Electronic Journal of Differential Equations, Vol. 2005(2005), No. 25, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

POSITIVE SOLUTIONS TO A SEMILINEAR HIGHER-ORDER ODE ON THE HALF-LINE

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Abstract. We study a semilinear non-autonomous ordinary differential equation (ODE) of order n. Explicit conditions for the existence of n linearly independent and positive solutions on the positive half-line are obtained. Also we establish lower solution estimates.

1. Introduction and statement of main result

The problem of existence of positive of solutions to a higher order nonlinear nonautonomous ordinary differential equations (ODEs) continues to attract the attention of many specialists, despite its long history, cf. [1, 2, 3, 7, 8, 10] and references therein. It is still one of the most burning problems of theory of ODEs, because of the absence of its complete solution. Let $p_k(t)$ $(t \geq 0\ k = 1, \dots n)$ be real continuous scalar-valued functions defined and bounded on $[0, \infty)$ and $p_0 \equiv 1$. Let $F : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function. In the present paper we investigate the semilinear equation

$$\sum_{k=0}^{n} p_k(t) \frac{d^{n-k}x(t)}{dt^{n-k}} = F(t, x) \quad (t > 0,\ x = x(t)) \tag{1.1}$$

with the initial conditions

$$x^{(j)}(0) = x_j \in \mathbb{R} \quad (j = 0, \dots, n-1). \tag{1.2}$$

A solution of problem (1.1), (1.2) is a function $x(\cdot)$ defined on $[0, \infty)$, having continuous derivatives up to the n-th order. In addition, $x(\cdot)$ satisfies (1.2) and (1.1) for all $t > 0$. The existence of solutions is assumed.

As it is well-known, the existence of positive solutions on the half-line for such equations is proved mainly in the case when p_k are constants, cf. [7, 8, 4]. In [6] the positivity conditions were derived for a class of semilinear nonautonomous equations in the divergent form. In [9], the following remarkable result is established:

2000 *Mathematics Subject Classification.* 34C10, 34C11.
Key words and phrases. Ordinary differential equations; nonlinear equations; positive solutions on the half-line.

Submitted September 28, 2004. Published March 6, 2005.
Supported by the Kamea fund of the Israel.

Solutions to the equation

$$\sum_{k=0}^{n} p_k(t) \frac{d^{n-k} v(t)}{dt^{n-k}} = 0 \quad (t > 0) \tag{1.3}$$

do not oscillate, if the roots of the polynomial

$$P(t,z) = \sum_{k=0}^{n} p_{n-k}(t) z^k \quad (z \in \mathbb{C}, t \geq 0)$$

are real and not intersecting. In the present paper, under some "close" conditions we prove that the nonlinear equation (1.1) has n linearly independent positive solutions. Besides, we generalize the corresponding result from [6].

Let polynomial $P(z,t)$ have the purely real roots $\rho_k(t)$ $(k = 1, \dots, n)$ with the property

$$\rho_k(t) \geq -\mu \ \ (t \geq 0;\ k = 1, \dots, n) \tag{1.4}$$

for some $\mu > 0$. Put

$$q_m(t) = \sum_{k=0}^{m} p_k(t) C_{n-k}^{m-k} (-1)^k \mu^{m-k} \quad (m = 1, \dots, n),\ q_0 \equiv 1. \tag{1.5}$$

and

$$d_0 = 1, d_{2k} = \sup_k q_{2k}(t) \quad \text{and} \quad d_{2k-1} = \inf_k q_{2k-1}(t)\ (k = 1, \dots, [n/2]),$$

where $[x]$ is the integer part of $x > 0$ and $C_n^k = \frac{n!}{k!(n-k)!}$.

Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *Let all the roots of the polynomial*

$$\tilde{Q}(z) := \sum_{k=0}^{n} (-1)^k d_k z^{n-k} \tag{1.6}$$

be real and nonnegative. In addition, let

$$F(y,t) \geq 0 \quad (y,t \geq 0). \tag{1.7}$$

Then (1.1) *has on* $(0,\infty)$ n *linearly independent positive solutions* $x_1, \dots, x_n$, *satisfying the inequalities*

$$x_j(t) \geq \text{const}\, e^{(-\mu+\tilde{r}_1)t} \geq 0 \quad (j = 1, \dots, n;\ \ t \geq t_0 > 0),$$

where $\tilde{r}_1 \geq 0$ *is the smallest root of* $\tilde{Q}(z)$.

This theorem is proved in the next two sections.

Example. Consider the equation

$$\frac{d^2x}{dt^2} + p_1(t)\frac{dx}{dt} + p_2(t)x = F(t,x) \quad (t > 0). \tag{1.8}$$

Assume that $p_1(t), p_2(t) \geq 0$ and $p_1^2(t) > 4p_2(t)$ $(t \geq 0)$. Put

$$p_1^+ = \sup_{t \geq 0}\ p_1(t).$$

Since $\rho_1(t) + \rho_2(t) = -p_1(t)$, we can take $\mu = p_1^+$. Hence,

$$q_1(t) = 2p_1^+ - p_1(t),\ q_2(t) = p_1^+\ (p_1^+ - p_1(t)) + p_2(t)$$

and

$$d_1 = \inf_t q_1(t) = p_1^+, \quad d_2 = \sup_t q_2(t).$$

If, in addition, $(p_1^+)^2 > 4d_2$ and (1.7) holds, then due to Theorem 1.1, equation (1.8) has 2 positive linearly independent solutions satisfying inequalities (1.1) with $n = 2$ and

$$-\mu + \tilde{r}_1 = -p_1^+/2 - \sqrt{(p_1^+)^2/4 - d_2}\,.$$

2. Preliminaries

Let $a_k(t)$ ($t \geq 0;\ k = 1, \dots, n$) be continuous scalar-valued functions defined and bounded on $[0, \infty)$, and $a_0 \equiv 1$. Consider the equation

$$\sum_{k=0}^{n} (-1)^k a_k(t) \frac{d^{n-k}u(t)}{dt^{n-k}} = 0 \quad (t > 0). \tag{2.1}$$

Put

$$c_{2k} := \sup_{t \geq 0} a_{2k}(t), \quad c_{2k-1} := \inf_{t \geq 0} a_{2k-1}(t) \quad (k = 1, \dots, [n/2]).$$

Lemma 2.1. *Assume all the roots of the polynomial*

$$Q(z) = \sum_{k=0}^{n} (-1)^k c_k z^{n-k} \ \ (c_0 = 1,\ z \in \mathbb{C})$$

be real and nonnegative. Then a solution u of (2.1) with the initial conditions

$$u^{(j)}(0) = 0, j = 0, \dots, n-2;\ u^{(n-1)}(0) = 1 \tag{2.2}$$

satisfies the inequalities

$$u^{(j)}(t) \geq e^{r_1 t} \sum_{k=0}^{j} C_k^j \frac{r_1^{j-k} t^{n-1-k}}{(n-1-k)!} \geq 0 \quad (j = 0, \dots, n-1;\ t > 0),$$

where $r_1 \geq 0$ is the smallest root of $Q(z)$.

Proof. We have

$$b_k(t) := (-1)^k (c_k - a_k(t)) \geq 0 \quad (k = 1, \dots, n).$$

Rewrite equation (2.1) in the form

$$\sum_{k=0}^{n} (-1)^k c_k \frac{d^{n-k}u}{dt^{n-k}} = \sum_{1}^{n} b_k(t) \frac{d^{n-k}u}{dt^{n-k}}\,. \tag{2.3}$$

Denote

$$G(t) = \frac{1}{2i\pi} \int_C \frac{e^{zt}dz}{Q(z)},$$

where C is a smooth contour surrounding all the zeros of $Q(z)$. That is, G is the Green functions for the autonomous equation

$$\sum_{k=0}^{n} (-1)^k c_k \frac{d^{n-k}w(t)}{dt^{n-k}} = 0. \tag{2.4}$$

Put

$$y(t) \equiv \sum_{k=0}^{n} c_k (-1)^k \frac{d^{n-k}u(t)}{dt^{n-k}}. \tag{2.5}$$

Then thanks to the variation of constants formula,

$$u(t) = w(t) + \int_0^t G(t-s)y(s)ds,$$

where $w(t)$ is a solution of (2.3). Since

$$G^{(j)}(t) = \frac{1}{2i\pi}\int_C \frac{z^j e^{zt}dz}{(z-r_1)\dots(z-r_n)}\,,$$

where $r_1 \le \dots \le r_n$ are the roots of $Q(z)$ with their multiplicities, due to [5, Lemma 1.11.2], we get

$$G^{(j)}(t) = \frac{1}{(n-1)!}\Big[\frac{d^{n-1}z^j e^{zt}}{dz^{n-1}}\Big]_{z=\theta}$$

with $\theta \in [r_1, r_n]$. Hence,

$$G^{(j)}(t) = \sum_{k=0}^{j} \frac{j!e^{\theta t}\theta^{j-k}t^{n-1-k}}{(j-k)!(n-1-k)!k!} \ge \sum_{k=0}^{j} \frac{j!e^{r_1 t}r_1^{j-k}t^{n-1-k}}{(j-k)!(n-1-k)!k!} \ge 0\,. \tag{2.6}$$

According to the initial conditions (2.2), we can write $w(t) = G(t)$. So

$$u(t) = G(t) + \int_0^t G(t-s)y(s)\,ds\,. \tag{2.7}$$

For $j \le n-2$ we have $G^{(j)}(0) = 0$ and

$$\begin{aligned}\frac{d^j}{dt^j}\int_0^t G(t-s)y(s)ds &= \frac{d}{dt}\int_0^t G^{(j-1)}(t-s)y(s)ds\\ &= \int_0^t G^{(j)}(t-s)y(s)ds \quad (j = 0,\dots,n-1)\,.\end{aligned}$$

Hence thanks to (2.3) and (2.5),

$$\begin{aligned}y(t) &= \sum_1^n b_k(t)[G^{(n-k)}(t) + \int_0^t G^{(n-k)}(t-s)y(s)ds]\\ &= K(t,t) + \int_0^t K(t,t-s)y(s)\,ds\,,\end{aligned} \tag{2.8}$$

where

$$K(t,\tau) = \sum_1^n b_k(t)G^{(n-k)}(\tau) \quad (t,\tau \ge 0).$$

According to (2.6), $K(t,\tau) \ge 0$ $(t,\tau \ge 0)$. Put $h(t) = K(t,t)$. Let V be the Volterra operator with the kernel $K(t,t-s)$. Then thanks to (2.8) and the Neumann series,

$$y(t) = h(t) + \sum_1^\infty (V^k h)(t) \ge h(t) \ge 0.$$

Hence (2.7) yields,

$$\begin{aligned}u^{(j)}(t) &= G^{(j)}(t) + \int_0^t G^{(j)}(t-s)y(s)ds\\ &\ge G^{(j)}(t) + \int_0^t G^{(j)}(t-s)K(s,s)ds\\ &\ge G^{(j)}(t) \quad (j = 0,\dots,n-1).\end{aligned}$$

This inequality and (2.6) prove the lemma. □

Recall that a scalar valued function $W(t,\tau)$ defined for $t \ge \tau \ge 0$ is *the Green function to equation* (2.1) if it satisfies that equation for $t > \tau$ and the initial conditions

$$\lim_{t\downarrow\tau} \frac{\partial^k W(t,\tau)}{\partial t^k} = 0 \quad (k = 0,\dots,n-2)$$
$$\lim_{t\downarrow\tau} \frac{\partial^{n-1} W(t,\tau)}{\partial t^{n-1}} = 1\,.$$

Lemma 2.2. *Assume all the roots of polynomial $Q(z)$ are real and nonnegative. Then the Green function to equation* (2.1) *and its derivatives up to* $(n-1)$ *order are nonnegative. Moreover,*

$$\frac{\partial^j W(t,\tau)}{\partial t^j} \ge e^{r_1(t-\tau)} \sum_{k=0}^{j} C_j^k \frac{r_1^{j-k}(t-\tau)^{n-1-k}}{(n-1-k)!} \ge 0 \quad (j = 0,\dots,n-1;\ t > \tau \ge 0),$$

Proof. For a $\tau > 0$, take the initial conditions

$$u^{(j)}(\tau) = 0, \quad j = 0,\dots,n-2; \quad u^{(n-1)}(\tau) = 1.$$

Then the corresponding solution $u(t)$ to (2.1) is equal to $W(t,\tau)$. Repeating the argument in the proof of Lemma 2.1, we have

$$\frac{\partial^j W(t,\tau)}{\partial t^j} \ge G^{(j)}(t-\tau) + \int_\tau^t G^{(j)}(t-\tau-s)K(s,s-\tau)ds \ge G^{(j)}(t-\tau).$$

According to (2.6) this proves the lemma. □

3. Proof of Theorem 1.1

In (1.3) put $v(t) = e^{-\mu t}u(t)$. Then

$$0 = e^{\mu t}\sum_{k=0}^{n} p_k(t)\frac{d^{n-k}e^{-\mu t}u}{dt^{n-k}} = \sum_{k=0}^{n} p_k(t)(\frac{d}{dt}-\mu)^{n-k}u.$$

That is, (1.3) is reduced to the equation

$$P(t,\frac{d}{dt}-\mu)u \equiv \sum_{k=0}^{n} p_k(t)(\frac{d}{dt}-\mu)^{n-k}u = 0\,. \tag{3.1}$$

However,

$$\begin{aligned}
P(t,z-\mu) &= \sum_{k=0}^{n} p_k(t)(z-\mu)^{n-k} \\
&= \sum_{k=0}^{n} p_k(t)\sum_{j=0}^{n-k} C_{n-k}^{j}(-\mu)^j z^{n-k-j} \\
&= \sum_{k=0}^{n} p_k(t)\sum_{m=k}^{n} C_{n-k}^{m-k}(-\mu)^{m-k} z^{n-m} \\
&= \sum_{m=0}^{n} z^{n-m}\sum_{k=0}^{m} p_k(t)C_{n-k}^{m-k}(-\mu)^{k-m}\,.
\end{aligned}$$

So

$$P(t, z-\mu) = \sum_{m=0}^{n} (-1)^m q_m(t) z^{n-m},$$

where $q_m(t)$ are defined by (1.5). Take into account that

$$P(t, z-\mu) = \prod_{k=1}^{n} (z - \rho_k(t) - \mu) = \prod_{k=1}^{n} (z - \tilde{\rho}_k(t)),$$

where according to (1.4), $\tilde{\rho}_k(t) \equiv \rho_k(t) + \mu \geq 0$. Hence it follows that $q_m(t)$ are nonnegative and we can apply Lemma 2.1 to (3.1). Due to Lemma 2.1 and the substitution $v(t) = e^{-\mu t} u(t)$, we have the following statement.

Lemma 3.1. *Assume condition* (1.4) *and that all the roots of the polynomial* $\tilde{Q}(z)$ *defined by* (1.6) *are real and nonnegative. Then the Green function* $\tilde{W}(t, \tau)$ *for equation* (1.3) *is positive and*

$$\frac{\partial^j e^{\mu(t-\tau)} \tilde{W}(t, \tau)}{\partial t^j} \geq e^{\tilde{r}_1 (t-\tau)} \sum_{k=0}^{j} C_j^k \frac{\tilde{r}_1^{j-k} (t-\tau)^{n-1-k}}{(n-1-k)!} \geq 0$$

for $j = 0, \dots, n-1$; $t > \tau \geq 0$. *In particular,*

$$\tilde{W}(t, \tau) \geq e^{(-\mu + \tilde{r}_1)(t-\tau)} \frac{(t-\tau)^{n-1}}{(n-1)!} \quad (t > \tau \geq 0). \tag{3.2}$$

Lemma 3.2. *Assume the hypothesis of Theorem 1.1. Let* $v(t)$ *be a positive solution of the linear non-autonomous problem* (1.2)–(1.3)*. Then a solution* $x(t)$ *of problem* (1.1)–(1.2) *is also positive. Moreover,* $x(t) \geq v(t)$, $t \geq 0$.

Proof. Thanks to the Variation of Constants Formula, (1.1) can be rewritten as

$$x(t) = v(t) + \int_0^t \tilde{W}(t, s) F(s, x(s)) ds.$$

Since $\tilde{W}(t, s)$ is positive due to the previous lemma, there is a sufficiently small $t_0 \geq 0$, such that $x(t) \geq 0$, $t \leq t_0$. Hence, $x(t) \geq v(t)$, $t \leq t_0$. Extending this inequality to all $t \geq 0$, we prove the lemma. □

Proof of Theorem 1.1. Take n solutions $x_k(t)$ $(k = 1, \dots, n)$ of (1.1) satisfying the conditions

$$x_k^{(j)}(\epsilon k) = 0, \quad (j = 0, \dots, n-2), \quad x_k^{(n-1)}(\epsilon k) = 1$$

with an arbitrary $\epsilon > 0$. It can be directly checked that these solutions are linearly independent.

Now take n solutions $v_k(t)$ $(k = 1, \dots, n)$ of (1.3) satisfying the same conditions

$$v_k^{(j)}(\epsilon k) = 0, \quad (j = 0, \dots, n-2), \quad v_k^{(n-1)}(\epsilon k) = 1.$$

Then due to Lemma 3.1,

$$v_k(t) \equiv \tilde{W}(t, \epsilon k) \geq 0 \quad (t \geq \epsilon k).$$

Now the required result is due to Lemma 3.2. □

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