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BOUNDEDNESS AND ASYMPTOTIC STABILITY IN A CHEMOTAXIS MODEL WITH INDIRECT SIGNAL PRODUCTION AND LOGISTIC SOURCE

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Abstract. This article concerns the chemotaxis-growth system with indirect signal production

$$\begin{gathered}u_t=\Delta u-\nabla\cdot(u\nabla v)+\mu u(1-u),\quad x\in\Omega,\ t>0,\\ 0=\Delta v-v+w,\quad x\in\Omega,\ t>0,\\ w_t=-\delta w+u,\quad x\in\Omega,\ t>0,\end{gathered}$$

on a smooth bounded domain $\Omega\subset\mathbb{R}^n$ $(n\geq 1)$ with homogeneous Neumann boundary condition, where the parameters $\mu,\delta>0$. It is proved that if $n\leq 2$ and $\mu>0$, for all suitably regular initial data, this model possesses a unique global classical solution which is uniformly-in-time bounded. While in the case $n\geq 3$, we show that if μ is sufficiently large, this system possesses a global bounded solution. Furthermore, the large time behavior and rates of convergence have also been considered under some explicit conditions.

1. Introduction

In this article, we consider the chemotaxis model with indirect signal production

$$\begin{gathered}u_t=\Delta u-\nabla\cdot(u\nabla v)+\mu u(1-u),\quad x\in\Omega,\ t>0,\\ 0=\Delta v-v+w,\quad x\in\Omega,\ t>0,\\ w_t=-\delta w+u,\quad x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=0,\quad x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad w(x,0)=w_0(x),\quad x\in\Omega,\end{gathered}\tag{1.1}$$

where $\Omega\subset\mathbb{R}^n(n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\partial/\partial\nu$ denotes the derivative with respect to the outer normal of $\partial\Omega$, the parameters $\mu>0$ and $\delta>0$. System (1.1) describes the spread and aggregative behavior of the Mountain Pine Beetle (MPB) in a forest habitat, u represents the density of flying MPB, v denotes the concentration of beetle pheromone, w is the density of nesting MPB. We can refer to [28] for more details about the biological background.

Chemotaxis is the biased movement of the cells as a response to gradients of the concentration of the chemical signal substance, which plays an important role in

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numerous biological fields such as cell sorting, wound healing, pattern formation, embryonic morphogenesis and bacteria aggregation [7, 10, 16, 28]. The renowned chemotaxis system was put forward by Keller and Segel in 1970 [17] to describe the collective behavior of cells. And since then, many mathematicians have widely studied different types chemotaxis system for a variety of chemotaxis processes [4], the prototypical chemotaxis system with logistic source under homogeneous Neumann boundary condition reads as follows:

$$\begin{gathered} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^{\gamma}, \quad x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, \quad x \in \Omega,\ t > 0, \end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary, $\chi > 0, \gamma > 1$ and $\kappa, \mu, \tau \geq 0$, u and v is the density of the cells and the concentration of the chemical signal substance, respectively. In the case $\tau = 0$ and $\gamma = 2$, if either $n \leq 2$ or $\mu > \frac{n-2}{n}\chi$, Tello and Winkler [33] proved that the model (1.2) possesses a global bounded classical solution. Moreover, the global existence and boundedness of weak solutions to (1.2) was proved in [33, 40]. When the second equation in (1.2) is replaced by $0 = v - m(t) + u$ with $m(t) := \frac{1}{|\Omega|} \int_\Omega u(x,t)dx$, Winkler [39] proved that the solution of (1.2) blows up in finite time if $n \geq 5$ and $\gamma < \frac{3}{2} + \frac{1}{2n-2}$. On the other hand, in the case $\tau = 1$ and $\gamma = 2$, if either $n = 2$ or $n \geq 3$ and $\mu > 0$ is suitably large, then the solution of system (1.2) is globally bounded [38]. Model (1.2) has also been extensively studied by many other authors [2, 5, 6, 18, 24, 31, 41, 42, 43].

In model (1.2), we know that the chemical signal is directly produced by cells themselves, however, in some realistic biological processes the chemical signal production through the intermediate stages, and such indirect signal production mechanisms may have new properties [13]. Before stating our main results in the present paper, we consider the system

$$\begin{gathered} u_t = \Delta u - \nabla \cdot (u \nabla v) + \mu(u - u^{\alpha}), \quad x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + w, \quad x \in \Omega,\ t > 0, \\ w_t = -\delta w + u, \quad x \in \Omega,\ t > 0, \end{gathered} \tag{1.3}$$

model (1.3) has been intensively studied by many authors. For instance, if $\mu = 0$ and the second equation in (1.3) is replaced by $0 = \Delta v - \frac{1}{|\Omega|} \int_\Omega w(x,t)dx + w$, in the case $n = 2$, Tao and Winkler [32] proved that under the assumption $\int_\Omega u_0 < 8\pi\delta$ with $\delta > 0$ the solution remain uniformly bounded, if $\int_\Omega u_0 > 8\pi\delta$ with $\delta \geq 0$, the corresponding solution blow-up in infinite time. Moreover, if $n = 2, \alpha > 1$, the boundedness of solution in (1.3) was showed in [20]. If $\mu > 0, \alpha = 2$, Hu and Tao [15] obtained that the solution of (1.3) is uniformly-in-time bounded in three dimensional setting, while under the assumption that $\mu > \frac{1}{8\delta^2}$ the solution will exponentially converges to $(1, \frac{1}{\delta}, \frac{1}{\delta})$ as $t \to \infty$. Model (1.3) has also been intensively studied by many other authors [9, 12, 25, 26, 34, 35].

Inspired by the arguments in [15, 32], we mainly consider the boundedness and asymptotic behavior of solutions to (1.1). In this paper, we suppose that the initial data (u_0, w_0) satisfies

$$\begin{gathered} u_0 \in C^0(\overline{\Omega}), \quad u_0 \geq 0\ (\not\equiv 0) \quad \text{in } \overline{\Omega}, \\ w_0 \in C^1(\overline{\Omega}), \quad w_0 \geq 0 \quad \text{in } \overline{\Omega}. \end{gathered} \tag{1.4}$$

Our main results are stated as follows. The first of our results addresses the corresponding low-dimensional framework.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \le 2)$ be a bounded domain with smooth boundary and $\mu, \delta > 0$. Suppose that* (1.4) *holds, then* (1.1) *possesses a unique global classical solution (u, v, w) which is uniformly bounded in the sense that*
$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } t > 0$$
with some constant $C > 0$.

Next, we consider the global boundedness of solution to (1.1) in the case $n \ge 3$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$ be a bounded domain with smooth boundary and $\delta > 0$. Assume that μ is sufficiently large and* (1.4) *holds, then* (1.1) *possesses a unique global classical solution (u, v, w) which is uniformly bounded.*

Finally, motivated by the ideas in [15, 36], we shall investigate asymptotic behavior of solutions to (1.1) under the condition $\mu > 1/(16\delta^2)$.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n (n \ge 1)$ be a bounded domain with smooth boundary and $\delta > 0$. Suppose that (u, v, w) is a global bounded classical solution of* (1.1)*, and the initial data (u_0, w_0) satisfies* (1.4)*. Assume that*
$$\mu > \frac{1}{16\delta^2},$$
then there exist $C > 0$ and $\kappa > 0$ such that
$$\|u - 1\|_{L^\infty(\Omega)} + \|v - \frac{1}{\delta}\|_{L^\infty(\Omega)} + \|w - \frac{1}{\delta}\|_{L^\infty(\Omega)} \le Ce^{-\kappa t} \quad \text{for all } t > 0.$$

Remark 1.4. Under the assumption $\mu > 1/(8\delta^2)$, Hu and Tao [15] proved that the asymptotic behavior of the solutions to (1.3) in the case $\alpha = 2$. However, for the simplification (1.1), this condition $\mu > 1/(8\delta^2)$ in [15] can be relaxed to $\mu > 1/(16\delta^2)$.

The rest of this article is organized as follows. In the next section, we give the local existence of a solution to (1.1) and some important inequalities for later proofs. Section 3, we main consider the global boundedness of solution to (1.1) in the case $n = 2$. In the case when $n = 1$ and $n \ge 3$, the global boundedness of solution will be shown in Section 4. Section 5 is concern with the asymptotic behavior of the solution to system (1.1).

2. Preliminaries

We first state the following result related to local existence of a classical solution to (1.1), for the detail proof, we refer the reader to [27, 30].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n (n \ge 1)$ be a bounded domain with smooth boundary and $\mu, \delta > 0$. Assume that* (1.4) *holds, then there exist $T_{\max} \in (0, \infty]$ and a unique triple (u, v, w) of nonnegative functions:*
$$\begin{gathered} u \in C^0\left(\overline{\Omega} \times [0, T_{\max})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{\max})\right), \\ v \in C^{2,0}\left(\overline{\Omega} \times [0, T_{\max})\right), \\ w \in C^{0,1}\left(\overline{\Omega} \times [0, T_{\max})\right), \end{gathered}$$
which solves (1.1) *classically in $\Omega \times (0, T_{\max})$ and has the following extensibility property*
$$\text{if } T_{\max} < \infty \text{ then } \|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\max}. \tag{2.1}$$

The Gagliardo-Nirenberg inequality will be used in our later proofs. For details, we refer the reader to [11, 23].

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary, and let $p \geq 1$ and $r \in (0, p)$. Then there exists a constant $C_{GN} > 0$ such that*

$$\|\varphi\|_{L^p(\Omega)} \leq C_{GN} \left(\|\nabla \varphi\|^{\lambda}_{L^2(\Omega)} \|\varphi\|^{1-\lambda}_{L^r(\Omega)} + \|\varphi\|_{L^r(\Omega)} \right)$$

for all $\varphi \in W^{1,2}(\Omega) \cap L^r(\Omega)$ holds with $\lambda \in (0,1)$ satisfies

$$\frac{n}{p} = \lambda \left(\frac{n}{2} - 1 \right) + \frac{n}{r}(1 - \lambda),$$

that is,

$$\lambda = \frac{\frac{n}{r} - \frac{n}{p}}{1 - \frac{n}{2} + \frac{n}{r}}.$$

Finally, we give the following property on the total mass of the first equation in (1.1) which can easily be achieved.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and $\mu, \delta > 0$. Then there exists a constant $K > 0$ such that*

$$\int_\Omega u(\cdot, t) \leq K \quad \text{for all } t \in (0, T_{\max}). \tag{2.2}$$

Proof. Integrating the first equation in (1.1) with respect to $x \in \Omega$, we have

$$\frac{d}{dt} \int_\Omega u = \mu \int_\Omega u - \mu \int_\Omega u^2 \quad \text{for all } t \in (0, T_{\max}).$$

Using Hölder's inequality we can deduce that

$$\frac{d}{dt} \int_\Omega u \leq \mu \int_\Omega u - \frac{\mu}{|\Omega|} \Big(\int_\Omega u \Big)^2 \quad \text{for all } t \in (0, T_{\max}),$$

which implies

$$\|u(\cdot, t)\|_{L^1(\Omega)} \leq K := \max\{\|u_0\|_{L^1(\Omega)}, |\Omega|\} \quad \text{for all } t \in (0, T_{\max}).$$

Hence, this proof can be completed. □

3. A priori estimate for $n = 2$

In this section, we consider the global boundedness of solutions to (1.1) with $n = 2$. Firstly, to prove Lemma 3.2, we shall need the following basic inequality.

Lemma 3.1 ([15, Lemma 2.3])**.** *Let $\mu > 0$ and $\delta > 0$. Then there exists $M := M(\mu, \delta) > 0$ such that*

$$\mu s + \big(\frac{2}{\delta} - \mu\big) s^2 + (\mu + 1) s \ln s - \mu s^2 \ln s \leq M \quad \text{for all } s > 0.$$

Next, we start to establish a bound for $\int_\Omega w^2$.

Lemma 3.2. *Let $\mu, \delta > 0$ and suppose that (1.4) holds. Then there exists a constant $C > 0$ such that the classical solution of (1.1) satisfying*

$$\int_\Omega w^2 \leq C \quad \text{for all } t \in (0, T_{\max}). \tag{3.1}$$

Proof. Testing the first equation in (1.1) by $\ln u + 1$ and integrating we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u\ln u \\ &= \int_\Omega (\ln u+1)\Delta u - \int_\Omega (\ln u+1)\nabla\cdot(u\nabla v) + \mu\int_\Omega (\ln u+1)u(1-u) \\ &= -\int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega \nabla u\cdot\nabla v + \mu\int_\Omega (\ln u+1)u(1-u) \\ &\le -\int_\Omega u\Delta v + \mu\int_\Omega (\ln u+1)u(1-u) \\ &\le \int_\Omega uw + \mu\int_\Omega u + \mu\int_\Omega u\ln u - \mu\int_\Omega u^2 - \mu\int_\Omega u^2\ln u \end{aligned} \tag{3.2}$$

for all $t\in(0,T_{\max})$. Next, testing the third equation in (1.1) by w and integrating with respect to $x\in\Omega$, we see that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 = -\delta\int_\Omega w^2 + \int_\Omega uw \quad \text{for all } t\in(0,T_{\max}). \tag{3.3}$$

Combining (3.2) and (3.3) and adding $\int_\Omega u\ln u$ to both sides of this, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big\{\int_\Omega u\ln u + \frac{1}{2}\int_\Omega w^2\Big\} + \int_\Omega u\ln u \\ &\le 2\int_\Omega uw - \delta\int_\Omega w^2 + \mu\int_\Omega u + (\mu+1)\int_\Omega u\ln u - \mu\int_\Omega u^2 - \mu\int_\Omega u^2\ln u \end{aligned} \tag{3.4}$$

for all $t\in(0,T_{\max})$. We will deal with the first term on the right hand of (3.4). Applying Young's inequality we obtain

$$2\int_\Omega uw \le \frac{\delta}{2}\int_\Omega w^2 + \frac{2}{\delta}\int_\Omega u^2. \tag{3.5}$$

Therefore, combining (3.4) and (3.5) we have

$$\begin{aligned} &\frac{d}{dt}\Big\{\int_\Omega u\ln u + \frac{1}{2}\int_\Omega w^2\Big\} + \Big\{\int_\Omega u\ln u + \frac{\delta}{2}\int_\Omega w^2\Big\} \\ &\le \mu\int_\Omega u + (\mu+1)\int_\Omega u\ln u + \big(\frac{2}{\delta}-\mu\big)\int_\Omega u^2 - \mu\int_\Omega u^2\ln u \end{aligned} \tag{3.6}$$

for all $t\in(0,T_{\max})$. Letting $y(t) := \int_\Omega u\ln u + \frac{1}{2}\int_\Omega w^2$ and using Lemma 3.1, we have

$$y'(t) + C_1 y(t) \le C_2 \quad \text{for all } t\in(0,T_{\max}),$$

where $C_1 := \min\{1,\delta\}$ and $C_2 := M|\Omega|$ with M is defined in Lemma 3.1. Therefore, by the definition of $y(t)$ and together with the basic inequality $-s\ln s \le 1/e$ for all $s>0$, we obtain (3.1). □

To obtain (3.8) below, we give the following important lemma.

Lemma 3.3. *Let $n=2$ and suppose that (1.4) holds. Then for any $1<q<\infty$, one can find a constant $C>0$ such that*

$$\int_\Omega |\nabla v|^q \le C \quad \text{for all } t\in(0,T_{\max}). \tag{3.7}$$

Proof. Noting that v satisfies

$$-\Delta v+v=w,\quad x\in\Omega,\ t>0,$$
$$\frac{\partial v}{\partial\nu}=0,\quad x\in\partial\Omega,\ t>0.$$

From (3.1) and elliptic equation theory, for some $C_1>0$ and $C_2>0$, we infer that

$$\|v\|_{W^{2,2}(\Omega)}\le C_1\|w\|_{L^2(\Omega)}\le C_2\quad\text{for all } t\in(0,T_{\max}).$$

Since $n=2$ and according to Sobolev embedding theory, for any $1<q<\infty$, we see that (3.7). □

Finally, our aim is to sutdy the boundedness of $\int_\Omega u^p+\int_\Omega w^{p+1}$.

Lemma 3.4. *Let $n=2$, $\mu,\delta>0$, and suppose that* (1.4) *holds. Then for any $p>1$, there exists a constant $C>0$ such that*

$$\int_\Omega u^p+\int_\Omega w^{p+1}\le C\quad\textit{for all } t\in(0,T_{\max}).\tag{3.8}$$

Proof. Testing the first equation in (1.1) by u^{p-1}, integrating by parts and using (3.7) and Young's inequality, for some $C_1>0$, we obtain

$$\begin{aligned}
&\frac1p\frac{d}{dt}\int_\Omega u^p\\
&=-(p-1)\int_\Omega u^{p-2}|\nabla u|^2+(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v+\mu\int_\Omega u^p-\mu\int_\Omega u^{p+1}\\
&\le-\frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2+\frac{p-1}{2}\int_\Omega u^p|\nabla v|^2+\mu\int_\Omega u^p-\mu\int_\Omega u^{p+1}\\
&\le-\frac{2(p-1)}{p^2}\int_\Omega|\nabla u^{p/2}|^2+\frac{p-1}{2}\int_\Omega u^{p+\frac12}+\frac{p-1}{2}\int_\Omega|\nabla v|^{4p+2}\\
&\quad+\mu\int_\Omega u^p-\mu\int_\Omega u^{p+1}\\
&\le-\frac{2(p-1)}{p^2}\int_\Omega|\nabla u^{p/2}|^2+\frac{p-1}{2}\int_\Omega u^{p+\frac12}+\mu\int_\Omega u^p-\mu\int_\Omega u^{p+1}+C_1
\end{aligned}\tag{3.9}$$

for all $t\in(0,T_{\max})$. Now we will cope with the second term on the right-hand side of (3.9). According to Lemma 2.2 and (2.2), there exist $C_2>0$ and $C_3>0$ such that

$$\begin{aligned}
&\int_\Omega u^{p+\frac12}=\|u^{p/2}\|^{\frac{2p+1}{p}}_{L^{\frac{2p+1}{p}}(\Omega)}\\
&\le C_2\|\nabla u^{p/2}\|^{\frac{2p+1}{p}\cdot\theta}_{L^2(\Omega)}\|u^{p/2}\|^{\frac{2p+1}{p}\cdot(1-\theta)}_{L^{2/p}(\Omega)}+C_2\|u^{p/2}\|^{\frac{2p+1}{p}}_{L^{2/p}(\Omega)}\\
&\le C_3\|\nabla u^{p/2}\|^{\frac{2p+1}{p}\cdot\theta}_{L^2(\Omega)}+C_3
\end{aligned}\tag{3.10}$$

for all $t\in(0,T_{\max})$, where $\theta=\frac{2p-1}{2p+1}\in(0,1)$, thus

$$\frac{2p+1}{p}\theta=\frac{2p-1}{p}=2-\frac1p<2.$$

Applying Young's inequality, for some $C_4>0$, we can derive

$$\frac{p-1}{2}\int_\Omega u^{p+\frac12}\le\frac{2(p-1)}{p^2}\|\nabla u^{p/2}\|^2_{L^2(\Omega)}+C_4.\tag{3.11}$$

Combining (3.9) and (3.11), for some $C_5 > 0$, we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \le \mu\int_\Omega u^p - \mu\int_\Omega u^{p+1} + C_5 \tag{3.12}$$

for all $t \in (0, T_{\max})$. Adding $\int_\Omega u^p$ to both sides of (3.12) we obtain

$$\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le (\mu p+1)\int_\Omega u^p - \mu p\int_\Omega u^{p+1} + C_5 p \tag{3.13}$$

for all $t \in (0, T_{\max})$. Next applying Young's inequality, for some $C_6 > 0$, we have

$$\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p + \frac{\mu p}{2}\int_\Omega u^{p+1} \le C_6$$

for all $t \in (0, T_{\max})$, which implies

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max}), \tag{3.14}$$

$$\int_t^{t+\tau}\int_\Omega u^{p+1} \le C \quad \text{for all } t \in (0, T_{\max} - \tau) \tag{3.15}$$

where $\tau := \min\{1, \frac{1}{2}T_{\max}\}$. Testing the third equation in (1.1) by w^p and integrating yields

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega w^{p+1} + \delta\int_\Omega w^{p+1} = \int_\Omega uw^p \le \frac{\delta}{2}\int_\Omega w^{p+1} + \Big(\frac{2}{\delta}\Big)^p\int_\Omega u^{p+1}$$

namely,

$$\frac{d}{dt}\int_\Omega w^{p+1} + \frac{\delta(p+1)}{2}\int_\Omega w^{p+1} \le (p+1)\big(\frac{2}{\delta}\big)^p\int_\Omega u^{p+1} \tag{3.16}$$

for all $t \in (0, T_{\max})$, which implies $\|w(\cdot,t)\|_{L^{p+1}(\Omega)}$ is bounded for all $t \in (0, T_{\max})$ because of (3.15). Hence, with (3.14), we have (3.8). $\square$

4. A priori estimate for $n = 1$ and $n \ge 3$

In this section, we establish the boundedness of $\int_\Omega u^p + \int_\Omega w^{p+1}$ in the case $n = 1$ and $n \ge 3$. To obtain our goals, we first establish a differential inequality.

Lemma 4.1. *Let $p > 1$ and $\mu, \delta > 0$, and suppose that* (1.4) *holds. Then we have*

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\Big\} + \delta\int_\Omega w^{p+1} + \frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{p/2}|^2 \\ &\le \frac{p-1}{p}\int_\Omega u^p w + \int_\Omega uw^p + \mu\int_\Omega u^p - \mu\int_\Omega u^{p+1} \end{aligned} \tag{4.1}$$

for all $t \in (0, T_{\max})$.

Proof. Multiplying the first equation in (1.1) by u^{p-1}, integrating by parts and combining the second equation in (1.1), we obtain

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \\
&= -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + (p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v + \mu\int_\Omega u^p - \mu\int_\Omega u^{p+1} \\
&= -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{p/2}|^2 + \frac{p-1}{p}\int_\Omega \nabla u^p\cdot\nabla v + \mu\int_\Omega u^p - \mu\int_\Omega u^{p+1} \\
&= -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{p/2}|^2 - \frac{p-1}{p}\int_\Omega u^p\cdot\Delta v + \mu\int_\Omega u^p - \mu\int_\Omega u^{p+1} \\
&\le -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{p/2}|^2 + \frac{p-1}{p}\int_\Omega u^p w + \mu\int_\Omega u^p - \mu\int_\Omega u^{p+1}
\end{aligned}
\tag{4.2}
$$

for all $t\in(0,T_{\max})$. Next, testing the third equation in (1.1) by w^p and integrating, we have

$$
\frac{1}{p+1}\frac{d}{dt}\int_\Omega w^{p+1} + \delta\int_\Omega w^{p+1} = \int_\Omega uw^p \quad \text{for all } t\in(0,T_{\max}). \tag{4.3}
$$

Therefore, combining (4.2) and (4.3) we obtain (4.1). □

Next, we will establish a bound for $\int_\Omega u^p + \int_\Omega w^{p+1}$ in the case $n=1$.

Lemma 4.2. *Let $n=1$ and $\mu,\delta>0$, and suppose that (1.4) holds. Then for any $p>1$, there exists a constant $C>0$ such that*

$$
\int_\Omega u^p + \int_\Omega w^{p+1} \le C \quad \textit{for all } t\in(0,T_{\max}). \tag{4.4}
$$

Proof. Using Young's inequality, there exists $C_1>0$ such that

$$
\frac{p-1}{p}A^pB + AB^p \le \frac{\delta}{2}B^{p+1} + C_1A^{p+1} \quad \text{for all } p>1 \text{ and } A,B\ge 0. \tag{4.5}
$$

Therefore, combining (4.1) and (4.5) and applying Young's inequality, for some $C_2>0, C_3>0$, we can obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\Big\} + \delta\int_\Omega w^{p+1} + \frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{p/2}|^2 \\
&\le C_2\int_\Omega u^{p+1} + \frac{\delta}{2}\int_\Omega w^{p+1} + C_3,
\end{aligned}
\tag{4.6}
$$

that is,

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\Big\} + \frac{\delta}{2}\int_\Omega w^{p+1} + \frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{p/2}|^2 \\
&\le C_2\int_\Omega u^{p+1} + C_3
\end{aligned}
\tag{4.7}
$$

for all $t\in(0,T_{\max})$. Now we deal with the term $C_2\int_\Omega u^{p+1}$ in (4.7). Since $n=1$, using Lemma 2.2 there exists $C_{GN}>0$ such that

$$
\begin{aligned}
\int_\Omega u^{p+1} &= \|u^{p/2}\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\
&\le C_{GN}\|\nabla u^{p/2}\|_{L^2(\Omega)}^{\frac{2(p+1)}{p}\cdot\theta}\|u^{p/2}\|_{L^{2/p}(\Omega)}^{\frac{2(p+1)}{p}\cdot(1-\theta)} + C_{GN}\|u^{p/2}\|_{L^{2/p}(\Omega)}^{\frac{2(p+1)}{p}},
\end{aligned}
\tag{4.8}
$$

where

$$\theta := \frac{p^2}{p^2+2p+1} \in (0,1).$$

We can easily be checked that

$$\frac{2(p+1)}{p}\cdot\theta = \frac{2(p+1)}{p}\frac{p^2}{(p+1)^2} = \frac{2p}{p+1} < 2.$$

Therefore, using (2.2) and Young's inequality, for some $C_4>0$, we can derive

$$C_2\int_\Omega u^{p+1} \le \frac{2(p-1)}{p^2}\|\nabla u^{p/2}\|^2_{L^2(\Omega)} + C_4. \tag{4.9}$$

Combining (4.7) and (4.9) we have

$$\frac{d}{dt}\Big\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\Big\} + \frac{\delta}{2}\int_\Omega w^{p+1} + \frac{2(p-1)}{p^2}\int_\Omega |\nabla u^{p/2}|^2 \le C_5 \tag{4.10}$$

for all $t\in(0,T_{\max})$, where $C_5=C_3+C_4$. Applying the Poincaré inequality, for some $C_6>0$, we obtain

$$\int_\Omega \varphi^2 \le C_6\int_\Omega |\nabla\varphi|^2 + C_6\Big(\int_\Omega |\varphi|^{p/2}\Big)^p \quad \text{for all } \varphi\in W^{1,2}(\Omega). \tag{4.11}$$

Therefore, combining (4.11) and (2.2) there exists $C_7>0$ such that

$$\frac{2(p-1)}{p^2}\int_\Omega |\nabla u^{p/2}|^2 \ge \frac{2(p-1)}{C_7p^2}\int_\Omega u^p - \frac{2(p-1)}{p^2}K^p \quad \text{for all } t\in(0,T_{\max}). \tag{4.12}$$

From (4.10) and (4.12) we have

$$\frac{d}{dt}\Big\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\Big\} + \Big\{\frac{2(p-1)}{C_7p^2}\int_\Omega u^p + \frac{\delta}{2}\int_\Omega w^{p+1}\Big\} \le C_8 \tag{4.13}$$

for all $t\in(0,T_{\max})$, where $C_8 = C_5 + \frac{2(p-1)}{p^2}K^p$. Letting $y(t) := \frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}$ and $C_9 = \min\{\frac{2(p-1)}{C_7 p}, \frac{\delta(p+1)}{2}\}$ we obtain

$$y'(t) + C_9 y(t) \le C_8 \quad \text{for all } t\in(0,T_{\max}).$$

By the definition of $y(t)$, for some $C_{10}>0$ we can derive

$$\int_\Omega u^p \le C_{10} \quad\text{and}\quad \int_\Omega w^{p+1}\le C_{10} \quad \text{for all } t\in(0,T_{\max}). \tag{4.14}$$

The proof is complete □

Next, we consider the boundedness of $\int_\Omega u^p + \int_\Omega w^{p+1}$ in the case $n\ge 3$.

Lemma 4.3. *Let $\Omega\subset\mathbb{R}^n (n\ge3)$ be a bounded domain with smooth boundary and $\delta>0$, and suppose that (1.4) holds. Then for $p>1$ and μ sufficiently large, there exists a constant $C>0$ such that*

$$\int_\Omega u^p + \int_\Omega w^{p+1} \le C \quad \textit{for all } t\in(0,T_{\max}). \tag{4.15}$$

Proof. According to (4.1) and (4.5) and using Young's inequality, for some $C_1 > 0, C_2 > 0$, we have

$$\begin{aligned} &\frac{d}{dt}\Big\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\Big\} + \delta\int_\Omega w^{p+1} + \frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{p/2}|^2 \\ &\le \frac{p-1}{p}\int_\Omega u^p w + \int_\Omega u w^p + \mu\int_\Omega u^p - \mu\int_\Omega u^{p+1} \\ &\le \frac{\delta}{2}\int_\Omega w^{p+1} - \big(\frac{\mu}{2} - C_1\big)\int_\Omega u^{p+1} + C_2 \end{aligned} \tag{4.16}$$

for all $t \in (0, T_{\max})$. Since μ is sufficiently large, we can derive $\frac{\mu}{2} - C_1 > 0$. Thus, we know that

$$\frac{d}{dt}\Big\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\Big\} + \frac{\delta}{2}\int_\Omega w^{p+1} + \frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{p/2}|^2 \le C_2 \tag{4.17}$$

for all $t \in (0, T_{\max})$. Using the Poincaré inequality and (2.2), there exists $C_3 > 0$ such that

$$\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{p/2}|^2 \ge \frac{4(p-1)}{C_3 p^2}\int_\Omega u^p - \frac{4(p-1)}{p^2}K^p \quad \text{for all } t \in (0, T_{\max}). \tag{4.18}$$

Thus, combining (4.17) and (4.18) we obtain

$$\frac{d}{dt}\Big\{\frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}\Big\} + \Big\{\frac{4(p-1)}{C_3 p^2}\int_\Omega u^p + \frac{\delta}{2}\int_\Omega w^{p+1}\Big\} \le C_4, \tag{4.19}$$

where $C_4 = C_2 + \frac{4(p-1)}{p^2}K^p$. Letting $y(t) := \frac{1}{p}\int_\Omega u^p + \frac{1}{p+1}\int_\Omega w^{p+1}$ we obtain

$$y'(t) + C_5 y(t) \le C_4 \quad \text{for all } t \in (0, T_{\max}),$$

where $C_5 = \min\Big\{\frac{4(p-1)}{C_3 p}, \frac{\delta(p+1)}{2}\Big\}$. By an ODE comparison argument and the definition of $y(t)$, we can derive (4.15). $\square$

Finally, we will give a boundedness criterion for the solutions of (1.1).

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^n (n \ge 1)$ be a bounded domain with smooth boundary and (1.4) hold. If there exist $p \ge 2$ such that $p > n-1$ and that*

$$\int_\Omega u^p + \int_\Omega w^{p+1} \le C(p) \quad \textit{for all } t \in (0, T_{\max}) \tag{4.20}$$

with a constant $C(p) > 0$, then $T_{\max} = \infty$ and

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty \quad \textit{for all } t > 0. \tag{4.21}$$

Proof. Noting that v satisfies

$$\begin{gathered} -\Delta v + v = w, \quad x \in \Omega,\ t > 0, \\ \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0. \end{gathered}$$

According to (4.20) and elliptic regularity theory, one can find $c_1 > 0$ such that

$$\|v(\cdot, t)\|_{W^{2,p+1}(\Omega)} \le c_1 \quad \text{for all } t \in (0, T_{\max}).$$

Applying Sobolev embedding theory [14], fix $p > n-1$, for some $c_2 > 0$ independent of p, we can derive

$$\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \le c_2 \quad \text{for all } t \in (0, T_{\max}). \tag{4.22}$$

From the well-known Moser-Alikakos iteration [1, 8], testing the first equation of (1.1) by $u^{p-1}(p \geq 2)$ and integrating by parts over Ω, using (4.22) and Young's inequality we have

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \mu\int_\Omega u^{p+1} \\ &= (p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v + \mu\int_\Omega u^p \\ &\leq c_2(p-1)\int_\Omega u^{p-1}|\nabla u| + \mu(p-1)\int_\Omega u^p \\ &\leq \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \big(\frac{c_2^2}{2}+\mu\big)(p-1)\int_\Omega u^p \end{aligned} \tag{4.23}$$

for all $t \in (0, T_{\max})$. Using the following interpolation inequality [29], we can find $c_3 > 0$ independent p such that

$$\|\varphi\|_{L^2(\Omega)}^2 \leq \varepsilon\|\nabla\varphi\|_{L^2(\Omega)}^2 + c_3\left(1+\varepsilon^{-\frac{n}{2}}\right)\|\varphi\|_{L^1(\Omega)}^2 \quad \text{for all } \varepsilon > 0. \tag{4.24}$$

Letting $c_4 = \frac{c_2^2}{2} + \mu + 1$, combining (4.23) and (4.24) we conclude

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p + \frac{2(p-1)}{p}\int_\Omega |\nabla u^{p/2}|^2 \leq c_4 p(p-1)\int_\Omega u^p \\ &\leq \frac{2(p-1)}{p}\int_\Omega |\nabla u^{p/2}|^2 + c_5 p(p-1)(1+p^n)\Big(\int_\Omega u^{p/2}\Big)^2 \end{aligned} \tag{4.25}$$

for all $t \in (0, T_{\max})$, where $c_5 = c_3 c_4 \max\{1, (c_4/2)^{n/2}\}$. Therefore, the inequality (4.25) can be rewritten as

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \leq c_5 p(p-1)(1+p^n)\Big(\int_\Omega u^{p/2}\Big)^2$$

for all $t \in (0, T_{\max})$, together with the fact $(1+p^n) \leq (1+p)^n$ we have

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \leq c_5 p(p-1)(1+p)^n\Big(\int_\Omega u^{p/2}\Big)^2$$

for all $t \in (0, T_{\max})$. Thus,

$$\frac{d}{dt}\Big[e^{p(p-1)t}\int_\Omega u^p\Big] \leq c_5 e^{p(p-1)t} p(p-1)(1+p)^n\Big(\int_\Omega u^{p/2}\Big)^2 \tag{4.26}$$

for all $t \in (0, T_{\max})$. Integrating (4.26) over $[0, t]$ for $0 < t < T_{\max}$, we obtain

$$\int_\Omega u^p(x,t) \leq \int_\Omega u_0^p(x) + c_5(1+p)^n \sup_{0\leq t\leq T_{\max}}\Big(\int_\Omega u^{p/2}(x,t)\Big)^2. \tag{4.27}$$

We define

$$G(p) := \max\Big\{\|u_0\|_{L^\infty(\Omega)}, \sup_{0\leq t\leq T_{\max}}\Big(\int_\Omega u^p(x,t)\Big)^{1/p}\Big\}. \tag{4.28}$$

Then, from (4.27) and (4.28) we obtain

$$G(p) \leq [c_6(1+p)^n]^{1/p} G(\frac{p}{2}) \quad \text{for all } p \geq 2,$$

where $c_6 = |\Omega| + c_5$. Taking $p = 2^j$, $j = 1, 2, \cdots$, we obtain

$$G(2^j) \leq c_6^{2^{-j}}\left(1+2^j\right)^{2^{-j}n} G\left(2^{j-1}\right) \leq \cdots$$

$$
\begin{aligned}
&\le c_6^{2^{-j}+\dots+2^{-1}}\left(1+2^j\right)^{2^{-j}n}\cdots(1+2)^{2^{-1}n}G(1)\\
&\le c_6[2^{j2^{-j}n}(2^{-j}+1)^{2^{-j}n}]\cdots[2^{2^{-1}n}(2^{-1}+1)^{2^{-1}n}]G(1)\\
&\le c_6 2^{[j2^{-j}+(j-1)2^{-(j-1)}+\dots+2^{-1}]n}\cdot 2^{[2^{-j}+2^{-(j-1)}+\dots+2^{-1}]n}G(1)\\
&\le c_6 2^{3n}G(1).
\end{aligned}
$$

Letting $j\to\infty$ and using (2.2) we infer that

$$
\|u(\cdot,t)\|_{L^\infty}\le c_6 2^{3n}G(1)\le C:=c_6 2^{3n}\max\{\|u_0\|_{L^\infty(\Omega)},\|u_0\|_{L^1(\Omega)}\} \tag{4.29}
$$

for all $t\in(0,T_{\max})$. Therefore, using Lemma 2.1, (4.29), and the extensibility criterion we can obtain (4.21). $\square$

Proof of Theorems 1.1 and 1.2. The results follow by using Lemmata 3.4, 4.2 and 4.3 and the boundedness criterion of Lemma 4.4. $\square$

5. Asymptotic behavior of solutions

In this section, we consider the asymptotic behavior of solutions to (1.1) under the assumption that μ is positive and properly large. Then the solution (u,v,w) will exponentially converges to the constants stationary solution $(1,1/\delta,1/\delta)$. Firstly, we give the following lemma which plays an important role for later proofs.

Lemma 5.1 ([3, Lemma 3.1]). *Suppose that $f:(1,\infty)$ is a uniformly continuous nonnegative function such that $\int_1^\infty f(t)dt<\infty$. Then $f(t)\to 0$ as $t\to\infty$.*

To proof of Theorem 1.3, we shall need the following basic property. For a detail proof, we refer the reader to [22, 36].

Lemma 5.2. *Let $a_1,a_2,a_3,a_4,a_5\in\mathbb{R}$ satisfy*

$$
a_1>0,\quad a_2>0,\quad a_1a_2a_3-\frac{a_1a_5^2}{4}-\frac{a_2a_4^2}{4}>0. \tag{5.1}
$$

Then there exists $\varepsilon>0$ such that

$$
a_1x^2+a_2y^2+a_3z^2+a_4xz+a_5yz\ge\varepsilon(x^2+y^2+z^2) \tag{5.2}
$$

for any $x,y,z\in\mathbb{R}$.

Now, we use an energy functional to establish the asymptotic behavior of solutions to system (1.1), and an idea from [15, 37, 44].

Lemma 5.3. *Let $\delta>0$ and (u,v,w) be a non-negative global bounded classical solution of (1.1). Assume that the initial data (u_0,w_0) satisfies (1.4). If*

$$
\mu>\frac{1}{16\delta^2}, \tag{5.3}
$$

then there exist $\epsilon>0$ and $\beta>0$, such that the functions

$$
E(t):=\int_\Omega(u-1-\ln u)+\frac{\beta}{2}\int_\Omega\left(w-\frac{1}{\delta}\right)^2, \tag{5.4}
$$

$$
F(t):=\int_\Omega(u-1)^2+\int_\Omega\left(v-\frac{1}{\delta}\right)^2+\int_\Omega\left(w-\frac{1}{\delta}\right)^2 \tag{5.5}
$$

satisfy

$$
E(t)\ge 0\quad\textit{for all } t>0, \tag{5.6}
$$

$$\frac{d}{dt}E(t) \le -\epsilon F(t) \quad \textit{for all } t > 0. \tag{5.7}$$

Proof. We define

$$\begin{aligned} A(t) &:= \int_\Omega (u - 1 - \ln u) \quad \text{for all } t > 0, \\ B(t) &:= \frac{1}{2}\int_\Omega \big(w - \frac{1}{\delta}\big)^2 \quad \text{for all } t > 0. \end{aligned}$$

Then we can rewritten (5.4) as

$$E(t) = A(t) + \beta \cdot B(t) \quad \text{for all } t > 0.$$

Next we prove (5.7). By the straightforward calculations we have

$$\begin{aligned} \frac{d}{dt}A(t) &= \int_\Omega u_t - \frac{1}{u}u_t \\ &= \int_\Omega \Big[\Delta u - \nabla\cdot(u\nabla v) + \mu u(1-u) \\ &\quad - \frac{1}{u}(\Delta u - \nabla\cdot(u\nabla v) + \mu u(1-u))\Big] \\ &= -\int_\Omega \frac{\Delta u}{u} + \int_\Omega \frac{1}{u}\cdot\nabla\cdot(u\nabla v) - \mu\int_\Omega (u-1)^2 \\ &= -\int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega \frac{1}{u}\nabla u\cdot\nabla v - \mu\int_\Omega (u-1)^2 \\ &\le \frac{1}{4}\int_\Omega |\nabla v|^2 - \mu\int_\Omega (u-1)^2. \end{aligned} \tag{5.8}$$

To estimate $\int_\Omega |\nabla v|^2$, testing the second equation of (1.1) by $\big(v - \frac{1}{\delta}\big)$ and integrating by parts we have

$$\begin{aligned} 0 &= \int_\Omega \Delta v\big(v - \frac{1}{\delta}\big) + \int_\Omega \big(w - \frac{1}{\delta}\big)\big(v - \frac{1}{\delta}\big) - \int_\Omega \big(v - \frac{1}{\delta}\big)^2 \\ &= -\int_\Omega |\nabla v|^2 + \int_\Omega \big(w - \frac{1}{\delta}\big)\big(v - \frac{1}{\delta}\big) - \int_\Omega \big(v - \frac{1}{\delta}\big)^2, \end{aligned} \tag{5.9}$$

that is

$$\int_\Omega |\nabla v|^2 = \int_\Omega \big(w - \frac{1}{\delta}\big)\big(v - \frac{1}{\delta}\big) - \int_\Omega \big(v - \frac{1}{\delta}\big)^2. \tag{5.10}$$

Combining (5.8) and (5.10), we obtain

$$\frac{d}{dt}A(t) \le \frac{1}{4}\int_\Omega \big(w - \frac{1}{\delta}\big)\big(v - \frac{1}{\delta}\big) - \frac{1}{4}\int_\Omega \big(v - \frac{1}{\delta}\big)^2 - \mu\int_\Omega (u-1)^2. \tag{5.11}$$

Rewriting the third equation in system (1.1) as

$$w_t = -\delta w + u = -\delta\big(w - \frac{1}{\delta}\big) + (u-1). \tag{5.12}$$

Multiplying the equation (5.12) by $\big(w - \frac{1}{\delta}\big)$ and using Young's inequality, we derive

$$\frac{d}{dt}B(t) = -\delta\int_\Omega \big(w - \frac{1}{\delta}\big)^2 + \int_\Omega (u-1)\big(w - \frac{1}{\delta}\big). \tag{5.13}$$

Therefore, combining (5.11) and (5.13) we infer that

$$\frac{d}{dt}E(t) \le -\mu\int_\Omega (u-1)^2 - \frac{1}{4}\int_\Omega \big(v - \frac{1}{\delta}\big)^2 - \delta\beta\int_\Omega \big(w - \frac{1}{\delta}\big)^2$$

$$+\frac{1}{4}\int_\Omega \big(w-\frac{1}{\delta}\big)\big(v-\frac{1}{\delta}\big)+\beta\int_\Omega (u-1)\big(w-\frac{1}{\delta}\big) \quad \text{for all } t>0.$$

Now we will prove that there exists $\epsilon>0$ such that

$$\frac{d}{dt}E(t)\le -\epsilon\Big\{\int_\Omega (u-1)^2+\int_\Omega \big(v-\frac{1}{\delta}\big)^2+\delta\int_\Omega \big(w-\frac{1}{\delta}\big)^2\Big\}. \tag{5.14}$$

We set $a_1:=\mu$, $a_2:=1/4$, $a_3:=\delta\beta$, $a_4:=-\beta$, $a_5:=-1/4$, $x:=u-1$, $y:=v-\frac{1}{\delta}$, and $z:=w-\frac{1}{\delta}$. From (5.3) and $\mu,\delta>0$, we obtain

$$a_1=\mu>0,\quad a_2=\frac{1}{4}>0,$$
$$a_1a_2a_3-\frac{a_1a_5^2}{4}-\frac{a_2a_4^2}{4}=\frac{\mu\delta\beta}{4}-\frac{\mu}{64}-\frac{\beta^2}{16}=\frac{16\mu\delta\beta-4\beta^2-\mu}{64}>0.$$

Using Lemma 5.2, we obtain there exists $\epsilon>0$ such that (5.14) holds. The proof is complete. □

In view of Lemma 5.3 we have the asymptotic behavior of u,v and w.

Lemma 5.4. *Let (u,v,w) be a global bounded classical solution of* (1.1) *and the initial data (u_0,w_0) satisfy* (1.4). *Then we have the following asymptotic behavior*

$$\|u-1\|_{L^\infty(\Omega)}+\|v-\frac{1}{\delta}\|_{L^\infty(\Omega)}+\|w-\frac{1}{\delta}\|_{L^\infty(\Omega)}\to 0 \quad \textit{as } t\to\infty. \tag{5.15}$$

Proof. From u,v,w are bounded in $\Omega\times(0,\infty)$, applying standard parabolic regularity theory [19] (see also [3, 21]), there exist $\theta\in(0,1)$ and $C>0$ such that

$$\|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])}+\|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])}+\|w\|_{C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])}\le C \tag{5.16}$$

for all $t\ge 1$. Integrating (5.7) over $(1,\infty)$ and thanks to $E(t)\ge 0$ we have

$$\int_1^\infty F(t)dt\le \frac{1}{\epsilon}E(t)<\infty.$$

According to (5.16) we infer that $F(t)$ is uniformly continuous in $(1,\infty)$. Therefore, from Lemma 5.1 we obtain

$$F(t)=\int_\Omega (u-1)^2+\int_\Omega \big(v-\frac{1}{\delta}\big)^2+\int_\Omega \big(w-\frac{1}{\delta}\big)^2\to 0 \quad \text{as } t\to\infty. \tag{5.17}$$

Using the Gagliardo-Nirenberg interpolation inequality with some $C_{GN}>0$ we obtain

$$\|\varphi\|_{L^\infty(\Omega)}\le C_{GN}\|\varphi\|^{\frac{n}{n+2}}_{W^{1,\infty}(\Omega)}\|\varphi\|^{\frac{2}{n+2}}_{L^2(\Omega)} \quad \text{for all } \varphi\in W^{1,\infty}(\Omega). \tag{5.18}$$

Applying (5.16) and collecting (5.17)-(5.18), we have (5.15). □

To obtain the rate of convergence for u,v and w, we give the following lemma.

Lemma 5.5. *Let (u,v,w) be a nonnegative global bounded classical solution of* (1.1) *and the initial data (u_0,w_0) fulfill* (1.4). *Then one can find $C>0$ and $\kappa>0$ such that*

$$\|u-1\|_{L^\infty(\Omega)}+\|v-\frac{1}{\delta}\|_{L^\infty(\Omega)}+\|w-\frac{1}{\delta}\|_{L^\infty(\Omega)}\le Ce^{-\kappa t} \quad \textit{for all } t>0. \tag{5.19}$$

Proof. Applying the L'Hôpital theorem and letting $H(s) := s - \ln s$ for $s > 0$ we obtain

$$\lim_{s\to 1} \frac{H(s) - H(1)}{(s-1)^2} = \lim_{s\to 1} \frac{H'(s)}{2(s-1)} = \frac{1}{2}. \tag{5.20}$$

Therefore, using (5.20) and Lemma 5.4, there exists $t_1 > 0$ such that

$$\frac{1}{4}\int_\Omega (u-1)^2 \le A(t) = \int_\Omega (u - 1 - \ln u) \le \int_\Omega (u-1)^2 \quad \text{for all } t > t_1. \tag{5.21}$$

Hence, by the definitions of $E(t)$ and $F(t)$, there exists $C_1 > 0$ such that

$$E(t) \le C_1 F(t) \quad \text{for all } t > t_1. \tag{5.22}$$

Plugging (5.22) into (5.7) we obtain

$$\frac{d}{dt}E(t) \le -\epsilon F(t) \le -\frac{\epsilon}{C_1}E(t) \quad \text{for all } t > t_1, \tag{5.23}$$

which implies there exist $C_2 > 0$ and $l > 0$ such that

$$E(t) \le C_2 e^{-lt} \quad \text{for all } t > t_1. \tag{5.24}$$

Hence, by (5.21) we can find some $C_3 > 0$ such that

$$\int_\Omega (u-1)^2 + \int_\Omega \big(v - \frac{1}{\delta}\big)^2 + \int_\Omega \big(w - \frac{1}{\delta}\big)^2 \le C_3 E(t) \le C_2 C_3 e^{-lt} \tag{5.25}$$

for all $t > t_1$. Applying the Gagliardo-Nirenberg interpolation inequality (5.18) with the regularity of u, v and w, we can obtain there exist $C_4 > 0$ and $\kappa > 0$ such that

$$\|u-1\|_{L^\infty(\Omega)} + \|v - \frac{1}{\delta}\|_{L^\infty(\Omega)} + \|w - \frac{1}{\delta}\|_{L^\infty(\Omega)} \le C_4 e^{-\kappa t} \quad \text{for all } t > t_1. \tag{5.26}$$

Furthermore, choosing C_4 large enough such that (5.26) holds for all $t > 0$. The proof is complete. □

Proof of Theorem 1.3. The theorem follows immediately from Lemma 5.5. □

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