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ASYMPTOTIC BEHAVIOUR OF NONLINEAR WAVE EQUATIONS IN A NONCYLINDRICAL DOMAIN BECOMING UNBOUNDED

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Abstract. We study the asymptotic behaviour for the solution of nonlinear wave equations in a noncylindrical domain, becoming unbounded in some directions, as the time t goes to infinity. If the limit of the source term is independent of these directions and t, the wave converges to the solution of an elliptic problem defined on a lower dimensional domain. The rate of convergence depends on the limit behaviour of the source term and on the coefficient of the nonlinear term.

1. Introduction

In recent years, there is much interest in evolution problems set in time-dependent domains. These problems arise in many real world applications when the spatial domain of the considered phenomena depends strongly on time, see for instance the survey paper [14] and the references cited therein.

Let us denote the points in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ as

$$x = (X_1, X_2) = (x_1, \dots, x_{n_1}, x'_1, \dots, x'_{n_2}),$$

where n_1 and n_2 are positive integers. Then we consider a time-dependent family of bounded subsets in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ defined as

$$\Omega_t := (-\ell_0 - \ell t, \ell_0 + \ell t)^{n_1} \times \omega, \quad t \geq 0,$$

where ω is a bounded open subset of $\mathbb{R}^{n_2}$ with sufficiently smooth boundary, $\ell_0 > 0$ and the speed of expansion ℓ is constant. In $\mathbb{R}^+ \times \mathbb{R}^{n_1+n_2}$, we obtain the noncylindrical domain and its lateral boundary

$$Q_t := \cup_{0<s<t}\{s\} \times \Omega_s, \quad \Sigma_t := \cup_{0<s<t}\{s\} \times \partial\Omega_s, \quad t > 0.$$

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We are interested in the asymptotic behaviour, as $t \to +\infty$, of the solution of the following nonlinear wave equation set in Q_t,

$$\begin{gathered} u'' - \Delta u + \beta u' + \gamma(t)|u|^\rho u = f(t,x), \quad \text{in } Q_t, \\ u(t,x) = 0, \quad \text{on } \Sigma_t, \\ u(0,x) = u^0(x), \quad u'(0,x) = u^1(x), \quad \text{in } \Omega_0, \end{gathered} \tag{1.1}$$

where the prime stands for the time derivative, Δ is the Laplace operator, β is a positive constant and γ is a nonnegative function.

This study is motivated by some recent works on the asymptotic behaviour of the solutions of boundary value problems in a domain Ω_ℓ, when the size of Ω_ℓ becomes unbounded in some directions, as the parameter $\ell \to +\infty$ (independently of the time). See for instance [3, 4, 5, 11] for elliptic and parabolic problems and [2, 10] for hyperbolic problems. In the paper at hand, we give to ℓt the same role of the parameter ℓ in these papers.

The existence and uniqueness of solutions for wave problems in noncylindrical domains was considered by several authors, see [16, 17, 6, 7, 8, 9, 18] and related works. To focus on the asymptotic behaviour, we considered Problem (1.1) whose existence and uniqueness can be established by arguing as in [9].

Many works dealt with the asymptotic behaviour in time for the solutions of evolution problems in noncylindrical domains. Using the multiplier method, Bardos and Chen [1] proved that the energy of the linear wave equation decays when the domain is timelike and expanding. Nakao and Narazaki [18] and Rabello [19] studied the decay of the energy for weak solutions of nonlinear wave problems in expanding domains. There idea relays on the penalization method, introduced by Lions [16]. Another method consists in considering a suitable change of variables that transforms the noncylindrical domain to a cylindrical one, establish energy estimates for the new problem, then derive the desired energy estimates for the noncylindrical problem, see for instance [13, 15]. The drawback of this method is that the differential operator of the transformed problem is, in general, more complicated.

In this work, we study the problem directly in the noncylindrical domain, without any change of variables. The idea is based on the use of some special cut-off functions, depending on (t, X_1), to obtain local estimates of the difference between the wave and its limit. This technique was recently introduced by Guesmia [12] for a parabolic problem in a noncylindrical domain, see also [5]. Roughly speaking, if $f(t,x)$ converges to some $f_\infty(X_2)$ and $\gamma(t)$ converges to 0, faster enough in a sense to be made precise later, we obtain the convergence $u(t) \to u_\infty$ in interior regions of the domain Q_t. Here u_∞ is the solution of an elliptic problem defined on ω. Then, the rate convergence $u(t) \to u_\infty$ is analysed and improved under some assumptions.

The main features of this work can be summarized as follows:

• In [13, 18, 19], the size of the domain is assumed to remain bounded as $t \to +\infty$ and the limit of the solution of the considered problem is zero. This situation arises when the decay in the energy of the solution, due to the expansion of the domain and damping terms, overtakes the contribution of the source term. In this work, Ω_t becomes unbounded in n_1 directions and the limit of the solution, in interior regions of the domain, is not necessarily zero, as $t \to +\infty$. To the best of our knowledge, the asymptotic behaviour of such problems has not been considered before.

• In contrast with [12], the source term f in this work depends on all the variables $(t, x) \in \mathbb{R}^+ \times (-\ell_0 - \ell t, \ell_0 + \ell t)^{n_1} \times \omega$ and not only on $X_2 \in \omega$.

The rest of this article is organized as follows: In the next section, we state an existence and uniqueness result for $u(t)$, solution of Problem (1.1). Then we define u_∞, the candidate limit $u(t)$ as $t \to +\infty$, and the cut-off functions needed in the sequel. In section 3, we give an energy estimate for $u(t)$ as well as a local energy estimate for the difference $u(t) - u_\infty$. In the last section, we give the convergence results and discuss some particular cases where the rate of convergence is exponential.

2. Preliminaries

2.1. **Existence and uniqueness of solutions.** First, let us state our assumptions:

• Concerning the speed of expansion, in the n_1 first directions, it satisfies

$$0 \le \ell \le 1. \tag{2.1}$$

This ensures that Σ_t satisfies the so-called timelikness condition

$$|\nu_t| \le |\nu_x| \quad \text{on } \Sigma_t, \text{ for } t > 0,$$

where $\nu_1 = (\nu_t, \nu_x)$ is the unit outward normal to Σ_t and $|\cdot|$ denotes the usual Euclidian norm.

• The nonlinear term in Problem (1.1) is subject to the following assumptions (Recall that $x \in \mathbb{R}^{n_1+n_2}$)

$$0 < \rho \le \frac{2}{(n_1+n_2)-2}, \text{ if } n_1 + n_2 > 2, \quad 0 < \rho \le \infty \text{ if } n_1 = n_2 = 1, \tag{2.2}$$

$$\gamma \ge 0, \quad \gamma' \le 0, \quad \gamma, \gamma' \in L^\infty(0,t). \tag{2.3}$$

• The initial data and the source term satisfy

$$u^0 \in H_0^2(\Omega_0), \quad u^1 \in H_0^1(\Omega_0), \quad f \in H^1(0,t; L^2(\Omega_s)). \tag{2.4}$$

Then we have the following existence and uniqueness result.

Theorem 2.1. *Let $t > 0$. Under the assumptions eqreftlike–(2.4) there exists a unique solution for Problem (1.1), in the sense that*

$$u \in L^\infty(0,t; H_0^1(\Omega_s) \cap H^2(\Omega_s)), \quad u' \in L^\infty(0,t; H^1(\Omega_s)), \quad u'' \in L^2(0,t; L^2(\Omega_s))$$

and we can take u' as a test function, i.e. the following identity holds

$$\int_{\Omega_s} (u'' - \Delta u + \beta u' + \gamma(s)|u|^\rho u) u'(s) dx = \int_{\Omega_s} f(s) u'(s) dx,$$

for a.e. $s \in (0,t)$.

Proof. To express Ω_s using the notation of [9], we consider $K(s) = 1 + \frac{\ell}{\ell_0} s$. Then Ω_s can also defined as

$$\Omega_s = \{(X_1, X_2) \in \mathbb{R}^{n_1} \times \omega \mid X_1 = K(s) Y_1, Y_1 \in (-\ell_0, \ell_0)^{n_1}\}, \quad s \in (0,t).$$

The rest of the proof becomes similar to the proof of [9, Theorem 3.1], hence it is omitted. □

2.2. **Limit problem.** We set

$$\nabla_{X_1}u=(\partial_{x_1}u,\dots,\partial_{x_{n_1}}u)^T,\quad \nabla_{X_2}u=(\partial_{x'_1}u,\dots,\partial_{x'_{n_2}}u)^T,$$
$$\nabla u=\begin{pmatrix}\nabla_{X_1}u\\ \nabla_{X_2}u\end{pmatrix}),\quad \nabla_{x,t}u=\begin{pmatrix}u'\\ \nabla u\end{pmatrix}$$

and we assume that the source term becomes independent of the variables (t, X_1), i.e.

$$f(t,X_1,X_2)\to f_\infty(X_2),\quad \text{as } t\to+\infty,$$

for some

$$f_\infty\in L^2(\omega). \tag{2.5}$$

To handle the nonlinear term, in the estimations below, we need to assume that

$$\gamma(t)\to 0\quad \text{as } t\to+\infty.$$

The sense of these two convergences will be made precise below.

Passing formally to the limit in (1.1), one expects the limit problem to become independent of (t, X_1), as $t\to+\infty$. More precisely, the candidate limit of $u(t)$, as $t\to+\infty$, is the solution of the elliptic problem defined on ω,

$$\begin{gathered}-\Delta_{X_2}u_\infty=f_\infty\quad \text{in } \omega,\\ u_\infty=0\quad \text{on } \partial\omega,\end{gathered} \tag{2.6}$$

where $\Delta_{X_2}:=\partial^2_{x'_1}+\dots+\partial^2_{x'_{n_2}}$. It is well known that Problem (2.6) has a unique solution $u\in H^1_0(\omega)$ and one can check easily that

$$|\nabla_{X_2}u_\infty|_{L^2(\omega)}\le|f_\infty|_{L^2(\omega)}. \tag{2.7}$$

Remark 2.2. By the Sobolev embedding theorem (Recall that $\omega\subset\mathbb{R}^{n_2}$), we have:
- if $n_2\in\{1,2\}$, then $H^1(\omega)\subset L^{\rho+2}(\omega)$ for $0<\rho\le\infty$.
- if $n_2\ge 3$, then due to (2.2) we have $0<\rho\le\frac{2}{(n_1+n_2)-2}$ which implies that $0<\rho\le\frac{2}{n_2-2}$, hence $H^1(\omega)\subset L^{\rho+2}(\omega)$.

Therefore, under assumption (2.2), it holds that

$$|u_\infty|_{L^{\rho+2}(\omega)}\le C_S|\nabla u_\infty|_{L^2(\omega)},$$

for $n_2\ge 1$ and some constant C_S depending only on ω. Combining this inequality with (2.7) we have

$$|u_\infty|_{L^{\rho+2}(\omega)}\le C_S|f_\infty|_{L^2(\omega)}. \tag{2.8}$$

2.3. **Special cut-off functions.** To estimate the converge of $u(t)$ towards u_∞, we consider the functions

$$\begin{gathered}w(t,X_1,X_2):=u(t,X_1,X_2)-u_\infty(X_2),\\ F(t,X_1,X_2):=f(t,X_1,X_2)-f_\infty(X_2),\end{gathered}$$

for $(X_1,X_2)\in\Omega_t$ and $t\ge 0$. Since u_∞ depends only on X_2, then the function w satisfies the equation

$$w''-\Delta w+\beta w'+\gamma|u|^\rho u=F\quad \text{in } Q_t, \tag{2.9}$$

with the initial conditions

$$w(0,x)=u^0(x)-u_\infty(X_2),\quad w'(0,x)=u^1(x).$$

Observe that if $u_\infty \neq 0$ on Σ_t, then $w \neq 0$ on Σ_t. As a consequence $w(t) \notin H_0^1(\omega)$, hence it is not a valid test function for equation (2.9). This motivates the consideration of the next cut-off functions.

For a fixed $t > 1$, let m be a integer such that $0 \leq m \leq t-1$. On one hand, we consider the sequence of sets

$$S_m^t := \{(s, X_1) : t-m < s < t,\ |x_i| < \ell_0 + \ell(m-t+s),\ \text{for } i = 1, \dots, n_1\}.$$

This sequence is increasing in m, i.e. $S_m^t \subset S_{m+1}^t$, and satisfies

$$S_m^t \subset \cup_{t-m<s<t}\{s\} \times (-\ell_0 - \ell s, \ell_0 + \ell s)^{n_1} \subset (t-m, t) \times \mathbb{R}^{n_1}.$$

On the other hand, we consider a sequence of smooth cut-off functions, depending on (s, X_1),

$$\varrho_m = \varrho_m(s, X_1) : (0,t) \times \mathbb{R}^{n_1} \rightarrow \mathbb{R}$$

and satisfying

$$\begin{gathered}\varrho_m = \begin{cases} 1 & \text{in } S_m^t, \\ 0 & \text{in } \{(0,t) \times \mathbb{R}^{n_1}\} \setminus S_{m+1}^t, \end{cases} \\ 0 \leq \varrho_m \leq 1, \quad |\nabla_{X_1} \varrho_m|, |\varrho_m'| \leq \theta, \end{gathered}$$

where θ is a constant independent of t and m. We have in particular $\varrho_m(0, X_1) = 0$ and $\varrho_m = 0$ near the lateral boundary Σ_t. The supports of $\nabla_{X_1} \varrho_m$ and ϱ_m' are included in $S_{m+1}^t \setminus S_m^t$.

3. Energy Estimates

In this section, we establish some useful lemmas needed in the sequel. The first one gives an estimation for u and its derivatives.

Lemma 3.1. *Under the assumptions* (2.1)–(2.4), *the solution of Problem* (1.1) *satisfies,*

$$\begin{aligned} &\int_{\Omega_t} |u'(t)|^2 + |\nabla u(t)|^2 + \frac{\gamma(t)}{\rho+2}|u(t)|^{\rho+2} dx + \int_{Q_t} \beta |u'|^2 + \frac{2|\gamma'|}{\rho+2}|u|^{\rho+2}\, dx\, ds \\ &\leq C_0\Big(1 + |f|^2_{L^2(Q_t)}\Big), \quad \text{for } t > 0, \end{aligned}$$

where C_0 is a positive constant independent of t.

Proof. Since the solutions u satisfies $u = 0$ on Σ_t, then all the tangential derivatives of u are also vanishing on Σ_t, so $\nabla_{x,t} u = \frac{\partial u}{\partial \nu}\ \nu$, on Σ_t, which implies that

$$u' = \frac{\partial u}{\partial \nu}\nu_t, \quad \nabla u = \frac{\partial u}{\partial \nu}\nu_x, \quad \text{on } \Sigma_t.$$

Thanks to Theorem 2.1, we can take u' as a test function and arguing as in [1], we obtain

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_t} |u'(t)|^2 + |\nabla u(t)|^2 + \frac{\gamma(t)}{\rho+2}|u(t)|^{\rho+2} dx + \int_{Q_t} \beta |u'|^2 - \frac{\gamma'}{\rho+2}|u|^{\rho+2}\, dx\, ds \\ &= \frac{1}{2}\int_{\Omega_0} |u^1|^2 + |\nabla u^0|^2 + \frac{\gamma(0)}{\rho+2}|u^0|^{\rho+2} dx + \int_{Q_t} f u'\, dx\, ds \\ &\quad + \frac{1}{2}\int_{\Sigma_t} (\frac{\partial u}{\partial \nu})^2\ \nu_t\ (|\nu_x|^2 - \nu_t^2)\ d\sigma, \end{aligned}$$

for $t > 0$. Using the fact that $|\nu_t| \leq |\nu_x|$ on Σ_t and noting that $\nu_t \leq 0$ for expanding domains, we infer that the boundary integral term in the right-hand side is nonpositive. Then applying Young's inequality $fu' \leq \frac{\beta}{2}(u')^2 + \frac{1}{2\beta}f^2$, we obtain

$$\begin{aligned}&\int_{\Omega_t} |u'(t)|^2 + |\nabla u(t)|^2 + \frac{\gamma(t)}{\rho+2}|u(t)|^{\rho+2}dx + \int_{Q_t} \beta|u'|^2 + \frac{2|\gamma'|}{\rho+2}|u|^{\rho+2}\,dx\,ds\\ &\leq \int_{\Omega_0} |u^1|^2 + |\nabla u^0|^2 + \frac{\gamma(0)}{\rho+2}|u^0|^{\rho+2}dx + \frac{1}{\beta}\int_{Q_t} f^2\,dx\,ds.\end{aligned}$$

This completes the proof. □

The second lemma, gives an estimation for the difference $u(t) - u_\infty$ in interior regions of Ω_t and Q_t. For simplicity, we set

$$D(t,x) := |w'(t,x)|^2 + |\nabla w(t,x)|^2 + \gamma(t)|u(t,x)|^{\rho+2}, \quad \text{for } x \in \Omega_t,\ t \geq 0. \tag{3.1}$$

Then we have the following energy inequality.

Lemma 3.2. *Under assumptions* (2.1)–(2.5)*, the solutions of Problem* (1.1) *and Problem* (2.6) *satisfy*

$$\begin{aligned}&\int_{\Omega_t} D(t)\varrho_m^2(t)dx + \int_{S_m^t\times\omega} D\ \ dx\,ds\\ &\leq C_1\int_{(S_{m+1}^t\setminus S_m^t)\times\omega} D\,dx\,ds + C_1\int_{S_{m+1}^t\times\omega} F^2 + \gamma|u_\infty|^{\rho+2}\,dx\,ds, \quad \text{for a.e. } t > 0,\end{aligned}$$

for some positive constant C_1 independent of t.

Proof. To derive local energy estimates, we use ϱ_m and its proprieties.

- *A local energy identity.* Let us multiply (2.9) by $2w\varrho_m^2$, it yields

$$\begin{aligned}&\frac{\partial}{\partial s}(\beta\varrho_m^2 w^2 + 2\varrho_m^2 ww') - 2\beta\varrho_m'\varrho_m w^2 - 2\varrho_m^2|w'|^2 - 4\varrho_m'\varrho_m ww' + 2\gamma|u|^\rho uw\varrho_m^2\\ &+ 2\varrho_m^2|\nabla w|^2 - 2\nabla\cdot(\varrho_m^2 w\nabla w) + 4\varrho_m w(\nabla\varrho_m\cdot\nabla w) = 2w\varrho_m^2 F.\end{aligned}$$

Then, multiplying (2.9) by $2\alpha w'\varrho_m^2$, for some constant $\alpha > 0$, yields

$$\begin{aligned}&\frac{\partial}{\partial s}\Big(\alpha\varrho_m^2|w'|^2 + \alpha\varrho_m^2|\nabla w|^2 + \frac{2\alpha\gamma}{\rho+2}|u|^{\rho+2}\varrho_m^2\Big)\\ &- 2\alpha\varrho_m'\varrho_m|w'|^2 + 2\alpha\beta\varrho_m^2|w'|^2 - \frac{2\alpha\gamma'}{\rho+2}|u|^{\rho+2}\varrho_m^2 - \frac{4\alpha\gamma}{\rho+2}|u|^{\rho+2}\varrho_m'\varrho_m\\ &- 2\alpha\varrho_m'\varrho_m|\nabla w|^2 - 2\alpha\nabla\cdot(\varrho_m^2 w'\nabla w) + 4\alpha\varrho_m w'(\nabla\varrho_m\cdot\nabla w) = 2\alpha w'\varrho_m^2 F.\end{aligned}$$

Summing the above identities, we obtain

$$\begin{aligned}&\frac{\partial}{\partial s}\Big(\beta\varrho_m^2 w^2 + 2\varrho_m^2 ww' + \alpha\varrho_m^2|w'|^2 + \alpha\varrho_m^2|\nabla w|^2 + \frac{2\alpha\gamma}{\rho+2}|u|^{\rho+2}\varrho_m^2\Big)\\ &- 2\varrho_m^2|w'|^2 + 2\alpha\beta\varrho_m^2|w'|^2 + 2\varrho_m^2|\nabla w|^2 - 2\alpha\varrho_m'\varrho_m|\nabla w|^2\\ &+ 2\gamma|u|^{\rho+2}\varrho_m^2 - 2\gamma|u|^\rho uu_\infty\varrho_m^2 - \frac{2\alpha\gamma'}{\rho+2}|u|^{\rho+2}\varrho_m^2 - \frac{4\alpha\gamma}{\rho+2}|u|^{\rho+2}\varrho_m'\varrho_m\\ &- 2\beta\varrho_m'\varrho_m w^2 - 4\varrho_m'\varrho_m ww' - 2\alpha\varrho_m'\varrho_m|w'|^2 - 2\nabla\cdot(\varrho_m^2 w\nabla w) + 4\varrho_m w(\nabla\varrho_m\cdot\nabla w)\\ &- 2\alpha\nabla\cdot(\varrho_m^2 w'\nabla w) + 4\alpha\varrho_m w'(\nabla\varrho_m\cdot\nabla w) = 2w\varrho_m^2 F + 2\alpha w'\varrho_m^2 F.\end{aligned}$$

Collecting the terms with derivatives of ϱ in the right-hand side of the above identity, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial s}\Big(\beta\varrho_m^2 w^2 + 2\varrho_m^2 ww' + \alpha\varrho_m^2|w'|^2 + \alpha\varrho_m^2|\nabla w|^2 + \frac{2\alpha\gamma}{\rho+2}|u|^{\rho+2}\varrho_m^2\Big)\\
&2(\alpha\beta-1)\varrho_m^2|w'|^2 + 2\varrho_m^2|\nabla w|^2 + 2\big(\gamma - \frac{\alpha\gamma'}{\rho+2}\big)|u|^{\rho+2}\varrho_m^2\\
&= 2\beta\varrho_m'\varrho_m w^2 + 4\varrho_m'\varrho_m ww' + 2\alpha\varrho_m'\varrho_m|w'|^2 + 2\alpha\varrho_m'\varrho_m|\nabla w|^2\\
&\quad - 4\varrho_m w(\nabla\varrho_m\cdot\nabla w) - 4\alpha\varrho_m w'(\nabla\varrho_m\cdot\nabla w) + 2\alpha\nabla\cdot(\varrho_m^2 w'\nabla w)\\
&\quad + 2\nabla\cdot(\varrho_m^2 w\nabla w) + \frac{4\alpha}{\rho+2}|u|^{\rho+2}\gamma\varrho_m'\varrho_m + 2\gamma(|u|^\rho u)u_\infty\varrho_m^2\\
&\quad + 2w\varrho_m^2F + 2\alpha w'\varrho_m^2F.
\end{aligned}$$

Integrating on Q_t and taking into account the fact that $\varrho_m = 0$ for $t = 0$ and on Σ_t, we end up with the identity

$$\begin{aligned}
&\int_{\Omega_t}\Big(\beta w^2(t) + 2ww'(t) + \alpha|w'(t)|^2 + |\nabla w(t)|^2 + \frac{2\alpha\gamma(t)}{\rho+2}|u(t)|^{\rho+2}\Big)\varrho_m^2(t)dx\\
&+ \int_{Q_t} 2(\alpha\beta-1)\varrho_m^2|w'|^2 + 2\varrho_m^2|\nabla w|^2 + 2\big(\gamma - \frac{\alpha\gamma'}{\rho+2}\big)|u|^{\rho+2}\varrho_m^2\,dx\,ds\\
&= \int_{Q_t} 2\beta\varrho_m'\varrho_m w^2 + 4\varrho_m'\varrho_m ww' + 2\alpha\varrho_m'\varrho_m|w'|^2 + 2\alpha\varrho_m'\varrho_m|\nabla w|^2\\
&\quad + \frac{4\alpha\gamma}{\rho+2}|u|^{\rho+2}\varrho_m'\varrho_m\,dx\,ds - \int_{Q_t} 4\varrho_m w(\nabla\varrho_m\cdot\nabla w) - 4\alpha\varrho_m w'(\nabla\varrho_m\cdot\nabla w)\,dx\,ds\\
&\quad + \int_{Q_t} 2\gamma(|u|^\rho u)u_\infty\varrho_m^2\,dx\,ds + \int_{Q_t} 2w\varrho_m^2F + 2\alpha w'\varrho_m^2F\,dx\,ds.
\end{aligned}$$

- *Estimate for the left-hand side of* (3.2). Using the inequality

$$2ww' \ge -\Big(\beta w^2 + \frac{1}{\beta}|w'|^2\Big),$$

then choosing $\alpha > 1/\beta$, we obtain

$$\beta\varrho_m^2 w^2 + 2\varrho_m^2 ww' + \alpha\varrho_m^2|w'|^2 + \alpha\varrho_m^2|\nabla w|^2 \ge \delta_0\varrho_m^2|w'|^2 + \alpha\varrho_m^2|\nabla w|^2,$$

where $\delta_0 = (\alpha - \frac{1}{\beta}) > 0$. Integrating on Q_t , and taking into account that $\gamma' \le 0$, we deduce that the left-hand side of (3.2) is bounded below by

$$\begin{aligned}
&\int_{\Omega_t}\Big(\delta_0|w'(t)|^2 + \alpha|\nabla w(t)|^2 + \frac{2\alpha\gamma(t)}{\rho+2}|u(t)|^{\rho+2}\Big)\varrho_m^2(t)dx\\
&+ 2\int_{Q_t}\Big(\beta\delta_0|w'|^2 + |\nabla w|^2 + \big(\gamma + \frac{\alpha|\gamma'|}{\rho+2}\big)|u|^{\rho+2}\Big)\varrho_m^2\,dx\,ds.
\end{aligned}$$

- *Estimate for the right-hand side of* (3.2). Given that the supports of ϱ_m' and $|\nabla\varrho_m|$ are included in the set $S_{m+1}^t\backslash S_m^t$, the right-hand side of (3.2) can be estimated above by

$$\begin{aligned}
&c_0\int_{(S_{m+1}^t\backslash S_m^t)\times\omega} |w'|^2 + |w|^2 + |\nabla w|^2 + \gamma|u|^{\rho+2}\,dx\,ds\\
&+ \int_{Q_t} 2\gamma(|u|^\rho u)u_\infty\varrho_m^2\,dx\,ds + \int_{Q_t} 2w\varrho_m^2F + 2\alpha w'\varrho_m^2F\,dx\,ds.
\end{aligned}$$

Here and in the sequel, c_i denotes positive constants depending (at most) on θ, α and ω, but not on t. To estimate the second integral, containing $(|u|^\rho u)u_\infty$, we apply Young's inequality $ab \le \frac{\varepsilon a^p}{p} + \frac{1}{\varepsilon^{q/p}}\frac{b^q}{q}$ for $p = \frac{\rho+2}{\rho+1}$, $q = \rho+2$ and $\varepsilon \in (0,1)$. We obtain

$$(|u|^\rho u)u_\infty \le \frac{(\rho+1)\varepsilon}{\rho+2}|u|^{\rho+2} + \frac{1}{(\rho+2)\varepsilon^{(\rho+1)}}|u_\infty|^{\rho+2}.$$

The same inequality, for $p = q = 2$, yields

$$\begin{gathered} 2w\varrho_m^2 F + 2\alpha w' F \le \varepsilon(w^2 + |w'|^2) + \frac{1+\alpha^2}{\varepsilon}F^2, \\ 2ww' \le w^2 + |w'|^2, \\ 2w|\nabla w| \le w^2 + |\nabla w|^2. \end{gathered}$$

Then, the right-hand side of (3.2) is bounded above by

$$\begin{aligned} &c_0 \int_{(S_{m+1}^t \setminus S_m^t)\times\omega} |w'|^2 + |w|^2 + |\nabla w|^2 + \gamma|u|^{\rho+2}\,dx\,ds \\ &+ c_1\varepsilon \int_{Q_t} (|w'|^2 + |w|^2 + \gamma|u|^{\rho+2})\varrho_m^2\,dx\,ds + \frac{c_1}{\varepsilon^{(\rho+1)}}\int_{Q_t}(F^2 + \gamma|u_\infty|^{\rho+2})\varrho_m^2\,dx\,ds. \end{aligned}$$

Since ω is bounded, then Poincaré's inequality in the X_2-direction yields

$$\int_{\Omega_t} |w(t)|^2\varrho_m^2(t)dx \le c_\omega^2 \int_{\Omega_t} |\nabla_{X_2} w(t)|^2 \varrho_m^2(t)dx \le c_\omega^2 \int_{\Omega_t} |\nabla w(t)|^2\varrho_m^2(t)dx,$$

where c_ω is the Poincaré constant. Thus the right-hand side of (3.2) is bounded above by

$$\begin{aligned} &c_2 \int_{(S_{m+1}^t \setminus S_m^t)\times\omega} |w'|^2 + |\nabla w|^2 + \gamma|u|^{\rho+2}\,dx\,ds \\ &+ c_2\varepsilon \int_{Q_t} (|w'|^2 + |\nabla w|^2 + \gamma|u|^{\rho+2})\varrho_m^2\,dx\,ds + \frac{c_2}{\varepsilon^{(\rho+1)}}\int_{Q_t}(F^2 + \gamma|u_\infty|^{\rho+2})\varrho_m^2\,dx\,ds. \end{aligned}$$

- *End of proof.* The estimations of the two sides of (3.2) yields

$$\begin{aligned} &\int_{\Omega_t} \Big(\delta_0|w'(t)|^2 + \alpha|\nabla w(t)|^2 + \frac{2\alpha\gamma(t)}{\rho+2}|u(t)|^{\rho+2}\Big)\varrho_m^2(t)dx \\ &+ 2\int_{Q_t}\Big(\beta\delta_0|w'|^2 + |\nabla w|^2 + \big(\gamma + \frac{\alpha|\gamma'|}{\rho+2}\big)|u|^{\rho+2}\Big)\varrho_m^2\,dx\,ds \\ &\le c_2 \int_{(S_{m+1}^t \setminus S_m^t)\times\omega} |w'|^2 + |\nabla w|^2 + \gamma|u|^{\rho+2}\,dx\,ds \\ &\quad + c_2\varepsilon\int_{Q_t}\big(|w'|^2 + |\nabla w|^2 + \gamma|u|^{\rho+2}\big)\varrho_m^2\,dx\,ds \\ &\quad + \frac{c_2}{\varepsilon^{(\rho+1)}}\int_{Q_t}\big(F^2 + \gamma|u_\infty|^{\rho+2}\big)\varrho_m^2\,dx\,ds. \end{aligned}$$

For ε small enough, we end up with

$$\begin{aligned} &\int_{\Omega_t}\big(|w'(t)|^2 + |\nabla w(t)|^2 + \gamma(t)|u(t)|^{\rho+2}\big)\varrho_m^2(t)dx \\ &+ \int_{Q_t}(|w'|^2 + |\nabla w|^2 + \gamma|u|^{\rho+2})\varrho_m^2\,dx\,ds \end{aligned}$$

$$\le c_3 \int_{(S^t_{m+1}\setminus S^t_m)\times\omega} |w'|^2 + |\nabla w|^2 + \gamma|u|^{\rho+2}\,dx\,ds$$
$$+ c_3 \int_{Q_t} (F^2 + \gamma|u_\infty|^{\rho+2})\varrho_m^2\,dx\,ds.$$

This completes the proof. □

Remark 3.3. Thanks to Inequality (2.8), we obtain

$$\int_{S^t_{m+1}\times\omega} \gamma|u_\infty|^{\rho+2}\varrho_m^2\,dx\,ds = |u_\infty|^{\rho+2}_{L^{\rho+2}(\omega)} \int_{S^t_{m+1}} \gamma\varrho_m^2 dX_1 ds$$
$$\le C_S^{\rho+2}|f_\infty|^{\rho+2}_{L^2(\omega)} \int_{S^t_{m+1}} \gamma\varrho_m^2 dX_1 ds$$

and since $0 \le \varrho_m \le 1$, we obtain

$$\int_{Q_t} \gamma|u_\infty|^{\rho+2}\varrho_m^2\ \ dx\,ds \le C_S^{\rho+2}|f_\infty|^{\rho+2}_{L^2(\omega)} 2^{n_1}(\ell_0+\ell t)^{n_1} \int_{t-m-1}^t \gamma(s)ds.$$

Thus

$$\int_{S^t_{m+1}\times\omega} \gamma|u_\infty|^{\rho+2}\varrho_m^2\,dx\,ds \le C_2(\ell_0+\ell t)^{n_1} \int_{t-m-1}^t \gamma(s)ds \tag{3.2}$$

where C_2 is a constant independent of t and m.

4. Main Results

In this section, we establish the convergence $u(t) \to u_\infty$, in bounded interior region of Ω_t and Q_t, under some assumptions involving the asymptotic behaviour of f and γ as $t \to +\infty$.

4.1. **Convergence theorems.** Let us consider the nonnegative real function

$$g_0(t) := \sum_{j=1}^{[t]-1} (k^j \int_{S^t_{j+1}\times\omega} |f-f_\infty|^2 + \gamma|u_\infty|^{\rho+2}\,dx\,ds), \quad t \ge 2, \tag{4.1}$$

where $[\cdot]$ denotes the integer part and $k := C_1/(1+C_1)$, ($C_1 > 0$is the constant considered in Lemma 3.2). Then, we have the following convergence on $S_1^t \times \omega$.

Theorem 4.1. *Assume* (2.1)–(2.5) *and*

$$g_0(t) \to 0, \quad \text{as } t \to +\infty, \tag{4.2}$$
$$t|f|^2_{L^2(Q_t)} = o(e^{\mu_0 t}), \quad \text{as } t \to +\infty \tag{4.3}$$

where $\mu_0 := \ln(1+\frac{1}{C_1})$. *Then we have*

$$u' \to 0, \quad \nabla_{X_1}u \to 0, \quad \nabla_{X_2}u \to \nabla_{X_2}u_\infty \quad \text{in } L^2(S_1^t\times\omega), \tag{4.4}$$
$$\gamma^{\frac{1}{\rho+2}}u \to 0 \quad \text{in } L^{\rho+2}(S_1^t\times\omega), \tag{4.5}$$

as $t \to +\infty$. *Moreover, if* $f = f_\infty$ *and* $\gamma = 0$, *the above convergences are exponential.*

Proof. The main idea is an iteration technique on the increasing sequence of sets $S^t_m \times \omega$. First, we observe that

$$\int_{(S^t_{m+1}\setminus S^t_m)\times\omega} D\,dx\,ds = \int_{S^t_{m+1}\times\omega} D\ \,dx\,ds - \int_{S^t_m\times\omega} D\ \,dx\,ds$$

and therefore Lemma 3.2 yields in particular

$$(1+C_1)\int_{S^t_m\times\omega} D\ \,dx\,ds \le C_1\int_{S^t_{m+1}\times\omega} D\,dx\,ds + C_1\int_{S^t_{m+1}\times\omega} F^2+\gamma|u_\infty|^{\rho+2}\,dx\,ds.$$

Since $k=\dfrac{C_1}{1+C_1}$, then $0<k<1$ and we can rewrite the precedent inequality as

$$\int_{S^t_m\times\omega} D\ \,dx\,ds \le k\int_{S^t_{m+1}\times\omega} D\,dx\,ds + k\int_{S^t_{m+1}\times\omega} F^2+\gamma|u_\infty|^{\rho+2}\,dx\,ds. \tag{4.6}$$

This is an inequality that we can iterate for $m=1,\dots,[t]-1$. It follows that

$$\begin{aligned}
\int_{S^t_1\times\omega} D\ \,dx\,ds &\le k\int_{S^t_2\times\omega} D\ \,dx\,ds + k\int_{S^t_2\times\omega}\big(F^2+\gamma|u_\infty|^{\rho+2}\big)\,dx\,ds\\
&\le k^2\int_{S^t_3\times\omega} D\ \,dx\,ds + \sum_{j=1}^{2}\Big(k^j\int_{S^t_{1+j}\times\omega} F^2+\gamma|u_\infty|^{\rho+2}\,dx\,ds\Big)\\
&\dots\\
&\le k^{[t]-1}\int_{S^t_{[t]}\times\omega} D\ \,dx\,ds + \sum_{j=1}^{[t]-1}\Big(k^j\int_{S^t_{1+j}\times\omega} F^2+\gamma|u_\infty|^{\rho+2}\,dx\,ds\Big).
\end{aligned}$$

Note that $t-1<[t]\le t$ and $\mu_0=-\ln k>0$. Then $k^{[t]-1}=e^{([t]-1)\ln k}=e^{-\mu_0([t]-1)}$ and it follows that

$$\begin{aligned}
&\int_{S^t_1\times\omega} D\ \,dx\,ds\\
&\le c_5e^{-\mu_0 t}\int_{S^t_{[t]}\times\omega} D\,dx\,ds + \sum_{j=1}^{[t]-1}\Big(k^j\int_{S^t_{1+j}\times\omega} F^2+\gamma|u_\infty|^{\rho+2}\,dx\,ds\Big).
\end{aligned} \tag{4.7}$$

To estimate the first integral term in the right-hand side of (4.7), we write

$$\begin{aligned}
\int_{S^t_{[t]}\times\omega} D\,dx\,ds &\le \int_{Q_t} D\,dx\,ds\\
&\le \int_{Q_t} |u'|^2+|\nabla u|^2+|\nabla_{X_2}u_\infty|^2+\gamma|u|^{\rho+2}\,dx\,ds\\
&\le \int_{Q_t} |u'|^2+|\nabla u|^2+\gamma|u|^{\rho+2}\,dx\,ds\\
&\quad + |\nabla_{X_2}u_\infty|^2_{L^2(\omega)}\int_0^t\Big(\int_{(-\ell_0-\ell s,\ell_0+\ell s)^{n_1}} dX_1\Big)ds.
\end{aligned}$$

Taking into account Lemma 3.1 and (2.7), it follows that

$$\begin{aligned}
\int_{S^t_{[t]}\times\omega} D\ \,dx\,ds &\le c_6t(1+|f|^2_{L^2(Q_t)}) + \frac{2^{n_1}}{\ell(n_1+1)}|f_\infty|^2_{L^2(\omega)}(\ell_0+\ell t)^{n_1+1}\\
&\le c_7(t^{n_1+1}|f_\infty|^2_{L^2(\omega)}+t|f|^2_{L^2(Q_t)})
\end{aligned}$$

for large t. Substituting this in (4.7) and expending the expression of $D(t,x)$, we obtain

$$\begin{aligned} &\int_{S_1^t\times\omega} |u'|^2 + |\nabla_{X_1} u|^2 + |\nabla_{X_2}(u-u_\infty)|^2 + \gamma|u|^{\rho+2}\,dx\,ds \\ &\le c_8\big(t^{n_1+1}|f_\infty|^2_{L^2(\omega)} + t|f|^2_{L^2(Q_t)}\big)e^{-\mu_0 t} + g_0(t) \end{aligned} \tag{4.8}$$

where g_0 is the function given by (4.1). Since (4.2) and (4.3) ensure that the left-hand side of (4.8) tends to zero, as $t\to+\infty$, then the convergences (4.4) and (4.5) follow.

If $f=f_\infty$ and $\gamma=0$ then $g_0=0$ and $|f|^2_{L^2(Q_t)}$ grows polynomially in time, hence the claimed exponential convergences are a consequence of (4.8). This completes the proof. □

Remark 4.2. (i) The source term f satisfies (4.3) for example when $|f|_{L^2(\Omega_t)}$ is bounded or grows polynomially in time.

(ii) The function g_0 satisfies (4.2) if the convergences $f(t)\to f_\infty$, $\gamma(t)\to 0$, as $t\to+\infty$, are strong enough. Some examples are given below.

(iii) If $f_\infty=0$, and by consequence $u_\infty=0$, then g_0 does not depend on γ. In this case, Theorem 4.1 holds without any convergence assumption of $\gamma(t)$ towards 0.

The next corollary gives the convergence on the domain Ω_1.

Corollary 4.3. *Under assumptions* (2.1)–(2.5)*,* (4.2) *and* (4.3)*, we have*

$$u'(t)\to 0,\quad \nabla_{X_1}u(t)\to 0,\quad \nabla_{X_2}u(t)\to\nabla_{X_2}u_\infty \quad \text{in } L^2(\Omega_1),$$
$$\gamma(t)^{\frac{1}{\rho+2}}u(t)\to 0\quad \text{in } L^{\rho+2}(\Omega_1),$$

as $t\to+\infty$. Moreover, if $f=f_\infty$ and $\gamma=0$, the above convergences are exponential.

Proof. Using Lemma 3.2, we have in particular for $m=1$,

$$\begin{aligned} \int_{\Omega_1} D(t)dx &\le \int_{\Omega_t} D(t)\varrho_1^2(t)dx \\ &\le C_1\int_{S_2^t\times\omega} D\,dx\,ds + C_1\int_{S_2^t\times\omega} F^2+\gamma|u_\infty|^{\rho+2}\,dx\,ds. \end{aligned}$$

Then we can estimate the integral $\int_{S_2^t\times\omega} D\,dx\,ds$ by using the above iteration technique for $m=2,\dots,[t]-1$. Arguing as in the proof of Theorem 4.1, we end up with

$$\int_{\Omega_1} D(t)dx \le c_9(t^{n_1+1}|f_\infty|^2_{L^2(\omega)} + t|f|^2_{L^2(Q_t)})e^{-\mu_0 t} + g_0(t).$$

Hence the corollary follows. □

4.2. **Convergence in arbitrary interior regions.** The assumptions (4.2) and (4.3) can be considerably weakened to involve only the asymptotic behaviours of f and γ for large t. Moreover, we show that the above convergences hold for an arbitrary interior bounded region of Ω_t and Q_t.

Let O be a bounded subset of $\mathbb{R}^{n_1}\times\omega$ and a be a positive constant. Since Ω_t is increasing in time and becomes unbounded in the X_1 direction, as $t\to+\infty$, then there exists $m_0>a$ such that

$$(t-a,t)\times O \Subset (t-m_0,t)\times\Omega_{m_0}, \tag{4.9}$$

and we can check that

$$(t-m_0,t)\times\Omega_{m_0}\Subset S^t_{2m_0}\times\omega,\quad \text{for } t>2m_0.$$

Let us consider the function

$$g_{m_0}(t):=\sum_{j=2m_0+1}^{[t/2]}\Big(k^j\int_{S^t_{1+j}\times\omega}|f-f_\infty|^2+\gamma|u_\infty|^{\rho+2}\,dx\,ds\Big). \tag{4.10}$$

Then, we have the following convergences on $(t-a,t)\times O$.

Theorem 4.4. *Under the assumptions* (2.1)–(2.5) *and*

$$g_{m_0}(t)\to 0 \text{ and } t|f|^2_{L^2(Q_t)}=o(e^{\frac{\mu_0}{2}t}),\quad \text{as } t\to+\infty, \tag{4.11}$$

we have

$$u'\to 0,\quad \nabla_{X_1}u\to 0,\quad \nabla_{X_2}u\to\nabla_{X_2}u_\infty\quad \text{in } L^2((t-a,t)\times O),$$
$$\gamma^{\frac{1}{\rho+2}}u\to 0\quad \text{in } L^{\rho+2}((t-a,t)\times O),$$

as $t\to+\infty$. Moreover, if $f=f_\infty$ and $\gamma=0$, the above convergences are exponential.

Proof. Let us take $t>4m_0+2$, i.e., $[t/2]>2m_0$. Since $(t-a,t)\times O\subset\subset S^t_{2m_0}\times\omega$, then iterating Inequality (4.6) for $m=2m_0,\dots,[t/2]-1$, we obtain

$$\begin{aligned}
&\int_{(t-a,t)\times O}D\ \,dx\,ds\\
&\le\int_{S^t_{2m_0}\times\omega}D\ \,dx\,ds\\
&\le k^{[t/2]-2m_0}\int_{S^t_{[\frac{t}{2}]}\times\omega}D\ \,dx\,ds+\sum_{j=2m_0+1}^{[\frac{t}{2}]}(k^{j-2m_0}\int_{S^t_j\times\omega}F^2+\gamma|u_\infty|^{\rho+2}\,dx\,ds)
\end{aligned}$$

hence

$$\int_{(t-a,t)\times O}D\,dx\,ds\le c_{10}\Big((t^{n_1+1}|f_\infty|^2_{L^2(\omega)}+t|f|^2_{L^2(Q_t)})e^{-\frac{\mu_0}{2}t}+g_{m_0}(t)\Big) \tag{4.12}$$

where $c_{10}>0$ and g_{m_0} is defined by (4.10). Under the assumption (4.11), the right-hand side tends to zero, as $t\to+\infty$, and the theorem follows. □

Remark 4.5. In contrast with g_0 defined in (4.1), by a sum that involves the values of $f-f_\infty$ and γ on $S^t_{[t]}\times\omega$ (which is identical to Q_t if t is an integer), the function g_{m_0} involves only the values of $f-f_\infty$ and γ on $S^t_{[t/2]+1}\times\omega$, included in the strip $(\frac{t}{2}-1,t)\times\mathbb{R}^{n_1}\times\omega$.

Corollary 4.6. *Under the assumptions* (2.1)–(2.5) *and* (4.11)*, we have*

$$u'(t)\to 0,\quad \nabla_{X_1}u(t)\to 0,\quad \nabla_{X_2}u(t)\to\nabla_{X_2}u_\infty\quad \text{in } L^2(O),$$
$$\gamma^{\frac{1}{\rho+2}}u(t)\to 0\quad \text{in } L^{\rho+2}(O),$$

as $t\to+\infty$. Moreover, if $f=f_\infty$ and $\gamma=0$, the above convergences are exponential.

Proof. Using Lemma 3.2, we have for $m = 2m_0$ and $t > 2m_0 + 1$

$$\begin{aligned}\int_O D(t)dx &\le \int_{\Omega_t} D(t)\varrho_{2m_0}(t)dx\\ &\le C_1 \int_{S^t_{2m_0+1}\times\omega} D\,dx\,ds + C_1 \int_{S^t_{2m_0+1}\times\omega} F^2 + \gamma|u_\infty|^{\rho+2}\,dx\,ds.\end{aligned}$$

The integral $\int_{S^t_{2m_0+1}\times\omega} D\,dx\,ds$ in the right-hand side can be estimated as above by iteration for $m = 2m_0 + 1, \dots, [t/2] - 1$. The rest of the proof is similar to the proof of Theorem 4.1 and hence is omitted. □

4.3. **Exponential convergence.** We give now some assumptions on the asymptotic behaviour of γ and f for large t, other than the trivial case $f = f_\infty$ and $\gamma = 0$, that ensure an exponential rate of convergences.

Theorem 4.7. *Assume* (2.1)–(2.5), *and that*

$$\gamma(t),\ |f(t) - f_\infty|^2_{L^2(\Omega_t)} \le K_2 e^{-\mu_1 t}, \tag{4.13}$$

for large t and some positive constants K_2 and μ_1. Then we have

$$\begin{gathered}|u'|_{L^2((t-a,t)\times O)},\ |\nabla_{X_1} u|_{L^2((t-a,t)\times O)},\ |\nabla_{X_2}(u - u_\infty)|_{L^2((t-a,t)\times O)} \le Me^{-\mu' t},\\ |\gamma^{\frac{1}{\rho+2}} u|_{L^{\rho+2}((t-a,t)\times O)} \le M\ e^{-\frac{2\mu'}{\rho+2} t},\end{gathered}$$

for some positive constants M and μ', such that $0 < \mu' < \min\{\mu_0/2, \mu_1\}/2$.

Proof. On one hand, $|f|^2_{L^2(Q_t)}$ grows polynomially since (4.13) yields

$$\begin{aligned}|f|^2_{L^2(Q_t)} &\le 2\int_0^t |f_\infty|^2_{L^2(\omega)}\Big(\int_{(-\ell_0-\ell s,\ell_0+\ell s)^{n_1}} dX_1 + 2K_2 e^{-\mu_1 s}\Big)ds\\ &\le c_{11} t^{n_1+1}\end{aligned} \tag{4.14}$$

for large t. On the other hand, by Remark 3.3 we derive

$$\begin{aligned}&\int_{S^t_{1+j}\times\omega} F^2 + \gamma|u_\infty|^{\rho+2}\,dx\,ds\\ &\le \int_{t-(1+j)}^t \int_{\Omega_s} F^2\,dx\,ds + C_2(\ell t + \ell_0)^{n_1}\int_{t-(1+j)}^t \gamma(s)ds\\ &\le K_2(1 + C_2(\ell t + \ell_0)^{n_1})\int_{t-(1+j)}^t e^{-\mu_1 s}ds\\ &\le K_2(1 + C_2(\ell t + \ell_0)^{n_1})(1 + j)e^{-\mu_1 t} \times e^{\mu_1(1+j)}\\ &\le c_{12} t^{n_1+1} e^{-\mu_1 t} \times e^{\mu_1 j},\end{aligned}$$

for large t. Since $k^j = e^{-\mu_0 j}$ then we have

$$k^j \int_{S^t_{1+j}\times\omega} F^2 + \gamma|u_\infty|^2\,dx\,ds \le c_{12} t^{n_1+1} e^{-\mu_1 t} \times e^{(\mu_1-\mu_0)j},$$

for $2m_0 + 1 \le j \le [t/2]$. Summing the above inequalities from $2m_0 + 1$ to $[t/2]$, we obtain

$$g_{m_0}(t) \le c_{12} t^{n_1+1} e^{-\mu_1 t} \sum_{j=2m_0+1}^{[t/2]} e^{(\mu_1-\mu_0)j}.$$

If $\mu_1 < \mu_0$, then the sum term in the right-hand is bounded independently of t. If $\mu_1 \geq \mu_0$, then

$$\sum_{j=2m_0+1}^{[t/2]} e^{(\mu_1-\mu_0)j} \leq c_{13} t e^{(\mu_1-\mu_0)\frac{t}{2}}.$$

Therefore, in both cases it holds that

$$g_{m_0}(t) \leq c_{14} t^{n_1+2} e^{-\min\{\frac{\mu_0+\mu_1}{2},\mu_1\}t}, \tag{4.15}$$

for large t. The estimations (4.14) and (4.15) means that Assumption (4.11) is satisfied.

Going back to (4.12) we derive that

$$\begin{aligned}
&\int_{(t-a,t)\times O} D(t,x)\,dx\,ds \\
&\leq c_{10}(t^{n_1+1}|f_\infty|^2_{L^2(\omega)} + c_{11}t^{n_1+2})e^{-\frac{\mu_0}{2}t} + c_{14}t^{n_1+2}e^{-\min\{\frac{\mu_0+\mu_1}{2},\mu_1\}t}.
\end{aligned}$$

Expending the expression of $D(t,x)$, we end up with

$$\begin{aligned}
&\int_{(t-a,t)\times O} |u'|^2 + |\nabla_{X_1}u|^2 + |\nabla_{X_2}(u-u_\infty)|^2 + \gamma|u|^{\rho+2}\,dx\,ds \\
&\leq c_{15}t^{n_1+2}\ e^{-\min\{\frac{\mu_0}{2},\mu_1\}t}.
\end{aligned}$$

This completes the proof. □

Remark 4.8. (i) Under assumption (4.13), the convergences in Corollary 4.6 are also exponential.

(ii) Theorem 4.7 also holds if we replace the assumption (4.13) by the following one

$$\int_{t-1}^{t} \gamma(s)ds,\ \int_{t-1}^{t}\int_{\Omega_s} |f-f_\infty|^2\,dx\,ds \leq K_3 e^{-\mu_2 t},$$

for large t and some positive constants K_3 and μ_2.

Remark 4.9. As long as the existence result of Theorem 2.1 holds, we can obtain the same results as in this article for more general domains, e.g.

$$\Omega_t = \Big(\prod_{i=1}^{n_1}(-\alpha_i(t),\beta_i(t))\Big)\times\omega, \quad t \geq 0,$$

where $\alpha_i(t)$ and $\beta_i(t)$ are smooth functions satisfying

$$\beta_i(0)+\alpha_i(0) > 0 \text{ and } \alpha_i(t),\beta_i(t) \to +\infty, \quad \text{as } t \to +\infty$$

and their derivatives satisfy

$$0 < \alpha_i'(t),\ \beta_i'(t) < 1, \quad \text{for } i = 1,\dots,n_1.$$

Of course, the definitions of S_m^t and ϱ_m must be adapted to this case.

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References

[1] C. Bardos, G. Chen; Control and stabilization for the wave equation. III: Domain with moving boundary. *SIAM J. Control Optim.*, 19:123–138, 1981.

[2] B. Brighi, S. Guesmia; Asymptotic behaviour of solutions of hyperbolic problems on a cylindrical domain. *Discrete Contin. Dyn. Syst. suppl*, (Special):160–169, 2007.

[3] M. Chipot; *ℓ goes to plus infinity.* Birkhäuser, 2002.

[4] M. Chipot, A. Rougirel; On the asymptotic behaviour of the solution of parabolic problems in cylindrical domains of large size in some directions. *Discrete Contin. Dyn. Syst. Ser. B*, 1(3):319–338, 2001.

[5] M. Chipot, K. Yeressian; Exponential rates of convergence by an iteration technique. *C. R. Math. Acad. Sci. Paris, Sér. I*, 346:21–26, 2008.

[6] J. Cooper, C. Bardos; A nonlinear wave equation in a dependent domain. *J. Math. Anal. Appl.*, 42:29–60, 1973.

[7] J. Cooper, L. A. Medeiros; The Cauchy problem for non linear wave equations in domains with moving boundary. *Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser.*, 26:829–838, 1972.

[8] J. Ferreira; Nonlinear hyperbolic-parabolic partial differential equation in noncylindrical domain. *Rend. Circ. Mat. Palermo (2)*, 44(1):135–146, 1995.

[9] J. Ferriera, N. A. Lar'kin; Global solvability of a mixed problem for a nonlinear hyperbolic-parabolic equation in noncylindrical domains. *Portugal. Math.*, 53(4):381–395, 1996.

[10] S. Guesmia; Some results on the asymptotic behaviour for hyperbolic problems in cylindrical domains becoming unbounded. *J. Math. Anal. Appl.*, 341(2):1190–1212, 2008.

[11] S. Guesmia; Some convergence results for quasilinear parabolic boundary value problems in cylindrical domain of large size. *Nonlinear Anal.*, 70(9):3320–3331, 2009.

[12] S. Guesmia; Large time and space size behaviour of the heat equation in noncylindrical domains. *Archiv der Mathematik*, 101(3):293–299, 2013.

[13] T. G. Ha, J. Y. Park; Global existence and uniform decay of a damped Klein-Gordon equation in a noncylindrical domain. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 74(2):577–584, 2011.

[14] E. Knobloch, R. Kerchetnikov; Problems on time-varying domains: Formulation, dynamics, and challenges. *Acta Applicandae Math.*, 137(1):123–157, Dec. 2015.

[15] N. A. Lar'kin, M. H. Simões; Nonlinear wave equation with a nonlinear boundary damping in a noncylindrical domain. *Matematica Contempornea*, 23:19–34, 2002.

[16] J. L. Lions; *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod-Gautier Villars, 1969.

[17] L. A. Medeiros; Nonlinear wave equations in domains with variable boundary. *Arch. Ration. Mech. Anal.*, 47:47–58, 1972.

[18] M. L. Nakao, T. Narazaki; Existence and decay of solutions of some nonlinear wave equations in noncylindrical domains. *Math. Reports, Kyushu Univ.*, 11(2):117–125, 1978.

[19] T. N. Rabello; Decay of solutions of a nonlinear hyperbolic system in noncylindrical domain. *Int. J. Math. Math. Sci.*, 17(3): 561–570, 1994.

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