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FUNDAMENTAL SOLUTIONS OF TWO MULTIDIMENSIONAL ELLIPTIC EQUATIONS

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Abstract. We construct fundamental solutions for two-multidimensional elliptic equations. The solutions are written in explicit form via hypergeometric Gauss functions for $\lambda = 0$ and via confluent Horn functions for $\lambda \neq 0$. It is proved that the fundamental solutions found possess power-type singularity ρ^{2-n} as $\rho \to 0$.

1. Introduction

The practical value of mixed-type equations was first highlighted by Chaplygin [5] in 1902. Investigations of boundary-value problems for mixed-type equations was begun by Tricomi in his works [28, 29]. He stated and solved the first boundary problem for the equation

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Holmgren [20] simplified the solution of Tricomi problems for the equation

$$y^m\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

In his doctoral thesis Gellerstedt [15] solved the Tricomi problem for the equation

$$y^m\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - cu = F(x, y),$$

and, in [13, 14] he generalized Tricomi's results.

A systematic study of mixed-type equations attracted authors' attention since the middle of 40s of the past century after indication by Frankl of the possibility of their application in transonic and supersonic gas dynamics and hydrodynamics [8, 9, 10].

Essential contribution to the development of the theory of mixed-type equations was made by the mathematicians Germain, Bader [3], Bitsadze [4], Babenko [2], Volkodavov [31], Keldysh [23], Vekua [30] and others.

During recent years, the main interests focus shifted towards practical applications of mathematical models in various fields of sciences [16, 11]. The monograph

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[16] contains many examples of applications of mathematical models in biology, chemistry, and population genetics. In paper [11], the results on g-differential equations are applied to some mathematical models.

In [27] the parametric Stokes phenomena studied for Gauss hypergeometric differential equation from the viewpoint of the alien calculus. In the present article, for the degenerate elliptic equation

$$L_\lambda(u) \equiv x_n^m \sum_{i=1}^{n-1} \Big(\frac{\partial^2 u}{\partial x_i^2} + \lambda^2 u \Big) + \frac{\partial^2 u}{\partial x_n^2} = 0, \quad (m > 0,\ n > 2), \tag{1.1}$$

and for the elliptic equation

$$T_\lambda(u) \equiv e^{x_n} \sum_{i=1}^{n-1} \Big(\frac{\partial^2 u}{\partial x_i^2} + \lambda^2 u \Big) + \frac{\partial^2 u}{\partial x_n^2} = 0, \quad (n > 2) \tag{1.2}$$

fundamental solutions are constructed. The fundamental solutions are written in explicit form via hypergeometric Gauss functions for $\lambda = 0$ and confluent Horn functions for $\lambda \neq 0$. It is proved that the fundamental solutions obtained possess power-type singularity (further we will use power singularity term) ρ^{2-n} as $\rho \to 0$.

2. Finding fundamental solutions of equation (1.1)

For $\lambda = 0$, the equation (1.1) takes the form

$$L_0(u) \equiv x_n^m \sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial x_n^2} = 0, \quad (m > 0, n > 2). \tag{2.1}$$

Following the works [17, 18, 19, 12], we search for a solution for equation (2.1) in the form

$$u(x_1, x_2, \dots, x_n) = P\,\omega(\xi), \tag{2.2}$$

where

$$P = \rho^{-(\mu+n-2)}, \quad \mu = \frac{m}{m+2}, \quad \xi = \frac{\rho^2 - \rho_1^2}{\rho^2}, \tag{2.3}$$

$$\rho^2 = \sum_{i=1}^{n-1} \Big(x_i - x_i^{(0)} \Big)^2 + \frac{4}{(m+2)^2} \Big(x_n^{\frac{m+2}{2}} - \Big(x_n^{(0)} \Big)^{\frac{m+2}{2}} \Big)^2,$$

$$\rho_1^2 = \sum_{i=1}^{n-1} \Big(x_i - x_i^{(0)} \Big)^2 + \frac{4}{(m+2)^2} \Big(x_n^{\frac{m+2}{2}} + \Big(x_n^{(0)} \Big)^{\frac{m+2}{2}} \Big)^2,$$

while $\omega(\xi)$ is a function yet unknown.

By substituting the function (2.2) into (2.1), we obtain the equation

$$\xi(1-\xi)\omega_{\xi\xi} + \Big(\mu - \Big(\frac{\mu+n-2}{2} + \frac{\mu}{2} + 1 \Big) \xi \Big) \omega_\xi - \frac{\mu+n-2}{2} \frac{\mu}{2} \omega = 0. \tag{2.4}$$

Comparing (2.4) with Gauss equation

$$\xi(1-\xi)\omega_{\xi\xi} + \big(\delta - (\alpha+\beta+1)\xi \big) \omega_\xi - \alpha\beta\omega = 0, \tag{2.5}$$

which in a neighborhood of the point $\xi = 0$ has two linearly independent solutions

$$\omega_1 = F(\alpha, \beta; \delta; \xi), \quad \omega_2 = \xi^{1-\delta} F(\alpha - \delta + 1, \beta - \delta + 1; 2 - \delta; \xi),$$

where

$$F(\alpha,\beta;\delta;\xi)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\delta)_k}\frac{\xi^k}{k!},$$

being hypergeometric Gauss function [7], and taking into account (2.3), we conclude that for equation (2.4) the following functions are solutions

$$\omega_1=F\Big(\frac{\mu+n-2}{2},\frac{\mu}{2};\mu;\frac{\rho^2-\rho_1^2}{\rho^2}\Big),$$
$$\omega_2=\Big(\frac{\rho^2-\rho_1^2}{\rho^2}\Big)^{1-\mu}F\Big(\frac{\mu+n-2}{2}+1-\mu,\frac{\mu}{2}+1-\mu;2-\mu;\frac{\rho^2-\rho_1^2}{\rho^2}\Big).$$

Consequently, solutions of (2.1) are given by the functions

$$q_{01}(M,M_0)=C_1\rho^{-(\mu+n-2)}F\Big(\frac{\mu+n-2}{2},\frac{\mu}{2};\mu;\frac{\rho^2-\rho_1^2}{\rho^2}\Big), \tag{2.6}$$
$$q_{02}(M,M_0)=C_2\rho^{-(\mu+n-2)}\Big(\frac{\rho^2-\rho_1^2}{\rho^2}\Big)^{1-\mu}F\Big(\frac{n-\mu}{2},1-\frac{\mu}{2};2-\mu;\frac{\rho^2-\rho_1^2}{\rho^2}\Big), \tag{2.7}$$

where C_1 and C_2 are some constants.

Using transformation formula in [7],

$$F(\alpha,\beta;\delta;\xi)=(1-\xi)^{-\alpha}F\Big(\alpha,\delta-\beta;\delta;\frac{\xi}{\xi-1}\Big) \tag{2.8}$$

we write the solutions (2.6), (2.7) in the form

$$q_{01}(M,M_0)=C_1\rho_1^{-(\mu+n-2)}F\Big(\frac{\mu+n-2}{2},\frac{\mu}{2};\mu;\frac{\rho_1^2-\rho^2}{\rho_1^2}\Big), \tag{2.9}$$
$$\begin{aligned}&q_{02}(M,M_0)\\&=C_{21}\rho_1^{-(\mu+n-2)}\Big(\frac{\rho_1^2-\rho^2}{\rho_1^2}\Big)^{1-\mu}F\Big(\frac{n-\mu}{2},1-\frac{\mu}{2};2-\mu;\frac{\rho_1^2-\rho^2}{\rho_1^2}\Big).\end{aligned} \tag{2.10}$$

Note (see [26]) that one can pass straightforwardly to solutions (2.9), (2.10) if one seeks a solution in the form

$$u(x_1,x_2,\dots,x_n)=P\,\omega(\sigma),$$

where

$$P=\rho_1^{-(\mu+n-2)},\quad \mu=\frac{m}{m+2},\quad \sigma=\frac{\rho^2}{\rho_1^2}.$$

Now let us consider the case $\lambda\neq 0$. Following the papers [17, 18], we seek a solution of equation (1.1) in the form

$$u(x_1,x_2,\dots,x_n)=P\omega(\xi,\eta), \tag{2.11}$$

where

$$P=\rho^{-(\mu+n-2)},\quad \xi=\frac{\rho^2-\rho_1^2}{\rho^2},\quad \eta=\frac{\lambda^2\rho^2}{4}, \tag{2.12}$$

while $\omega(\xi,\eta)$ is a function yet unknown.

By substituting the function (2.11) into equation (1.1), we obtain

$$\begin{gathered}\xi(1-\xi)\omega_{\xi\xi}+\xi\eta\omega_{\xi\eta}+(\mu-(\frac{\mu+n-2}{2}+\frac{\mu}{2}+1)\xi)\omega_\xi\\ +\frac{\mu}{2}\eta\omega_\eta-\frac{\mu+n-2}{2}\frac{\mu}{2}\omega=0,\\ \eta\omega_{\eta\eta}-\xi\omega_{\xi\eta}+(1-\frac{\mu+n-2}{2})\omega_\eta+\omega=0.\end{gathered}\tag{2.13}$$

Comparing (2.13) with the system

$$\begin{gathered}\xi(1-\xi)\omega_{\xi\xi}+\xi\eta\omega_{\xi\eta}+\big(\delta-(\alpha+\beta+1)\xi\big)\omega_\xi+\beta\eta\omega_\eta-\alpha\beta\omega=0,\\ \eta\omega_{\eta\eta}-\xi\omega_{\xi\eta}+(1-\alpha)\omega_\eta+\omega=0,\end{gathered}\tag{2.14}$$

which possesses in vicinity of the point $\xi=0$ the two linearly independent solutions

$$\omega_1=H_3(\alpha,\beta;\delta;\xi,\eta),\quad \omega_2=\xi^{1-\delta}H_3(\alpha-\delta+1,\beta-\delta+1;2-\delta;\xi,\eta),\tag{2.15}$$

where

$$H_3(\alpha,\beta;\delta;\xi,\eta)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\alpha)_{k-l}(\beta)_k}{(\delta)_k}\frac{\xi^k}{k!}\frac{\eta^l}{l!}\tag{2.16}$$

is confluent with the Horn-Kummer function in H_3 [6], we conclude that the following functions are solutions of system (2.13),

$$\omega_1=H_3\Big(\frac{\mu+n-2}{2},\frac{\mu}{2};\mu;\xi,\eta\Big)=H_3\Big(\frac{\mu+n-2}{2},\frac{\mu}{2};\mu;\frac{\rho^2-\rho_1^2}{\rho^2},\frac{\lambda^2\rho^2}{4}\Big),\tag{2.17}$$

$$\begin{aligned}\omega_2&=\xi^{1-\mu}H_3\Big(\frac{\mu+n-2}{2}+1-\mu,\frac{\mu}{2}+1-\mu;2-\mu;\xi,\eta\Big)\\ &=\Big(\frac{\rho^2-\rho_1^2}{\rho^2}\Big)^{1-\mu}H_3\Big(\frac{n-\mu}{2},1-\frac{\mu}{2};2-\mu;\frac{\rho^2-\rho_1^2}{\rho^2},\frac{\lambda^2\rho^2}{4}\Big).\end{aligned}\tag{2.18}$$

Therefore, the solutions of equation (1.1) are given by the functions

$$q_{\lambda1}(M,M_0)=C_3\rho^{-(\mu+n-2)}H_3\Big(\frac{\mu+n-2}{2},\frac{\mu}{2};\mu;\frac{\rho^2-\rho_1^2}{\rho^2},\frac{\lambda^2\rho^2}{4}\Big),\tag{2.19}$$

$$\begin{aligned}q_{\lambda2}(M,M_0)&=C_4\rho^{-(\mu+n-2)}\Big(\frac{\rho^2-\rho_1^2}{\rho^2}\Big)^{1-\mu}\\ &\quad\times H_3\Big(\frac{n-\mu}{2},1-\frac{\mu}{2};2-\mu;\frac{\rho^2-\rho_1^2}{\rho^2},\frac{\lambda^2\rho^2}{4}\Big),\end{aligned}\tag{2.20}$$

where C_3 and C_4 are some constants.

Applying the transformation formula in [21],

$$H_3(\alpha,\beta;\delta;\xi,\eta)=(1-\xi)^{-\alpha}H_3\Big(\alpha,\delta-\beta;\delta;\frac{\xi}{\xi-1};\eta(1-\xi)\Big),\tag{2.21}$$

one can write solutions (2.19) and (2.20) in the form

$$q_{\lambda1}(M,M_0)=C_3\rho_1^{-(\mu+n-2)}H_3\Big(\frac{\mu+n-2}{2},\frac{\mu}{2};\mu;\frac{\rho_1^2-\rho^2}{\rho_1^2},\frac{\lambda^2\rho_1^2}{4}\Big),\tag{2.22}$$

$$\begin{aligned}&q_{\lambda2}(M,M_0)\\ &=C_{41}\rho_1^{-(\mu+n-2)}\Big(\frac{\rho_1^2-\rho^2}{\rho_1^2}\Big)^{1-\mu}H_3\Big(\frac{n-\mu}{2},1-\frac{\mu}{2};2-\mu;\frac{\rho_1^2-\rho^2}{\rho_1^2},\frac{\lambda^2\rho_1^2}{4}\Big).\end{aligned}\tag{2.23}$$

Note that one can arrive straightforwardly at the solutions (2.22), (2.23) if one seeks solution in the form

$$u(x_1, x_2, \dots, x_n) = P\,\omega(\sigma, \tau),$$

where

$$P = \rho_1^{-(\mu+n-2)}, \quad \mu = \frac{m}{m+2}, \quad \sigma = \frac{\rho^2}{\rho_1^2} \quad \tau = \frac{\lambda^2\rho^2}{4}.$$

Let us consider some properties of solutions.

Lemma 2.1. *Solutions* (2.6)*,* (2.7) *and* (2.19)*,* (2.20) *satisfy the following conditions*

$$\frac{\partial q_{01}(M, M_0)}{\partial x_n}\Big|_{x_n=0} = 0, \quad q_{02}(M, M_0)\Big|_{x_n=0} = 0,$$
$$\frac{\partial q_{\lambda 1}(M, M_0)}{\partial x_n}\Big|_{x_n=0} = 0, \quad q_{\lambda 2}(M, M_0)\big|_{x_n=0} = 0.$$

The proof os the above lemma is to a straightforward calculation.

Lemma 2.2. *Solutions* (2.6)*,* (2.7) *and* (2.19)*,* (2.20) *possess power singularity* ρ^{2-n} *as* $\rho \to 0$*.*

Proof. It can be carried out on the basis of analytic continuation formulae for the hypergeometric Gauss function [1] and confluent Horn-Kumner function [22],

$$\begin{aligned} F(\alpha, \beta; \delta; \xi) &= \frac{\Gamma(\delta)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\delta-\alpha)}(-\xi)^{-\alpha} F\Big(\alpha, 1+\alpha-\delta; 1+\alpha-\beta; \frac{1}{\xi}\Big) \\ &\quad + \frac{\Gamma(\delta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\delta-\beta)}(-\xi)^{-\beta} F\Big(\beta, 1+\beta-\delta; 1+\beta-\alpha; \frac{1}{\xi}\Big), \end{aligned}$$

$$\begin{aligned} &H_3(\alpha, \beta; \delta; \xi, \eta) \\ &= \frac{\Gamma(\delta)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\delta-\alpha)}(-\xi)^{-\alpha} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\alpha)_{k-l}(1+\alpha-\beta)_{k-l}}{(1+\alpha-\delta)_{k-l}} \frac{(1/\xi)^k}{k!} \frac{(-\xi\eta)^l}{l!} \\ &\quad + \frac{\Gamma(\delta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\delta-\beta)}(-\xi)^{-\beta} \Xi_2\Big(\beta, 1+\beta-\delta; 1+\beta-\alpha; \frac{1}{\xi}; -\eta\Big), \end{aligned}$$

where

$$\Xi_2(\alpha, \beta; \delta; \xi, \eta) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\delta)_{k+l}} \frac{\xi^k}{k!} \frac{\eta^l}{l!}$$

is a Humbert function [22]. □

Lemma 2.2 implies the following result.

Theorem 2.3. *The functions* (2.6) *and* (2.7) *are fundamental solutions of equation* (1.1) *for* $\lambda = 0$*, while functions* (2.19) *and* (2.20) *are for* $\lambda \neq 0$*.*

3. Fundamental solutions of (1.2)

For $\lambda = 0$, equation (1.2) takes the form

$$T_0(u) \equiv e^{x_n} \sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial x_n^2} = 0, \quad (m > 0, n > 2). \tag{3.1}$$

We seek a solution of equation (3.1) in the form

$$u(x_1, x_2, \dots, x_n) = P\omega(\sigma), \tag{3.2}$$

where

$$P = \rho_1^{-(n-1)}, \quad \sigma = \frac{\rho^2}{\rho_1^2}, \tag{3.3}$$

$$\rho^2 = \sum_{i=1}^{n-1} \Big(x_i - x_i^{(0)}\Big)^2 + 4\Big(e^{\frac{x_n}{2}} - e^{\frac{x_n^{(0)}}{2}}\Big)^2,$$

$$\rho_1^2 = \sum_{i=1}^{n-1} \Big(x_i - x_i^{(0)}\Big)^2 + 4\Big(e^{\frac{x_n}{2}} + e^{\frac{x_n^{(0)}}{2}}\Big)^2,$$

while $\omega(\sigma)$ is a function yet unknown.

Substituting function (3.2) into (3.1), we obtain the equation

$$\sigma(1-\sigma)\omega_{\sigma\sigma} + (\frac{n}{2} - \Big(\frac{n-1}{2} + \frac{1}{2} + 1\Big)\sigma\Big)\omega_\sigma - \frac{n-1}{2}\frac{1}{2}\omega = 0. \tag{3.4}$$

By the changeof variable $\chi = 1 - \sigma$ this equation can be represented in the form

$$\chi(1-\chi)\omega_{\chi\chi} + \Big(1 - \Big(\frac{n-1}{2} + \frac{1}{2} + 1\Big)\chi\Big)\omega_\chi - \frac{n-1}{2}\frac{1}{2}\omega = 0. \tag{3.5}$$

Equation (3.5) has two independent solutions [1],

$$\omega_1 = F\Big(\frac{n-1}{2}, \frac{1}{2}; 1; \chi\Big),$$

$$\begin{aligned} \omega_2 &= F\Big(\frac{n-1}{2}, \frac{1}{2}; 1; \chi\Big)\ln\chi \\ &\quad + \sum_{k=0}^{\infty} \frac{\big(\frac{n-1}{2}\big)_k \big(\frac{1}{2}\big)_k}{(k!)^2} \Big(\psi\Big(\frac{n-1}{2} + k\Big) + \psi\Big(\frac{1}{2} + k\Big) - 2\psi(1+k)\Big)\chi^k, \end{aligned}$$

where

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

is logarithmic derivative of Euler gamma-function.

Consequently, solutions of equation (3.1) are given by the functions

$$q_{01}(M, M_0) = C_1\rho_1^{-(n-1)} F\Big(\frac{n-1}{2}, \frac{1}{2}; 1; \frac{\rho_1^2 - \rho^2}{\rho_1^2}\Big), \tag{3.6}$$

$$\begin{aligned} q_{02}(M, M_0) &= C_2\rho_1^{-(n-1)}\Big(F\Big(\frac{n-1}{2}, \frac{1}{2}; 1; \frac{\rho_1^2 - \rho^2}{\rho_1^2}\Big)\ln\frac{\rho_1^2 - \rho^2}{\rho_1^2} \\ &\quad + \sum_{k=0}^{\infty} \frac{\big(\frac{n-1}{2}\big)_k \big(\frac{1}{2}\big)_k}{(k!)^2} \Big(\psi\Big(\frac{n-1}{2} + k\Big) + \psi\Big(\frac{1}{2} + k\Big) \\ &\quad - 2\psi(1+k)\Big)\Big(\frac{\rho_1^2 - \rho^2}{\rho_1^2}\Big)^k\Big). \end{aligned} \tag{3.7}$$

Now consider the case $\lambda \neq 0$. We seek a solution of equation (1.2) in the form

$$u(x_1, x_2, \dots, x_n) = P\,\omega(\sigma, \tau), \tag{3.8}$$

where

$$P=\rho_1^{-(\mu+n-2)},\quad \sigma=\frac{\rho^2}{\rho_1^2},\quad \tau=\frac{\lambda^2\rho_1^2}{4}, \tag{3.9}$$

while $\omega(\sigma,\tau)$ is a function yet unknown.

Substituting function (3.8) into equation (1.2), we get

$$\begin{gathered}\sigma(1-\sigma)\omega_{\sigma\sigma}-(1-\sigma)\tau\omega_{\sigma\tau}+(\frac{n}{2}-(\frac{n-1}{2}\\ +\frac{1}{2}+1)\sigma)\omega_\sigma+\frac{1}{2}\tau\omega_\tau-\frac{n-1}{2}\frac{1}{2}\omega=0,\\ \tau\omega_{\tau\tau}+(1-\sigma)\omega_{\sigma\tau}+(1-\frac{n-1}{2})\omega_\tau+\omega=0.\end{gathered} \tag{3.10}$$

The change of variable $\chi=1-\sigma$ allows us to write the present system in the form

$$\begin{gathered}\chi(1-\chi)\omega_{\chi\chi}+\chi\tau\omega_{\chi\tau}+(1-(\frac{n-1}{2}+\frac{1}{2}+1)\chi)\omega_\chi+\frac{1}{2}\tau\omega_\tau-\frac{n-1}{2}\frac{1}{2}\omega=0,\\ \tau\omega_{\tau\tau}-\chi\omega_{\chi\tau}+(1-\frac{n-1}{2})\omega_\tau+\omega=0.\end{gathered} \tag{3.11}$$

From (2.17) and (2.18) it is evident that, for $\mu=1$, the solutions of the system coincide. Moreover, the solutions (2.15) of system (2.14) coincide not only for $\delta=1$, but also for any natural δ. Indeed, let $\delta=p$ be a natural number. Then the functions

$$\begin{aligned}\Phi_1&=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\delta)}H_3(\alpha,\beta;\delta;\xi,\eta)\\ &=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\delta)}\sum_{k=0}^\infty\sum_{l=0}^\infty\frac{(\alpha)_{k-l}(\beta)_k}{(\delta)_k}\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\ &=\sum_{k=0}^\infty\sum_{l=0}^\infty\frac{\Gamma(\alpha+k-l)\Gamma(\beta+k)}{\Gamma(\delta+k)}\frac{\xi^k}{k!}\frac{\eta^l}{l!}\end{aligned}$$

and

$$\begin{aligned}\Phi_2&=\frac{\Gamma(\alpha+1-\delta)\Gamma(\beta+1-\delta)}{\Gamma(2-\delta)}\xi^{1-\delta}H_3(\alpha-\delta+1,\beta-\delta+1;2-\delta;\xi,\eta)\\ &=\frac{\Gamma(\alpha+1-\delta)\Gamma(\beta+1-\delta)}{\Gamma(2-\delta)}\xi^{1-\delta}\sum_{k=0}^\infty\sum_{l=0}^\infty\frac{(\alpha+1-\delta)_{k-l}(\beta+1-\delta)_k}{(2-\delta)_k}\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\ &=\xi^{1-\delta}\sum_{k=0}^\infty\sum_{l=0}^\infty\frac{\Gamma(\alpha+1-\delta+k-l)\Gamma(\beta+1-\delta+k)}{\Gamma(2-\delta+k)}\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\ &=\sum_{k=0}^\infty\sum_{l=0}^\infty\frac{\Gamma(\alpha+1-\delta+k-l)\Gamma(\beta+1-\delta+k)}{\Gamma(2-\delta+k)}\frac{\xi^{1-\delta+k}}{k!}\frac{\eta^l}{l!}\end{aligned}$$

are equal to each other. To find the second solution in this case, suppose that α and β are nonnegative integers. In a way analogous to that in [1], we consider the limit

$$\begin{aligned}&\lim_{\delta\to p}\frac{\Phi_1-\Phi_2}{\delta-p}\\ &=\frac{\partial}{\partial\delta}(\Phi_1-\Phi_2)\Big|_{\delta=p}\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\alpha+k-l)\Gamma(\beta+k)\Gamma'(p+k)}{\Gamma(p+k)\Gamma(p+k)}\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\
&\quad+\xi^{1-p}\ln\xi\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\alpha+1-p+k-l)\Gamma(\beta+1-p+k)}{\Gamma(2-p+k)}\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\
&\quad+\xi^{1-p}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\alpha+1-p+k-l)\Gamma(\beta+1-p+k)}{\Gamma(2-p+k)}\\
&\quad\times\Big(\frac{\Gamma'(\alpha+1-p+k-l)}{\Gamma(\alpha+1-p+k-l)}+\frac{\Gamma'(\beta+1-p+k)}{\Gamma(\beta+1-p+k)}\Big)\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\
&\quad-\lim_{\delta\to p}\xi^{1-\delta}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\alpha+1-\delta+k-l)\Gamma(\beta+1-\delta+k)}{\Gamma(2-\delta+k)}\frac{\Gamma'(2-\delta+k)}{\Gamma(2-\delta+k)}\frac{\xi^k}{k!}\frac{\eta^l}{l!}.
\end{aligned}
$$

The first of the series can be written in the form

$$
\begin{aligned}
&-\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\alpha+k-l)\Gamma(\beta+k)\Gamma'(p+k)}{\Gamma(p+k)\Gamma(p+k)}\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\
&=-\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(p)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\alpha)_{k-l}(\beta)_k}{(p)_k}\psi(p+k)\frac{\xi^k}{k!}\frac{\eta^l}{l!}.
\end{aligned}
$$

The second series is equal to

$$
\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(p)}\ln\xi\, H_3(\alpha,\beta;p;\xi,\eta).
$$

We rewrite the third series as

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\alpha+k-l)\Gamma(\beta+k)}{\Gamma(p+k)}\Big(\frac{\Gamma'(\alpha+k-l)}{\Gamma(\alpha+k-l)}+\frac{\Gamma'(\beta+k)}{\Gamma(\beta+k)}\Big)\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\
&=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(p)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\alpha)_{k-l}(\beta)_k}{(p)_k}\Big(\psi(\alpha+k-l)+\psi(\beta+k)\Big)\frac{\xi^k}{k!}\frac{\eta^l}{l!}
\end{aligned}
$$

because $\frac{1}{\Gamma(2-p+k)}=0$ for $k=0,1,\dots,p-2$. From the formula

$$
\frac{\Gamma'(1-z)}{(\Gamma(1-z))^2}=\frac{\Gamma'(z)}{\Gamma(z)\Gamma(1-z)}+\cos\pi z\Gamma(z),
$$

we obtain

$$
\lim_{\delta\to p}\frac{\Gamma'(2-\delta+k)}{(\Gamma(2-\delta+k))^2}=(-1)^{p-k-1}\Gamma(p-k-1).
$$

It follows that the fourth addend can be rewritten in the form

$$
\begin{aligned}
&-\lim_{\delta\to p}\xi^{1-\delta}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\alpha+1-\delta+k-l)\Gamma(\beta+1-\delta+k)}{\Gamma(2-\delta+k)}\frac{\Gamma'(2-\delta+k)}{\Gamma(2-\delta+k)}\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\
&\quad-\lim_{\delta\to p}\xi^{1-\delta}\sum_{k=0}^{p-2}\sum_{l=0}^{\infty}\frac{\Gamma(\alpha+1-\delta+k-l)\Gamma(\beta+1-\delta+k)}{\Gamma(2-\delta+k)}\frac{\Gamma'(2-\delta+k)}{\Gamma(2-\delta+k)}\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\
&\quad-\lim_{\delta\to p}\xi^{1-\delta}\sum_{k=p-1}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\alpha+1-\delta+k-l)\Gamma(\beta+1-\delta+k)}{\Gamma(2-\delta+k)}\frac{\Gamma'(2-\delta+k)}{\Gamma(2-\delta+k)}\frac{\xi^k}{k!}\frac{\eta^l}{l!}
\end{aligned}
$$

$$
\begin{aligned}
&= -\xi^{1-p}\sum_{k=0}^{p-1}\sum_{l=0}^{\infty}\Gamma(\alpha+1-p+k-l)\Gamma(\beta+1-p+k)(-1)^{p-k-1}\Gamma(p-k-1)\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\
&\quad -\xi^{1-p}\sum_{k=p-1}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\alpha+1-p+k-l)\Gamma(\beta+1-p+k)}{\Gamma(2-p+k)}\frac{\Gamma'(2-p+k)}{\Gamma(2-p+k)}\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\
&= -\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{p-k-1}\Gamma(\alpha+1-p+k-l)\Gamma(\beta+1-p+k)\Gamma(p-k-1)\frac{\xi^{1-p+k}}{k!}\frac{\eta^l}{l!}\\
&\quad -\sum_{k=p-1}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\alpha+1-p+k-l)\Gamma(\beta+1-p+k)}{\Gamma(2-p+k)}\frac{\Gamma'(2-p+k)}{\Gamma(2-p+k)}\frac{\xi^{1-p+k}}{k!}\frac{\eta^l}{l!}\\
&= \sum_{k=1}^{p-1}\sum_{l=0}^{\infty}(-1)^{k-1}(k-1)!\Gamma(\alpha-k-l)\Gamma(\beta-k)\frac{\xi^{-k}}{(p-k-1)!}\frac{\eta^l}{l!}\\
&\quad -\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\alpha+k-l)\Gamma(\beta+k)}{\Gamma(p+k)}\psi(1+k)\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\
&= \sum_{k=1}^{p-1}\sum_{l=0}^{\infty}(-1)^{k-1}(k-1)!\Gamma(\alpha-k-l)\Gamma(\beta-k)\frac{\xi^{-k}}{(p-k-1)!}\frac{\eta^l}{l!}\\
&\quad -\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(p)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\alpha)_{k-l}(\beta)_k}{(p)_k}\psi(1+k)\frac{\xi^k}{k!}\frac{\eta^l}{l!}.
\end{aligned}
$$

Consequently, when α and β are not negative integers, the second solution has the form

$$
\begin{aligned}
\omega_2 &= H_3(\alpha,\beta;p;\xi,\eta)\ln\xi+\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\alpha)_{k-l}(\beta)_k}{(p)_k}\Big(\psi(\alpha+k-l)\\
&\quad +\psi(\beta+k)-\psi(p+k)-\psi(1+k)\Big)\frac{\xi^k}{k!}\frac{\eta^l}{l!}\\
&\quad +\frac{\Gamma(p)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k=1}^{p-1}\sum_{l=0}^{\infty}(-1)^{k-1}(k-1)!\Gamma(\alpha-k-l)\Gamma(\beta-k)\frac{\xi^{-k}}{(p-k-1)!}\frac{\eta^l}{l!}.
\end{aligned}
$$

Thus, a solution of system (3.10) is given by the functions

$$
\omega_1 = H_3(\frac{n-1}{2},\frac{1}{2};1;\frac{\rho_1^2-\rho 2^2}{\rho^2},\frac{\lambda^2 r_1^2}{4}), \tag{3.12}
$$

and

$$
\begin{aligned}
\omega_2 &= H_3\Big(\frac{n-1}{2},\frac{1}{2};1;\frac{\rho_1^2-\rho^2}{\rho_1^2},\frac{\lambda^2r_1^2}{4}\Big)\ln(\Big(frac\rho_1^2-\rho^2\rho_1^2\Big)\\
&\quad +\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\frac{n-1}{2})_{k-l}(\frac{1}{2})_k}{(1)_k}\Big(\psi(\frac{n-1}{2}+k-l)+\psi(\frac{1}{2}+k)-2\psi(1+k)\Big) \qquad (3.13)\\
&\quad \times\frac{(\frac{\rho_1^2-\rho^2}{\rho_1^2})^k}{k!}\frac{(\frac{\lambda^2r_1^2}{4})^l}{l!}.
\end{aligned}
$$

Next we derive that a solution of equation (1.2) is provided by the functions

$$q_{\lambda 1}(M,M_0)=C_1\rho_1^{-(n-1)}H_3(\frac{n-1}{2},\frac{1}{2};1;\frac{\rho_1^2-\rho^2}{\rho_1^2},\frac{\lambda^2r_1^2}{4}), \tag{3.14}$$

and

$$\begin{aligned} q_{\lambda 2}(M,M_0)=C_2\rho_1^{-(n-1)}\Big(&H_3\Big(\frac{n-1}{2},\frac{1}{2};1;\frac{\rho_1^2-\rho^2}{\rho_1^2},\frac{\lambda^2r_1^2}{4}\Big)\ln(\frac{\rho_1^2-\rho^2}{\rho_1^2})\\ &+\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\frac{n-1}{2})_{k-l}(\frac{1}{2})_k}{(1)_k}\Big(\psi(\frac{n-1}{2}+k-l)+\psi(\frac{1}{2}+k)\\ &-2\psi(1+k)\Big)\frac{(\frac{\rho_1^2-\rho^2}{\rho_1^2})^k}{k!}\frac{(\frac{\lambda^2r_1^2}{4})^l}{l!}\Big). \end{aligned} \tag{3.15}$$

Let us consider some properties of the solutions.

Lemma 3.1. *The solutions* (3.6) *and* (3.14) *satisfy*

$$\frac{\partial q_{01}(M,M_0)}{\partial x_n}\Big|_{x_n=0}=0,\quad \frac{\partial q_{\lambda 1}(M,M_0)}{\partial x_n}\Big|_{x_n=0}=0.$$

The proof of the above lemma is a straightforward calculation.

Lemma 3.2. *The solutions* (3.6) *and* (3.14) *possess a singularity* ρ^{2-n} *as* $\rho\to 0$.

The proof of the above lemma is carried out analogously to that in Lemma 2. We omit it. Lemma 3.2 implies the following result.

Theorem 3.3. *The functions* (3.6) *and* (3.14) *are fundamental solutions of equations* (1.2) *with* $\lambda=0$ *and with* $\lambda\neq 0$*, respectively.*

Conclusions. Nigmedzyanova [25] obtained a fundamental solution of the equation (1.1) by using generalized shift operator technique [24]. To this end, by the change of variables

$$\xi_i=x_i,\quad j=\overline{1,n-1},\quad \xi_n=\frac{2}{m+2}x_n^{\frac{m+2}{2}}$$

she reduced equation (1.1) to the form

$$\sum_{i=1}^{n-1}\frac{\partial^2u}{\partial\xi_i^2}+\frac{\partial^2u}{\partial\xi_n^2}+\frac{m}{m+2}\frac{1}{\xi_n}\frac{\partial u}{\partial\xi_n}+\lambda^2u=0. \tag{3.16}$$

Clearly, for no value of m, equation (1.1) is reducible to equation (3.16). Therefore, in the present article, along with equation (1.1) equation (1.2) was considered.

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