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REMARK ON DUFFING EQUATION WITH DIRICHLET BOUNDARY CONDITION

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Abstract. In this note, we prove the existence of a solution to the semilinear second order ordinary differential equation

$$u''(x) + r(x)u' + g(x, u) = f(x)\,,$$
$$x(0) = x(\pi) = 0\,,$$

using a variational method and critical point theory.

1. Introduction

We denote H the Sobolev space of absolutely continuous functions $u : (0, \pi) \to \mathbb{R}$ such that $u' \in L^2(0, \pi)$ and $u(0) = u(\pi) = 0$. Let us consider the nonlinear problem

$$\begin{gathered} u''(x) + r(x)u' + g(x, u) = f(x)\,, \quad x \in [0, \pi]\,, \\ u(0) = u(\pi) = 0\,, \end{gathered} \tag{1.1}$$

where $r \in H$, the nonlinearity $g : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is Caratheodory's function and $f \in L^1(0, \pi)$.

A physical example of this equation is the forced pendulum equation. In articles [1, 2] the authors assume that the friction coefficient r is nondecreasing and the nonlinearity g satisfies the condition

$$\frac{g(x, u) - g(x, v)}{u - v} \le k < 1\,.$$

They prove the uniqueness of the solution. In this work, we prove the existence of a solution to the problem (1.1) under more general condition

$$G(x, s) \le \frac{1}{2}\big(1 - \varepsilon + \frac{1}{4}r^2 + \frac{1}{2}r'\big)s^2 + c\,, \quad x \in [0, \pi]\,, s \in \mathbb{R}\,,$$

where $G(x, s) = \int_0^s g(x, t)\,dt$, $c > 0$, and $\varepsilon \in (0, 1)$.

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2. Preliminaries

Notation: We shall use the classical space $C^k(0,\pi)$ of functions whose k-th derivative is continuous and the space $L^p(0,\pi)$ of measurable real-valued functions whose p-th power of the absolute value is Lebesgue integrable. We use the symbols $\|\cdot\|$, and $\|\cdot\|_p$ to denote the norm in H and in $L^p(0,\pi)$, respectively.

By a solution to (1.1) we mean a function $u\in C^1(0,\pi)$ such that u' is absolutely continuous, u satisfies the boundary conditions and the equation (1.1) is satisfied a.e. in $(0,\pi)$.

For simplicity's sake we denote $R(x) = e^{\int_0^x \frac{1}{2} r(\xi)\,d\xi}$ and multiply (1.1) by the function $R(x)$. We put $w(x)=R(x)u(x)$ and obtain for w an equivalent Dirichlet problem

$$\begin{gathered} w''(x) - \big(\frac{1}{4}r^2(x) + \frac{1}{2}r'(x)\big)w(x) + R(x)g(x,\frac{w}{R(x)}) = R(x)f(x)\,,\\ w(0)=w(\pi)=0\,. \end{gathered} \tag{2.1}$$

We study (2.1) by using variational methods. More precisely, we investigate the functional $J: H\to\mathbb{R}$, which is defined by

$$J(w) = \frac{1}{2}\int_0^\pi \big[(w')^2 + \big(\frac{1}{4}r^2 + \frac{1}{2}r'\big)w^2\big]\,dx - \int_0^\pi \big[R^2G(x,\frac{w}{R}) - Rfw\big]\,dx\,, \tag{2.2}$$

where

$$G(x,s) = \int_0^s g(x,t)\,dt\,.$$

We say that w is a critical point of J, if

$$\langle J'(w), v\rangle = 0 \quad \text{for all } v\in H\,.$$

We see that every critical point $w\in H$ of the functional J satisfies

$$\int_0^\pi \big[w'v' + \big(\frac{1}{4}r^2 + \frac{1}{2}r'\big)wv\big]\,dx - \int_0^\pi \big[Rg(x,\frac{w}{R})v - Rfv\big]\,dx = 0$$

for all $v\in H$, and w is a weak solution to (2.1), and vice versa. The usual regularity argument for ODE proves immediately (see Fučík [3]) that any weak solution to (2.1) is also a solution in the sense mentioned above.

We suppose that there are $c>0$ and $\varepsilon\in(0,1)$ such that

$$G(x,s) \le \frac{1}{2}\big(1-\varepsilon + \frac{1}{4}r^2(x) + \frac{1}{2}r'(x)\big)s^2 + c \quad x\in[0,\pi]\,,\ s\in\mathbb{R}\,. \tag{2.3}$$

Remark 2.1. The condition (2.3) is satisfied for example if $g(x,s)=(1-\varepsilon)s$ and $\frac{1}{4}r^2 + \frac{1}{2}r' \ge 0$. It is easy to find a function r which is not nondecreasing on $[0,\pi]$ and which satisfies $\frac{1}{4}r^2 + \frac{1}{2}r' \ge 0$. For example $r(x) = -x+\pi+\sqrt{2}$.

3. Main result

Theorem 3.1. *Under the assumption* (2.3)*, Problem (2.1) has at least one solution in H.*

Proof. First we prove that J is a weakly coercive functional; i. e.,

$$\lim_{\|w\|\to\infty} J(w) = \infty \quad \text{for all } w\in H.$$

Because of the compact imbedding of H into $C(0,\pi)$, ($\|w\|_{C(0,\pi)} \leq c_1\|w\|$), and the assumption (2.3) we obtain

$$\begin{aligned} J(w) &= \frac{1}{2}\int_0^\pi \big[(w')^2 + \big(\frac{1}{4}r^2 + \frac{1}{2}r'\big)w^2\big]\,dx - \int_0^\pi \big[R^2G(x,\frac{w}{R}) - Rfw\big]\,dx \\ &\geq \frac{1}{2}\|w\|^2 - \frac{1}{2}(1-\varepsilon)\|w\|_2^2 - \|R^2\|_1 c - \|Rf\|_1 c_1\|w\|\,. \end{aligned} \tag{3.1}$$

Because of Poincare's inequality $\|w\|_2 \leq \|w\|$ and (3.1) we have

$$J(w) \geq \frac{\varepsilon}{2}\|w\|^2 - c\,\|R^2\|_1 - c_1\|Rf\|_1\,. \tag{3.2}$$

Then (3.2) implies $\lim_{\|w\|\to\infty} J(w) = \infty$.

Next we prove that J is a weakly sequentially lower semi-continuous functional on H. Consider an arbitrary $w_0 \in H$ and a sequence $\{w_n\} \subset H$ such that $w_n \rightharpoonup w_0$ in H. Due to compact imbedding H into $C(0,\pi)$ we have $w_n \to w_0$ in $C(0,\pi)$. This and the continuity $g(x,t)$ in the variable t imply

$$\begin{aligned} &\frac{1}{2}\int_0^\pi \big(\frac{1}{4}r^2 + \frac{1}{2}r'\big)w_n^2\,dx - \int_0^\pi \big[R^2G(x,\frac{w_n}{R}) - Rfw_n\big]\,dx \to \\ &\frac{1}{2}\int_0^\pi \big(\frac{1}{4}r^2 + \frac{1}{2}r'\big)w_0^2\,dx - \int_0^\pi \big[R^2G(x,\frac{w_0(x)}{R}) - Rfw_0\big]\,dx\,. \end{aligned} \tag{3.3}$$

Due to the weak sequential lower semi-continuity of the Hilbert norm $\|\cdot\|$ (i.e. $\liminf_{n\to\infty}\|w_n\| \geq \|w_0\|$) and (3.3), we have

$$\liminf_{n\to\infty} J(w_n) \geq J(w_0)\,.$$

The weak sequential lower semi-continuity and the weak coercivity of the functional J imply (see Struwe [4]) the existence of a critical point of the functional J; i.e., a weak solution to the equation (2.1) and, consequently, to equation (1.1). □

References

[1] P. Amster: *Nonlinearities in a second order ODE*, Electron. J. Diff. Eqns., Conf. 06, 2001, pp. 13-21.

[2] P. Amster, M. C. Mariani: *A second order ODE with a nonlinear final condition*, Electron. J. Diff. Eqns., Vol. **2001** (2001), No. 75, pp. 1-9.

[3] S. Fučík: *Solvability of Nonlinear Equations and Boundary Value Problems*, D. Reidel Publ. Company, Holland 1980.

[4] M. Struwe: *Variational Methods*, Springer, Berlin, (1996).

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