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EXISTENCE OF SOLUTIONS FOR A NON-ISOTHERMAL NAVIER-STOKES-ALLEN-CAHN SYSTEM WITH THERMO-INDUCED COEFFICIENTS

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Abstract. This article aims to study the existence of solutions for a non-isothermal Navier-Stokes-Allen-Cahn system with thermo-induced coefficients. The system can be considered as a model describing the motion of a mixture of two viscous incompressible fluids with viscosity, thermal conductivity and interfacial thickness being temperature-dependent. This is a more general system than the previous ones considered in the literature, involving temperature dependence on all main coefficients. The strong non-linear couplings between those equations because of the temperature dependence brings new mathematical difficulties that only allows working in two dimensions.

1. Introduction

We consider a non-isothermal Navier-Stokes-Allen-Cahn system where certain coefficients are temperature-dependent. The system can be considered as a model describing the motion of a mixture of two viscous incompressible fluids with viscosity, thermal conductivity and interfacial thickness being temperature-dependent, and matched densities. This system consists of Navier-Stokes equations coupled with a phase-field equation given by a convective Allen-Cahn equation and an energy transport equation. More precisely, we investigate the existence of solutions to the problem

$$\begin{aligned} &u_t + u\cdot\nabla u - \nabla\cdot(\nu(\theta)Du) + \nabla p \\ &= \lambda\Big(-\nabla\cdot(\varepsilon(\theta)\nabla\phi) + \frac{1}{\varepsilon(\theta)}F'(\phi)\Big)\nabla\phi - \alpha\Delta\theta\nabla\theta, \end{aligned} \tag{1.1}$$

$$\nabla\cdot u = 0, \tag{1.2}$$

$$\phi_t + u\cdot\nabla\phi = \gamma\Big(\nabla\cdot\Big(\varepsilon(\theta)\nabla\phi\Big) - \frac{1}{\varepsilon(\theta)}F'(\phi)\Big), \tag{1.3}$$

$$\theta_t + u\cdot\nabla\theta = \nabla\cdot(k(\theta)\nabla\theta), \tag{1.4}$$

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in $\Omega \times (0,\infty)$, where Ω is a bounded domain of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We complement the system with the initial and boundary conditions

$$\begin{gathered} u(0)=u_0, \quad \phi(0)=\phi_0, \quad \theta(0)=\theta_0, \quad \text{in } \Omega, \\ u=0, \quad \frac{\partial \phi}{\partial \eta}=0, \quad \frac{\partial \theta}{\partial \eta}=0, \quad \text{on } \partial\Omega \times (0,\infty), \end{gathered} \tag{1.5}$$

where η stands for the exterior normal to the boundary $\partial\Omega$.

Here, u, p and θ denote the mean velocity of the fluid mixture, the pressure, and the temperature; the phase-field variable ϕ represents the volume fraction of the two components. $Du = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$ corresponds to the symmetric part of the velocity gradient. $\nu > 0$ is the viscosity of the mixture, $\lambda > 0$ is the surface tension parameter, $\alpha > 0$ is associated to the thermal expansion coefficient, $\varepsilon > 0$ is a parameter related to the interfacial thickness, $F(\phi)$ is the potential energy density, $\gamma > 0$ is the relaxation time of the interface and $k > 0$ is the thermal conductivity. We assume that ν, ε and k are temperature-dependent.

Isothermal Navier-Stokes-Allen-Cahn systems have been extensively investigated in the literature, see, for instance, [13, 25, 14, 11, 19] and references therein. Concerning non-isothermal Navier-Stokes-Allen-Cahn systems, there are few theoretical results available. This is because it is not a trivial question to include temperature dependence in a way such that the obtained models are at the same time thermodynamically consistent and mathematically tractable. For instance, [23] introduced a non-isothermal model for a mixture of two fluids with thermo-induced Marangoni effects. The mathematical analysis of that system was carried out in [27, 28]. See [6] for the well-posedness of the one-dimensional non-isentropic compressible Navier-Stokes-Allen-Cahn system with temperature-dependent heat conductivity and [26] for the case with phase variable dependent viscosity. It is worth mentioning the works [8, 9, 10, 15] where a non-isothermal Navier-Stokes-Cahn-Hilliard system was analysed.

The system under consideration was previously studied by the authors in two different situations. When only the viscosity coefficient is temperature-dependent, the problem was considered in [16]. The well-posedness of the model was established showing the existence of global weak solutions in dimensions two and three, and existence and uniqueness of global strong solutions in dimension two, and local strong solutions in dimension three. No restriction on the size of the initial data was required. Later, in [17], it was also allowed that the thermal conductivity to be temperature-dependent. The existence and uniqueness of local strong solutions in two and three dimensions for any initial data were proved. Moreover, the existence of global weak solutions and the existence and uniqueness of global strong solutions in dimension two were obtained when the initial temperature is suitably small.

As far as we know, there are no studies about the phase-field equation with an interfacial thickness that develops with temperature. However, there are some physical situations where the thickness of the interface between fluid phases could depend on the temperature. We can mention [3], where is presented an atomistically informed parametrization of a phase-field model to describe the anisotropic mobility of liquid-solid interfaces in silicon. In this model, the interfacial mobility parameter of the phase-field describing the relaxation dynamics of the interface depends on temperature and also on interface orientation. See also [4], whereby using the method of molecular dynamics, the thickness of the interface between the fluid phases is determined as a function of temperature.

From the mathematical point of view, a closer study about a interfacial thickness variation is the analysis of sharp interface limit for phase-field models. In those cases, the thickness of the diffuse interface tends to zero. We can mention [20] for the Allen-Cahn equation, [2] for the Stokes-Allen-Cahn system, [5] for the Cahn-Hilliard equation, and [1] for the Navier-Stokes-Cahn-Hilliard system.

The goal of this article is to consider the general system (1.1)-(1.5) involving temperature dependence on all main coefficients. The strong non-linear couplings between those equations due to the temperature dependence brings new mathematical challenges that only allow working in dimension two. Moreover, the existence of solutions will be local in time; however, no restriction on the size of initial conditions is imposed. Observe that lower and higher order estimates cannot be obtained in a separate way. In order to prove the existence of local solutions, a novel higher order differential inequality for the shifted functions is constructed and combined with a small time argument, which was inspired by [18].

This article is organized as follows. In Section 2, some notation, assumptions, and the main theorem of this paper are introduced. In Section 3, the phase-field equation with thermo-induced interfacial thickness is analysed. In Section 4, the main theorem about existence of local in time solutions to (1.1)-(1.5) is proved.

2. Notation and main result

Before we state the main result of this paper, we fix the notation used through the text.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$ and let $|\Omega|$ denote the measure of the set Ω. By $L^p = L^p(\Omega)$, with $1 \le p \le \infty$, we denote the standard Lebesgue spaces and $H^m = W^{m,2}(\Omega)$, $0 \le m < +\infty$, the Sobolev spaces. $H_0^1 = H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the H^1-norm. The norms in L^2 or $(L^2)^2$ will be denoted by $\|\cdot\|$ and the inner product in $\mathbb{R}^2$ and the tensor product in $\mathbb{R}^{2^2}$ are denoted by $(\cdot,\cdot)$; the norms on H^m and $(H^m)^2$ are indicated by $\|\cdot\|_{H^m}$. We write $u \in L^2$, even when u is a vector field, meaning that all of its components are in L^2, and so on, since no confusion arises. Given two Banach spaces X, Y, we define the norm in $X \cap Y$ by $\|\cdot\|_{X\cap Y} = \|\cdot\|_X + \|\cdot\|_Y$.

For the mean velocity, we introduce the standard functional spaces of divergence-free vector fields

$$\mathcal{V}(\Omega) = \{v \in (C_0^\infty(\Omega))^2 : \nabla\cdot v = 0 \text{ in } \Omega\},$$
$$H = \overline{\mathcal{V}(\Omega)}^{(L^2)^2}, \quad V = \overline{\mathcal{V}(\Omega)}^{(H_0^1)^2}.$$

The duality product between V and V' will be denoted by $\langle\cdot,\cdot\rangle$, and by Poincaré and Korn inequalities, $\|\nabla v\|$ and $\|Dv\|$ are equivalent norms in V. Let P be the orthogonal projection from $(L^2)^2$ onto H. We shall denote by w_i and λ_i, respectively, the eigenfunctions and eigenvalues of the Stokes operator $A = -P\Delta$ defined in $V \cap H^2$. We observe that w_i are orthogonal and complete in the spaces H, V and $V \cap H^2$, see [24].

Let us highlight some inequalities that will be frequently used in the text. We recall that we are considering the two-dimensional case. The Ladyzhenskaya inequality

$$\|v\|_{L^4}^2 \le \sqrt{2}\|\nabla v\|\,\|v\|, \quad \forall v \in H_0^1, \tag{2.1}$$

and the Gagliardo-Nirenberg interpolation inequality

$$\|D^j v\|_{L^p} \le C\|v\|^{\alpha}_{\mathrm{W}^{m,r}}\|v\|^{1-\alpha}_{L^q}, \quad \forall v \in L^q \cap \mathrm{W}^{m,r},\ 1 \le q,\ r \le \infty \tag{2.2}$$

where

$$\frac{1}{p} = \frac{j}{2} + \alpha\Big(\frac{1}{r} - \frac{m}{2}\Big) + (1-\alpha)\frac{1}{q}, \quad \frac{j}{m} \le \alpha \le 1,$$

for some constant $C > 0$ with the only exception: if $1 < r < \infty$ and $m - j - \frac{2}{r}$ is a non-negative integer, then the above inequality holds only for $\frac{j}{m} \le \alpha < 1$ (see, e.g., [24, 12, 21]).

For the potential energy, we consider the following conditions: $F \in C^2(\mathbb{R})$ and satisfies

$$\lim_{s\to\pm\infty} F'(s) = \pm\infty, \quad \lim_{s\to\pm\infty} F''(s) = +\infty, \tag{2.3}$$

$$F'(1) \ge 0, \quad F'(-1) \le 0. \tag{2.4}$$

We observe that the classical double-well potential $F(s) = \frac{1}{4}(s^2-1)^2$ satisfies (2.3)-(2.4). We note that (2.3) implies the existence of some positive constants $C_i, i = 1,2,3$ such that

$$-C_1 \le F''(s), \quad -C_2 \le F(s) \le F'(s)s + C_3 \quad \text{for all } s \in \mathbb{R}, \tag{2.5}$$

where $F(s) = \int_0^s F'(r)dr$. Moreover, assumption (2.4) guarantees the maximum principle for the phase-field equation, and with this, F, F', F'' will be bounded.

We assume that the viscosity $\nu \in C^1(\mathbb{R})$, the interfacial thickness $\varepsilon \in C^2(\mathbb{R})$, and the thermal conductivity $k \in C^2(\mathbb{R})$ satisfy

$$0 < \nu(s), \quad 0 < \varepsilon(s), \quad 0 < k(s), \quad \forall s \in \mathbb{R}.$$

However, as the maximum principle for the temperature will be true, we can transform equations (1.1), (1.3), and (1.4) into equivalent ones. This can be done by suitably modifying ν, ε, and k outside the interval $[-\|\theta_0\|_{L^\infty}, \|\theta_0\|_{L^\infty}]$, see, e.g., [18, 28]. Therefore, we can assume that ν, ε, and k satisfy

$$0 < \nu_0 \le \nu(s), \quad 0 < \varepsilon_0 \le \varepsilon(s), \quad 0 < k_0 \le k(s) \quad \text{for all } s \in \mathbb{R},$$
$$\nu, \varepsilon, k, \nu', \varepsilon', k', \varepsilon'', k'' \text{ are bounded for all } s \in \mathbb{R}.$$

From now on, without loss of generality, we assume the constants $\lambda = \alpha = \gamma = 1$. Throughout this paper, C will denote a positive constant which may vary from line to line. Now we state the main result of this paper.

Theorem 2.1. *Given $u_0 \in V$, $\phi_0, \theta_0 \in H^2$, with $\frac{\partial\phi_0}{\partial\eta} = \frac{\partial\theta_0}{\partial\eta} = 0$ on $\partial\Omega$ and $\|\phi_0\|_{L^\infty} \le 1$, there exists $T^* > 0$ such that problem* (1.1)-(1.4) *with initial and boundary conditions* (1.5) *has at least one solution (u, ϕ, θ) satisfying*

$$u \in L^\infty(0,T^*;H) \cap L^2(0,T^*;V), \quad u_t \in L^{\frac{4}{3}}(0,T^*;V'),$$
$$\phi, \theta \in L^\infty(0,T^*;H^1 \cap L^\infty) \cap L^2(0,T^*;H^2), \quad \phi_t, \theta_t \in L^2(0,T^*;L^2),$$
$$|\phi| \le 1, \quad |\theta| \le \|\theta_0\|_{L^\infty} \quad a.e.\ \Omega \times (0,T^*),$$
$$\begin{aligned}&\langle u_t, v\rangle + (u\cdot\nabla u, v) + (\nu(\theta)Du, Dv)\\ &= \Big(\big(-\nabla\cdot(\varepsilon(\theta)\nabla\phi) + \frac{1}{\varepsilon(\theta)}F'(\phi)\big)\nabla\phi, v\Big) - (\Delta\theta\nabla\theta, v), \forall v \in V,\end{aligned}$$
$$\phi_t + u\cdot\nabla\phi = \nabla\cdot(\varepsilon(\theta)\nabla\phi) - \frac{1}{\varepsilon(\theta)}F'(\phi),$$

$$\theta_t + u \cdot \nabla\theta = \nabla \cdot (k(\theta)\nabla\theta),$$

for a.e. $(0,T^*) \times \Omega$, *and it fulfills the boundary conditions* $\frac{\partial\phi}{\partial\eta} = \frac{\partial\theta}{\partial\eta} = 0$ *on* $\partial\Omega \times (0,T^*)$, *as well as the initial conditions in* (1.5).

We have stated the Theorem with the minimal regularity required to obtain the existence of solutions to system (1.1)-(1.5). However, with some additional and long computations we can show that the solutions are more regular; indeed, $u \in L^\infty(0,T^*;V) \cap L^2(0,T^*;H^2)$ and $\phi,\theta \in L^\infty(0,T^*;H^2) \cap L^2(0,T^*;H^3)$. This can be done similarly as in the proof of [17, Theorem 4.3] (cf. Step 1), so we leave the details to interested readers.

3. Phase-field equation with thermo-induced interfacial thickness

In this section, we investigate the existence of strong solutions to the phase-field equation with thermo-induced interfacial thickness. We are going to apply this existence result when treating the whole system. In this way, we can assume enough regularity for u and θ. In particular, as u will be coming from the Galerkin approximations, we assume that u is a smooth divergence-free vector field vanishing on $\partial\Omega$ and that $\theta \in L^\infty(0,T;H^1 \cap L^\infty) \cap L^2(0,T;H^2)$ with $\theta_t \in L^2(0,T;L^2)$.

Proposition 3.1. *Let* $T > 0$ *and* $\phi_0 \in H^1 \cap L^\infty$ *such that* $\|\phi_0\|_{L^\infty} \leq 1$. *Then, the problem*

$$\begin{gathered}\phi_t + u \cdot \nabla\phi = \nabla \cdot (\varepsilon(\theta)\nabla\phi) - \frac{1}{\varepsilon(\theta)}F'(\phi) \quad \text{in } \Omega \times (0,T), \\ \phi(0) = \phi_0 \ \text{in } \Omega, \quad \frac{\partial\phi}{\partial\eta} = 0 \ \text{on } \partial\Omega \times (0,T),\end{gathered} \tag{3.1}$$

has a unique solution $\phi \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$ *with* $\phi_t \in L^2(0,T;L^2)$ *and* $|\phi| \leq 1$ *a.e.* $\Omega \times (0,T)$.

Proof. We start by introducing an approximate function

$$F_1'(s) = \begin{cases} sF''(1) + F'(1) - F''(1), & s \geq 1, \\ F'(s), & -1 \leq s \leq 1, \\ sF''(-1) + F'(-1) + F''(-1), & s \leq -1. \end{cases} \tag{3.2}$$

Let $F_1(s) = \int_0^s F_1'(r)\,dr$ and observe that properties (2.5) are true for all $s \in \mathbb{R}$ and F_1'' is bounded. See, e.g. [7, 13, 16]. Once the maximum principle for the unique solution ϕ_1 of (3.1) with F_1 instead of F is true, there will follow that $F_1(\phi_1) = F(\phi_1)$. Thenceforth, ϕ_1 will be a solution of (3.1), which is unique. To not overburden the notation, we will omit the subscript on F_1 and ϕ_1.

To show the existence of solutions we employ the Faedo-Galerkin method. Let α_k and $\psi_k, k \in \mathbb{N}$, be the eigenvalues and eigenfunctions of $-\Delta$ with homogeneous Neumann boundary conditions.

For each $m \in \mathbb{N}$, we look for a function $\phi_m(x,t) = \sum_{k=1}^m b_k^m(t)\psi_k(x)$ such that, for $k = 1,\dots,m$,

$$(\phi_m', \psi_k) + (u \cdot \nabla\phi_m, \psi_k) + (\varepsilon(\theta)\nabla\phi_m, \nabla\psi_k) + \Big(\frac{1}{\varepsilon(\theta)}F'(\phi_m), \psi_k\Big) = 0, \tag{3.3}$$

$$\phi_m(0) = \phi_0^m, \tag{3.4}$$

where $\phi_0^m \in span\{w_1, \dots, w_m\}$ and $\phi_0^m \to \phi_0$ in H^1 as $m \to \infty$. Then, (3.3)-(3.4) becomes a nonlinear system of ordinary differential equations for $b^m = (b_1^m, \dots, b_m^m)$,

$$\begin{aligned} \frac{d}{dt} b_k^m(t) = &- \sum_{i=1}^{m} b_i^m(t)(u \cdot \nabla\psi_i, \psi_k) - \sum_{i=1}^{m} b_i^m(t)(\varepsilon(\theta)\nabla\psi_i, \nabla\psi_k) \\ &- \Big(\frac{1}{\varepsilon(\theta)} F'(\phi_m), \psi_k\Big) := f^k(t, b^m), \end{aligned} \tag{3.5}$$

$$b_k^m(0) = \ k^{th} \text{ component of } \phi_0^m. \tag{3.6}$$

As $\varepsilon_0 \le \varepsilon(\cdot) \le \varepsilon_1$ and F'' is a bounded function, it is not difficult to see that $f(t, b^m) = (f^1, \dots, f^m)$ is locally Lipschitz continuous with respect to b^m. Therefore, (3.5)-(3.6) has a unique maximal solution in $[0, T_m)$ with $T_m \le T$ such that $b_k^m \in H^1(0, T_m)$. To show that $T_m = T$, we multiply (3.3) by b_k^m and sum for $k = 1, \dots, m$, to find

$$\frac{1}{2}\frac{d}{dt}\|\phi_m\|^2 + \varepsilon_0\|\nabla\phi_m\|^2 + \int_\Omega \frac{1}{\varepsilon(\theta)} F'(\phi_m)\phi_m dx \le 0,$$

where we have used that u is divergence-free and vanishes on the boundary.

Integrating from 0 to t and using properties (2.5) of F give us

$$\|\phi_m\|^2 + \varepsilon_0 \int_0^T \|\nabla\phi_m\|^2 dt \le CT|\Omega| + \|\phi_0\|^2, \tag{3.7}$$

where C is independent of m. Thus, we conclude that $T_m = T$ and that $\{\phi_m\}$ is bounded in $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ independently of m.

Next, we show that $\{\phi_m\}$ is bounded in $L^2(0, T; H^2)$. By multiplying (3.3) by $\alpha_k b_k^m$, and summing from $k = 1, \dots, m$, we discover that

$$(\phi_m', -\Delta\phi_m) - (u \cdot \nabla\phi_m, \Delta\phi_m) + (\nabla \cdot (\varepsilon(\theta)\nabla\phi_m), \Delta\phi_m) - \Big(\frac{1}{\varepsilon(\theta)} F'(\phi_m), \Delta\phi_m\Big) = 0,$$

from which it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla\phi_m\|^2 + \varepsilon_0\|\Delta\phi_m\|^2 \\ &\le (u \cdot \nabla\phi_m, \Delta\phi_m) - (\varepsilon'(\theta)\nabla\theta \cdot \nabla\phi_m, \Delta\phi_m) - \Big(\frac{1}{\varepsilon(\theta)} F'(\phi_m), \Delta\phi_m\Big). \end{aligned} \tag{3.8}$$

We proceed to estimate the right-hand side of the above inequality. First, note that

$$\begin{aligned} |(u \cdot \nabla\phi_m, \Delta\phi_m)| &\le \|u\|_{L^\infty}\|\nabla\phi_m\|\|\Delta\phi_m\| \\ &\le \delta\varepsilon_0\|\Delta\phi_m\|^2 + C\|\nabla\phi_m\|^2, \end{aligned}$$

for any $\delta > 0$ that will be chosen later and C depends on $\|u\|_{L^\infty}$.

For the next term, we recall that ε' is a bounded function and use Gagliardo-Nirenberg interpolation inequality (2.2) together with elliptic estimates to arrive at

$$\begin{aligned} |(\varepsilon'(\theta)\nabla\theta \cdot \nabla\phi_m, \Delta\phi_m)| &\le C\|\nabla\theta\|_{L^4}\|\nabla\phi_m\|_{L^4}\|\Delta\phi_m\| \\ &\le \delta\varepsilon_0\|\Delta\phi_m\|^2 + C\|\nabla\theta\|_{H^1}\|\nabla\theta\|\|\nabla\phi_m\|_{H^1}\|\nabla\phi_m\| \\ &\le 2\delta\varepsilon_0\|\Delta\phi_m\|^2 + C(\|\theta\|_{H^2}^2 + 1)\|\nabla\phi_m\|^2 + C\|\theta\|_{H^2}^2, \end{aligned}$$

where C depends on $\|\theta\|_{L^\infty(0,T;H^1)}$ and we have used that $\{\phi_m\}$ is bounded in $L^\infty(0, T; L^2)$.

We define $\beta_1 = \max\{|F''(1)|, |F''(-1)|\}$ and

$$\beta_2 = \max\Big\{\max_{-1\le s\le 1}\{|F'(s)|\}, |F'(1)-F''(1)|, |F'(-1)+F''(-1)|\Big\}.$$

Since $\varepsilon_0 \le \varepsilon(\cdot)$, by using (3.2), we have that

$$\begin{aligned} \big|\Big(\frac{1}{\varepsilon(\theta)}F'(\phi_m), \Delta\phi_m\Big)\big| &\le C\int_\Omega (\beta_1|\phi_m| + \beta_2)|\Delta\phi_m|\,dx \\ &\le C\|\phi_m\|\|\Delta\phi_m\| + C\|\Delta\phi_m\| \\ &\le \delta\varepsilon_0\|\Delta\phi_m\|^2 + C, \end{aligned} \tag{3.9}$$

where we have used (3.7).

By plugging the previous estimates into (3.8) and taking δ small enough, we arrive at

$$\frac{d}{dt}\|\nabla\phi_m\|^2 + \varepsilon_0\|\Delta\phi_m\|^2 \le C(\|\theta\|_{H^2}^2+1)\|\nabla\phi_m\|^2 + C\|\theta\|_{H^2}^2 + C.$$

Since $(\|\theta\|_{H^2}^2+1)$ is integrable on $[0,T]$, by Gronwall Lemma, we conclude that

$$\|\nabla\phi_m\|^2 \le Ce^{TC}(\|\nabla\phi_0\|^2 + T + 1),$$

and so

$$\|\nabla\phi_m\|^2 + \int_0^T \|\Delta\phi_m\|^2 dt \le C(T,\theta,\phi_0,u). \tag{3.10}$$

We note that the constant C is an increasing function on T, so it can be chosen independent of T for T in any finite interval. This estimate and (3.7) imply that $\{\phi_m\}$ is bounded independently of m in $L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$.

To guarantee some strong convergence for $\{\phi_m\}$, we estimate $\{(\phi_m)_t\}$. For this, we multiply (3.3) by $(b_k^m)'$ and sum from $k=1,\dots,m$, to obtain

$$\begin{aligned} &\|(\phi_m)_t\|^2 + (u\cdot\nabla\phi_m, (\phi_m)_t) + (\varepsilon(\theta)\nabla\phi_m, \nabla(\phi_m)_t) \\ &+ \Big(\frac{1}{\varepsilon(\theta)}F'(\phi_m), (\phi_m)_t\Big) = 0. \end{aligned} \tag{3.11}$$

Observe that

$$\begin{aligned} (\varepsilon(\theta)\nabla\phi_m, \nabla(\phi_m)_t) &= \int_\Omega \varepsilon(\theta)\frac{1}{2}\frac{\partial}{\partial t}|\nabla\phi_m|^2 dx \\ &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}(\varepsilon(\theta)|\nabla\phi_m|^2)dx - \frac{1}{2}\int_\Omega \varepsilon'(\theta)\theta_t|\nabla\phi_m|^2 dx, \end{aligned}$$

so we can rewrite (3.11) as

$$\begin{aligned} &\|(\phi_m)_t\|^2 + \frac{1}{2}\frac{d}{dt}\|(\varepsilon(\theta))^{1/2}\nabla\phi_m\|^2 \\ &= -(u\cdot\nabla\phi_m, (\phi_m)_t) + \frac{1}{2}\int_\Omega \varepsilon'(\theta)\theta_t|\nabla\phi_m|^2 dx - \Big(\frac{1}{\varepsilon(\theta)}F'(\phi_m), (\phi_m)_t\Big). \end{aligned} \tag{3.12}$$

We estimate the right-hand side of (3.12) term by term. For the first term,

$$|(u\cdot\nabla\phi_m, (\phi_m)_t)| \le \|u\|_{L^\infty}\|\nabla\phi_m\|\|(\phi_m)_t\| \le \delta\|(\phi_m)_t\|^2 + C,$$

for any $\delta > 0$ to be chosen later and we have used (3.10). Next, as ε' is bounded, Gagliardo-Nirenberg inequality (2.2), and elliptic estimates yield

$$\frac{1}{2}\int_\Omega \varepsilon'(\theta)\theta_t|\nabla\phi_m|^2 dx \le C\|\theta_t\|\|\nabla\phi_m\|_{L^4}^2$$

$$
\begin{aligned}
&\le C\|\theta_t\|\|\phi_m\|_{H^2}\|\nabla\phi_m \\
&\le C\|\Delta\phi_m\|^2 + C\|\theta_t\|^2 + C,
\end{aligned}
$$

where we have used that $\{\phi_m\}$ is bounded in $L^\infty(0,T;H^1)$. In a similar way to (3.9), it follows

$$
\Big|\Big(\frac{1}{\varepsilon(\theta)}F'(\phi_m),(\phi_m)_t\Big)\Big| \le \delta\|(\phi_m)_t\|^2 + C.
$$

Therefore, taking δ small enough, (3.12) can be estimated by

$$
\|(\phi_m)_t\|^2 + \frac{d}{dt}\|(\varepsilon(\theta))^{1/2}\nabla\phi_m\|^2 \le C\|\Delta\phi_m\|^2 + C\|\theta_t\|^2 + C.
$$

By integrating in time, we infer that $\{(\phi_m)_t\}$ is bounded in $L^2(0,T;L^2)$ independently of m and, by the compactness lemma [22, Cor. 4], that there exists a subsequence (relabeled the same) of $\{\phi_m\}$ that converges strongly to a function ϕ in $L^2(0,T;H^1)$.

Since F'' is bounded, $F'(\phi_m)\to F'(\phi)$ strongly in $L^2(0,T;L^2)$. Thus, we can pass to the limit as $m\to\infty$ in (3.3) and infer the existence of a strong solution ϕ of (3.1).

The proof of the maximum principle, i.e., if the initial datum satisfies $\|\phi_0\|_{L^\infty}\le 1$ then $|\phi|\le 1$ a.e. $\Omega\times(0,T)$, can be done as usual. We multiply equation (3.1) by the positive part $(\phi-1)_+$ and integrate in Ω to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\phi-1)_+\|^2 + \int_\Omega \varepsilon(\theta)|\nabla(\phi-1)_+|^2dx + \int_\Omega \frac{1}{\varepsilon(\theta)}(F'(\phi)-F'(1))(\phi-1)_+dx \\
&= -\int_\Omega \frac{1}{\varepsilon(\theta)}F'(1)(\phi-1)_+dx \le 0
\end{aligned}
$$

because $F'(1)\ge 0$ by (2.4). Next, we use the Mean Value Theorem to arrive at

$$
\frac{1}{2}\frac{d}{dt}\|(\phi-1)_+\|^2 + \int_\Omega \frac{1}{\varepsilon(\theta)}F''(\gamma)(\phi-1)(\phi-1)_+dx \le 0.
$$

Finally, by (2.5) and the fact that $\varepsilon_0\le\varepsilon(\cdot)$ we deduce that

$$
\frac{1}{2}\frac{d}{dt}\|(\phi-1)_+\|^2 \le \frac{C_1}{\varepsilon_0}\|(\phi-1)_+\|^2.
$$

Gronwall Lemma and the fact that $(\phi_0-1)_+=0$ imply that $(\phi-1)_+=0$ and therefore $\phi\le 1$. We proceed in analogous way by multiplying by the negative part $(\phi+1)_-$ to obtain that $\phi\ge -1$.

To finish, let $\phi_1,\phi_2\in L^\infty(0,T;H^1\cap L^\infty)\cap L^2(0,T;H^2)$ be two solutions of (3.1). Then $\phi_1-\phi_2$ solves

$$
(\phi_1-\phi_2)_t + u\cdot\nabla(\phi_1-\phi_2) = \nabla\cdot(\varepsilon(\theta)\nabla(\phi_1-\phi_2)) - \frac{1}{\varepsilon(\theta)}(F'(\phi_1)-F'(\phi_2))
$$

together with $\frac{\partial}{\partial\eta}(\phi_1-\phi_2)=0$ on $\partial\Omega\times(0,T)$ and $(\phi_1-\phi_2)(0)=0$ in Ω.

Multiplying the above equation by $\phi_1-\phi_2$ and integrating in Ω, we see

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\phi_1-\phi_2\|^2 + \varepsilon_0\|\nabla(\phi_1-\phi_2)\|^2 &\le -\Big(\frac{1}{\varepsilon(\theta)}(F'(\phi_1)-F'(\phi_2)),\phi_1-\phi_2\Big) \\
&\le C\|F'(\phi_1)-F'(\phi_2)\|\|\phi_1-\phi_2\| \\
&\le C\|\phi_1-\phi_2\|^2,
\end{aligned}
$$

where we used that $\nabla \cdot u = 0$, the Mean Value Theorem for F' and the fact that F'' is bounded.

Gronwall Lemma gives the uniqueness. This completes the proof. $\square$

If we assume more regularity on the initial datum and that $\theta \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3)$, the solution ϕ to (3.1) is more regular.

Proposition 3.2. *Under assumptions of Proposition 3.1 and, moreover, $\phi_0 \in H^2$ satisfying $\frac{\partial \phi_0}{\partial \eta} = 0$ on $\partial\Omega$, the solution ϕ of (3.1) satisfies*

$$\phi \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3).$$

Proof. We derive further a priori estimates for the approximate solution. Multiplying (3.3) by $\alpha_k^2 b_k^m$ and adding from $k = 1,\dots,m$, it follows that

$$((\phi_m)_t, \Delta^2\phi_m) + (u\cdot\nabla\phi_m, \Delta^2\phi_m) + (\varepsilon(\theta)\nabla\phi_m, \nabla\Delta^2\phi_m) + \Big(\frac{1}{\varepsilon(\theta)}F'(\phi_m), \Delta^2\phi_m\Big) = 0.$$

Note that $\frac{\partial(\Delta\phi_m)}{\partial\eta} = 0$ and $\frac{\partial(\partial_t\phi_m)}{\partial\eta} = 0$ on $\partial\Omega$. Hence, we have that

$$\begin{aligned}
((\phi_m)_t, \Delta^2\phi_m) &= \frac{1}{2}\frac{d}{dt}\|\Delta\phi_m\|^2,\\
(u\cdot\nabla\phi_m, \Delta^2\phi_m) &= -(\nabla u\nabla\phi_m, \nabla\Delta\phi_m) - (D^2\phi_m u, \nabla\Delta\phi_m),\\
(\varepsilon(\theta)\nabla\phi_m, \nabla\Delta^2\phi_m) &= -(\varepsilon'(\theta)\nabla\theta\cdot\nabla\phi_m, \Delta^2\phi_m) - (\varepsilon(\theta)\Delta\phi_m, \Delta^2\phi_m)\\
&= (\varepsilon''(\theta)\nabla\theta\nabla\theta\cdot\nabla\phi_m, \nabla\Delta\phi_m) + (\varepsilon'(\theta)D^2\theta\nabla\phi_m, \nabla\Delta\phi_m)\\
&\quad + (\varepsilon'(\theta)D^2\phi_m\nabla\theta, \nabla\Delta\phi_m) + (\varepsilon'(\theta)\nabla\theta\Delta\phi_m, \nabla\Delta\phi_m)\\
&\quad + (\varepsilon(\theta)\nabla\Delta\phi_m, \nabla\Delta\phi_m),\\
\big(\frac{1}{\varepsilon(\theta)}F'(\phi_m), \Delta^2\phi_m\big) &= \big(\frac{\varepsilon'(\theta)}{\varepsilon^2(\theta)}\nabla\theta F'(\phi_m), \nabla\Delta\phi_m\big) - \big(\frac{1}{\varepsilon(\theta)}F''(\phi_m)\nabla\phi_m, \nabla\Delta\phi_m\big).
\end{aligned}$$

Therefore, as $\varepsilon_0 \le \varepsilon(\cdot)$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta\phi_m\|^2 + \varepsilon_0\|\nabla\Delta\phi_m\|^2\\
&\le (\nabla u\nabla\phi_m, \nabla\Delta\phi_m) + (D^2\phi_m u, \nabla\Delta\phi_m) - (\varepsilon''(\theta)\nabla\theta\nabla\theta\cdot\nabla\phi_m, \nabla\Delta\phi_m)\\
&\quad - (\varepsilon'(\theta)D^2\theta\nabla\phi_m, \nabla\Delta\phi_m) - (\varepsilon'(\theta)D^2\phi_m\nabla\theta, \nabla\Delta\phi_m)\\
&\quad - (\varepsilon'(\theta)\nabla\theta\Delta\phi_m, \nabla\Delta\phi_m) - \Big(\frac{\varepsilon'(\theta)}{\varepsilon^2(\theta)}\nabla\theta F'(\phi_m), \nabla\Delta\phi_m\Big)\\
&\quad + \Big(\frac{1}{\varepsilon(\theta)}F''(\phi_m)\nabla\phi_m, \nabla\Delta\phi_m\Big) := \sum_{i=1}^{8} I_i.
\end{aligned} \tag{3.13}$$

We estimate the right-hand side of (3.13) term by term by using Hölder, Young, Gagliardo-Nirenberg interpolation (2.2), and elliptic inequalities together with the fact that $\{\phi_m\}$ is bounded independently of m in $L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$ which was already proved in the previous Proposition.

We denote by δ a positive constant that will be chosen later. Then, we have

$$\begin{gathered}
I_1 \le \delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C,\\
I_2 \le \delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C\|\Delta\phi_m\|^2 + C,
\end{gathered}$$

where C depends on u. Since ε'' is bounded,

$$\begin{aligned} I_3 &\le \delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C\|\nabla\theta\|_{L^8}^4\|\phi_m\|_{H^2}\|\nabla\phi_m\| \\ &\le \delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C\|\theta\|_{H^2}^4\left(\|\Delta\phi_m\| + \|\phi_m\|\right) \\ &\le \delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C\|\Delta\phi_m\|^2 + C, \end{aligned}$$

where C depends on $\|\theta\|_{L^\infty(0,T;H^2)}$. Next, since ε' is bounded,

$$\begin{aligned} I_4 &\le \delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C\|\theta\|_{H^3}\|\theta\|_{H^2}\|\nabla\phi_m\|_{H^1}\|\nabla\phi_m\| \\ &\le \delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C\|\theta\|_{H^3}\left(\|\Delta\phi_m\| + \|\phi_m\|\right) \\ &\le \delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C\|\Delta\phi_m\|^2 + C\|\theta\|_{H^3}^2 + C. \end{aligned}$$

Similarly,

$$\begin{aligned} I_5 &\le \delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C\|\theta\|_{H^2}^2\|\phi_m\|_{H^3}\|\phi_m\|_{H^2} \\ &\le \delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C\left(\|\nabla\Delta\phi_m\| + \|\Delta\phi_m\| + \|\phi_m\|\right)\left(\|\Delta\phi_m\| + \|\phi_m\|\right) \\ &\le 3\delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C\|\Delta\phi_m\|^2 + C \end{aligned}$$

and

$$I_6 \le 2\delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C\|\Delta\phi_m\|^2 + C.$$

As in (3.9), using that ε' is bounded,

$$\begin{aligned} I_7 &\le C\int_\Omega |\nabla\theta|\left(\beta_1|\phi_m| + \beta_2\right)|\nabla\Delta\phi_m|dx \\ &\le C\|\nabla\theta\|_{L^4}\|\phi_m\|_{L^4}\|\nabla\Delta\phi_m\| + C\|\nabla\theta\|\|\nabla\Delta\phi_m\| \\ &\le 2\delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C. \end{aligned}$$

Finally, as F'' is bounded,

$$I_8 \le \delta\varepsilon_0\|\nabla\Delta\phi_m\|^2 + C.$$

By taking δ small enough and plugging the previous estimates into (3.13) we arrive at

$$\frac{d}{dt}\|\Delta\phi_m\|^2 + \varepsilon_0\|\nabla\Delta\phi_m\|^2 \le C\|\Delta\phi_m\|^2 + C\|\theta\|_{H^3}^2 + C.$$

As $\theta \in L^2(0,T;H^3)$, it follows that

$$\|\Delta\phi_m\|^2 + \varepsilon_0\int_0^T \|\nabla\Delta\phi_m\|^2\,dt \le C(T,\theta,\phi_0,u). \tag{3.14}$$

We note again that the constant C is an increasing function on T, so it can be chosen independent of T for T in any finite interval. Hence, $\phi_m \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3)$. This completes the proof. $\square$

4. Proof of Theorem 2.1

To show the existence of a local solution to (1.1)-(1.5), we will use the semi-Galerkin method. We split the proof into four steps. In the first step, we introduce the approximate problem. In the second step, we show the existence of a local in time solution for the approximate problem. However, the time of existence depends on the approximate parameter. So, in the third step, we obtain a new differential inequality for some norms of the approximate solution which allows concluding the existence of small-time independent of the parameter. Finally, in the last step, we pass to the limit in the approximate problem showing the existence of a local solution to (1.1)-(1.5).

Step 1: Approximate problem. Let w_i and $\lambda_i, i \in \mathbb{N}$, be the eigenfunctions and eigenvalues of the Stokes operator A. For $m \in \mathbb{N}$, we denote by P_m the orthogonal projection from H onto $H_m = \operatorname{span}\{w_1, \dots, w_m\}$. Fixed $T > 0$ and for $m \in \mathbb{N}$, we consider the following approximate problem of finding

$$u_m(x,t) = \sum_{i=1}^{m} g_i^m(t) w_i(x), \quad \phi_m \text{ and } \theta_m,$$

satisfying

$$\begin{aligned} &(u_m', v) + (u_m \cdot \nabla u_m, v) + (\nu(\theta_m) D u_m, Dv) \\ &= \Big(\big(-\nabla \cdot (\varepsilon(\theta_m) \nabla \phi_m) + \frac{1}{\varepsilon(\theta_m)} F'(\phi_m) \big) \nabla \phi_m, v \Big) - (\Delta \theta_m \nabla \theta_m, v), \\ &\forall\, v \in H_m, \end{aligned} \tag{4.1}$$

$$(\phi_m)_t + u_m \cdot \nabla \phi_m = \nabla \cdot (\varepsilon(\theta_m) \nabla \phi_m) - \frac{1}{\varepsilon(\theta_m)} F'(\phi_m) \quad \text{in } \Omega \times (0,T), \tag{4.2}$$

$$(\theta_m)_t + u_m \cdot \nabla \theta_m = \nabla \cdot \big(k(\theta_m) \nabla \theta_m \big) \text{ in } \Omega \times (0,T), \tag{4.3}$$

$$u_m(0) = P_m u_0, \quad \phi_m(0) = \phi_0, \quad \theta_m(0) = \theta_0 \quad \text{in } \Omega, \tag{4.4}$$

$$\frac{\partial \phi_m}{\partial \eta} = \frac{\partial \theta_m}{\partial \eta} = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{4.5}$$

Step 2: Existence of local solutions to the approximate problem. We have the following result.

Lemma 4.1. *There exist $0 < T_m \le T$ and $u_m \in H^1(0, T_m; H_m)$, $\phi_m, \theta_m \in L^\infty(0, T_m; H^2) \cap L^2(0, T_m; H^3)$ such that (u_m, ϕ_m, θ_m) is the unique solution to (4.1)-(4.5).*

Proof. The existence of a unique local solution to this approximate problem can be done by using the Schauder Fixed Point Theorem. Indeed, given $u_m \in C([0,T]; H_m)$ with $u_m(t) = \sum_{i=1}^m g_i^m(t) w_i$ and $\big(\sum_{i=1}^m |g_i^m(t)|^2\big)^{1/2} \le M$, where M is a large enough constant that will be chosen later, there exists a unique $\theta_m \in L^\infty(0,T; H^2) \cap L^2(0,T; H^3)$ with $(\theta_m)_t \in L^2(0,T; L^2)$ and $\|\theta_m\|_{L^\infty} \le \|\theta_0\|_{L^\infty}$ solution to (4.3). The well-posedness to (4.3) can be obtained by the Faedo-Galerkin method in the spirit of Section 3, see also [18, 28]. Moreover, it holds that

$$\|\theta_m\|_{L^\infty(0,T;H^2) \cap L^2(0,T;H^3)} \le C_{m,M}. \tag{4.6}$$

By Propositions 3.1 and 3.2, there exists a unique $\phi_m \in L^\infty(0,T; H^2) \cap L^2(0,T; H^3)$ with $(\phi_m)_t \in L^2(0,T; L^2)$ and $\|\phi_m\|_{L^\infty} \le 1$ solution to (4.2). Moreover, from (3.7), (3.10) and (3.14) it follows that

$$\|\phi_m\|_{L^\infty(0,T;H^2) \cap L^2(0,T;H^3)} \le C_{m,M}. \tag{4.7}$$

Once θ_m, ϕ_m are determined, we turn to look for functions $\widehat{u}_m = \sum_{i=1}^m \hat{g}_i^m(t) w_i(x)$ satisfying (4.1), which is a system of m non-linear ordinary differential equations for $\{\hat{g}_i^m(t)\}_{i=1}^m$. From the assumptions on the coefficients ν and ε, it is standard to show the local well-posedness of the initial value problem using the classical theory of ordinary differential equations. To see that $\widehat{u}_m$ is defined on the whole interval $[0,T]$, we take $\widehat{u}_m$ as a test function in (4.1) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widehat{u}_m\|^2+\nu_0\|D\widehat{u}_m\|^2 &\le -(\nabla\cdot(\varepsilon(\theta_m)\nabla\phi_m)\nabla\phi_m,\widehat{u}_m)\\ &\quad+\Big(\frac{1}{\varepsilon(\theta_m)}F'(\phi_m)\nabla\phi_m,\widehat{u}_m\Big)-(\Delta\theta_m\nabla\theta_m,\widehat{u}_m).\end{aligned}\tag{4.8}$$

We proceed to estimate the right-hand side of (4.8) term by term. For the first term, by using that ε and ε' are bounded, Gagliardo-Nirenberg inequality (2.2), and elliptic estimates, we have

$$\begin{aligned}&-\big((\varepsilon'(\theta_m)\nabla\theta_m\cdot\nabla\phi_m)\nabla\phi_m,\widehat{u}_m\big)-(\varepsilon(\theta_m)\Delta\phi_m\nabla\phi_m,\widehat{u}_m)\\ &\le C\|\widehat{u}_m\|_{L^\infty}\|\nabla\theta_m\|\|\nabla\phi_m\|_{L^4}^2+C\|\widehat{u}_m\|_{L^\infty}\|\Delta\phi_m\|\|\nabla\phi_m\|\\ &\le\frac{\nu_0}{4}\|D\widehat{u}_m\|^2+C_m\|\nabla\theta_m\|^2\|\phi_m\|_{H^2}^2+C_m\|\Delta\phi_m\|^2\|\nabla\phi_m\|^2.\end{aligned}$$

Using the fact that $\varepsilon_0\le\varepsilon(\cdot)$ and that $\|\phi_m\|_{L^\infty}\le 1$ to estimate F', it follows

$$\Big(\frac{1}{\varepsilon(\theta_m)}F'(\phi_m)\nabla\phi_m,\widehat{u}_m\Big)\le\frac{\nu_0}{4}\|D\widehat{u}_m\|^2+C\|\nabla\phi_m\|^2.$$

Finally, rewriting the last term we have that

$$-(\Delta\theta^m\nabla\theta^m,\widehat{u}^m)=(\nabla\theta^m\otimes\nabla\theta^m,\nabla\widehat{u}^m)\le\frac{\nu_0}{4}\|D\widehat{u}_m\|^2+C\|\nabla\theta_m\|^4.$$

Therefore, (4.8) becomes

$$\begin{aligned}&\frac{d}{dt}\|\widehat{u}_m\|^2+\frac{\nu_0}{2}\|D\widehat{u}_m\|^2\\ &\le C_m\big(\|\nabla\theta_m\|^2\|\phi_m\|_{H^2}^2+\|\Delta\phi_m\|^2\|\nabla\phi_m\|^2+\|\nabla\phi_m\|^2+\|\nabla\theta_m\|^4\big).\end{aligned}$$

Integration in time, (4.6) and (4.7) allow us to conclude that

$$\|\widehat{u}_m(t)\|^2+\nu_0\int_0^t\|D\widehat{u}_m\|^2dt\le\|u_0\|^2+C_{m,M}T.\tag{4.9}$$

Hence, $\widehat{u}_m\in L^\infty(0,T;H)\cap L^2(0,T;V)$.

In this way, we obtain a well-defined operator

$$\begin{array}{ccccc}\Phi_T^m: C([0,T];H_m) & \to & \big(L^\infty(0,T;H^2)\cap L^2(0,T;H^3)\big)^2 & \to & H^1(0,T;H_m)\\ u_m & \mapsto & (\phi_m,\theta_m) & \mapsto & \widehat{u}_m\end{array}.$$

We next show that the operator Φ_T^m is continuous. As it was shown in [28, pp. 432-433], we can assure that θ_m is continuous with respect to u_m. More precisely, let $\overline{\theta}_m=\theta_m^1-\theta_m^2$ and $\overline{u}_m=u_m^1-u_m^2$, then it holds

$$\|\overline{\theta}_m\|_{L^\infty(0,T;L^2)\cap L^2(0;T;H^1)}\le C_m\|\overline{u}_m\|_{L^2(0,T;H)}.$$

To show that ϕ_m is continuous with respect to u_m, let ϕ_m^1 and ϕ_m^2 be two solutions to (4.2). So, $\overline{\phi}_m=\phi_m^1-\phi_m^2$ satisfies

$$\begin{aligned}&\partial_t\overline{\phi}_m+(u_m^1-u_m^2)\cdot\nabla\phi_m^1+u_m^2\cdot\nabla\overline{\phi}_m\\ &=\nabla\cdot\big(\varepsilon(\theta_m^1)\nabla\overline{\phi}_m\big)+\nabla\cdot\big((\varepsilon(\theta_m^1)-\varepsilon(\theta_m^2))\nabla\phi_m^2\big)-\frac{1}{\varepsilon(\theta_m^1)}F'(\phi_m^1)+\frac{1}{\varepsilon(\theta_m^2)}F'(\phi_m^2).\end{aligned}$$

Multiplying by $\overline{\phi}_m$, integrating over Ω and denoting $\frac{1}{\varepsilon(s)} = G(s)$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\overline{\phi}_m\|^2 + \varepsilon_0\|\nabla\overline{\phi}_m\|^2 \\
&\le \big(\overline{u}_m\cdot\nabla\overline{\phi}_m, \phi_m^1\big) - \big((\varepsilon(\theta_m^1) - \varepsilon(\theta_m^2))\nabla\phi_m^2, \nabla\overline{\phi}_m\big) \\
&\quad + \|F''\|_{L^\infty}\int_\Omega |G(\theta_m^1)||\overline{\phi}_m|^2 dx + \|G'\|_{L^\infty}\int_\Omega |F'(\phi_m^2)||\overline{\theta}_m||\overline{\phi}_m| dx,
\end{aligned}
$$

where we have used the Mean Value Theorem to estimate the last two terms as follows

$$
\begin{aligned}
&-\Big(\frac{1}{\varepsilon(\theta_m^1)}F'(\phi_m^1) - \frac{1}{\varepsilon(\theta_m^2)}F'(\phi_m^2), \overline{\phi}_m\Big) \\
&= -\big(G(\theta_m^1)(F'(\phi_m^1) - F'(\phi_m^2)), \overline{\phi}_m\big) - \big((G(\theta_m^1) - G(\theta_m^2))F'(\phi_m^2), \overline{\phi}_m\big) \\
&\le \|F''\|_{L^\infty}\int_\Omega |G(\theta_m^1)||\overline{\phi}_m|^2 dx + \|G'\|_{L^\infty}\int_\Omega |F'(\phi_m^2)||\overline{\theta}_m||\overline{\phi}_m| dx.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\overline{\phi}_m\|^2 + \varepsilon_0\|\nabla\overline{\phi}_m\|^2 \\
&\le \|\overline{u}_m\|\|\nabla\overline{\phi}_m\|\|\phi_m^1\|_{L^\infty} + C\|\overline{\phi}_m\|^2 + C\|\overline{\theta}_m\|^2 + \|\varepsilon'\|_{L^\infty}\int_\Omega |\overline{\theta}_m||\nabla\phi_m^2||\nabla\overline{\phi}_m|\, dx \\
&\le \frac{\varepsilon_0}{4}\|\nabla\overline{\phi}_m\|^2 + C(\|\overline{u}_m\|^2 + \|\overline{\phi}_m\|^2 + \|\overline{\theta}_m\|^2 + \|\overline{\theta}_m\|_{L^4}\|\nabla\phi_m^2\|_{L^4}\|\nabla\overline{\phi}_m\|) \\
&\le \frac{\varepsilon_0}{2}\|\nabla\overline{\phi}_m\|^2 + C(\|\overline{u}_m\|^2 + +\|\overline{\phi}_m\|^2 + \|\overline{\theta}_m\|_{H^1}^2)
\end{aligned}
$$

and conclude that

$$
\frac{d}{dt}\|\overline{\phi}_m\|^2 + \varepsilon_0\|\nabla\overline{\phi}_m\|^2 \le C(\|\overline{\phi}_m\|^2 + \|\overline{u}_m\|^2 + \|\overline{\theta}_m\|_{H^1}^2).
$$

Gronwall Lemma and the continuity of θ_m with respect to u_m yield

$$
\|\overline{\phi}_m\|^2 \le C(\|\overline{u}_m\|_{L^2(0,T;H)}^2 + \|\overline{\theta}_m\|_{L^2(0,T;H^1)}^2) \le C\|\overline{u}_m\|_{L^2(0,T;H)}^2,
$$

which implies the continuity of ϕ_m with respect to u_m.

It is not difficult to see the continuity of the solution $\widehat{u}_m$ to the system of ordinary differential equations (4.1) with respect to ϕ_m and θ_m. Indeed, we can consider $\overline{\widehat{u}}_m = \widehat{u}_m^1 - \widehat{u}_m^2$, where $\widehat{u}_m^i, i = 1,2$ are two solutions to (4.1). Writing the system that $\overline{\widehat{u}}_m$ satisfies and taking $\overline{\widehat{u}}_m$ as test function, we can deduce that

$$
\|\overline{\widehat{u}}_m\|_{L^\infty(0,T;H)}^2 \le C_m(\|\overline{\phi}_m\|_{L^2(0,T;H^1)}^2 + \|\overline{\theta}_m\|_{L^2(0,T;H^1)}^2).
$$

We only sketch how to deal with the higher-order nonlinear term since the other terms are easier. Notice that, by integrating by parts twice, we can rewrite

$$
\begin{aligned}
&-(\nabla\cdot(\varepsilon(\theta_m^1)\nabla\phi_m^1)\nabla\phi_m^1, \overline{\widehat{u}}_m) \\
&= (\varepsilon(\theta_m^1)\nabla\phi_m^1 D^2\phi_m^1, \overline{\widehat{u}}_m) + (\varepsilon(\theta_m^1)\nabla\phi_m^1\otimes\nabla\phi_m^1, \nabla\overline{\widehat{u}}_m) \\
&= (\varepsilon(\theta_m^1)\frac{1}{2}\nabla|\nabla\phi_m^1|^2, \overline{\widehat{u}}_m) + (\varepsilon(\theta_m^1)\nabla\phi_m^1\otimes\nabla\phi_m^1, \nabla\overline{\widehat{u}}_m) \\
&= -\frac{1}{2}(\varepsilon'(\theta_m^1)\nabla\theta_m^1|\nabla\phi_m^1|^2, \overline{\widehat{u}}_m) + (\varepsilon(\theta_m^1)\nabla\phi_m^1\otimes\nabla\phi_m^1, \nabla\overline{\widehat{u}}_m),
\end{aligned}
$$

where we have used that $\nabla \cdot \overline{\overline{u}}_m = 0$. We proceed similarly for $i = 2$. Hence, rearranging terms,

$$\begin{aligned}
&-(\nabla \cdot (\varepsilon(\theta_m^1)\nabla\phi_m^1)\nabla\phi_m^1, \overline{\overline{u}}_m) + (\nabla \cdot (\varepsilon(\theta_m^2)\nabla\phi_m^2)\nabla\phi_m^2, \overline{\overline{u}}_m) \\
&= -\frac{1}{2}((\varepsilon'(\theta_m^1) - \varepsilon'(\theta_m^2))\nabla\theta_m^1|\nabla\phi_m^1|^2, \overline{\overline{u}}_m) - \frac{1}{2}(\varepsilon'(\theta_m^2)\nabla\overline{\theta}_m|\nabla\phi_m^1|^2, \overline{\overline{u}}_m) \\
&\quad - \frac{1}{2}(\varepsilon'(\theta_m^2)\nabla\theta_m^2\nabla\overline{\phi}_m \cdot \nabla\phi_m^1, \overline{\overline{u}}_m) - \frac{1}{2}(\varepsilon'(\theta_m^2)\nabla\theta_m^2\nabla\phi_m^2 \cdot \nabla\overline{\phi}_m, \overline{\overline{u}}_m) \\
&\quad + ((\varepsilon(\theta_m^1) - \varepsilon(\theta_m^2))\nabla\phi_m^1 \otimes \nabla\phi_m^1, \nabla\overline{\overline{u}}_m) + (\varepsilon(\theta_m^2)\nabla\overline{\phi}_m \otimes \nabla\phi_m^1, \nabla\overline{\overline{u}}_m) \\
&\quad + (\varepsilon(\theta_m^2)\nabla\phi_m^2 \otimes \nabla\overline{\phi}_m, \nabla\overline{\overline{u}}_m).
\end{aligned}$$

Since ε' and ε'' are bounded and, thus, ε and ε' are Lipschitz continuous functions, the terms on the right-hand side can be estimated by using Hölder, Young and Gagliardo-Nirenberg inequalities. For instance, for the first term

$$\begin{aligned}
|((\varepsilon'(\theta_m^1) - \varepsilon'(\theta_m^2))\nabla\theta_m^1|\nabla\phi_m^1|^2, \overline{\overline{u}}_m)| &\le C\|\overline{\theta}_m\|_{L^4}\|\nabla\theta_m^1\|_{L^4}\|\nabla\phi_m^1\|_{L^4}^2\|\overline{\overline{u}}_m\|_{L^\infty} \\
&\le C_m\|\overline{\theta}_m\|_{H^1}\|\theta_m^1\|_{H^2}\|\phi_m^1\|_{H^2}\|\overline{\overline{u}}_m\|_{L^2} \\
&\le C_{m,M}(\|\overline{\theta}_m\|_{H^1}^2 + \|\overline{\overline{u}}_m\|_{L^2}^2),
\end{aligned}$$

where we have used (4.6), (4.7) and the fact that all norms on a finite-dimensional vector space are equivalent. The remainder terms are treated similarly.

Thenceforth, Φ_T^m is a continuous operator. Since H_m is a finite dimensional space, $H^1(0,T;H_m)$ is compact in $C([0,T];H_m)$. Moreover, as Φ_T^m is a bounded operator from $C([0,T];H_m)$ into $H^1(0,T;H_m)$, we conclude that Φ_T^m is a compact operator from $C([0,T];H_m)$ into itself.

From (4.9), by taking $M^2 = \|u_0\|^2/2$, we can choose T_m small enough such that we are able to apply the Schauder Fixed Point Theorem and conclude that there exists (u_m, ϕ_m, θ_m) solution to the approximate problem (4.1)-(4.5) defined on $[0, T_m]$. The proof of the uniqueness of the solution is standard, so we omit the details. □

Step 3: Local existence time independent of m. We will construct a new differential inequality and combine it with a small data argument for the shifted approximate solution $\widetilde{u}_m(t) = u_m(t) - u_m(0)$, $\widetilde{\phi}_m(t) = \phi_m(t) - \phi_0$ and $\widetilde{\theta}_m(t) = \theta_m(t) - \theta_0$ to show the existence of small time $T^* > 0$ such the solution (u_m, ϕ_m, θ_m) to (4.1)-(4.5) is defined on $[0, T^*]$.

First, we rewrite the approximate problem in terms of the shifted approximate solution. Starting with the equation for the mean velocity,

$$\begin{aligned}
&(\widetilde{u}_m', v) + ((\widetilde{u}_m + u_m(0)) \cdot \nabla\widetilde{u}_m, v) + (\nu(\widetilde{\theta}_m + \theta_0)D\widetilde{u}_m, Dv) \\
&= -((\widetilde{u}_m + u_m(0)) \cdot \nabla u_m(0), v) - (\nu(\widetilde{\theta}_m + \theta_0)Du_m(0), Dv) \\
&\quad + \Big(\big(-\nabla \cdot (\varepsilon(\widetilde{\theta}_m + \theta_0)\nabla(\widetilde{\phi}_m + \phi_0)) + \frac{1}{\varepsilon(\widetilde{\theta}_m + \theta_0)}F'(\widetilde{\phi}_m + \phi_0)\big)\nabla(\widetilde{\phi}_m + \phi_0), v\Big) \\
&\quad - (\Delta\widetilde{\theta}_m\nabla(\widetilde{\theta}_m + \theta_0), v) - (\Delta\theta_0\nabla(\widetilde{\theta}_m + \theta_0), v), \forall\, v \in H_m, \\
&\qquad \widetilde{u}_m(0) = 0.
\end{aligned} \tag{4.10}$$

For the phase-field, we have that

$$\begin{aligned} &\partial_t\widetilde{\phi}_m + (\widetilde{u}_m + u_m(0))\cdot\nabla\widetilde{\phi}_m - \nabla\cdot(\varepsilon(\widetilde{\theta}_m+\theta_0)\nabla\widetilde{\phi}_m) \\ &= \nabla\cdot(\varepsilon(\widetilde{\theta}_m+\theta_0)\nabla\phi_0) - (\widetilde{u}_m+u_m(0))\cdot\nabla\phi_0 - \frac{1}{\varepsilon(\widetilde{\theta}_m+\theta_0)}F'(\widetilde{\phi}_m+\phi_0) \end{aligned} \tag{4.11}$$

with $\widetilde{\phi}_m(0)=0$ and homogeneous Neumann boundary condition.

For the temperature, it holds

$$\begin{aligned} &\partial_t\widetilde{\theta}_m + \widetilde{u}_m\cdot\nabla(\widetilde{\theta}_m+\theta_0) - \nabla\cdot(k(\widetilde{\theta}_m+\theta_0)\nabla\widetilde{\theta}_m) \\ &= \nabla\cdot(k(\widetilde{\theta}_m+\theta_0)\nabla\theta_0) - u_m(0)\cdot\nabla(\widetilde{\theta}_m+\theta_0), \end{aligned} \tag{4.12}$$

with $\widetilde{\theta}_m(0)=0$ and homogeneous Neumann boundary condition.

Next, we take $v=\widetilde{u}_m$ in (4.10),

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\widetilde{u}_m\|^2 + \nu_0\|D\widetilde{u}_m\|^2 \\ &\leq -((\widetilde{u}_m+u_m(0))\cdot\nabla u_m(0),\widetilde{u}_m) - (\nu(\widetilde{\theta}_m+\theta_0)Du_m(0),D\widetilde{u}_m) \\ &\quad - (\nabla\cdot(\varepsilon(\widetilde{\theta}_m+\theta_0)\nabla\widetilde{\phi}_m)\nabla\widetilde{\phi}_m,\widetilde{u}_m) - (\nabla\cdot(\varepsilon(\widetilde{\theta}_m+\theta_0)\nabla\phi_0)\nabla\widetilde{\phi}_m,\widetilde{u}_m) \\ &\quad - (\nabla\cdot(\varepsilon(\widetilde{\theta}_m+\theta_0)\nabla\widetilde{\phi}_m)\nabla\phi_0,\widetilde{u}_m) - (\nabla\cdot(\varepsilon(\widetilde{\theta}_m+\theta_0)\nabla\phi_0)\nabla\phi_0,\widetilde{u}_m) \\ &\quad + \Big(\frac{F'(\widetilde{\phi}_m+\phi_0)}{\varepsilon(\widetilde{\theta}_m+\theta_0)}\nabla\widetilde{\phi}_m,\widetilde{u}_m\Big) + \Big(\frac{F'(\widetilde{\phi}_m+\phi_0)}{\varepsilon(\widetilde{\theta}_m+\theta_0)}\nabla\phi_0,\widetilde{u}_m\Big) \\ &\quad - (\Delta\widetilde{\theta}_m\nabla(\widetilde{\theta}_m+\theta_0),\widetilde{u}_m) - (\Delta\theta_0\nabla(\widetilde{\theta}_m+\theta_0),\widetilde{u}_m) \\ &:= \sum_{i=1}^{10} I_i. \end{aligned} \tag{4.13}$$

We estimate the right-hand side of (4.13) by using Hölder, Young, Ladyzhenskaya (2.1), Gagliardo-Nirenberg interpolation (2.2), and elliptic inequalities. Recall that ϕ_m and θ_m satisfy the maximum principle, i.e., $\|\phi_m\|_{L^\infty}\leq 1$ and $\|\theta_m\|_{L^\infty}\leq\|\theta_0\|_{L^\infty}$, so $\|\widetilde{\phi}_m\|\leq C\|\phi_0\|_{L^\infty}$ and $\|\widetilde{\theta}_m\|\leq C\|\theta_0\|_{L^\infty}$; hence,

$$\|\nabla\widetilde{\phi}_m\|_{L^4} \leq C\|\widetilde{\phi}_m\|_{H^2}^{1/2}\|\widetilde{\phi}_m\|_{L^\infty}^{1/2} \leq C(\|\Delta\widetilde{\phi}_m\|^{1/2}+1).$$

A similar estimate holds for $\widetilde{\theta}_m$. Moreover, it holds $\|u_m(0)\|_V\leq\|u_0\|_V$. We denote by δ a positive constant that will be chosen later. Then we have

$$\begin{aligned} I_1 &\leq \|u_0\|_V\|\widetilde{u}_m\|_{L^4}^2 + C\|u_0\|_V^2\|\widetilde{u}_m\|_{L^4} \\ &\leq C(\|\nabla\widetilde{u}_m\|\|\widetilde{u}_m\| + \|\nabla\widetilde{u}_m\|) \\ &\leq 2\delta\nu_0\|D\widetilde{u}_m\|^2 + C\|\widetilde{u}_m\|^2 + C. \end{aligned}$$

For the next term, we use that ν is bounded,

$$I_2 \leq C\|u_0\|_V\|D\widetilde{u}_m\| \leq \delta\nu_0\|D\widetilde{u}_m\|^2 + C.$$

Rewriting I_3 and using that ε and ε' are bounded, it follows that

$$\begin{aligned} I_3 &= -((\varepsilon'(\widetilde{\theta}_m+\theta_0)\nabla(\widetilde{\theta}_m+\theta_0)\cdot\nabla\widetilde{\phi}_m)\nabla\widetilde{\phi}_m,\widetilde{u}_m) - (\varepsilon(\widetilde{\theta}_m+\theta_0)\Delta\widetilde{\phi}_m\nabla\widetilde{\phi}_m,\widetilde{u}_m) \\ &\leq C(\|\nabla\widetilde{\theta}_m\|_{L^4}+\|\nabla\theta_0\|_{L^4})\|\nabla\widetilde{\phi}_m\|_{L^4}^2\|\widetilde{u}_m\|_{L^4} + C\|\Delta\widetilde{\phi}_m\|\|\nabla\widetilde{\phi}_m\|_{L^4}\|\widetilde{u}_m\|_{L^4} \\ &\leq C(\|\Delta\widetilde{\theta}_m\|^{1/2}+1)(\|\Delta\widetilde{\phi}_m\|+1)\|\nabla\widetilde{u}_m\|^{1/2}\|\widetilde{u}_m\|^{1/2} \end{aligned}$$

$$
\begin{aligned}
&\quad + C\|\Delta\widetilde{\phi}_m\|(\|\Delta\widetilde{\phi}_m\|^{1/2}+1)\|\nabla\widetilde{u}_m\|^{1/2}\|\widetilde{u}_m\|^{1/2}\\
&= C\|\Delta\widetilde{\theta}_m\|^{1/2}\|\Delta\widetilde{\phi}_m\|\|\nabla\widetilde{u}_m\|^{1/2}\|\widetilde{u}_m\|^{1/2} + C\|\Delta\widetilde{\theta}_m\|^{1/2}\|\nabla\widetilde{u}_m\|^{1/2}\|\widetilde{u}_m\|^{1/2}\\
&\quad + C\|\Delta\widetilde{\phi}_m\|\|\nabla\widetilde{u}_m\|^{1/2}\|\widetilde{u}_m\|^{1/2} + C\|\nabla\widetilde{u}_m\|^{1/2}\|\widetilde{u}_m\|^{1/2}\\
&\quad + C\|\Delta\widetilde{\phi}_m\|^{3/2}\|\nabla\widetilde{u}_m\|^{1/2}\|\widetilde{u}_m\|^{1/2}\\
&\le 3\delta\nu_0\|D\widetilde{u}_m\|^2 + 3\delta\varepsilon_0\|\Delta\widetilde{\phi}_m\|^2 + 2\delta k_0\|\Delta\widetilde{\theta}_m\|^2 + C\|\widetilde{u}_m\|^2(\|D\widetilde{u}_m\|^2+1) + C.
\end{aligned}
$$

Similarly for the next three terms

$$
\begin{aligned}
I_4 &= -((\varepsilon'(\widetilde{\theta}_m+\theta_0)\nabla(\widetilde{\theta}_m+\theta_0)\cdot\nabla\phi_0)\nabla\widetilde{\phi}_m,\widetilde{u}_m) - (\varepsilon(\widetilde{\theta}_m+\theta_0)\Delta\phi_0\nabla\widetilde{\phi}_m,\widetilde{u}_m)\\
&\le C\|\nabla\widetilde{\theta}_m\|_{L^4}\|\nabla\widetilde{\phi}_m\|_{L^4}\|\widetilde{u}_m\|_{L^4} + C\|\nabla\widetilde{\phi}_m\|_{L^4}\|\widetilde{u}_m\|_{L^4}\\
&\le C(\|\Delta\widetilde{\theta}_m\|^{1/2}+1)(\|\Delta\widetilde{\phi}_m\|^{1/2}+1)\|\nabla\widetilde{u}_m\|^{1/2}\|\widetilde{u}_m\|^{1/2}\\
&\quad + C(\|\Delta\widetilde{\phi}_m\|^{1/2}+1)\|\nabla\widetilde{u}_m\|\\
&\le 3\delta\nu_0\|D\widetilde{u}_m\|^2 + 3\delta\varepsilon_0\|\Delta\widetilde{\phi}_m\|^2 + 2\delta k_0\|\Delta\widetilde{\theta}_m\|^2 + C\|\widetilde{u}_m\|^2(\|D\widetilde{u}_m\|^2+1) + C,
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= -((\varepsilon'(\widetilde{\theta}_m+\theta_0)\nabla(\widetilde{\theta}_m+\theta_0)\cdot\nabla\widetilde{\phi}_m)\nabla\phi_0,\widetilde{u}_m) - (\varepsilon(\widetilde{\theta}_m+\theta_0)\Delta\widetilde{\phi}_m\nabla\phi_0,\widetilde{u}_m)\\
&\le C\|\widetilde{\theta}_m\|_{H^2}^{1/2}\|\widetilde{\phi}_m\|_{H^2}^{1/2}\|\nabla\widetilde{u}_m\|^{1/2}\|\widetilde{u}_m\|^{1/2} + C\|\widetilde{\phi}_m\|_{H^2}^{1/2}\|\nabla\widetilde{u}_m\|\\
&\quad + C\|\Delta\widetilde{\phi}_m\|\|\widetilde{u}_m\|^{1/2}\|\nabla\widetilde{u}_m\|^{1/2}\\
&\le 4\delta\nu_0\|D\widetilde{u}_m\|^2 + 4\delta\varepsilon_0\|\Delta\widetilde{\phi}_m\|^2 + 2\delta k_0\|\Delta\widetilde{\theta}_m\|^2 + C\|\widetilde{u}_m\|^2(\|D\widetilde{u}_m\|^2+1) + C,
\end{aligned}
$$

and

$$
\begin{aligned}
I_6 &= -((\varepsilon'(\widetilde{\theta}_m+\theta_0)\nabla(\widetilde{\theta}_m+\theta_0)\cdot\nabla\phi_0)\nabla\phi_0,\widetilde{u}_m) - (\varepsilon(\widetilde{\theta}_m+\theta_0)\Delta\phi_0\nabla\phi_0,\widetilde{u}_m)\\
&\le \delta k_0\|\Delta\widetilde{\theta}_m\|^2 + 3\delta\nu_0\|\nabla\widetilde{u}_m\|^2 + C\|\widetilde{u}_m\|^2 + C.
\end{aligned}
$$

Since $\|\widetilde{\phi}_m\| \le C\|\phi_0\|_{L^\infty}$, $\|\widetilde{\theta}_m\| \le C\|\theta_0\|_{L^\infty}$ and $\varepsilon_0 \le \varepsilon(\cdot)$ it follows that

$$
\begin{gathered}
I_7 \le C\|\nabla\widetilde{\phi}_m\|\|\widetilde{u}_m\| \le C\|\nabla\widetilde{\phi}_m\|^2 + C\|\widetilde{u}_m\|^2,\\
I_8 \le C\|\widetilde{u}_m\|^2 + C.
\end{gathered}
$$

We observe that I_9 will be canceled later. So, for the last term

$$
\begin{aligned}
I_{10} &\le \|\Delta\theta_0\|(\|\nabla\widetilde{\theta}_m\|_{L^4} + \|\nabla\theta_0\|_{L^4})\|\widetilde{u}_m\|_{L^4}\\
&\le C(\|\Delta\widetilde{\theta}_m\|^{1/2}+1)\|\nabla\widetilde{u}_m\|\\
&\le \delta k_0\|\Delta\widetilde{\theta}_m\|^2 + \delta\nu_0\|D\widetilde{u}_m\|^2 + C.
\end{aligned}
$$

Now, we multiply (4.11) by $-\Delta\widetilde{\phi}_m$, integrate over Ω and sum up, to arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\widetilde{\phi}_m\|^2 + \varepsilon_0\|\Delta\widetilde{\phi}_m\|^2\\
&\le ((\widetilde{u}_m + u_m(0))\cdot\nabla\widetilde{\phi}_m,\Delta\widetilde{\phi}_m) + ((\widetilde{u}_m + u_m(0))\cdot\nabla\phi_0,\Delta\widetilde{\phi}_m)\\
&\quad - (\varepsilon'(\widetilde{\theta}_m+\theta_0)\nabla(\widetilde{\theta}_m+\theta_0)\cdot\nabla\widetilde{\phi}_m,\Delta\widetilde{\phi}_m)\\
&\quad - (\varepsilon'(\widetilde{\theta}_m+\theta_0)\nabla(\widetilde{\theta}_m+\theta_0)\nabla\phi_0,\Delta\widetilde{\phi}_m) - (\varepsilon(\widetilde{\theta}_m+\theta_0)\Delta\phi_0,\Delta\widetilde{\phi}_m)\\
&\quad + \Big(\frac{F'(\widetilde{\phi}_m+\phi_0)}{\varepsilon(\widetilde{\theta}_m+\theta_0)},\Delta\widetilde{\phi}_m\Big) := \sum_{i=11}^{16} I_i.
\end{aligned}
\tag{4.14}
$$

We proceed to estimate (4.14) term by term in an analogous way as the previous estimates. We start with

$$\begin{aligned}
I_{11} &\le (\|\widetilde{u}_m\|_{L^4} + \|u_m(0)\|_{L^4})\|\nabla\widetilde{\phi}_m\|_{L^4}\|\Delta\widetilde{\phi}_m\| \\
&\le C(\|\nabla\widetilde{u}_m\|^{1/2}\|\widetilde{u}_m\|^{1/2} + 1)(\|\Delta\widetilde{\phi}_m\|^{1/2} + 1)\|\Delta\widetilde{\phi}_m\| \\
&\le C\|\Delta\widetilde{\phi}_m\|^{3/2}\|\nabla\widetilde{u}_m\|^{1/2}\|\widetilde{u}_m\|^{1/2} + C\|\Delta\widetilde{\phi}_m\|\|\nabla\widetilde{u}_m\|^{1/2}\|\widetilde{u}_m\|^{1/2} \\
&\quad + C\|\Delta\widetilde{\phi}_m\|^{3/2} + C\|\Delta\widetilde{\phi}_m\| \\
&\le 4\delta\varepsilon_0\|\Delta\widetilde{\phi}_m\|^2 + \delta\nu_0\|D\widetilde{u}_m\|^2 + C\|\widetilde{u}_m\|^2(\|D\widetilde{u}_m\|^2 + 1) + C,
\end{aligned}$$

and similarly

$$\begin{aligned}
I_{12} &\le C(\|\nabla\widetilde{u}_m\|^{1/2}\|\widetilde{u}_m\|^{1/2} + 1)\|\Delta\widetilde{\phi}_m\| \\
&\le \delta\varepsilon_0\|\Delta\widetilde{\phi}_m\|^2 + \delta\nu_0\|D\widetilde{u}_m\|^2 + C\|\widetilde{u}_m\|^2 + C.
\end{aligned}$$

Since ε' is bounded, it follows

$$\begin{aligned}
I_{13} &\le (\|\nabla\widetilde{\theta}_m\|_{L^4} + \|\nabla\theta_0\|_{L^4})\|\nabla\widetilde{\phi}_m\|_{L^4}\|\Delta\widetilde{\phi}_m\| \\
&\le C(\|\nabla\widetilde{\theta}_m\|_{H^1}^{1/2}\|\nabla\widetilde{\theta}_m\|^{1/2} + 1)(\|\Delta\widetilde{\phi}_m\|^{1/2} + 1)\|\Delta\widetilde{\phi}_m\| \\
&\le C(\|\Delta\widetilde{\theta}_m\|^{1/2} + 1)\|\nabla\widetilde{\theta}_m\|^{1/2}(\|\Delta\widetilde{\phi}_m\|^{3/2} + \|\Delta\widetilde{\phi}_m\|) + C\|\Delta\widetilde{\phi}_m\|^{3/2} + C\|\Delta\widetilde{\phi}_m\| \\
&\le 6\delta\varepsilon_0\|\Delta\widetilde{\phi}_m\|^2 + \delta k_0\|\Delta\widetilde{\theta}_m\|^2 + C\|\nabla\widetilde{\theta}_m\|^2(\|\Delta\widetilde{\theta}_m\|^2 + 1) + C.
\end{aligned}$$

By using again that ε' is bounded, one has

$$\begin{aligned}
I_{14} &\le (\|\nabla\widetilde{\theta}_m\|_{L^4} + \|\nabla\theta_0\|_{L^4})\|\nabla\phi_0\|_{L^4}\|\Delta\widetilde{\phi}_m\| \\
&\le C(\|\Delta\widetilde{\theta}_m\|^{1/2} + 1)\|\Delta\widetilde{\phi}_m\| \\
&\le \delta\varepsilon_0\|\Delta\widetilde{\phi}_m\|^2 + \delta k_0\|\Delta\widetilde{\theta}_m\|^2 + C.
\end{aligned}$$

As ε is bounded,

$$I_{15} \le \delta\varepsilon_0\|\Delta\widetilde{\phi}_m\|^2 + C.$$

For the last term, we use that $\|\widetilde{\phi}_m\| \le C$ and that $\varepsilon_0 \le \varepsilon(\cdot)$ to obtain

$$I_{16} \le \delta\varepsilon_0\|\Delta\widetilde{\phi}_m\|^2 + C.$$

We multiply (4.12) by $-\Delta\widetilde{\theta}_m$, integrate over Ω and sum up, to arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\widetilde{\theta}_m\|^2 + k_0\|\Delta\widetilde{\theta}_m\|^2 \\
&\le (\widetilde{u}_m \cdot \nabla(\widetilde{\theta}_m + \theta_0), \Delta\widetilde{\theta}_m) + (u_m(0) \cdot \nabla(\widetilde{\theta}_m + \theta_0), \Delta\widetilde{\theta}_m) \\
&\quad - (k'(\widetilde{\theta}_m + \theta_0)\nabla(\widetilde{\theta}_m + \theta_0) \cdot \nabla\widetilde{\theta}_m, \Delta\widetilde{\theta}_m) \\
&\quad - (k'(\widetilde{\theta}_m + \theta_0)\nabla(\widetilde{\theta}_m + \theta_0) \cdot \nabla\theta_0, \Delta\widetilde{\theta}_m) - (k(\widetilde{\theta}_m + \theta_0)\Delta\theta_0, \Delta\widetilde{\theta}_m) \\
&:= \sum_{i=17}^{21} I_i.
\end{aligned} \tag{4.15}$$

We note that $I_{17} = -I_9$, so it will be canceled later.

$$\begin{aligned}
I_{18} &\le \|u_m(0)\|_{L^4}(\|\nabla\widetilde{\theta}_m\|_{L^4} + \|\nabla\theta_0\|_{L^4})\|\Delta\widetilde{\theta}_m\| \\
&\le C(\|\Delta\widetilde{\theta}_m\|^{1/2} + 1)\|\Delta\widetilde{\theta}_m\|
\end{aligned}$$

$$\leq 2\delta k_0\|\Delta\widetilde{\theta}_m\|^2 + C.$$

Since k' is bounded, in a similar way as for I_{13}, we have that

$$\begin{aligned}
I_{19} &\leq (\|\nabla\widetilde{\theta}_m\|_{L^4} + \|\nabla\theta_0\|_{L^4})\|\nabla\widetilde{\theta}_m\|_{L^4}\|\Delta\widetilde{\theta}_m\| \\
&\leq C(\|\nabla\widetilde{\theta}_m\|_{H^1}^{1/2}\|\nabla\widetilde{\theta}_m\|^{1/2} + 1)(\|\Delta\widetilde{\theta}_m\|^{1/2} + 1)\|\Delta\widetilde{\theta}_m\| \\
&\leq 5\delta k_0\|\Delta\widetilde{\theta}_m\|^2 + C\|\nabla\widetilde{\theta}_m\|^2(\|\Delta\widetilde{\theta}_m\|^2 + 1) + C.
\end{aligned}$$

Using again that k' is bounded, it follows, analogously to I_{18},

$$I_{20} \leq \delta k_0\|\Delta\widetilde{\theta}_m\|^2 + C.$$

Finally, as k is bounded,

$$I_{21} \leq \delta k_0\|\Delta\widetilde{\theta}_m\|^2 + C.$$

By adding (4.13), (4.14) and (4.15), using estimates for I_1 to I_{21}, and taking δ small enough, we infer that

$$\frac{d}{dt}A(t) + B(t) \leq CA(t)(B(t) + 1) + C \tag{4.16}$$

where

$$\begin{gathered}
A(t) = \|\widetilde{u}_m\|^2 + \|\nabla\widetilde{\phi}_m\|^2 + \|\nabla\widetilde{\theta}_m\|^2, \\
B(t) = \nu_0\|D\widetilde{u}_m\|^2 + \varepsilon_0\|\Delta\widetilde{\phi}_m\|^2 + k_0\|\Delta\widetilde{\theta}_m\|^2.
\end{gathered}$$

We will prove that

$$A(t) < \frac{1}{2C} \text{ for } t \in [0, T^*], \tag{4.17}$$

where

$$T^* < \frac{1}{4C(\frac{1}{2} + C)}. \tag{4.18}$$

Since $A(0) = 0$, (4.17) holds for small times. Suppose by contradiction that there exists $T_m^* \in [0, T^*)$ such that

$$A(t) < \frac{1}{2C} \text{ in } [0, T_m^*) \quad \text{and} \quad A(T_m^*) = \frac{1}{2C}. \tag{4.19}$$

Using (4.19) into (4.16), it follows that

$$\frac{d}{dt}A(t) + B(t) \leq C\frac{1}{2C}\,(B(t) + 1) + C,$$

which implies

$$\frac{d}{dt}A(t) + \frac{1}{2}B(t) \leq \frac{1}{2} + C.$$

Consequently, by (4.18), for $t \in [0, T_m^*]$,

$$A(t) \leq (\frac{1}{2} + C)T^* < \frac{1}{4C} < \frac{1}{2C},$$

leading to a contradiction with the definition of T_m^* given by (4.19). Therefore, (4.17) holds for any $t \in [0, T^*]$ and T^* is independent of m.

Step 4: Passing to the limit as $m \to +\infty$. From the previous step, it follows that

$$\begin{gathered}
\{u_m\} \text{ is bounded in } L^\infty(0, T^*; H) \cap L^2(0, T^*; V), \\
\{\phi_m\}, \{\theta_m\} \text{ are bounded in } L^\infty(0, T^*; H^1 \cap L^\infty) \cap L^2(0, T^*; H^2),
\end{gathered}$$

independently of m.

To pass to the limit on the nonlinear terms, it is necessary to obtain some strong convergences. To this end, observe that from (4.1)

$$\begin{aligned}\|(u_m)_t\|_{V'} \le C(&\|u_m\|_{L^4}^2 + \|Du_m\| + \|\nabla\phi_m\|_{L^4}\|D^2\phi_m\| \\ &+ \|\nabla\phi_m\|_{L^4}^2 + \|\nabla\phi_m\| + \|\nabla\theta_m\|_{L^4}^2).\end{aligned}$$

By Gagliardo-Nirenberg interpolation inequality (2.2) and the fact that $\|\phi_m\|_{L^\infty} \le 1$ and $\|\theta_m\|_{L^\infty} \le \|\theta_0\|_{L^\infty}$, it follows that

$$\|\nabla\phi_m\|_{L^4}^2 \le C\|\phi_m\|_{H^2}\|\phi_m\|_{L^\infty} \le C\|\phi_m\|_{H^2} \quad \text{and} \quad \|\nabla\theta_m\|_{L^4}^2 \le C\|\theta_m\|_{H^2}.$$

Ladyzhenskaya inequality (2.1) together with the previous estimates, give us

$$\|(u_m)_t\|_{V'} \le C\big(\|u_m\|_V + \|\nabla u_m\|\|u_m\| + \|\phi_m\|_{H^2}^{3/2} + \|\phi_m\|_{H^2} + \|\theta_m\|_{H^2}\big).$$

Therefore, $\{(u_m)_t\}$ is bounded in $L^{\frac{4}{3}}(0,T^*;V')$ independently of m.

From (4.2), Ladyzhenskaya inequality (2.1), Gagliardo-Nirenberg interpolation inequality (2.2), $\|\phi_m\|_{L^\infty} \le 1$, and the fact that ε and ε' are bounded, we see that

$$\begin{aligned}\|(\phi_m)_t\| &\le \|u_m\|_{L^4}\|\nabla\phi_m\|_{L^4} + \|\nabla\phi_m\|_{L^4}\|\nabla\theta_m\|_{L^4} + \|\Delta\phi_m\| + C \\ &\le C\big(\|u_m\|\|\nabla u_m\| + \|\phi_m\|_{H^2} + \|\theta_m\|_{H^2} + 1\big),\end{aligned}$$

hence, $\{(\phi_m)_t\}$ is bounded in $L^2(0,T^*;L^2)$. Similarly, from (4.3), it follows that

$$\|(\theta_m)_t\| \le C\big(\|u_m\|\|\nabla u_m\| + \|\theta_m\|_{H^2} + 1\big).$$

Thus, $\{(\theta_m)_t\}$ is bounded in $L^2(0,T^*;L^2)$.

By the uniform estimates in m and the compactness lemma, there exist functions $u \in L^\infty(0,T^*;H) \cap L^2(0,T^*;V)$, $\phi,\theta \in L^\infty(0,T^*;H^1 \cap L^\infty) \cap L^2(0,T^*;H^2)$ such that, up subsequences, as $m \to \infty$,

$$\begin{gathered} u_m \rightharpoonup u \quad \text{in } L^2(0,T^*;V); \\ u_m \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T^*;H); \\ u_m \to u \quad \text{in } L^2(0,T^*;H); \\ \phi_m,\theta_m \rightharpoonup \phi,\theta \quad \text{in } L^2(0,T^*;H^2); \\ \phi_m,\theta_m \overset{*}{\rightharpoonup} \phi,\theta \quad \text{in } L^\infty(0,T^*;H^1); \\ \phi_m,\theta_m \to \phi,\theta \quad \text{in } L^2(0,T^*;H^1) \cap C([0,T^*];L^p),\ 1 \le p < \infty. \end{gathered}$$

All the previous convergences allow us to pass to the limit in (4.1)-(4.5), noting that $\nu(\theta_m) \to \nu(\theta)$, $\varepsilon(\theta_m) \to \varepsilon(\theta)$, $k(\theta_m) \to k(\theta)$, and $F'(\phi_m) \to F'(\phi)$ strongly in $C([0,T^*];L^p), 1 \le p < \infty$.

We only show the convergence of the terms involving $\varepsilon(\theta_m)$ in the velocity equation (4.1), since the convergence of the other terms is standard. We observe that

$$-\big(\nabla\cdot(\varepsilon(\theta)\nabla\phi)\nabla\phi, w_j\big) = \big(\varepsilon(\theta)\nabla\phi\otimes\nabla\phi, \nabla w_j\big) + \big(\varepsilon(\theta)\nabla\phi D^2\phi, w_j\big).$$

For any $\psi \in C_0^\infty([0,T^*])$, we can write

$$\begin{aligned}&\int_0^{T^*}\Big(-\big(\nabla\cdot(\varepsilon(\theta)\nabla\phi)\nabla\phi, w_j\big) + \big(\nabla\cdot(\varepsilon(\theta_m)\nabla\phi_m)\nabla\phi_m, w_j\big)\Big)\psi(t)dt \\ &= \int_0^{T^*}\big(\varepsilon(\theta)\nabla\phi\otimes\nabla\phi - \varepsilon(\theta_m)\nabla\phi_m\otimes\nabla\phi_m, \nabla w_j\big)\psi(t)dt\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{T^*}\big(\varepsilon(\theta)\nabla\phi D^2\phi-\varepsilon(\theta_m)\nabla\phi_m D^2\phi_m,w_j\big)\psi(t)dt\\
&:=J_1+J_2.
\end{aligned}
$$

For the first term, as ε is bounded, we have that

$$
\begin{aligned}
|J_1|&\le\int_0^{T^*}\big|\big((\varepsilon(\theta)-\varepsilon(\theta_m))\nabla\phi\otimes\nabla\phi,\nabla w_j\big)\psi(t)\big|dt\\
&\quad+\int_0^{T^*}\big|\big(\varepsilon(\theta_m)(\nabla\phi-\nabla\phi_m)\otimes\nabla\phi,\nabla w_j\big)\psi(t)\big|dt\\
&\quad+\int_0^{T^*}\big|\big(\varepsilon(\theta_m)\nabla\phi_m\otimes(\nabla\phi-\nabla\phi_m),\nabla w_j\big)\psi(t)\big|dt\\
&\le C\|\varepsilon(\theta)-\varepsilon(\theta_m)\|_{L^\infty(0,T^*;L^2)}\|\phi\|^2_{L^2(0,T^*;H^2)}\\
&\quad+C\|\phi_m-\phi\|_{L^2(0,T^*;H^1)}\|\phi\|_{L^2(0,T^*;H^1)}\\
&\quad+C\|\phi_m\|_{L^2(0,T^*;H^1)}\|\phi_m-\phi\|_{L^2(0,T^*;H^1)}
\end{aligned}
$$

which converges to 0 as $m\to\infty$. For the second term,

$$
\begin{aligned}
|J_2|&\le\Big|\int_0^{T^*}\big(\varepsilon(\theta)\nabla\phi(D^2\phi-D^2\phi_m),w_j\big)\psi(t)dt\Big|\\
&\quad+\int_0^{T^*}\big|\big(\varepsilon(\theta)(\nabla\phi-\nabla\phi_m)D^2\phi_m,w_j\big)\psi(t)\big|dt\\
&\quad+\int_0^{T^*}\big|\big((\varepsilon(\theta)-\varepsilon(\theta_m))\nabla\phi_m D^2\phi_m,w_j\big)\psi(t)\big|dt\\
&:=J_{21}+J_{22}+J_{23}.
\end{aligned}
$$

Since ε is bounded and $\nabla\phi\in L^2(0,T^*;L^2)$, it follows that $J_{21}\to 0$ as $m\to\infty$. For the next term,

$$
J_{22}\le C\|\phi_m-\phi\|_{L^2(0,T^*;H^1)}\|\phi_m\|_{L^2(0,T^*;H^2)}
$$

so, $J_{22}\to 0$ as $m\to\infty$. Finally,

$$
J_{23}\le C\|\varepsilon(\theta)-\varepsilon(\theta_m)\|_{L^\infty(0,T^*;L^4)}\|\phi_m\|^2_{L^2(0,T^*;H^2)},
$$

which implies that $J_{23}\to 0$ as $m\to\infty$.

We treat now the other term in the velocity equation concerning with ε. We have that

$$
\begin{aligned}
&\Big|\int_0^{T^*}\Big(\frac{1}{\varepsilon(\theta)}F'(\phi)\nabla\phi,w_j\Big)\psi(t)-\Big(\frac{1}{\varepsilon(\theta_m)}F'(\phi_m)\nabla\phi_m,w_j\Big)\psi(t)dt\Big|\\
&\le\int_0^{T^*}\Big|\Big(\Big(\frac{1}{\varepsilon(\theta)}-\frac{1}{\varepsilon(\theta_m)}\Big)F'(\phi)\nabla\phi,w_j\Big)\psi(t)\Big|dt\\
&\quad+\int_0^{T^*}\Big|\Big(\frac{1}{\varepsilon(\theta_m)}(F'(\phi)-F'(\phi_m))\nabla\phi,w_j\Big)\psi(t)\Big|dt\\
&\quad+\int_0^{T^*}\Big|\Big(\frac{1}{\varepsilon(\theta_m)}F'(\phi_m)(\nabla\phi-\nabla\phi_m),w_j\Big)\psi(t)\Big|dt.
\end{aligned}
$$

Since $\varepsilon_0\le\varepsilon(\cdot)$, ε and F' are bounded (because $\|\phi_m\|_{L^\infty}\le 1$), we can pass to the limit as $m\to\infty$ and show that the integral converges to zero.

Thus, we conclude that (u, ϕ, θ) is a local solution to (1.1)-(1.5). The proof of Theorem 2.1 is complete. $\square$

5. Conclusion

We have considered a general system (1.1)-(1.5) involving temperature dependence on all main coefficients. We have proved the existence of local in time solutions in the two-dimensional case without any restriction on the size of initial conditions. This is due to the strong non-linear couplings between those equations due to the temperature dependence which brings new mathematical challenges. Since lower and higher order estimates cannot be obtained in a separate way, we have derived a novel higher order differential inequality for the shifted functions and combine it with a small time argument.

It will be interesting to consider the three-dimensional case. However, our argument fails in this case and other techniques should be developed.

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