has been treated in [3, 5, 10, 11] and references therein.

To derive equations (1.1) and (1.2), we consider the free energy

$$E(u) = C \iint J(x-y)(u(x)-u(y))^2 dxdy + \int F(u(x))dx, \tag{1.3}$$

where C is a constant, F is the primitive of f, and u represents the concentration of one of the species of a binary material.

Following [16], we consider the gradient flow for (1.3) in $H_0^{-1}(\Omega)$, where H_0^{-1} is the space of distributions in the dual space of H^1 and with mean value zero. We do this since the total energy, E, decreases along the trajectories, and the average of u should be conserved. We have

$$u_t = -\operatorname{grad}_{H_0^{-1}} E(u). \tag{1.4}$$

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Since the representative of $\operatorname{grad} E(u)$ in H_0^{-1} is

$$\operatorname{grad}_{H_0^{-1}} E(u) = -\triangle(J(x-y)dyu(x) - \int J(x-y)u(y)dy + f(u)),$$

(1.4) gives

$$\frac{\partial u}{\partial t} = \triangle(\int_\Omega J(x-y)dyu(x) - \int_\Omega J(x-y)u(y)dy + f(u)). \tag{1.5}$$

In (1.3), making the approximation

$$u(x) - u(y) \simeq \nabla u(x) \cdot (x-y),$$

and assuming J to be isotropic, equation (1.4) leads to

$$\frac{\partial u}{\partial t} = -\triangle(d\triangle u - f(u)), \tag{1.6}$$

which is the classical Cahn-Hilliard equation.

Equations (1.5) and (1.6) are important in the study of materials science for modelling certain phenomena such as spinodal decomposition, Ostwald ripening, and grain boundary motion.

There is a lot of work on equation (1.6) (see for example [1, 2, 4, 9, 12, 13, 15, 14, 20] and references therein). For equation (1.5), there are very few results. In [6] and [7], we discussed the Neumann and Dirichlet boundary problems for (1.5). Here, we consider the Cauchy problem, where $\Omega = R^n$ and $\int J = 1$. Note that the steady state solutions for (1.5) in a bounded domain with no flux boundary condition satisfy the equation in (1.2) without the constraint.

In this paper, we prove the global existence and uniqueness of solutions for equation (1.1). Also we prove the existence of nonconstant solutions for equation (1.2). The techniques used in the proof of the latter result can also be applied to the nonlocal phase field system discussed in [8].

We organize this paper as follows. In section 2, we establish the existence, uniqueness and continuous dependence on initial values for classical solutions of equation (1.1). In section 3, we prove that under certain conditions, there exists a discontinuous steady state solution for equation (1.2).

2. The Cauchy problem for the nonlocal Cahn-Hilliard equation

For $T > 0$, let $Q_T = \mathbb{R}^n \times (0,T)$. We make the following assumptions:

(D1) $f \in C^{2+\beta}(\mathbb{R})$ and $\varphi'(u) \geq c$ for some positive constants c and β,
(D2) $J \in C^{2+\beta}(\mathbb{R}^n)$, $\triangle J \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and $\int_{\mathbb{R}^n} J = 1$.

First, we prove the uniqueness and continuous dependence of solutions on initial data. We have

Proposition 2.1. *Let u_i $(i = 1,2)$ be two solutions of (1.1) with initial data u_{i0} $(i = 1,2)$. If conditions (D1)–(D2) are satisfied, if $u_i \in C([0,T], L^1(\mathbb{R}^n)) \cap L^\infty(Q_T)$, and if $u_{i0} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ $(i = 1,2)$, then*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^n} |u_1 - u_2| dx \leq C(T) \int_{\mathbb{R}^n} |u_{10} - u_{20}| dx \tag{2.1}$$

for some constant $C(T)$.

Proof. For any $\tau \in (0,T)$, and $\psi \in C^{2,1}(Q_\tau)$, with $\psi = 0$ for $|x|$ large enough, after multiplying (1.1) by ψ, integrating over $[0,\tau]\times\mathbb{R}^n$, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} u_i(x,\tau)\psi(x,\tau)dx\\&= \int_{\mathbb{R}^n} u_i(x,0)\psi(x,0)dx + \int_0^\tau\int_{\mathbb{R}^n}(u_i\psi_t + \varphi(u_i)\triangle\psi)dx\,dt - \int_0^\tau\int_{\mathbb{R}^n}\psi\triangle J * u_i dx\,dt.\end{aligned}$$

Set $z = u_1 - u_2$, $z_0 = u_{10} - u_{20}$, then the above equality gives

$$\begin{aligned}\int_{\mathbb{R}^n} z(x,\tau)\psi(x,\tau)dx = \int_{\mathbb{R}^n} z_0(x)\psi(x,0)dx + \int_0^\tau\int_{\mathbb{R}^n} z(x,t)(\psi_t + b(x,t)\triangle\psi)dx\,dt\\ - \int_0^\tau\int_{\mathbb{R}^n}\psi\triangle J * z(x,t)dx\,dt,\end{aligned} \tag{2.2}$$

where

$$b(x,t) = \begin{cases}\frac{\varphi(u_1)-\varphi(u_2)}{u_1-u_2} & \text{for } u_1 \neq u_2,\\ \varphi'(u_1) & \text{for } u_1 = u_2.\end{cases} \tag{2.3}$$

Let $g(x) \in C_0^\infty(\mathbb{R}^n)$ have compact support, $0 \le g(x) \le 1$, and take $\lambda > 0$.

We will choose ψ, above, to satisfy certain conditions. First, consider the following final value problem on a large ball $B_R(0)$

$$\begin{gathered}\frac{\partial\psi}{\partial t} = -b(x,t)\triangle\psi + \lambda\psi \quad \text{for } |x| < R,\ 0 < t < \tau\\ \psi = 0 \quad \text{on } |x| = R,\, 0 < t < \tau\\ \psi(x,\tau) = g(x) \quad |x| \le R.\end{gathered} \tag{2.4}$$

There exists a unique solution $\psi \in C^{2,1}(B_R(0)\times(0,\tau))$ of of (2.4) which satisfies the following properties:

$$0 \le \psi \le e^{\lambda(t-\tau)}, \tag{2.5}$$

$$\int_0^\tau\int_{B_R(0)} b(x,t)|\triangle\psi|^2 dx\,dt \le C, \tag{2.6}$$

$$\sup_{0\le t\le\tau}\int_{B_R(0)}|\nabla\psi|^2 dx \le C, \tag{2.7}$$

where the constant C depends only on g. To extend ψ to be zero outside of $B_R(0)$, we define $\xi_R \in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{gathered}0 \le \xi_R \le 1,\\ \xi_R = 1 \quad \text{if } |x| < R-1,\\ \xi_R = 0 \quad \text{if } |x| > R - \frac{1}{2},\\ |\nabla\xi_R(x)|,\ |\triangle\xi_R(x)| \le C\end{gathered} \tag{2.8}$$

for some constant C which does not depend on R.

Let $\gamma = \xi_R\psi$, where ψ satisfies (2.4) in $B_R(0)$ and is zero outside. Using γ instead of ψ in (2.2), we have

$$\begin{aligned}&\int_{\mathbb{R}^n} z(x,\tau)g\xi_R dx - \int_{\mathbb{R}^n} \xi_R(x)z_0(x)\psi(x,0)dx + \int\int_{Q_\tau}(\triangle J * z - \lambda z)\xi_R\psi dx\,dt\\ &= \int\int_{Q_\tau} b(x,t)z(x,t)(2\nabla\xi_R\cdot\nabla\psi + \psi\triangle\xi_R)dx\,dt \equiv G(z,R).\end{aligned}$$

Since u_1 and u_2 belong to $L^\infty(Q_T)$, and since b is positive, from estimates (2.5)-(2.7) and (2.8), we have

$$\begin{aligned}|G(z,R)| &\le \int_0^\tau\int_{B_R\setminus B_{R-1}}(b|u_1-u_2|((2|\nabla\xi_R\|\nabla\psi| + |\psi\|\triangle\xi_R|))\\ &\le C\int_0^\tau\int_{B_R\setminus B_{R-1}} b(|u_1|+|u_2|)(|\nabla\psi|+1)dx\,dt \qquad (2.9)\\ &\le C\int_0^\tau\int_{B_R\setminus B_{R-1}}(|u_1|+|u_2|)dx\,dt.\end{aligned}$$

Since u_1 and u_2 belong to $L^1(Q_T)$, letting $R\to\infty$ we have $G(z,R)\to 0$. This implies

$$\int_{\mathbb{R}^n} z(x,\tau)g(x)dx \le \int_{\mathbb{R}^n}|z_0(x)|e^{-\lambda\tau}dx + \int_0^\tau\int_{\mathbb{R}^n}(|\triangle J * z - \lambda z|e^{\lambda(t-\tau)}dx\,dt.$$

Letting $\lambda\to 0$ and $g(x)\to \mathit{sign}\, z^+(x,\tau)$, we obtain

$$\int_{\mathbb{R}^n}(u_1-u_2)^+dx \le \int_{\mathbb{R}^n}|u_{10}-u_{20}|dx + C\int_0^\tau\int_{\mathbb{R}^n}|u_1-u_2|dx\,dt. \qquad (2.10)$$

Interchanging u_1 and u_2 yields

$$\int_{\mathbb{R}^n}|u_1-u_2|dx \le \int_{\mathbb{R}^n}|u_{10}-u_{20}|dx + C\int_0^\tau\int_{\mathbb{R}^n}|u_1-u_2|dx\,dt.$$

Inequality (2.1) follows from the above inequality and Gronwall's inequality. □

Next we prove the existence of a solution to (1.1).

Theorem 2.2. *For any $T>0$, if $u_0(x)\in C_0^{2+\beta}(\mathbb{R}^n)$, and if φ and J satisfy assumptions $(D_1)-(D_2)$, then there exists a unique solution of (1.1) which belongs to $C^{2+\beta,\frac{2+\beta}{2}}(Q_T)\cap L^1(Q_T)\cap L^\infty(Q_T)$.*

Proof. Since $u_0(x)=0$ for $|x|$ large enough, we consider

$$\begin{gathered}\frac{\partial u}{\partial t} = \triangle(\varphi(u) - J*u) \quad \text{in } B_R(0)\times(0,T),\\ u(x,t) = 0 \quad \text{on } \partial B_R(0)\times(0,T),\\ u(x,0) = u_0(x).\end{gathered} \qquad (2.11)$$

From [7, Theorem 2.4], there exists a unique solution $u(x,t)\in C^{2+\beta,\frac{2+\beta}{2}}(B_R(0)\times(0,T))$ of (2.11). Let $u(x,t)=ve^t$ in (2.11), we have

$$e^tv_t + ve^t = \varphi'(u)e^t\triangle v + \varphi''(u)|\nabla v|^2e^{2t} - e^t\triangle J * v. \qquad (2.12)$$

Multiplying (2.12) by v and using $v\triangle v = \frac{1}{2}\triangle v^2 - |\nabla v|^2$, we obtain

$$\frac{1}{2}(v^2)_t + v^2 = \frac{1}{2}\varphi'(u)\triangle v^2 + \frac{1}{2}\varphi''(u)\nabla v\cdot\nabla v^2 e^t - \varphi'(u)|\nabla v|^2 - v\triangle(J*v). \quad (2.13)$$

If there exists $(P_0,t_0)\in B_R(0)\times(0,T]$ such that $v^2(P_0,t_0)=\max v^2$, then we have $\triangle v^2(P_0,t_0)\le 0$, $\nabla v^2(P_0,t_0)=0$, $\nabla v(P_0,t_0)=0$, $v_t^2(P_0,t_0)\ge 0$, and (2.13) yields

$$v^2(P_0,t_0)\le -\int_{B_R}\triangle J(P_0-y)v(y,t_0)dyv(P_0,t_0). \quad (2.14)$$

This yields

$$\max|v|\le M\int_{B_R}|v(y,t_0)|dy \quad (2.15)$$

for some constant M which does not depend on R.

Since $u=0$ is also a solution of (2.11) with initial data $u_0=0$, by [7, Theorem 2.5], we have

$$\int_{B_R}|u(x,t)-0|dx\le C(T)\int_{B_R}|u_0-0|dx \quad (2.16)$$

for some constant $C(T)$ which does not depend on R. Inequalities (2.15) and (2.16) imply

$$\max|v|\le C(T)\int_{B_R}|u_0|dx. \quad (2.17)$$

Since $u_0\in L^1(\mathbb{R}^n)$, we have

$$\max|v|\le B(T) \quad (2.18)$$

for some constant $B(T)$ which does not depend on R. This yields

$$\max|u|\le B(T)e^T \quad (2.19)$$

for some constant $B(T)$ which does not depend on R. We have proved the solution of (2.11) is uniformly bounded, i.e.,

$$\max_{B_R\times[0,T]}|u(x,t)|\le C$$

for any $R>0$, where C does not depend on R. A similar argument to that in the proof in [7, Theorem 2.2] yields

$$\|u_R\|_{2+\beta}\le C(K,T) \quad (2.20)$$

for any $R>K\equiv$ *constant*, where u_R is a solution of (2.11) in $B_R\times(0,T)$ and $C(K,T)$ is a constant which does not depend on R ($\|\cdot\|_{2+\beta}$ is a *Hölder* norm defined in [19]).

By employing the usual diagonal process, we can choose a sequence $\{R_i\}$ such that u_{R_i}, Du_{R_i}, and $D^2u_{R_i}$ converge to u, Du, and D^2u pointwise, and u satisfies equation (1.1). From (2.16) and (2.19), we also have $u\in L^1(Q_T)\cap L^\infty(Q_T)$.

Uniqueness follows from Proposition 2.1. □

3. Steady state solutions for the nonlocal Cahn-Hilliard equation

In this part, we study equation (1.2).

Proposition 3.1. *Suppose $\Omega \subset \mathbb{R}^n$ is a closed and bounded set, $J(x) \geq 0$ and is continuous on $\mathbb{R}^n$, supp$J \supset B_\delta(0)$ for some positive constant δ, and f is nondecreasing. Then the only continuous solution of equation (1.2) is zero.*

Proof. Without loss of generality, we assume that $f(0) = 0$. If $f(0) \neq 0$, we may use $f(u) - f(0)$ instead of $f(u)$ in (1.2).
Case 1: $C \leq 0$ in equation (1.2). If the conclusion is not true, since $\int u dx = 0$, and u is continuous on Ω, there exists $P_0 \in \Omega$ such that $u(P_0) = \max u(x) > 0$. Let

$$A = \{y \in \Omega : u(y) = \max u(x)\}.$$

We claim: There exist $P_0 \in \partial A$ and $r > 0$ such that $K := (\Omega \setminus A) \cap B_r(P_0)$ has positive measure. If this is not true, we have $meas(\Omega \setminus A) = 0$. This and $u(x) = \max u$ on A imply $\int_\Omega u = \int_A u > 0$. This contradicts $\int_\Omega u = 0$.

Since $\operatorname{supp} J \supset B_\delta(0)$ implies $\operatorname{supp} J(P_0 - \cdot) \supset B_\delta(P_0)$, choosing $r_1 = \min\{\delta, r\}$ gives

$$\operatorname{meas}(K \cap B_{r_1}(P_0)) > 0, \tag{3.1}$$

$$J(P_0 - y) > 0 \quad \text{on } K \cap B_{r_1}(P_0), \tag{3.2}$$

$$u(P_0) - u(y) > 0 \quad \text{on } K \cap B_{r_1}(P_0). \tag{3.3}$$

Inequalities (3.1)-(3.3) imply

$$\int_\Omega J(P_0 - y)(u(P_0) - u(y))dy \geq \int_{K \cap B_{r_1}(P_0)} J(P_0 - y)(u(P_0) - u(y))dy > 0.$$

This and $f(u(P_0)) \geq 0$ imply

$$\int_\Omega J(P_0 - y)u(P_0)dy - \int_\Omega J(P_0 - y)u(y)dy + f(u(P_0)) > 0, \tag{3.4}$$

contradicting (1.2).
Case 2: $C > 0$ in (1.2). In this case, taking P_0 such that $u(P_0) = \min u < 0$ leads to a contradiction in a similar way. $\square$

If $f'(u)$ changes sign, we make the following assumptions:

(E1) $\Omega = (-1, 1)$ if $\dim \Omega = 1$, $\Omega = (-1, 1) \times \Omega'$ if $\dim \Omega > 1$.

(E2) $J(x) = J(|x|)$, $J(x) \geq 0$, and

$$M \geq \sup_{x \in \Omega} \int_\Omega J(x - y)dy \geq \inf_{x \in \Omega} \int_\Omega J(x - y)dy \geq m > 0$$

for positive constants M and m.

(E3) $f \in C^1(R)$ is odd, $f(1) = 0$, there exist $\delta > 0$ and $a \in (0, 1)$ such that $f'(u) \geq \delta$ on $[a, \infty)$, and $f(-a) \geq (1 + a)M$.

(E4) $C = 0$ in (1.2).

Remark 3.2. Condition (E3) implies that $f(-1) = 0$, $f'(u) \geq \delta$ on $(-\infty, -a]$, and $-f(a) \geq (1 + a)M$.

Let $j(x) = \int_\Omega J(x - y)dy$. From (E_2), we have

$$m \leq j(x) \leq M. \tag{3.5}$$

Dividing (1.2) by $j(x)$, we consider

$$\begin{gathered} u(x) - \frac{1}{j(x)} \int_\Omega J(x-y)u(y)dy + \frac{f(u(x))}{j(x)} = 0, \\ \int_\Omega u(x)dx = 0. \end{gathered} \tag{3.6}$$

Theorem 3.3. *If assumptions $(E_1)-(E_4)$ are satisfied, then there exists a solution of equation (3.6) such that*

$$u(x) \begin{cases} \geq a & \text{for } x \in M_1 \equiv (0,1) \times \Omega', \\ \leq -a & \text{for } x \in M_2 \equiv (-1,0) \times \Omega'. \end{cases} \tag{3.7}$$

Moreover, we have

$$-1 \leq u(x) \leq 1. \tag{3.8}$$

Proof. Following [3], we let

$$B = \{u \in L^\infty(\Omega) : u(-x_1, x') = -u(x_1, x'),\ u(x) \in [a,1] \text{ for } x \in M_1\}.$$

The definition of B implies that $u(x) \in [-1, -a]$ for $x \in M_2$. Define

$$Tu(x) = u(x) + h[\frac{1}{j(x)} \int_\Omega J(x-y)u(y)dy - u(x) - \frac{1}{j(x)} f(u(x))].$$

We want to show $T : B \to B$ is a contraction map if h is small enough. In fact, since $j(x) = \int_\Omega J(x-y)dy$, with assumption (E2), we have $j(-x_1, x') = j(x_1, x')$. And if $u(x) \in B$, we have

$$\begin{aligned}
&T(u(-x_1, x')) \\
&= u(-x_1, x') + \frac{h}{j(-x_1, x')} \int_{-1}^1 \int_{\Omega'} J(-x_1 - y_1, x' - y')u(y_1, y')dy_1 dy' \\
&\quad - hu(-x_1, x') + \frac{h}{j(-x_1, x')} f(u(-x_1, x')) \\
&= -u(x_1, x') - \frac{h}{j(x_1, x')} \int_{-1}^1 \int_{\Omega'} J(-x_1 + z_1, x' - y')u(z_1, y')dz_1 dy' \\
&\quad + hu(x_1, x') - \frac{h}{j(x_1, x')} f(u(x_1, x')) \\
&= -u(x_1, x') - \frac{h}{j(x_1, x')} \int_{-1}^1 \int_{\Omega'} J(x_1 - z_1, x' - y')u(z_1, y')dz_1 dy' \\
&\quad + hu(x_1, x') - \frac{h}{j(x_1, x')} f(u(x_1, x')) \\
&= -(u(x_1, x') + \frac{h}{j(x_1, x')} \int_{-1}^1 \int_{\Omega'} J(x_1 - z_1, x' - y')u(z_1, y')dz_1 dy' \\
&\quad - hu(x_1, x') + \frac{h}{j(x_1, x')} f(u(x_1, x'))) \\
&= -T(u(x_1, x')).
\end{aligned} \tag{3.9}$$

Choose h small enough such that

$$h \frac{1}{j(x)} f'(u) < 1 - h \tag{3.10}$$

for $u \in [-1, -a] \cup [a, 1]$ and $x \in \Omega$.

This implies that $u - h[u + \frac{1}{j(x)} f(u)]$ is increasing in u on $[a, 1]$. Since $u(y) \geq a$ for $y \in M_1$, and $u(y) \geq -1$ for $y \in M_2$, we have for $x \in M_1$

$$\begin{aligned}
&Tu(x) \\
&= h \frac{1}{j(x)} \int_\Omega J(x-y)u(y)dy + u - h[u + \frac{1}{j(x)} f(u)] \\
&\geq h \frac{1}{j(x)} \int_\Omega J(x-y)u(y)dy + a - ha - h\frac{1}{j(x)} f(a) \\
&= h \frac{1}{j(x)} \int_{M_1} J(x-y)u(y)dy + h\frac{1}{j(x)} \int_{M_2} J(x-y)u(y)dy + a - ha - h\frac{1}{j(x)} f(a) \\
&\geq ha \frac{1}{j(x)} \int_{M_1} J(x-y)dy - h\frac{1}{j(x)} \int_{M_2} J(x-y)dy + a - ha - h\frac{1}{j(x)} f(a) \\
&= a - ha \frac{1}{j(x)} \int_{M_2} J(x-y)dy - h\frac{1}{j(x)} \int_{M_2} J(x-y)dy - \frac{h}{j(x)} f(a) \\
&\geq a - \frac{h}{j(x)} [(1+a) \int_{M_2} J(x-y)dy + f(a)] \geq a
\end{aligned}$$

by (E3). Also,

$$\begin{aligned}
Tu(x) &= h \frac{1}{j(x)} \int_\Omega J(x-y)u(y)dy + u - h[u + \frac{1}{j(x)} f(u)] \\
&\leq h \frac{1}{j(x)} \int_\Omega J(x-y)u(y)dy + 1 - h - h\frac{1}{j(x)} f(1) \leq 1
\end{aligned} \tag{3.11}$$

for $x \in M_1$. Estimates (3.9)-(3.11) imply that T maps B to B.

For $u, v \in B$, choosing h small enough so that $0 < 1 - h(1 + \delta \frac{1}{M}) < 1$, we have

$$\begin{aligned}
&\|Tu - Tv\|_\infty \\
&= \|(u-v) + \frac{h}{j(x)} \int_\Omega J(x-y)(u(y) - v(y))dy \\
&\quad - h(u(x) - v(x)) - \frac{h}{j(x)} (f(u) - f(v))\|_\infty \\
&= \|(1 - h - \frac{hf'(\theta u + (1-\theta)v)}{j(x)})(u-v) + \frac{h}{j(x)} \int_\Omega J(x-y)(u(y) - v(y))dy\|_\infty \\
&\leq (1 - h(1 + \delta \frac{1}{M}))\|u - v\|_\infty + h\|(u-v)\|_\infty \\
&\leq (1 - h\delta \frac{1}{M})\|u - v\|_\infty,
\end{aligned}$$

where $\theta(x) \in (0,1)$ for all $x \in \Omega$. Here we used (E3) and the fact that for any $x \in \Omega$ either $u(x)$, $v(x) \geq a$ or $u(x)$, $v(x) \leq -a$.

Therefore, T is a contraction map from B to B. There exists a unique fixed point $u(x)$ such that $Tu = u$. Estimates (3.8) follows from the definition of B. $\square$

Remark 3.4. If we just consider the solution to

$$\int_\Omega J(x-y)u(x)dy - \int_\Omega J(x-y)u(y)dy + f(u) = 0 \quad \text{in } \Omega \tag{3.12}$$

without the condition $\int_\Omega u dx = 0$, then the conditions that f is odd and $J(x) = J(|x|)$ are not necessary. In this case, we can use a similar method to that in [3] to prove the existence of a discontinuous solution under conditions (E2), (E3)', and (E4), where

(E3)' $f \in C^1(R)$, $f(-1) = f(c) = f(1) = 0$ for $c \in (-1, 1)$, there exist $\delta > 0$, $a \in (0, 1)$, $b \in (-1, 0)$ such that $f'(x) \geq \delta$ on $[a, \infty) \cup (-\infty, b)$, $f(a) \leq -(1 + a)M$, and $f(b) \geq (1 + b)M$, where M is defined in (E2).

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