Electronic Journal of Differential Equations, Vol. 2005(2005), No. 77, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A PROPERTY OF SOBOLEV SPACES ON COMPLETE RIEMANNIAN MANIFOLDS

OGNJEN MILATOVIC

Abstract. Let (M, g) be a complete Riemannian manifold with metric g and the Riemannian volume form $d\nu$. We consider the $\mathbb{R}^k$-valued functions $T \in [W^{-1,2}(M) \cap L^1_{\text{loc}}(M)]^k$ and $u \in [W^{1,2}(M)]^k$ on M, where $[W^{1,2}(M)]^k$ is a Sobolev space on M and $[W^{-1,2}(M)]^k$ is its dual. We give a sufficient condition for the equality of $\langle T, u\rangle$ and the integral of $(T \cdot u)$ over M, where $\langle \cdot, \cdot\rangle$ is the duality between $[W^{-1,2}(M)]^k$ and $[W^{1,2}(M)]^k$. This is an extension to complete Riemannian manifolds of a result of H. Brézis and F. E. Browder.

1. Introduction and main result

The setting. Let (M, g) be a C^∞ Riemannian manifold without boundary, with metric $g = (g_{jk})$ and $\dim M = n$. We will assume that M is connected, oriented, and complete. By $d\nu$ we will denote the Riemannian volume element of M. In any local coordinates $x^1, \dots, x^n$, we have $d\nu = \sqrt{\det(g_{jk})}\, dx^1 dx^2 \dots dx^n$.

By $L^2(M)$ we denote the space of real-valued square integrable functions on M with the inner product

$$(u, v) = \int_M (uv)\, d\nu.$$

Unless specified otherwise, in all function spaces below, the functions are real-valued.

In what follows, $C^\infty(M)$ denotes the space of smooth functions on M, $C_c^\infty(M)$ denotes the space of smooth compactly supported functions on M, $\Omega^1(M)$ denotes the space of smooth 1-forms on M, and $L^2(\Lambda^1 T^* M)$ denotes the space of square integrable 1-forms on M.

By $W^{1,2}(M)$ we denote the completion of $C_c^\infty(M)$ in the norm

$$\|u\|^2_{W^{1,2}} = \int_M |u|^2\, d\nu + \int_M |du|^2\, d\nu,$$

where $d : C^\infty(M) \to \Omega^1(M)$ is the standard differential.

Remark 1.1. It is well known (see, for example, Chapter 2 in [1]) that if (M, g) is a complete Riemannian manifold, then $W^{1,2}(M) = \{u \in L^2(M) \colon du \in L^2(\Lambda^1 T^* M)\}$.

2000 *Mathematics Subject Classification.* 58J05.
Key words and phrases. Complete Riemannian manifold; Sobolev space.

Submitted June 25, 2005. Published July 8, 2005.

By $W^{-1,2}(M)$ we denote the dual space of $W^{1,2}(M)$, and by $\langle\cdot,\cdot\rangle$ we will denote the duality between $W^{-1,2}(M)$ and $W^{1,2}(M)$.

In what follows, $[C_c^\infty(M)]^k$, $[L^2(M)]^k$, $[L^2(\Lambda^1T^*M)]^k$ and $[W^{1,2}(M)]^k$ denote the space of all ordered k-tuples $u=(u_1,u_2,\dots,u_k)$ such that $u_j\in C_c^\infty(M)$, $u_j\in L^2(M)$, $u_j\in L^2(\Lambda^1T^*M)$, $u_j\in W^{1,2}(M)$, respectively, for all $1\leq j\leq k$. For $u\in[W^{1,2}(M)]^k$, we will use the following notation:

$$du := (du_1, du_2, \dots, du_k), \tag{1.1}$$

$$|u| := (u_1^2+u_2^2+\dots+u_k^2)^{1/2}, \tag{1.2}$$

$$|du| := (|du_1|^2+|du_2|^2+\dots+|du_k|^2)^{1/2}, \tag{1.3}$$

where $|du_j|$ denotes the length of the cotangent vector du_j.

The space $[W^{1,2}(M)]^k$ is the completion of $[C_c^\infty(M)]^k$ in the norm

$$\|u\|^2_{[W^{1,2}(M)]^k}=\int_M |u|^2\,d\nu+\int_M |du|^2\,d\nu,$$

where $|u|$ and $|du|$ are as in (1.2) and (1.3) respectively.

Remark 1.2. As in Remark 1.1, if (M,g) is a complete Riemannian manifold, then $[W^{1,2}(M)]^k=\{u\in[L^2(M)]^k\colon du\in[L^2(\Lambda^1T^*M)]^k\}$.

Assumption (H1). Assume that

(1) $u=(u_1,u_2,\dots,u_k)\in[W^{1,2}(M)]^k$ and
(2) $T=(T_1,T_2,\dots,T_k)$, where $T_1,T_2,\dots,T_k\in W^{-1,2}(M)\cap L^1_{\rm loc}(M)$.

Here, the notation $T_j\in W^{-1,2}(M)\cap L^1_{\rm loc}(M)$ means that T_j is a.e. defined function belonging to $L^1_{\rm loc}(M)$ such that

$$\phi\mapsto\int_M T_j\phi\,d\nu,\quad \phi\in C_c^\infty(M),$$

extends continuously to $W^{1,2}(M)$.

For a.e. $x\in M$, denote

$$(T\cdot u)(x):=\sum_{j=1}^k T_j(x)u_j(x), \tag{1.4}$$

$$\langle T,u\rangle:=\sum_{j=1}^k\langle T_j,u_j\rangle, \tag{1.5}$$

where $\langle\cdot,\cdot\rangle$ on the right hand side of (1.5) denotes the duality between $W^{-1,2}(M)$ and $W^{1,2}(M)$.

We now state our main result.

Theorem 1.3. *Assume that (M,g) is a complete Riemannian manifold. Assume that $u=(u_1,u_2,\dots,u_k)$ and $T=(T_1,T_2,\dots,T_k)$ satisfy the assumption (H1). Assume that there exists a function $f\in L^1(M)$ such that*

$$(T\cdot u)(x)\geq f(x),\quad \textit{a.e. on } M. \tag{1.6}$$

Then $(T\cdot u)\in L^1(M)$ and

$$\langle T,u\rangle=\int_M (T\cdot u)(x)\,d\nu(x).$$

In the following Corollary, by $W^{1,2}(M,\mathbb{C})$, $W^{-1,2}(M,\mathbb{C})$ and $L^1_{\mathrm{loc}}(M,\mathbb{C})$ we denote the complex analogues of spaces $W^{1,2}(M)$, $W^{-1,2}(M)$ and $L^1_{\mathrm{loc}}(M)$. By $\langle\cdot,\cdot\rangle$ we denote the Hermitian duality between $W^{-1,2}(M,\mathbb{C})$ and $W^{1,2}(M,\mathbb{C})$.

Corollary 1.4. *Assume that (M,g) is a complete Riemannian manifold. Assume that $T\in W^{-1,2}(M,\mathbb{C})\cap L^1_{\mathrm{loc}}(M,\mathbb{C})$ and $u\in W^{1,2}(M,\mathbb{C})$. Assume that there exists a real-valued function $f\in L^1(M)$ such that*

$$\operatorname{Re}(T\bar{u})\ge f,\quad \text{a.e. on } M.$$

Then $\operatorname{Re}(T\bar{u})\in L^1(M)$ and

$$\operatorname{Re}\langle T,u\rangle=\int_M \operatorname{Re}(T\bar{u})\,d\nu.$$

Remark 1.5. Theorem 1.3 and Corollary 1.4 extend the corresponding results of H. Brézis and F. E. Browder [3] from $\mathbb{R}^n$ to complete Riemannian manifolds. The results of [3] were used, among other applications, in studying self-adjointness and m-accretivity in $L^2(\mathbb{R}^n,\mathbb{C})$ of Schrödinger operators with singular potentials; see, for example, H. Brézis and T. Kato [4]. Analogously, Theorem 1.3 and Corollary 1.4 can be used in the study of self-adjoint and m-accretive realizations (in the space $L^2(M,\mathbb{C})$) of Schrödinger-type operators with singular potentials, where M is a complete Riemannian manifold, as well as in the study of partial differential equations on complete Riemannian manifolds.

2. Proof of Theorem 1.3

We will adopt the arguments of H. Brézis and F. E. Browder [3] to the context of a complete Riemannian manifold. In what follows, $F\colon\mathbb{R}^k\to\mathbb{R}^l$ is a C^1 vector-valued function $F(y)=(F_1(y),F_2(y),\dots,F_l(y))$. By $dF(y)$ we will denote the derivative of F at $y=(y_1,y_2,\dots,y_k)$.

Lemma 2.1. *Assume that $F\in C^1(\mathbb{R}^k,\mathbb{R}^l)$, $F(0)=0$, and for all $y\in\mathbb{R}^k$,*

$$|dF(y)|\le C$$

where $C\ge 0$ is a constant.

Assume that $u=(u_1,u_2,\dots,u_k)\in[W^{1,2}(M)]^k$. Then $(F\circ u)\in[W^{1,2}(M)]^l$, and the following holds:

$$d(F\circ u)=\sum_{j=1}^k\frac{\partial F}{\partial u_j}du_j,\tag{2.1}$$

where

$$\frac{\partial F}{\partial u_j}=\Big(\frac{\partial F_1}{\partial y_j}(u),\frac{\partial F_2}{\partial y_j}(u),\dots,\frac{\partial F_l}{\partial y_j}(u)\Big).\tag{2.2}$$

(Here the notation $\frac{\partial F_s}{\partial y_j}(u)$, where $1\le s\le l$, denotes the composition of $\frac{\partial F_s}{\partial y_j}$ and u. The notation $d(F\circ u)$ denotes the ordered l-tuple $(d(F_1\circ u),d(F_2\circ u),\dots,d(F_l\circ u))$, where $d(F_s\circ u)$, $1\le s\le l$, is the differential of the scalar-valued function $F_s\circ u$ on M.

Proof. Let $u\in[W^{1,2}(M)]^k$. By definition of $[W^{1,2}(M)]^k$, the weak derivatives du_j, $1\le j\le k$, exist and $du_j\in L^2(M)$. By Lemma 7.5 in [6], it follows that for all

$1 \leq s \leq l$, the following holds:

$$d(F_s \circ u) = \sum_{j=1}^{k} \frac{\partial F_s}{\partial u_j} du_j,$$

where

$$\frac{\partial F_s}{\partial u_j} = \frac{\partial F_s}{\partial y_j}(u).$$

This shows (2.1).

Since dF is bounded and since $du_j \in L^2(\Lambda^1 T^*M)$, it follows that $d(F_s \circ u) \in L^2(\Lambda^1 T^*M)$ for all $1 \leq s \leq l$. Thus $d(F \circ u) \in [L^2(\Lambda^1 T^*M)]^l$. Moreover, since $u \in [W^{1,2}(M)]^k$ and

$$|F_s \circ u| = |F_s(u) - F_s(0)| \leq C_1 |u|,$$

where $C_1 \geq 0$ is a constant and $|u|$ is as in (1.2), it follows that $(F_s \circ u) \in L^2(M)$ for all $1 \leq s \leq l$. Thus $(F \circ u) \in [L^2(M)]^l$. Therefore, $(F \circ u) \in [W^{1,2}(M)]^l$, and the Lemma is proven. $\square$

Lemma 2.2. *Assume that u, $v \in W^{1,2}(M) \cap L^\infty(M)$. Then $(uv) \in W^{1,2}(M)$ and*

$$d(uv) = (du)v + u(dv). \tag{2.3}$$

Proof. By the remark after the equation (7.18) in [6], the equation (2.3) holds if the weak derivatives du, dv exist and if $uv \in L^1_{\text{loc}}(M)$ and $((du)v + u(dv)) \in L^1_{\text{loc}}(M)$. By the hypotheses of the Lemma, these conditions are satisfied, and, hence, (2.3) holds.

Furthermore, since u, $v \in W^{1,2}(M) \cap L^\infty(M)$, we have $(uv) \in L^2(M)$. By hypotheses of the Lemma and by (2.3) we have $d(uv) \in L^2(M)$. Thus $(uv) \in W^{1,2}(M)$, and the Lemma is proven. $\square$

In the next lemma, the statement "$f\colon \mathbb{R} \to \mathbb{R}$ is a piecewise smooth function" means that f is continuous and has piecewise continuous first derivative.

Lemma 2.3. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a piecewise smooth function with $f(0) = 0$ and $f' \in L^\infty(\mathbb{R})$. Let S denote the set of corner points of f. Assume that $u \in W^{1,2}(M)$. Then $(f \circ u) \in W^{1,2}(M)$ and*

$$d(f \circ u) = \begin{cases} f'(u)\, du & \text{for all } x \text{ such that } u(x) \notin S \\ 0 & \text{for all } x \text{ such that } u(x) \in S \end{cases}$$

Proof. By the remark in the second paragraph below the equation (7.24) in [6], the Lemma follows immediately from Theorem 7.8 in [6]. $\square$

The following Corollary follows immediately from Lemma 2.3.

Corollary 2.4. *Assume that $u \in W^{1,2}(M)$. Then $|u| \in W^{1,2}(M)$ and*

$$d|u| = \begin{cases} f'(u)\, du & \text{for all } x \text{ such that } u(x) \neq 0 \\ 0 & \text{for all } x \text{ such that } u(x) = 0 \end{cases},$$

where $f(t) = |t|$, $t \in \mathbb{R}$.

Remark 2.5. Let $f(t) = |t|$, $t \in \mathbb{R}$. Let c be a real number. By Lemma 7.7 in [6] and by Corollary 2.4, we can write $d|u| = h(u)du$ a.e. on M, where

$$h(t) = \begin{cases} f'(t) & \text{for all } t \neq 0 \\ c & \text{otherwise.} \end{cases}$$

Lemma 2.6. *Assume that u, $v \in W^{1,2}(M)$ and let*

$$w(x) := \min\{u(x), v(x)\}.$$

Then $w \in W^{1,2}(M)$ and

$$|dw| \leq \max\{|du|, |dv|\}, \quad \text{a.e. on } M,$$

where $|du(x)|$ denotes the norm of the cotangent vector $du(x)$.

Proof. We can write

$$w(x) = \frac{1}{2}(u(x) + v(x) - |u(x) - v(x)|).$$

Since u, $v \in W^{1,2}(M)$, by Corollary 2.4 we have $|u - v| \in W^{1,2}(M)$, and, thus, $w \in W^{1,2}(M)$. By Remark 2.5, we have

$$dw(x) = \frac{1}{2}(du(x) + dv(x) - (h(u - v)) \cdot (du(x) - dv(x))), \quad \text{a.e. on } M, \tag{2.4}$$

where h is as in Remark 2.5.

Considering $dw(x)$ on sets $\{x \colon u(x) > v(x)\}$, $\{x \colon u(x) < v(x)\}$ and $\{x \colon u(x) = v(x)\}$, and using (2.4), we get

$$|dw(x)| \leq \max\{|du(x)|, |dv(x)|\}, \quad \text{a.e. on } M.$$

This concludes the proof of the Lemma. □

Lemma 2.7. *Let $a > 0$. Let $u = (u_1, u_2, \dots, u_k)$ be in $[W^{1,2}(M)]^k$, let $v = (v_1, v_2, \dots, v_k)$ be in $[W^{1,2}(M) \cap L^\infty(M)]^k$, and let*

$$w := \Big((|u|^2 + a^2)^{-1/2} \min\{(|u|^2 + a^2)^{1/2} - a, (|v|^2 + a^2)^{1/2} - a\}\Big)\, u,$$

where $|u|$ is as in (1.2). Then $w \in [W^{1,2}(M) \cap L^\infty(M)]^k$ and

$$|dw| \leq 3 \max\{|du|, |dv|\}, \quad \text{a.e. on } M,$$

where $|du|$ is as in (1.3).

Proof. Let $\phi = (|u|^2 + a^2)^{-1/2} u$. Then $\phi = F \circ u$, where $F \colon \mathbb{R}^k \to \mathbb{R}^k$ is defined by

$$F(y) = (|y|^2 + a^2)^{-1/2} y, \quad y \in \mathbb{R}^k.$$

Clearly, $F \in C^1(\mathbb{R}^k, \mathbb{R}^k)$ and $F(0) = 0$. It easily checked that the component functions

$$F_s(y) = (|y|^2 + a^2)^{-1/2} y_s$$

satisfy

$$\frac{\partial F_s}{\partial y_j} = \begin{cases} -(|y|^2 + a^2)^{-3/2} y_s y_j & \text{for } s \neq j \\ (|y|^2 + a^2)^{-3/2} (|y|^2 - y_j^2 + a^2) & \text{for } s = j. \end{cases}$$

Therefore, for all $1 \leq s, j \leq k$, we have

$$\Big|\frac{\partial F_s}{\partial y_j}(y)\Big| \leq \frac{1}{a},$$

and, hence, F satisfies the hypotheses of Lemma 2.1. Thus, by Lemma 2.1 we have $(F\circ u)=\phi\in[W^{1,2}(M)]^k$.

We now write the formula for $d\phi=(d\phi_1,d\phi_2,\dots,d\phi_k)$. We have

$$d\phi=(|u|^2+a^2)^{-3/2}\Big((|u|^2+a^2)du-\Big(\sum_{j=1}^k u_j du_j\Big)u\Big), \tag{2.5}$$

where du is as in (1.1).

By (2.5), using triangle inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned} |d\phi| &\le (|u|^2+a^2)^{-3/2}\Big((|u|^2+a^2)|du|+\big|\sum_{j=1}^k u_j du_j\big||u|\Big)\\ &\le (|u|^2+a^2)^{-3/2}\big((|u|^2+a^2)|du|+|u||du||u|\big)\\ &\le (|u|^2+a^2)^{-3/2}\big((|u|^2+a^2)|du|+(|u|^2+a^2)|du|\big)\\ &= 2(|u|^2+a^2)^{-1/2}|du|, \quad \text{a.e. on } M, \end{aligned} \tag{2.6}$$

where $|du_j|$ is the norm of the cotangent vector du_j, and $|u|$ and $|du|$ are as in (1.2) and (1.3) respectively.

Let

$$\psi:=\min\{(|u|^2+a^2)^{1/2}-a,(|v|^2+a^2)^{1/2}-a\}.$$

Then

$$(|u|^2+a^2)^{1/2}-a=G\circ u \quad \text{and} \quad (|v|^2+a^2)^{1/2}-a=G\circ v,$$

where

$$G(y)=(|y|^2+a^2)^{1/2}-a, \quad y\in\mathbb{R}^k.$$

Clearly, $G\in C^1(\mathbb{R}^k,\mathbb{R})$ and $G(0)=0$, and

$$\frac{\partial G}{\partial y_j}=(|y|^2+a^2)^{-1/2}y_j,$$

It is easily seen that there exists a constant $C_2\ge 0$ such that $|dG(y)|\le C_2$ for all $y\in\mathbb{R}^k$. Hence, by Lemma 2.1 we have $(G\circ u)\in W^{1,2}(M)$ and $(G\circ v)\in W^{1,2}(M)$.

Thus, by Lemma 2.6 we have $\psi\in W^{1,2}(M)$, and

$$|d\psi|\le\max\big\{\big|d((|u|^2+a^2)^{1/2}-a)\big|,\big|d((|v|^2+a^2)^{1/2}-a)\big|\big\}, \quad \text{a.e. on } M.$$

Using triangle inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned} |d((|u|^2+a^2)^{1/2}-a)| &= \big|(|u|^2+a^2)^{-1/2}\big(\sum_{j=1}^k u_j du_j\big)\big|\\ &\le (|u|^2+a^2)^{-1/2}|u||du|\\ &\le |du|, \end{aligned} \tag{2.7}$$

where $|u|$ and $|du|$ are as in (1.2) and (1.3) respectively. As in (2.7), we obtain

$$|d((|v|^2+a^2)^{1/2}-a)|\le|dv|.$$

Therefore, we get

$$|d\psi|\le\max\{|du|,|dv|\}, \quad \text{a.e. on } M, \tag{2.8}$$

where $|d\psi|$ is the norm of the cotangent vector $d\psi$, and $|du|$ and $|dv|$ are as in (1.3).

By definition of ϕ we have $\phi\in[L^\infty(M)]^k$ and, by definition of ψ we have

$$\psi\le(|v|^2+a^2)^{1/2}-a.$$

Thus,

$$\psi \le |v|, \tag{2.9}$$

where $|v|$ is as in (1.2).

Since $v \in [L^\infty(M)]^k$, we have $\psi \in L^\infty(M)$. We have already shown that $\phi \in [W^{1,2}(M)]^k$ and $\psi \in W^{1,2}(M)$. By Lemma 2.2 (applied to the components $\psi\phi_j$, $1 \le j \le k$, of $\psi\phi$) we have $w = \psi\phi \in [W^{1,2}(M)]^k$ and

$$d(\psi\phi) = (d\psi)\phi + \psi(d\phi). \tag{2.10}$$

By (2.10), (2.6) and (2.8), we have a.e. on M:

$$\begin{aligned} |dw| &= |(d\psi)\phi + \psi(d\phi)| \\ &\le |d\psi||\phi| + |\psi||d\phi| \\ &\le (\max\{|du|, |dv|\})\,|\phi| + 2(|u|^2 + a^2)^{-1/2}|du||\psi| \\ &\le \max\{|du|, |dv|\} + 2|du| \\ &\le 3\max\{|du|, |dv|\}, \end{aligned}$$

where the third inequality holds since $|\phi| \le 1$ and $|\psi|(|u|^2 + a^2)^{-1/2} \le 1$. This concludes the proof of the Lemma. □

Lemma 2.8. *Let $T = (T_1, T_2, \dots, T_k)$ and $u = (u_1, u_2, \dots, u_k)$ be as in the hypotheses of Theorem 1.3. Additionally, assume that u has compact support and $u \in [L^\infty(M)]^k$. Then the conclusion of Theorem 1.3 holds.*

Proof. Since the vector-valued function $u = (u_1, u_2, \dots, u_k) \in [W^{1,2}(M)]^k$ is compactly supported, it follows that the functions u_j are compactly supported. Thus, using a partition of unity we can assume that u_j is supported in a coordinate neighborhood V_j. Thus we can use the Friedrichs mollifiers. Let $\rho_j > 0$ and $(u_j)^{\rho_j} := J^{\rho_j}u$, where J^{ρ_j} denotes the Friedrichs mollifying operator as in Section 5.12 of [2]. Then $(u_j)^{\rho_j} \in C_c^\infty(M)$, and, as $\rho_j \to 0+$, we have $(u_j)^{\rho_j} \to u_j$ in $W^{1,2}(M)$; see, for example, Lemma 5.13 in [2]. Thus

$$\langle T_j, (u_j)^{\rho_j}\rangle \to \langle T_j, u_j\rangle, \quad \text{as } \rho_j \to 0+, \tag{2.11}$$

where $\langle\cdot,\cdot\rangle$ is as on the right hand side of (1.5).

Since $(u_j)^{\rho_j} \in C_c^\infty(M)$ and $T_j \in L^1_{\rm loc}(M)$, we have

$$\langle T_j, (u_j)^{\rho_j}\rangle = \int_M (T_j \cdot (u_j)^{\rho_j})\, d\nu. \tag{2.12}$$

Next, we will show that

$$\lim_{\rho_j \to 0+} \int_M (T_j \cdot (u_j)^{\rho_j})\, d\nu = \int_M (T_j u_j)\, d\nu. \tag{2.13}$$

Since $u_j \in L^\infty(M)$ is compactly supported, by properties of Friedrichs mollifiers (see, for example, the proof of Theorem 1.2.1 in [5]) it follows that

(i) there exists a compact set K_j containing the supports of u_j and $u_j^{\rho_j}$ for all $0 < \rho_j < 1$, and
(ii) the following inequality holds for all $\rho_j > 0$:

$$\|u_j^{\rho_j}\|_{L^\infty} \le \|u_j\|_{L^\infty}. \tag{2.14}$$

Since $(u_j)^{\rho_j} \to u_j$ in $L^2(M)$ as $\rho_j \to 0+$, after passing to a subsequence we have

$$(u_j)^{\rho_j} \to u_j \quad \text{a.e. on } M, \quad \text{as } \rho_j \to 0+\,. \tag{2.15}$$

By (2.14) we have

$$|T_j(x)(u_j)^{\rho_j}(x)| \le |T_j(x)| \|u_j\|_{L^\infty}, \quad \text{a.e. on } M. \tag{2.16}$$

Since $T_j \in L^1_{\rm loc}(M)$, it follows that $T_j \in L^1(K_j)$.

By (2.15), (2.16) and since $T_j \in L^1(K_j)$, using dominated convergence theorem, we have

$$\lim_{\rho_j \to 0+} \int_M (T_j \cdot (u_j)^{\rho_j})\, d\nu = \lim_{\rho_j \to 0+} \int_{K_j} (T_j \cdot (u_j)^{\rho_j})\, d\nu = \int_{K_j} (T_j u_j)\, d\nu = \int_M (T_j u_j)\, d\nu,$$

and (2.13) is proven. Now, using (2.11), (2.12), (2.13) and the notations (1.4) and (1.5), we get

$$\begin{aligned} \langle T, u\rangle &= \sum_{j=1}^k \langle T_j, u_j\rangle \\ &= \sum_{j=1}^k \lim_{\rho_j \to 0+} \langle T_j, (u_j)^{\rho_j}\rangle \\ &= \sum_{j=1}^k \lim_{\rho_j \to 0+} \int_M (T_j \cdot (u_j)^{\rho_j})\, d\nu \\ &= \sum_{j=1}^k \int_M (T_j u_j)\, d\nu \;=\; \int_M (T \cdot u)\, d\nu. \end{aligned} \tag{2.17}$$

This concludes the proof of the Lemma. □

Proof of Theorem 1.3. Let $u \in [W^{1,2}(M)]^k$. By definition of $[W^{1,2}(M)]^k$ in Section 1, there exists a sequence $v^m \in [C_c^\infty(M)]^k$ such that $v^m \to u$ in $[W^{1,2}(M)]^k$, as $m \to +\infty$. In particular, $v^m \to u$ in $[L^2(M)]^k$, and, hence, we can extract a subsequence, again denoted by v^m, such that $v^m \to u$ a.e. on M.

Define a sequence λ^m by

$$\lambda^m := \Big(|u|^2 + \frac{1}{m^2}\Big)^{-1/2} \min\Big\{\Big(|u|^2 + \frac{1}{m^2}\Big)^{1/2} - \frac{1}{m}, \Big(|v^m|^2 + \frac{1}{m^2}\Big)^{1/2} - \frac{1}{m}\Big\},$$

where v^m is the chosen subsequence of v^m such that $v^m \to u$ a.e. on M, as $m \to +\infty$. Clearly, $0 \le \lambda^m \le 1$. Define

$$w^m := \lambda^m u. \tag{2.18}$$

We know that $u \in [W^{1,2}(M)]^k$ and $v^m \in [C_c^\infty(M)]^k$. Thus, by Lemma 2.7, for all $m = 1, 2, 3, \dots$, we have $w^m \in [W^{1,2}(M) \cap L^\infty(M)]^k$, and

$$|d(w^m)| \le 3 \max\{|du|, |d(v^m)|\}, \tag{2.19}$$

where $|du|$ is as in (1.2). Furthermore, for all $m = 1, 2, 3, \dots$, we have

$$|w^m(x)| \le |u(x)|, \tag{2.20}$$

where $|\cdot|$ is as in (1.2).

Since $u \in [L^2(M)]^k$, by (2.20) it follows that $\{w^m\}$ is a bounded sequence in $[L^2(M)]^k$. Since $v^m \to u$ in $[W^{1,2}(M)]^k$, it follows that the sequence $\{v^m\}$ is bounded in $[W^{1,2}(M)]^k$. In particular, the sequence $\{d(v^m)\}$ is bounded in

$[L^2(\Lambda^1 T^*M)]^k$. Hence, by (2.19) it follows that $\{d(w^m)\}$ is a bounded sequence in $[L^2(\Lambda^1 T^*M)]^k$. Therefore, $\{w^m\}$ is a bounded sequence in $[W^{1,2}(M)]^k$. By Lemma V.1.4 in [7] it follows that there exists a subsequence of $\{w^m\}$, which we again denote by $\{w^m\}$, such that w^m converges weakly to some $z \in [W^{1,2}(M)]^k$. This means that for every continuous linear functional $A \in [W^{-1,2}(M)]^k$, we have

$$A(w_m) \to A(z), \quad \text{as } m \to +\infty.$$

Since

$$[W^{1,2}(M)]^k \subset [L^2(M)]^k \subset [W^{-1,2}(M)]^k,$$

it follows that $w^m \to z$ in weakly $[L^2(M)]^k$.

We will now show that, as $m \to +\infty$, $w^m \to u$ in $[L^2(M)]^k$. By definition of w^m in (2.18) it follows that $w^m \to u$ a.e. on M. Since $u \in [L^2(M)]^k$, using (2.20) and dominated convergence theorem we get $w^m \to u$ in $[L^2(M)]^k$, as $m \to +\infty$.

In particular, $w^m \to u$ weakly in $[L^2(M)]^k$. Therefore, by the uniqueness of the weak limit (see, for example, the beginning of Section III.1.6 in [7]), we have $z = u$. Therefore, $w^m \to u$ weakly in $[W^{1,2}(M)]^k$.

Thus, since $T \in [W^{-1,2}(M)]^k$, we have

$$\langle T, w^m\rangle \to \langle T, u\rangle, \quad \text{as } m \to +\infty. \tag{2.21}$$

By the definition of λ^m and (2.18) it follows that

$$|w^m(x)| \le |v^m(x)|. \tag{2.22}$$

Since $v^m \in [C_c^\infty(M)]^k$, by (2.22) it follows that the functions w^m have compact support. We have shown earlier that $w^m \in [W^{1,2}(M) \cap L^\infty(M)]^k$. Thus, by Lemma 2.8, the following equality holds:

$$\langle T, w^m\rangle = \int_M (T \cdot w^m)\, d\nu. \tag{2.23}$$

Let f be as in the hypotheses of the Theorem. Then

$$T \cdot w^m = T \cdot (\lambda^m u) = \lambda^m (T \cdot u) \ge \lambda^m f \ge -|f|. \tag{2.24}$$

By (2.24) it follows that $T \cdot w^m + |f| \ge 0$. Consider the sequence $T \cdot w^m + |f|$. Since $f \in L^1(M)$ and $(T \cdot w^m) \in L^1(M)$, by Fatou's lemma we get

$$\int_M \liminf_{m\to+\infty} (T \cdot w^m + |f|)\, d\nu \le \liminf_{m\to+\infty} \int_M (T \cdot w^m + |f|)\, d\nu. \tag{2.25}$$

Since $w^m \to u$ a.e. on M as $m \to +\infty$, we have $T \cdot w^m \to T \cdot u$ a.e. on M as $m \to +\infty$. Thus, by (2.25) we have

$$\int_M (T \cdot u + |f|)\, d\nu \le \int_M |f|\, d\nu + \liminf_{m\to+\infty} \int_M (T \cdot w^m)\, d\nu,$$

and, hence, by (2.23) and (2.21) we have

$$\begin{aligned}
\int_M (T \cdot u + |f|)\, d\nu &\le \int_M |f|\, d\nu + \liminf_{m\to+\infty} \int_M (T \cdot w^m)\, d\nu \\
&= \int_M |f|\, d\nu + \liminf_{m\to+\infty} \langle T, w^m\rangle \\
&= \int_M |f|\, d\nu + \langle T, u\rangle.
\end{aligned}$$

Since $f \in L^1(M)$, we have $(T \cdot u + |f|) \in L^1(M)$, and, hence, $(T \cdot u) \in L^1(M)$. We have

$$|T \cdot w^m| = |\lambda^m (T \cdot u)| \leq |T \cdot u|,$$

and by definition of w^m, we get, as $m \to +\infty$,

$$T \cdot w^m \to T \cdot u, \quad \text{a.e. on } M.$$

Using dominated convergence theorem, we get

$$\lim_{m \to +\infty} \int_M (T \cdot w^m)\, d\nu = \int_M (T \cdot u)\, d\nu \tag{2.26}$$

By (2.26), (2.23) and (2.21), we get

$$\langle T, u \rangle = \int_M (T \cdot u)\, d\nu.$$

This concludes the proof of the Theorem. □

Proof of Corollary 1.4. Let $T_1 = \operatorname{Re} T$ and $T_2 = \operatorname{Im} T$. Let $u_1 = \operatorname{Re} u$ and $u_2 = \operatorname{Im} u$. Then $\operatorname{Re}\langle T, u \rangle = \langle T_1, u_1 \rangle + \langle T_2, u_2 \rangle$ and $\operatorname{Re}(T \cdot \bar{u}) = T_1 u_1 + T_2 u_2$. Thus, Corollary 1.4 follows from Theorem 1.3. □

References

[1] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer-Verlag, Berlin, 1998.

[2] M. Braverman, O. Milatovic, M. Shubin, Essential self-adjointness of Schrödinger type operators on manifolds, *Russian Math. Surveys*, **57**(4) (2002), 641–692.

[3] H. Brézis, F. E. Browder, Sur une propriété des espaces de Sobolev, *C. R. Acad. Sci. Paris Sr. A-B*, **287**, no. 3, (1978), A113–A115. (French).

[4] H. Brézis, T. Kato, Remarks on the Schrödinger operator with singular complex potentials, *J. Math. Pures Appl.*, **58**(9) (1979), 137–151.

[5] G. Friedlander, M. Joshi, *Introduction to the Theory of Distributions*, Cambridge University Press, 1998.

[6] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York, 1998.

[7] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1980.

Ognjen Milatovic

Department of Mathematics and Statistics, University of North Florida, Jacksonville, FL 32224, USA

E-mail address: `omilatov@unf.edu`