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EXISTENCE OF SUBHARMONIC SOLUTIONS TO A HYSTERESIS SYSTEM WITH SINUSOIDAL EXTERNAL INFLUENCE

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Abstract. We consider a system of ordinary differential equations with nonlinearity describing relay hysteresis under sinusoidal external influence. Theorems on sufficient conditions for the existence of subharmonic solutions to the system being investigated are established.

1. Introduction and statement of problem

Dynamics of ordinary differential equation systems with discontinuous right-hand sides exposed to external influence is of undoubted interest. The history of such investigations started long ago (see, for example, [13]). Stable modes in relay systems are examined by iterative methods in [15]. The latest results on the solutions to second-order differential equations with discontinuous right-hand side are published in [1, 4, 5, 6, 7, 10, 14, 18, 19, 22]; the periodic solutions are considered in [4, 6, 7, 10, 14, 22]. Applied problems for these equations are discussed in [16, 20]. The existence of periodic solutions to Hamiltonian systems with periodic influences is proved in [3]. Lavrent'ev's problem on separated flows in the case of non-periodic external influence is analyzed in [20]. The ordinary differential equation of second-order with superlinear convex nonlinearity is investigated in [21]. Problems related to control of elliptic type distributed systems with discontinuous nonlinearity are approached in [17]. The systems of ordinary differential equations with nonlinearity of non-ideal relay type and external continuous influence are studied in [8, 9, 25, 26, 27]. This work proceeds the researches above.

We consider the automatic control system of the form

$$\dot{X} = AX + BF(\sigma) + kBf(t), \quad \sigma = (\Gamma, X). \tag{1.1}$$

Here $X \in E^d$ (E^d is d-dimensional Euclidean space); A is a real-valued $(d \times d)$ matrix; B and Γ are real-valued $(d \times 1)$ matrices; $k \in \mathbb{R}$; $f(t) = \sin(\omega t + \varphi)$, $\omega, \varphi \in \mathbb{R}$; (Γ, X) means the scalar product of vectors Γ and X. Ambiguous function F is defined by the relations: $F(\sigma) = m_2$ while $\sigma > l_1$ and $F(\sigma) = m_1$ while $\sigma < l_2$, where $m_1 < m_2$, $l_1 < l_2$ ($m_i, l_i \in \mathbb{R}$, $i = 1, 2$).

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Hence function $F(\sigma)$ describes an asymmetric relay hysteresis loop being traversed counterclockwise in plane $(\sigma, F(\sigma))$. Nonlinearities of this kind are often used in applications (see, e.g., [11, 12, 24]).

Unlike [15], in this paper we do not suppose that system (1.1) is strong positive and matrix A of the system is Hurwitz. In [4, 14], nonlinearity F corresponds to the special case when $-m_1 = m_2$ and $l_1 = l_2 = 0$.

We pose the problem that is to find out the conditions on the parameters of the relay hysteresis system under which there exist the periodic modes similar to the dominant-lock mode or the subharmonic-lock mode [23]. The analogy consists only in the locking process, as it is not necessary for the autonomous system under considered assumptions to have the self-oscillating mode or even a periodic solution.

We shall say that a solution of system (1.1) is called *subharmonic* if the period of the forced oscillation be multiple to the period of the external influence.

Thus in this paper we consider the problem on the existence of the subharmonic solutions to the hysteresis systems of form (1.1) with sinusoidal external influence.

2. Approach to the problem

First we present an approach to solving the problem for system (1.1). To construct the forced oscillations of system (1.1), we use the general solution of the system in the Cauchy form

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}(BF(\sigma) + kBf(\tau))d\tau. \tag{2.1}$$

Moreover, we assume that there is $t = T_B$ such that $X(0) = X(T_B)$. Then it follows from the solution of (2.1) that initial vector $X_0 = X(0)$ can be defined by the following expression:

$$X_0 = \big(E - e^{AT_B}\big)^{-1} \int_0^{T_B} e^{A(T_B-\tau)}B(F(\sigma) + kf(\tau))d\tau. \tag{2.2}$$

Therefore, using (2.1) and (2.2), we can formally define T_B-periodic solution of (1.1) as follows:

$$\begin{aligned} X(t) = e^{At}\big(E - e^{AT_B}\big)^{-1} &\int_0^{T_B} e^{A(T_B-\tau)}B(F(\sigma(\tau)) + kf(\tau))d\tau \\ &+ \int_0^t e^{A(t-\tau)}B(F(\sigma(\tau)) + kf(\tau))d\tau. \end{aligned} \tag{2.3}$$

Notice that in this case we need to know the properties of functions $\sigma(t)$ and $f(t)$.

We use (2.3) to construct the transcendental equations with respect to the parameters of the periodic solution, which describes the forced oscillations of the system with the relay hysteresis given by function $F(\sigma)$.

Let points X_1 and X_2 belong to the periodic trajectory and $(\Gamma, X_1) = l_1$, $(\Gamma, X_2) = l_2$. In time T_B the image point returns to the initial position. Then we have

$$\begin{aligned} X_1 &= e^{A\tau_1}X_2 + \int_0^{\tau_1} e^{A(\tau_1-\tau)}B(m_2 + kf(\tau))d\tau, \\ X_2 &= e^{A(T_B-\tau_1)}X_1 + \int_{\tau_1}^{T_B} e^{A(T_B-\tau)}B(m_1 + kf(\tau))d\tau, \end{aligned} \tag{2.4}$$

where τ_1 is the time it takes the image point to transit from X_2 to X_1, τ_2 is the time for return transition from X_1 to X_2. Note that $\tau_2 = T_B - \tau_1$.

From (2.4), we have

$$\begin{aligned} X_1 &= \big(E - e^{AT_B}\big)^{-1}\Big(e^{A\tau_1}\int_{\tau_1}^{T_B} e^{A(T_B-\tau)}B(m_1 + kf(\tau))d\tau \\ &\quad + \int_0^{\tau_1} e^{A(\tau_1-\tau)}B(m_2 + kf(\tau))d\tau\Big) \\ &= \big(E - e^{AT_B}\big)^{-1}Q_1 \end{aligned} \tag{2.5}$$

and similarly

$$\begin{aligned} X_2 &= \big(E - e^{AT_B}\big)^{-1}\Big(e^{A(T_B-\tau_1)}\int_0^{\tau_1} e^{A(\tau_1-\tau)}B(m_2 + kf(\tau))d\tau \\ &\quad + \int_{\tau_1}^{T_B} e^{A(T_B-\tau)}B(m_1 + kf(\tau))d\tau\Big) \\ &= \big(E - e^{AT_B}\big)^{-1}Q_2. \end{aligned} \tag{2.6}$$

Using the switching conditions and equalities (2.5), (2.6), we construct the transcendental equations for seeking τ_1 and τ_2, namely,

$$\begin{aligned} l_1 &= (\Gamma, \big(E - e^{AT_B}\big)^{-1}Q_1), \\ l_2 &= (\Gamma, \big(E - e^{AT_B}\big)^{-1}Q_2). \end{aligned} \tag{2.7}$$

Let there exist parameters of system (1.1) such that equations (2.7) is solvable for $\tau_1 > 0$, $\tau_2 > 0$, where $\tau_1 + \tau_2 = T_B$. Also, let the solutions of (2.7) satisfy system (2.4), where X_1 and X_2 are defined by (2.5) and (2.6) respectively. Then it is possible to state that the problem at issue is solved.

Let us remark that the solutions of system (2.7) can be a countable set. Whence system (1.1), generally speaking, can have a lot of subharmonic solutions.

3. Real nonzero distinct roots for $d = 2$

Let us write down equations (2.7) for the case when $d = 2$ and characteristic equation $|A - \lambda E| = 0$ has two real nonzero distinct roots λ_1 and λ_2. We perform the nonsingular linear transformation of system (1.1) with the matrix of the special form [9, 25, 26, 27]. In this case, we have $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

$$e^{At} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}, \quad \big(E - e^{AT_B}\big)^{-1} = \begin{pmatrix} (1 - e^{\lambda_1 T_B})^{-1} & 0 \\ 0 & (1 - e^{\lambda_2 T_B})^{-1} \end{pmatrix}.$$

After the transformation, here we return to the original notations for the matrices. Let $Q_i = \begin{pmatrix} q_1^i \\ q_2^i \end{pmatrix}$, where $i = 1, 2$. Component q_1^1 is defined by the equation

$$\begin{aligned} q_1^1 &= \frac{m_1}{\lambda_1}e^{\lambda_1\tau_1}\left(-1 + e^{\lambda_1(T_B-\tau_1)}\right) + ke^{\lambda_1(T_B+\tau_1)}\Big(\frac{-\lambda_1}{\lambda_1^2+\omega^2}e^{-\lambda_1 T_B}\sin(\omega T_B + \varphi) \\ &\quad - \frac{\omega}{\lambda_1^2+\omega^2}e^{-\lambda_1 T_B}\cos(\omega T_B + \varphi) - \frac{-\lambda_1}{\lambda_1^2+\omega^2}e^{-\lambda_1\tau_1}\sin(\omega\tau_1 + \varphi) \\ &\quad + \frac{\omega}{\lambda_1^2+\omega^2}e^{-\lambda_1\tau_1}\cos(\omega\tau_1 + \varphi)\Big) - \frac{m_2}{\lambda_1}\left(1 - e^{\lambda_1\tau_1}\right) \end{aligned}$$

$$
\begin{aligned}
&+ ke^{\lambda_1\tau_1}\Big(\frac{-\lambda_1}{\lambda_1^2+\omega^2}e^{-\lambda_1\tau_1}\sin(\omega\tau_1+\varphi) - \frac{\omega}{\lambda_1^2+\omega^2}e^{-\lambda_1\tau_1}\cos(\omega\tau_1+\varphi) \\
&- \frac{-\lambda_1}{\lambda_1^2+\omega^2}\sin\varphi + \frac{\omega}{\lambda_1^2+\omega^2}\cos\varphi\Big).
\end{aligned}
$$

Component q_2^1 is defined by the similar equality

$$
\begin{aligned}
q_2^1 = {}& \frac{m_1}{\lambda_2}e^{\lambda_2\tau_1}\left(-1+e^{\lambda_2(T_B-\tau_1)}\right) + ke^{\lambda_2(T_B+\tau_1)}\Big(\frac{-\lambda_2}{\lambda_2^2+\omega^2}e^{-\lambda_2 T_B}\sin(\omega T_B+\varphi) \\
&- \frac{\omega}{\lambda_2^2+\omega^2}e^{-\lambda_2 T_B}\cos(\omega T_B+\varphi) - \frac{-\lambda_2}{\lambda_2^2+\omega^2}e^{-\lambda_2\tau_1}\sin(\omega\tau_1+\varphi) \\
&+ \frac{\omega}{\lambda_2^2+\omega^2}e^{-\lambda_2\tau_1}\cos(\omega\tau_1+\varphi)\Big) - \frac{m_2}{\lambda_2}\left(1-e^{\lambda_2\tau_1}\right) \\
&+ ke^{\lambda_2\tau_1}\Big(\frac{-\lambda_2}{\lambda_2^2+\omega^2}e^{-\lambda_2\tau_1}\sin(\omega\tau_1+\varphi) - \frac{\omega}{\lambda_2^2+\omega^2}e^{-\lambda_2\tau_1}\cos(\omega\tau_1+\varphi) \\
&- \frac{-\lambda_2}{\lambda_2^2+\omega^2}\sin\varphi + \frac{\omega}{\lambda_2^2+\omega^2}\cos\varphi\Big).
\end{aligned}
$$

Now, using the coefficients of the original system, we write down the first transcendental equation of (2.7) for $\gamma_1 = 0$. We can afford these additional assumptions owing to the choice of the linear transformation for the original system. Further, we are looking for the subharmonic solutions.

Here and elsewhere γ_i $(i = 1,2)$ are the components of vector $\Gamma = \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$. We emphasize especially that vector Γ is obtained as a consequence of applying this transformation. From here we obtain

$$
\begin{aligned}
\frac{l_1}{\gamma_2}\left(1-e^{\lambda_2 T_B}\right) = {}& \frac{1}{\lambda_2}(m_2-m_1)e^{\lambda_2\tau_1} + \frac{m_1}{\lambda_2}e^{\lambda_2 T_B} - \frac{m_2}{\lambda_2} + k\left(e^{\lambda_2 T_B}-1\right) \\
&\times\Big(\frac{\lambda_2}{\lambda_2^2+\omega^2}\sin(\omega\tau_1+\varphi) + \frac{\omega}{\lambda_2^2+\omega^2}\cos(\omega\tau_1+\varphi)\Big).
\end{aligned} \tag{3.1}
$$

The second equation for τ_2 can be obtained by similar way. Values τ_1 and τ_2 are related by $\tau_2 = T_B - \tau_1$, where T_B is the period of forced oscillations that, in particular, may be equal to the period of function $f(t)$. First we write out components q_1^2 and q_2^2 of vector Q_2,

$$
\begin{aligned}
q_1^2 = {}& -\frac{m_2}{\lambda_1}e^{\lambda_1(T_B-\tau_1)}\left(1-e^{\lambda_1\tau_1}\right) + ke^{\lambda_1 T_B}\Big(\frac{-\lambda_1}{\lambda_1^2+\omega^2}e^{-\lambda_1\tau_1}\sin(\omega\tau_1+\varphi) \\
&- \frac{\omega}{\lambda_1^2+\omega^2}e^{-\lambda_1\tau_1}\cos(\omega\tau_1+\varphi) - \frac{-\lambda_1}{\lambda_1^2+\omega^2}\sin\varphi + \frac{\omega}{\lambda_1^2+\omega^2}\cos\varphi\Big) \\
&+ \frac{m_1}{\lambda_1}\left(-1+e^{\lambda_1(T_B-\tau_1)}\right) + ke^{\lambda_1\tau_1}\Big(\frac{-\lambda_1}{\lambda_1^2+\omega^2}e^{-\lambda_1 T_B}\sin(\omega T_B+\varphi) \\
&- \frac{\omega}{\lambda_1^2+\omega^2}e^{-\lambda_1 T_B}\cos(\omega T_B+\varphi) - \frac{-\lambda_1}{\lambda_1^2+\omega^2}e^{-\lambda_1\tau_1}\sin(\omega\tau_1+\varphi) \\
&+ \frac{\omega}{\lambda_1^2+\omega^2}e^{-\lambda_1\tau_1}\cos(\omega\tau_1+\varphi)\Big)
\end{aligned}
$$

and

$$
q_2^2 = -\frac{m_2}{\lambda_2}e^{\lambda_2(T_B-\tau_1)}\left(1-e^{\lambda_2\tau_1}\right) + ke^{\lambda_2 T_B}\Big(\frac{-\lambda_2}{\lambda_2^2+\omega^2}e^{-\lambda_2\tau_1}\sin(\omega\tau_1+\varphi)
$$

$$
\begin{aligned}
&- \frac{\omega}{\lambda_2^2+\omega^2} e^{-\lambda_2\tau_1}\cos(\omega\tau_1+\varphi) - \frac{-\lambda_2}{\lambda_2^2+\omega^2}\sin\varphi + \frac{\omega}{\lambda_2^2+\omega^2}\cos\varphi\Big) \\
&+ \frac{m_1}{\lambda_2}\Big(-1+e^{\lambda_2(T_B-\tau_1)}\Big) + ke^{\lambda_2\tau_1}\Big(\frac{-\lambda_2}{\lambda_2^2+\omega^2}e^{-\lambda_2 T_B}\sin(\omega T_B+\varphi) \\
&- \frac{\omega}{\lambda_2^2+\omega^2}e^{-\lambda_2 T_B}\cos(\omega T_B+\varphi) - \frac{-\lambda_2}{\lambda_2^2+\omega^2}e^{-\lambda_2\tau_1}\sin(\omega\tau_1+\varphi) \\
&+ \frac{\omega}{\lambda_2^2+\omega^2}e^{-\lambda_2\tau_1}\cos(\omega\tau_1+\varphi)\Big).
\end{aligned}
$$

Then the second transcendental equation of (2.7) takes the form

$$
\begin{aligned}
&\frac{l_2}{\gamma_2}\left(1-e^{\lambda_2 T_B}\right) \\
&= \frac{1}{\lambda_2}(m_1-m_2)e^{\lambda_2(T_B-\tau_1)} + \frac{m_2}{\lambda_2}e^{\lambda_2 T_B} - \frac{m_1}{\lambda_2} \\
&\quad + k\Big(1-e^{\lambda_2(T_B-\tau_1)}\Big)\Big(\frac{\lambda_2}{\lambda_2^2+\omega^2}\sin(\omega\tau_1+\varphi) + \frac{\omega}{\lambda_2^2+\omega^2}\cos(\omega\tau_1+\varphi)\Big) \\
&\quad + k\Big(e^{\lambda_2 T_B} - e^{\lambda_2(\tau_1-T_B)}\Big)\Big(\frac{\lambda_2}{\lambda_2^2+\omega^2}\sin\varphi + \frac{\omega}{\lambda_2^2+\omega^2}\cos\varphi\Big).
\end{aligned}
\tag{3.2}
$$

Next we solve equation (3.1) with respect to the expression in its right side in brackets

$$
\begin{aligned}
&k\left(\frac{\lambda_2}{\lambda_2^2+\omega^2}\sin(\omega\tau_1+\varphi) + \frac{\omega}{\lambda_2^2+\omega^2}\cos(\omega\tau_1+\varphi)\right) \\
&= \left(e^{\lambda_2 T_B}-1\right)^{-1}\Big(\frac{l_1}{\gamma_2}\left(1-e^{\lambda_2 T_B}\right) - \frac{1}{\lambda_2}(m_2-m_1)e^{\lambda_2\tau_1} \\
&\quad - \frac{m_1}{\lambda_2}e^{\lambda_2 T_B} + \frac{m_2}{\lambda_2}\Big).
\end{aligned}
\tag{3.3}
$$

Exactly the same expression is in the fourth term of equation (3.2). We substitute expression (3.3) in equation (3.2), then denote $y = e^{\lambda_2\tau_1}$ (assuming that $\tau_1 > 0$) and group the coefficients at y^2, y^1 and y^0 respectively. We have

$$
ay^2 + by + c = 0, \tag{3.4}
$$

where coefficients a, b and c are determined by the following equations:

$$
\begin{gathered}
a = \frac{m_2-m_1}{\lambda_2\left(e^{\lambda_2 T_B}-1\right)} - \frac{k}{e^{\lambda_2 T_B}\sqrt{\lambda_2^2+\omega^2}}\sin(\varphi+\delta), \\
b = \frac{ke^{\lambda_2 T_B}}{\sqrt{\lambda_2^2+\omega^2}}\sin(\varphi+\delta) + \frac{l_2}{\gamma_2}\left(1-e^{\lambda_2 T_B}\right) - \Big(\frac{m_2}{\lambda_2}e^{\lambda_2 T_B} - \frac{m_1}{\lambda_2}\Big) + \frac{l_1}{\gamma_2} \\
+ \frac{1}{\lambda_2\left(e^{\lambda_2 T_B}-1\right)}\left(m_1 e^{\lambda_2 T_B} - m_2\right) - \frac{e^{\lambda_2 T_B}(m_2-m_1)}{\left(e^{\lambda_2 T_B}-1\right)\lambda_2}, \\
c = -\frac{1}{\lambda_2}e^{\lambda_2 T_B}(m_1-m_2) - \frac{l_1}{\gamma_2}e^{\lambda_2 T_B} + \frac{e^{\lambda_2 T_B}}{e^{\lambda_2 T_B}-1}\Big(\frac{m_2}{\lambda_2} - \frac{m_1}{\lambda_2}e^{\lambda_2 T_B}\Big),
\end{gathered}
\tag{3.5}
$$

where $\delta = \arctan(\omega/\lambda_2)$. If the period of the external influence defined by function $f(t)$ is known, then we also consider value T_B in (3.5) as known. This follows from the agreement to search the subharmonic solutions. We suppose that $T_B = nT$, where $n \in \mathbb{N}$ and T is the period of function $f(t)$. To determine y as positive solution of equation (3.4), we also assume that $a = a(T_B)$, $b = b(T_B)$ and $c = c(T_B)$. In particular, solving equation (3.4), where $T_B = T$, it is necessary to keep in mind that we are only interested in solutions y such that $\tau_1 = \lambda_2^{-1} \ln y < T$. In addition, roots of equation (3.4), i.e. $y_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, should be real. So, we have to impose the condition on the coefficients of equation (3.4)

$$b^2 - 4ac \geq 0. \tag{3.6}$$

Inequality (3.6) is a condition that determines the existence domains for solution τ_1 and, consequently, for the periodic solutions of the original system in its multidimensional parameter space. Remind that we are interested in the positive solution τ_1 satisfying the condition $0 < \tau_1 < T_B$. Therefore, if $\lambda_2 > 0$, then at least one of roots y_1 or y_2 should be greater than unity. If $\lambda_2 < 0$, then at least one of the same roots should be greater than zero and less than unity. In short, when condition (3.6) is valid, the following conditions should also hold:

$$\begin{gathered} \text{if } \lambda_2 > 0, \text{ then at least one of roots satisfies } y_i > 1, \\ \text{if } \lambda_2 < 0, \text{ then at least one of roots satisfies } 0 < y_i < 1. \end{gathered} \tag{3.7}$$

If root τ_1 is found, then by given T_B, one can find τ_2. This means that sufficient conditions (3.6) and (3.7) for parameters a, b and c of equation (3.4) and, consequently, for the parameters of the original system, guarantee the existence of a periodic mode (cyclic behavior). After substituting $T_B = nT$ and τ_1 in (2.5) or (2.6), we obtain uniquely the switching points of periodic solutions in the phase plane, namely, point X_1 that belongs to switching line $\sigma = l_1$ or respectively point X_2 that belongs to $\sigma = l_2$.

After replacing T_B by nT, the solution of equation (3.4) is associated with searching the periodic modes similar to the dominant-lock or subharmonic-lock ones.

Next we formulate the results obtained above as a theorem on the sufficient condition for the existence of periodic solutions to (1.1).

Theorem 3.1. *Let by a nonsingular transformation, the initial automatic control system be reduced to the form of system* (1.1)*, where matrix* $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$*, vector* $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$*,* $f(t) = \sin(\omega t + \varphi)$*, function* $F(\sigma)$ *describes relay hysteresis,* $\sigma = (\Gamma, X)$*,* $\Gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$ *and* $\gamma_1 = 0$*. Let conditions* (3.6),(3.7) *hold and equation* (3.4) *be solvable for* $\tau_1 > 0$*. Then the initial automatic control system has at least one* T_B*-periodic solution, where* $T_B = nT$*,* $n \in \mathbb{N}$*,* T *is the period of function* $f(t)$*.*

If the discriminant of equation (3.4) equals zero, then there exists one root y of (3.4). If either $y > 1$ for $\lambda_2 > 0$ or $y < 1$ for $\lambda_2 < 0$, then it means that there exists a unique solution τ_1 and, therefore, after substituting τ_1 in (2.5) and (2.6), we obtain switching points X_1 and X_2 of the periodic solution. Thus the following theorem holds.

Theorem 3.2. *Let the conditions of Theorem 3.1 be satisfied. Then the number of roots* y_i *of equation* (3.4) *determines the number of periodic solutions of* (1.1) *if conditions* (3.6) *and* (3.7) *hold. System* (1.1) *can not have more than two periodic solutions for* $d = 2$*.*

We can formulate a statement similar to Theorem 3.1 for the case when $\gamma_1 \neq 0$ and $\gamma_2 = 0$. Condition $\gamma_1 = 0$ (or $\gamma_2 = 0$) allows one to reduce the system of transcendental algebraic equations for searching τ_1 and τ_2, where $T_B = \tau_1 + \tau_2$ is given, to the simple quadratic equation that should have the roots satisfying condition (3.7). In the general case, it is impossible to obtain analytically the solution of (2.7) even for two-dimensional system (1.1). However, for $\gamma_1 = 0$ these equations permit one to set conditions on the existence of the periodic solutions describing the forced oscillations such that the frequency equals the frequency of the external influence or is $1/n$ part of this frequency.

4. Real nonzero multiple roots for $d = 2$

Let us consider the case when the roots of the characteristic equation are real nonzero multiple. Suppose that the initial automatic system is reduced to the system with matrix $A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$, vector $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ [2], and, as before, $f(t) = \sin(\omega t + \varphi)$. Let $\gamma_2 = 0$. Next we write down system (2.7). We get the matrix

$$\left(E - e^{AT_B}\right)^{-1} = \begin{pmatrix} \left(1 - e^{\lambda T_B}\right)^{-1} & 0 \\ te^{\lambda T_B}\left(1 - e^{\lambda T_B}\right)^{-2} & \left(1 - e^{\lambda T_B}\right)^{-1} \end{pmatrix}.$$

The first component of vector Q_1 has the form

$$\begin{aligned} q_1^1 = e^{\lambda(T_B+\tau_1)} &\int_{\tau_1}^{T_B} e^{-\lambda\tau}(m_1 + k\sin(\omega\tau + \varphi))d\tau \\ &+ e^{\lambda\tau_1}\int_0^{\tau_1} e^{-\lambda\tau}(m_2 + k\sin(\omega\tau + \varphi))d\tau, \end{aligned}$$

and its second component is defined as follows:

$$\begin{aligned} q_2^2 = e^{\lambda(T_B+\tau_1)} &\int_{\tau_1}^{T_B} \Big((T_B + \tau_1)e^{-\lambda\tau}(m_1 + k\sin(\omega\tau + \varphi)) \\ &- \tau e^{-\lambda\tau}(m_1 + k\sin(\omega\tau + \varphi))\Big)d\tau \\ &+ e^{\lambda\tau_1}\int_0^{\tau_1}\Big(\tau_1 e^{-\lambda\tau}(m_2 + k\sin(\omega\tau + \varphi)) - \tau e^{-\lambda\tau}(m_2 + k\sin(\omega\tau + \varphi))\Big)d\tau. \end{aligned}$$

This means that for $\gamma_2 = 0$ the first equation of (2.7) takes the form

$$l_1 = \gamma_1\left(1 - e^{\lambda T_B}\right)^{-1} q_1^1. \tag{4.1}$$

After the canonical transformation under the condition $T_B = nT$, $n \in \mathbb{N}$, equation (4.1) can be rewritten as

$$\begin{aligned} \frac{l_1}{\gamma_1}\left(1 - e^{\lambda T_B}\right) = \frac{1}{\lambda}(m_2 - m_1)e^{\lambda\tau_1} + \frac{m_1}{\lambda}e^{\lambda T_B} - \frac{m_2}{\lambda} + k\left(e^{\lambda T_B} - 1\right) \\ \times\Big(\frac{\lambda}{\lambda^2 + \omega^2}\sin(\omega\tau_1 + \varphi) + \frac{\omega}{\lambda^2 + \omega^2}\cos(\omega\tau_1 + \varphi)\Big). \end{aligned} \tag{4.2}$$

Note that equation (4.2) differs from equation (3.1) only the denotation of the eigenvalue, namely, λ_2 is replaced by λ. Now let us consider the second equation of (2.7). We have the first component of vector Q_2,

$$\begin{aligned} q_1^2 = e^{\lambda T_B} &\int_0^{\tau_1} e^{-\lambda\tau}(m_2 + k\sin(\omega\tau + \varphi))d\tau \\ &+ e^{\lambda T_B}\int_{\tau_1}^{T_B} e^{-\lambda\tau}(m_1 + k\sin(\omega\tau + \varphi))d\tau \end{aligned}$$

and the second component of vector Q_2,

$$\begin{aligned} q_2^2 = e^{\lambda T_B} \int_0^{\tau_1} \big(T_B e^{-\lambda\tau}(m_2 + k\sin(\omega\tau+\varphi)) - \tau e^{-\lambda\tau}(m_2 + k\sin(\omega\tau+\varphi))\big)\, d\tau \\ + e^{\lambda T_B} \int_{\tau_1}^{T_B} \big(T_B e^{-\lambda\tau}(m_1 + k\sin(\omega\tau+\varphi)) - \tau e^{-\lambda\tau}(m_1 + k\sin(\omega\tau+\varphi))\big)\, d\tau. \end{aligned}$$

Then taking into account the form of vector Γ, we obtain the second equation of (2.7) as follows:

$$\begin{aligned} &\frac{l_2}{\gamma_1}\left(1 - e^{\lambda T_B}\right) \\ &= \frac{1}{\lambda}(m_1 - m_2)e^{\lambda(T_B - \tau_1)} + \frac{m_2}{\lambda} e^{\lambda T_B} - \frac{m_1}{\lambda} \\ &\quad + k\left(1 - e^{\lambda(T_B-\tau_1)}\right)\left(\frac{\lambda}{\lambda^2+\omega^2}\sin(\omega\tau_1+\varphi) + \frac{\omega}{\lambda^2+\omega^2}\cos(\omega\tau_1+\varphi)\right) \\ &\quad + k\left(e^{\lambda T_B} - e^{\lambda(\tau_1 - T_B)}\right)\left(\frac{\lambda}{\lambda^2+\omega^2}\sin\varphi + \frac{\omega}{\lambda^2+\omega^2}\cos\varphi\right). \end{aligned} \tag{4.3}$$

Therefore, equation (4.3) differs from equation (3.2) by replacing λ_2 to λ. Then the equation of form (3.4) can be obtained from the system of transcendental equations (4.2), (4.3) if λ_2 is replaced by λ in formulas (3.5) for defining coefficients $a = a(T_B)$, $b = b(T_B)$, and $c = c(T_B)$. In this case, δ is replaced by $\arctan(\omega/\lambda)$ in the formulas for defining $a = a(T_B)$ and $b = b(T_B)$. It is also necessary to replace the root of the characteristic equation λ_2 by λ under (3.6) and (3.7). Then if (3.6) holds, we require the following:

$$\begin{aligned} &\text{if } \lambda > 0, \text{ then at least one of roots satisfies } y_i > 1, \\ &\text{if } \lambda < 0, \text{ then at least one of roots satisfies } y_i < 1. \end{aligned} \tag{4.4}$$

We now formulate an analogue of Theorem 3.1 on the sufficient condition for the existence of periodic solutions to (1.1).

Theorem 4.1. *Let by a nonsingular linear transformation, the initial automatic control system be reduced to the form*

$$\begin{gathered} \dot{x}_1 = \lambda x_1 + F(\sigma) + k\sin(\omega t + \varphi), \\ \dot{x}_2 = x_1 + \lambda x_2. \end{gathered}$$

Here function $F(\sigma)$ describes relay hysteresis, $\sigma = (\Gamma, X)$, where $\Gamma = \begin{pmatrix}\gamma_1 \\ \gamma_2\end{pmatrix}$ and $\gamma_2 = 0$. Let conditions (3.6), (4.4) hold and equation (3.4) be solvable for $\tau_1 > 0$. Then the initial automatic control system has at least one T_B-periodic solution, where $T_B = nT$, $n \in \mathbb{N}$, T is the period of function $f(t) = \sin(\omega t + \varphi)$.

Thus, in the case of the Jordan block, condition $\gamma_2 = 0$ makes it also possible to reduce the problem on existence of periodic solutions to the problem on resolvability of the algebraic equation obtained for the case of two distinct roots of the characteristic equation.

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