Electronic Journal of Differential Equations, Vol. **1995**(1995), No. 09, pp. 1–15.
ISSN 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp (login: ftp) 147.26.103.110 or 129.120.3.113

Singularity Formation in Systems of Non-strictly Hyperbolic Equations *

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Abstract

We analyze finite time singularity formation for two systems of hyperbolic equations. Our results extend previous proofs of breakdown concerning 2×2 non-strictly hyperbolic systems to $n \times n$ systems, and to a situation where, additionally, the condition of genuine nonlinearity is violated throughout phase space. The systems we consider include as special cases those examined by Keyfitz and Kranzer and by Serre. They take the form

$$u_t + (\phi(u)u)_x = 0,$$

where ϕ is a scalar-valued function of the n-dimensional vector u, and

$$u_t + \Lambda(u)u_x = 0,$$

under the assumption $\Lambda = \mathrm{diag}\,\{\lambda^1, \dots, \lambda^n\}$ with $\lambda^i = \lambda^i(u - u^i)$, where $u - u^i \equiv \{u^1, \dots, u^{i-1}, u^{i+1}, \dots, u^n\}$.

1 Introduction

In this paper we examine the formation of singularities in solutions to two $n \times n$ systems. The first of these is a conservation law

$$\mathbf{u}_t + \mathcal{F}_x(\mathbf{u}) = \mathbf{0} \tag{1}$$

which has $\mathcal{F} = \phi(\mathbf{u})\mathbf{u}$, and so the two vector fields $\mathcal{F}$ and $\mathbf{u}$ are parallel. We call this situation *radial*. The second system takes the form

$$\mathbf{u}_t + \Lambda(\mathbf{u})\mathbf{u}_x = \mathbf{0}. \tag{2}$$

Here Λ is a matrix-valued function of $\mathbf{u}$, $\Lambda = \mathrm{diag}\,\{\lambda^1(\mathbf{u}), \dots, \lambda^n(\mathbf{u})\}$. The fact that Λ is diagonal leads to the consideration of n weakly coupled equations, coupled through the dependence of the ${\lambda_i}'s$. These dependencies will be given the

* *1991 Mathematics Subject Classifications:* 35L45, 35L65, 35L67, 35L80.
Key words and phrases: Finite time breakdown, non-strict hyperbolicity, linear degeneracy.

Submitted: June 12, 1995. Published: June 28, 1995.

more explicit form $\lambda^i = \lambda^i(\mathbf{u} - u^i)$, where $\mathbf{u} - u^i \equiv \{u^1, \dots, u^{i-1}, u^{i+1}, \dots, u^n\}$ which we term *quasi-orthogonal.* We examine two special cases of this.

Each system has n eigenvalues some of which become equal on a submanifold Σ in phase space. They are therefore non-strictly hyperbolic. The principal distinguishing feature of the two systems turns out to be that while in (1) finite time breakdown can never take place on Σ, in (2) this can only take place there. The 2×2 counterpart of (1) has been studied from a related perspective to ours in [2], while (2) has been considered via compensated compactness in [9]. Our approach to system (1) in Section 2 is first to examine the structure of simple waves in the case of general ϕ, then to construct an invariant in the case that $\phi(\mathbf{u})$ has the simple dependence $\phi = \chi(\frac{1}{2}|\mathbf{u}|^2)$, and exploit its properties for general initial data. This leads to an approach for general initial data and with a larger class of functions, ϕ. In Section 3, we find a necessary condition for finite time breakdown of solutions to (2), while in Section 4 we demonstrate that this does indeed take place in the 2×2 case. The proof of this last result is somewhat different from previous 2×2 breakdown results ([3], [4], [5], [7]). Finally, in Section 5, we present some numerical results showing the singularity formation in the equation of Section 4.

2 Radial Flux $n \times n$ Systems

In this section we briefly examine the system of equations

$$\mathbf{u}_t + \mathcal{F}_x(\mathbf{u}) = \mathbf{0}, \tag{3}$$

where the flux function $\mathcal{F}(\mathbf{u})$ takes the particular form $\mathcal{F}(\mathbf{u}) = \phi(\mathbf{u})\mathbf{u}$. Here $\phi : \mathbb{R}^n \to \mathbb{R}$, and the flux lies parallel to the vector field $\mathbf{u}$, so for convenience we call this a *radial* flux function. Setting $\mathcal{A}(\mathbf{u}) = \nabla_u(\phi(\mathbf{u})\mathbf{u})$ gives

$$\mathcal{A}(\mathbf{u}) = \mathbf{u} \otimes \nabla_u \phi(\mathbf{u}) + \phi(\mathbf{u})\mathbf{I}. \tag{4}$$

The first term in (4) has rank one, which reduces the characteristic polynomial for $\mathcal{A}(\mathbf{u})$ to

$$\begin{aligned} |\lambda \mathbf{I} - \mathcal{A}(\mathbf{u})| &= |(\lambda - \phi(\mathbf{u}))\mathbf{I} - \mathbf{u} \otimes \nabla_u \phi(\mathbf{u})| \\ &= (\lambda - \phi(\mathbf{u}))^n - (\lambda - \phi(\mathbf{u}))^{n-1} tr((\mathbf{u} \otimes \nabla_u \phi(\mathbf{u})) \\ &= (\lambda - \phi(\mathbf{u}))^{n-1}(\lambda - \phi(\mathbf{u}) - \mathbf{u}.\nabla_u \phi(\mathbf{u})). \end{aligned} \tag{5}$$

Labeling the characteristic speeds by

$$\lambda_i = \begin{cases} \phi(\mathbf{u}), & 1 \le i \le n-1, \\ \phi(\mathbf{u}) + \mathbf{u}.\nabla_u \phi(\mathbf{u}), & i = n, \end{cases} \tag{6}$$

implies the corresponding right eigenvectors, $\mathbf{r}_i$, satisfy
$(\phi(\mathbf{u})\mathbf{I} - \mathcal{A}(\mathbf{u}))\mathbf{r}_i = -\mathbf{u} \otimes \nabla_u \phi(\mathbf{u})\mathbf{r}_i = -\mathbf{u}(\nabla_u \phi.\mathbf{r}_i), 1 \le i \le n-1$, and

$((\phi(\mathbf{u})+\mathbf{u}.\nabla_u\phi(\mathbf{u}))\mathbf{I}-\mathcal{A}(\mathbf{u}))\mathbf{r}_n = (\mathbf{u}.\nabla_u\phi(\mathbf{u})\mathbf{I}-\mathbf{u}\otimes\nabla_u\phi(\mathbf{u}))\mathbf{r}_n = (\mathbf{u}.\nabla_u\phi)\mathbf{r}_n - \mathbf{u}(\nabla_u\phi.\mathbf{r}_n)$.

Consequently, for $1 \le i \le n-1$, the $\mathbf{r}_i$'s can be chosen proportional to a set of mutually orthogonal vectors $\{(\nabla_u^\perp\phi)_i, 1 \le i \le n-1\} \equiv \nabla_u\phi^\perp$ perpendicular to $\nabla_u\phi$. $\mathbf{r}_n$ is proportional to $\mathbf{u}$ unless $\mathbf{u}.\nabla_u\phi = 0$, in which case $\mathbf{r}_n \in \nabla_u\phi^\perp$.

Similarly, one finds that the first $n-1$ left eigenvectors, $\mathbf{l}_i$, belong to the set $\mathbf{u}^\perp$ and that $\mathbf{l}_n$ is proportional to $\nabla_u\phi$ or $\mathbf{l}_n \in \mathbf{u}^\perp$ if $\mathbf{u}.\nabla_u\phi = 0$. The first $n-1$ characteristic fields satisfy $\mathbf{r}_i.\nabla_u\lambda_i \propto (\nabla_u^\perp\phi)_i.\nabla_u\phi = 0$ and are linearly degenerate, ([2]), while the nth characteristic field satisfies $\mathbf{r}_n.\nabla_u\lambda_n \propto \mathbf{u}.\nabla_u(\phi+\mathbf{u}.\nabla_u\phi)$. Set $\Upsilon = \{\mathbf{u} \in \mathbb{R}^n, \mathbf{u}.\nabla_u(\phi+\mathbf{u}.\nabla_u\phi) = 0\}$. Transforming to polar coordinates in $\mathbb{R}^n$, with $u_1 = r\cos\theta_1, u_j = r\prod_{k=1}^{j-1}\sin\theta_k\cos\theta_j$, $u_n = r\prod_{k=1}^{n-1}\sin\theta_k$, implies that $\mathbf{r}_n.\nabla_u\lambda_n \propto r\partial_r(\phi+r\partial_r\phi) = r\partial_r^2(r\phi)$, and so the nth characteristic field is genuinely nonlinear only when this term is nonzero.

By equation (6), all eigenvalues of $\mathcal{A}$ are equal where $\mathbf{u}.\nabla_u\phi(\mathbf{u}) \equiv r\phi_r = 0$. Following the terminology and notation of [2], we set $\Sigma = \{\mathbf{u} \in \mathbb{R}^n, \mathbf{u}.\nabla_u\phi = 0\}$ and observe that for $n = 2$ the system loses strict hyperbolicity on Σ, strict hyperbolicity being defined through the presence of real, distinct eigenvalues ([6]). For $n > 2$ the system becomes non-strictly hyperbolic everywhere since by (5) there are $n-1$ identical, real, eigenvalues for any $\mathbf{u}$. Some details of the behavior of solutions lying in $\Sigma\cap\Upsilon$ and $\Sigma\cap\mathbf{C}\Upsilon$ can be found in [8].

In the following Lemma, we consider the behavior of simple wave solutions, ([1]), to (3).

Lemma 2.1 *Let $\mathbf{u} \in C^1([0,T];C^1(\mathbb{R}))$ be a solution to (3) of the form $\mathbf{u}(t,x) = \mathbf{v}(\psi(t,x))$ where $\psi(x,t)$ is a scalar function of t and x. Then given data $\psi_0(x) = \psi(0,x)$, $||\psi_x||_\infty(t) \to \infty$ can occur in finite time only if there is a point x where $\mathbf{v}(\psi_0(x)) \notin \Sigma\cup\Upsilon$.*

Proof For such solutions, (3) reduces to

$$\mathbf{v}_\psi\psi_t + \phi(\mathbf{v})\mathbf{v}_\psi\psi_x + (\nabla_v\phi(\mathbf{v}).\mathbf{v}_\psi)\psi_x\mathbf{v} = \mathbf{0} \tag{7}$$

or

$$(\psi_t + \phi(\mathbf{v})\psi_x)\mathbf{v}_\psi + \psi_x\mathbf{v}\otimes\nabla_v\phi(\mathbf{v})\mathbf{v}_\psi = \mathbf{0}. \tag{8}$$

Consequently $\mathbf{v}_\psi$ is a right eigenvector of the matrix

$$\mathcal{A}(\mathbf{v}) = \mathbf{v}\otimes\nabla_v\phi(\mathbf{v}) + \phi(\mathbf{v})\mathbf{I} \tag{9}$$

having eigenvalue λ such that $\psi_t + \lambda\psi_x = 0$. Now using (6), λ takes on either the value $\phi(\mathbf{v})$ with corresponding right eigenvectors $\mathbf{v}_\psi \in \nabla_v\phi^\perp(\mathbf{v})$, or the value $\phi(\mathbf{v}) + \mathbf{v}.\nabla_v\phi(\mathbf{v})$ with eigenvector $\mathbf{v}_\psi \propto \mathbf{v}$.

In the first case, because of the linear degeneracy, linear waves maintain $\phi(\mathbf{v})$ constant on the hypersurface $\nabla_v\phi^\perp(\mathbf{v})$ while preventing singularity formation.

In the second case, $\phi(\mathbf{v}) + \mathbf{v}.\nabla_v\phi(\mathbf{v})$ remains constant in the radial, $\mathbf{v}$, direction, however singularities may form in finite time provided both $\mathbf{v}.\nabla_v\phi$ and $\mathbf{v}.\nabla_v(\phi+\mathbf{v}.\nabla_v\phi)$ are nonzero. This can be seen as follows. Suppose first that $\lambda = \phi$, and so $\mathbf{v}_\psi \in \nabla_v\phi^\perp(\mathbf{v})$. Then ψ, and consequently ϕ, remains constant along the (*straight*) characteristic $\frac{dx(t)}{dt} = \phi(\mathbf{v}(\psi(t,x(t))))$. Differentiating $\psi_t + \phi\psi_x = 0$ with respect to x, gives $\psi_{tx} + \phi\psi_{xx} + \phi_x\psi_x = 0$. However $\phi_x = \nabla_v\phi.\mathbf{v}_\psi\psi_x = 0$ since $\mathbf{v}_\psi \in \nabla_v\phi^\perp(\mathbf{v})$, and so ψ_x can only evolve linearly along the characteristic. It is simple to show (eg. [8]) that no other derivatives can blow up either in this case. Next suppose that $\lambda = \phi + \mathbf{v}.\nabla_v\phi$. Then $\psi_t + (\phi + \mathbf{v}.\nabla_v\phi)\psi_x = 0$, and ψ, therefore $\phi + \mathbf{v}.\nabla_v\phi$, remains constant along the (again, *straight*) characteristic $\frac{dx}{dt} = \phi + \mathbf{v}.\nabla_v\phi$. Differentiating with respect to x gives $\psi_{tx} + (\phi + \mathbf{v}.\nabla_v\phi)\psi_{xx} + \mathbf{v}_\psi.\nabla_v(\phi + \mathbf{v}.\nabla_v\phi)\psi_x^2 = 0$. Since all the terms in brackets depend only on ψ, these are constant on the characteristic, and finite time blow up of ψ will depend (together with the sign of the derivative of the initial data, ψ_{0x}) on the last term being nonzero. However by equation (7) it follows that for this value of λ, $\mathbf{v}_\psi(\mathbf{v}.\nabla_v\phi) = \mathbf{v}(\nabla_v\phi.\mathbf{v}_\psi)$. So $\mathbf{v}_\psi$ is parallel to $\mathbf{v}$ unless $\mathbf{v}.\nabla_v\phi = 0$, in which case $\mathbf{v}$ lies in Σ and then $\nabla_v\phi.\mathbf{v}_\psi = 0$, *ie.* either $\nabla_v\phi = \mathbf{0}$ or $\mathbf{v}_\psi \in \nabla_v\phi^\perp(\mathbf{v})$. If $\mathbf{v}_\psi \in \nabla_v\phi^\perp(\mathbf{v})$ and $\mathbf{v} \in \Sigma$, we can argue as in the previous paragraph to show no blow up occurs, and if $\nabla_v\phi = \mathbf{0}$, it is straightforward to show the same thing directly. We now assume $\mathbf{v} \notin \Sigma$. In this case, for nontrivial solutions, the coefficient of ψ_x^2 above will be nonzero whenever the term $\mathbf{v}.\nabla_v(\phi + \mathbf{v}.\nabla_v\phi)$ is nonzero, *ie.* $\mathbf{v} \notin \Upsilon$. This is simply the condition for genuine nonlinearity of the nth characteristic field above. Blow up is therefore possible only in this case, details of which can be supplied using standard techniques, ([7]). □

Remark. It can be seen from the above that in the case when $\mathbf{v} \in \Sigma$, then all n eigenvectors $\mathbf{v}_\psi$ must lie in the $n-1$ dimensional hyperplane $\nabla_v\phi^\perp(\mathbf{v})$. However it remains possible to construct a basis of eigenvectors and appropriate definition. Now we consider the possibility of introducing more general data than that in the above Lemma. We will assume that here $\phi(\mathbf{u}) = \chi(\frac{1}{2}|\mathbf{u}|^2)$. Our approach will be to extract a scalar conservation law from (3). This provides an invariant which we use to examine breakdown of solutions. In fact since the term $\mathcal{F}(\mathbf{u}) = \chi(\frac{1}{2}|\mathbf{u}|^2)\mathbf{u}$ in (3) is now a gradient, $\chi(\frac{1}{2}|\mathbf{u}|^2)\mathbf{u} = \nabla_u\Psi(\frac{1}{2}|\mathbf{u}|^2)$ where $\Psi' \equiv \chi$, there exists an entropy, $\eta = \frac{1}{2}|\mathbf{u}|^2$, for (3) together with an entropy flux, $\nu = \Psi - |\mathbf{u}|^2\chi$, such that $\eta_t + \nu_x = 0$, ([6]). Instead we choose another pair η, ν, with a more convenient functional relation to deduce breakdown.

Lemma 2.2 *Let* $\mathbf{u} \in C^1([0,T]; C^1(\mathbb{R}))$ *be a solution to (3), with* $\phi(\mathbf{u}) = \chi(\frac{1}{2}|\mathbf{u}|^2)$. *Then given data* $\mathbf{u}_0(x) = \mathbf{u}(0,x)$, $||\mathbf{u}_x||_\infty(t) \to \infty$ *can occur in finite time if there is a point* x *where* $\mathbf{u}_{0x} \notin \mathbf{u}_0^\perp$ *and* $\mathbf{u}_0 \notin \Sigma \cup \Upsilon$. *In particular, this will occur if* $(3\chi' + \chi''|\mathbf{u}_0|^2)\mathbf{u}_0.\mathbf{u}_{0x} < 0$.

Proof We attempt to extract a scalar conservation law from (3) having the form

$$\eta_t + f_x(\eta) = 0. \tag{10}$$

In other words, we require that $\nu = f(\eta)$. Once this is done, establishing breakdown becomes straightforward. Assuming it is possible to derive (10) from (3), then $\eta = \eta(\mathbf{u})$ and so (10) implies

$$\nabla_u \eta.\mathbf{u}_t + f'(\eta)\nabla_u \eta.\mathbf{u}_x = 0 \tag{11}$$

or

$$\nabla_u \eta.(\mathbf{u}_t + f'(\eta)\mathbf{u}_x) = 0. \tag{12}$$

But (3) implies

$$\nabla_u \eta.(\mathbf{u}_t + \nabla_u \mathcal{F}\mathbf{u}_x) = 0, \tag{13}$$

and so (12) and (13) show

$$\nabla_u \eta \nabla_u \mathcal{F} = \nabla_u \eta f'(\eta) \tag{14}$$

which means that $f'(\eta)$ is an eigenvalue $\lambda(\mathbf{u})$ of $\nabla_u \mathcal{F}$ having left eigenvector $\nabla_u \eta$. Now, since $f'(\eta) = \lambda(\mathbf{u})$, then

$$f''(\eta)\nabla_u \eta = \nabla_u \lambda \tag{15}$$

which implies that $\nabla_u \lambda$ is also a left eigenvector of $\nabla_u \mathcal{F}$ unless $f''(\eta) = 0$, and then $\eta = f'^{-1}(\lambda(\mathbf{u}))$.

By (15), if $\mathbf{r}$ and $\mathbf{l}$ are right and left eigenvectors corresponding to λ, then

$$\mathbf{r}.\nabla_u \lambda = f''(\eta)\mathbf{r}.\nabla_u \eta \propto f''(\eta)\,\mathbf{r}.\mathbf{l}\,.$$

So $f''(\eta) = 0 \Rightarrow \mathbf{u} \in \Upsilon$. (Note also that $\mathbf{r}.\mathbf{l} = 0 \Rightarrow \mathbf{u} \in \Sigma$.) Setting $g = f'^{-1}$, (10) together with $\eta = g(\lambda(\mathbf{u}))$ gives

$$g'(\lambda)\lambda_t + f'(\eta)g'(\lambda)\lambda_x = 0 \tag{16}$$

or, since $f'(\eta) = \lambda$,

$$\lambda_t + \lambda\lambda_x = 0. \tag{17}$$

Now in the case $\mathcal{F} = \chi(\frac{1}{2}|\mathbf{u}|^2)\mathbf{u}$, we have from (6) that

$$\lambda_i = \begin{cases} \chi(\frac{1}{2}|\mathbf{u}|^2), & 1 \le i \le n-1, \\ \chi(\frac{1}{2}|\mathbf{u}|^2) + \chi'(\frac{1}{2}|\mathbf{u}|^2)|\mathbf{u}|^2, & i = n, \end{cases} \tag{18}$$

and corresponding left eigenvectors $\mathbf{l}_i$ lie in the set $\mathbf{u}^\perp, 1 \leq i \leq n-1$, or are proportional to $\nabla_u \phi = \chi' \mathbf{u}$ for $i = n$, unless $\mathbf{u} \in \Sigma$, *ie.* $\chi' \neq 0$. For the above procedure to be possible for some $\lambda = \lambda_i$, we recall that $\nabla_u \lambda$ must be a left eigenvector corresponding to some eigenvector λ. Since by (18), all the λ_i have $\nabla_u \lambda_i$ proportional to $\mathbf{u}$, then it becomes possible to proceed only using λ_n. This leads to the result

$$\lambda_{nt} + \lambda_n \lambda_{nx} = 0 \tag{19}$$

with λ_n given by (18), which then implies ([6]) that on the characteristic $\frac{dx}{dt} = \lambda_n$,

$$\lambda_{nx} = \frac{\lambda_{n0x}}{1 + \lambda_{n0x} t} \tag{20}$$

where $\lambda_{n0} = \chi(\frac{1}{2}|\mathbf{u}_0|^2) + \chi'(\frac{1}{2}|\mathbf{u}_0|^2)|\mathbf{u}_0|^2$ and $\mathbf{u}_0(x) = \mathbf{u}(0, x)$. So $\lambda_{n0x} = (3\chi' + \chi''|\mathbf{u}_0|^2)\mathbf{u}_0.\mathbf{u}_{0x}$. However, recalling the definition of Υ, genuine nonlinearity requires the expression $\mathbf{u}.\nabla_u(\phi + \mathbf{u}.\nabla_u \phi)$ to be nonzero. With $\phi(\mathbf{u}) = \chi(\frac{1}{2}|\mathbf{u}|^2)$ this implies $(3\chi' + \chi''|\mathbf{u}|^2)|\mathbf{u}|^2 \neq 0$. So, for $\mathbf{u}_0 \notin \Sigma \cup \Upsilon, \mathbf{u}_{0x} \notin \mathbf{u}_0^\perp$ then $\lambda_{n0x} \neq 0$, and for $(3\chi' + \chi''|\mathbf{u}_0|^2)\mathbf{u}_0.\mathbf{u}_{0x} < 0$, then $\lambda_{n0x} < 0$ and finite time breakdown follows from (20). $\square$

With the previous Lemma as motivation, we turn to the final result of this section. This is to obtain more general conditions on ϕ under which breakdown can take place for arbitrary data. $\mathbf{u}$ will be represented in terms of polar coordinates, $\mathbf{u} = (r, \theta_1, \ldots, \theta_{n-1}),\ r = |\mathbf{u}|$.

Theorem 2.1 *Let $\mathbf{u} \in C^1([0,T]; C^1(\mathbb{R}))$ be a solution to (3), with $\phi(\mathbf{u}) = \mathcal{J}(r\mathcal{K}(\theta_1, \ldots, \theta_{n-1}))$, $\mathcal{J} \in C^2(\mathbb{R}), \mathcal{K} \in C^1(\mathbb{R}^{n-1})$. Then $||\mathbf{u}_x||_\infty(t) \to \infty$ in finite time if there is a point x where $(2\mathcal{J}' + \mathcal{J}'' r\mathcal{K})(r\mathcal{K})_x < 0$ at $t = 0$.*

Proof As before, we attempt to construct a convenient scalar conservation law. Rather than working with (14) and general ϕ, it turns out to be convenient to proceed as follows. Observe that the general form of equation (17) could, by (18), have been replaced by an equation of the form

$$\phi_t + h(\phi)\phi_x = 0 \tag{21}$$

for an appropriate function h, depending on the choice of λ. With this as a starting point, we attempt to find the most general conditions on $\phi(\mathbf{u})$ for which (21) can be derived for some function h.

Now by (3),

$$\mathbf{u}_t + \phi_x \mathbf{u} + \phi \mathbf{u}_x = \mathbf{0}. \tag{22}$$

Taking the scalar product of (22) with $\nabla_u \phi$ gives

$$\phi_t + \phi_x(\mathbf{u}.\nabla_u)\phi + \phi\phi_x = \mathbf{0}, \tag{23}$$

and for this to be of the form (21) requires that

$$(\mathbf{u}.\nabla_u)\phi + \phi = h(\phi). \tag{24}$$

We therefore solve the equation

$$(\mathbf{u}.\nabla_u)\phi = h(\phi) - \phi \equiv \mathcal{G}(\phi). \tag{25}$$

Define a curve Γ by $x = x(s)$, $\frac{d\mathbf{u}}{ds} = \mathbf{u}, x(0) = \gamma$. Then on Γ (consider t here as a parameter), $\frac{d\phi}{ds}(\mathbf{u}(t, x(s))) = (\frac{d\mathbf{u}}{ds}.\nabla_u)\phi = (\mathbf{u}.\nabla_u)\phi = \mathcal{G}(\phi)$. Solving for $\mathbf{u}$ on Γ gives

$$\mathbf{u}(t, x(s)) = \mathbf{u}(t, \gamma)e^s \tag{26}$$

where

$$\frac{d\phi}{ds} = \mathcal{G}(\phi). \tag{27}$$

Integrating (27) gives

$$\mathcal{H}(\phi(\mathbf{u}(t, x(s)))) = \mathcal{H}(\phi(\mathbf{u}(t, \gamma))) + s \tag{28}$$

where $\mathcal{H}' \equiv 1/\mathcal{G}$. Combining (26) with (28),

$$\mathcal{H}(\phi(\mathbf{u}(t, x(s)))) = \mathcal{H}(\phi(\mathbf{u}(t, x(s))e^{-s})) + s \tag{29}$$

implies, together with the result from (26) with $\mathbf{u}$ expressed in polar coordinates that $r(t, x(s)) = r(t, \gamma)e^s, \theta_i(t, x(s)) = \theta_i(t, \gamma), 1 \le i \le n-1$,

$$\mathcal{H}(\phi(\mathbf{u}(t, x(s)))) = \mathcal{H}(\phi(\mathbf{u}(t, x(s)))r(t, \gamma)/r(t, x(s))) + \ln(r(t, x(s))/r(t, \gamma)), \tag{30}$$

or

$$\begin{aligned} \phi(r, \theta_1, \ldots, \theta_{n-1}) &= \mathcal{H}^{-1} \circ (\mathcal{H} \circ \phi(r_0, \theta_1, \ldots, \theta_{n-1}) + \ln(r/r_0)) \\ &\equiv \mathcal{J}(r\mathcal{K}(\theta_1, \ldots, \theta_{n-1})), \end{aligned} \tag{31}$$

where we have set $r(t, \gamma) = r_0$, $\mathcal{J} = \mathcal{H}^{-1} \circ \ln$, and $\mathcal{K} = 1/r_0 \exp \mathcal{H} \circ \phi$. Taking $\mathcal{J}$ from (31) and using (25) gives

$$\begin{aligned} & r\frac{\partial}{\partial r}\phi = \mathcal{G}(\phi) \\ \Rightarrow \mathcal{J}'(r\mathcal{K}(\theta_1, \ldots, \theta_{n-1}))r\mathcal{K}(\theta_1, \ldots, \theta_{n-1}) &= \mathcal{G} \circ \mathcal{J}(r\mathcal{K}(\theta_1, \ldots, \theta_{n-1})) \end{aligned} \tag{32}$$

or

$$\mathcal{J}'(z)z = h(\mathcal{J}(z)) - \mathcal{J}(z) \tag{33}$$

which gives a functional relation between $\mathcal{J}$ and h. $\mathcal{K}$ is unconstrained. Thus we obtain a single conservation law of the form (21) provided ϕ has the structure given by (31), and then from (33), (21) becomes

$$\mathcal{J}_t + (\mathcal{J} + \mathcal{J}'z)\mathcal{J}_x = 0, \quad z = r\mathcal{K}. \tag{34}$$

Alternatively, multiplying by $(\mathcal{J} + \mathcal{J}'z)'$ and dividing by $\mathcal{J}'$ ($\neq 0$ if $\mathbf{u} \notin \Sigma$) gives

$$(\mathcal{J} + \mathcal{J}'z)_t + (\mathcal{J} + \mathcal{J}'z)(\mathcal{J} + \mathcal{J}'z)_x = 0 \tag{35}$$

which implies (cp. (20)) $(\mathcal{J} + \mathcal{J}'z)_x \to \infty$ in finite time provided $(\mathcal{J} + \mathcal{J}'z)_x < 0$ at $t = 0$. The result follows. □

3 Quasi-orthogonal $n \times n$ Systems

Here we consider systems of the form

$$\mathbf{u}_t + \Lambda(\mathbf{u})\mathbf{u}_x = \mathbf{0}, \tag{36}$$

with

$$\Lambda = \operatorname{diag}\{\lambda^1(\mathbf{u}-u^1),\ldots,\lambda^n(\mathbf{u}-u^n)\}, \tag{37}$$

where

$$\mathbf{u}-u^i = \{u^1,\ldots,u^{i-1},u^{i+1},\ldots,u^n\},\ 1 \le i \le n. \tag{38}$$

For simplicity, we make the additional hypothesis that the λ^i admit either the following additive structure

$$\lambda^i(\mathbf{u}-u^i) = \sigma(\mathbf{u}) - \nu^i(u^i) \tag{39}$$

where

$$\sigma(\mathbf{u}) = \sum_{j=1}^{n} \nu^j(u^j), \tag{40}$$

or the multiplicative structure

$$\lambda^i(\mathbf{u}-u^i) = \prod_{j\neq i}^{n} \mu^j(u^j). \tag{41}$$

Since the eigenvalues of Λ are $\lambda^1,\ldots,\lambda^n$, equality of any pair defines a (possibly *empty*) set Σ where (36) becomes non-strictly hyperbolic. The component u^i of $\mathbf{u}$ remains constant on the i-th characteristic, $dx^i/dt = \lambda^i(\mathbf{u}-u^i),\ 1 \le i \le n$, and so there exist at least n Riemann invariants for (36). The i-th right eigenvector, $\mathbf{r}_i$, satisfies $\mathbf{r}_i \propto \mathbf{e}_i$ where the set $\{\mathbf{e}_i,\ 1 \le i \le n\}$ makes up the standard Cartesian basis for $\mathbb{R}^n$, therefore by (38) $\mathbf{r}_i.\nabla_u\lambda^i = 0,\ 1 \le i \le n$. So the set Υ where the problem becomes linearly degenerate comprises the full phase space $\mathbb{R}^n$.

Lemma 3.1 *Let Λ be a C^1 function, $\Lambda : \mathbb{R}^n \to \mathbb{R}^{n^2}$, and let $\mathbf{u}(t,x) \in C^1([0,t^*);C^1(\mathbb{R}^n))$ be a solution to (36), with $\mathbf{u}(t,0) = \mathbf{u}_0(x),\ x \in \mathbb{R}$, for some maximal t^*. Then, under either (39) with (40), or (41), $t^* < \infty$ if and only if $\mathbf{u} : \mathbb{R}^n - \Sigma \to \Sigma$, as a map from $\mathbf{u}_0 \to \mathbf{u}(t,.)$. In addition, $\mathbf{u} : \Sigma \to \Sigma$ on any interval of existence.*

Proof Define the characteristic Γ_i by $x^i = x^i(t)$, $\frac{dx^i}{dt} = \lambda^i, x^i(0) = \alpha^i$, $1 \le i \le n$. Differentiation along Γ_i will be written as $D_i \equiv \partial/\partial t + \lambda^i\partial/\partial x$, from which it is immediate by (36) that $D_i u^i = 0,\ 1 \le i \le n$, *ie.* $u^i(t,x^i(t)) = u_0^i(\alpha^i)$, where $u_0^i(x) \equiv u^i(0,x)$.

Differentiating (36) with respect to x implies

$$D_i u_x^i + u_x^i \sum_{j\neq i}^n \frac{\partial \lambda^i}{\partial u^j} u_x^j = 0. \tag{42}$$

Also, for $i \neq j$,

$$D_i u^j = D_j u^j + (\lambda^i - \lambda^j) u_x^j = (\lambda^i - \lambda^j) u_x^j. \tag{43}$$

Consequently, unless $\lambda^i = \lambda^j$,

$$D_i u_x^i + u_x^i \sum_{j\neq i}^n \frac{\partial \lambda^i}{\partial u^j} \frac{D_i u^j}{\lambda^i - \lambda^j} = 0. \tag{44}$$

Adopting the additive assumptions (39), (40) reduces equation (44) to

$$D_i u_x^i + u_x^i \sum_{j\neq i}^n \nu^j(u^j)' \frac{D_i u^j}{\nu^j(u^j) - \nu^i(u^i)} = 0 \tag{45}$$

implying

$$D_i u_x^i + u_x^i D_i \sum_{j\neq i}^n \ln|\nu^j(u^j) - \nu^i(u^i)| = 0 \tag{46}$$

or

$$D_i(u_x^i \prod_{j\neq i}^n |\nu^j(u^j) - \nu^i(u^i)|) = 0. \tag{47}$$

The multiplicative condition (41) instead reduces (44) to

$$D_i u_x^i + u_x^i \sum_{j\neq i}^n (\frac{\partial}{\partial u^j} \prod_{k\neq i}^n \mu_k(u^k)) \frac{D_i u^j}{\prod_{l\neq i}^n \mu_l(u^l) - \prod_{m\neq j}^n \mu_m(u^m)} = 0 \tag{48}$$

and so, on simplifying,

$$D_i u_x^i + u_x^i \sum_{j\neq i}^n \mu_j(u^j)' \frac{D_i u^j}{\mu_j(u^j) - \mu_i(u^i)} \tag{49}$$

which takes the same form as (45). We therefore have, as with (47),

$$D_i(u_x^i \prod_{j\neq i}^n |\mu^j(u^j) - \mu^i(u^i)|) = 0. \tag{50}$$

Thus, both sets of hypotheses stated lead to analogous results, namely that on any characteristic, Γ_i, one obtains a relation of the form

$$u_x^i \prod_{j\neq i}^n |\kappa^j(u^j) - \kappa^i(u^i)| = u_{0x}^i \prod_{j\neq i}^n |\kappa^j(u_0^j) - \kappa^i(u_0^i)|,\ (t,x) \in \Gamma_i, \tag{51}$$

where κ^i represents either μ^i or ν^i. Accordingly, if $\kappa^j(u_0^j(\alpha^i)) = \kappa^i(u_0^i(\alpha^i))$ for some $j \neq i$, then $\kappa^j(u^j(t,x^i(t))) = \kappa^i(u^i(t,x^i(t)))$, $t \in (0,t^*)$ for some $t^* > 0$, by local continuity in time. On the other hand, if the right side of (51) is nonzero, then $u_x(t,x^i(t)) \to \infty$ if ever $\kappa^j(u^j(t,x^i(t))) \to \kappa^i(u^i(t,x^i(t)))$ for some $j \neq i$. Both sets of hypotheses allow this form of behavior only in Σ. If (39), (40) hold, then $\nu^i(u^i) = \nu^j(u^j)$, $j \neq i$, implies $\sigma(\mathbf{u}) - \lambda^i(\mathbf{u} - u^i) = \sigma(\mathbf{u}) - \lambda^j(\mathbf{u} - u^j)$, so $\lambda^i(\mathbf{u} - u^i) = \lambda^j(\mathbf{u} - u^j)$. If however (41) holds, then $\mu^i(u^i) = \mu^j(u^j)$. But $\lambda^j(\mathbf{u} - u^j)/\lambda^i(\mathbf{u} - u^i) = \prod_{l\neq j}^n \mu^l(u^l)/\prod_{k\neq i}^n \mu^k(u^k) = \mu^i(u^i)/\mu^j(u^j)$, and so again $\lambda^i(\mathbf{u} - u^i) = \lambda^j(\mathbf{u} - u^j)$. □

Remark. It is possible to obtain analogous results to the above under other conditions than (39)-(41). Either condition can however apply to the system considered in the next section, and so we do not generalize further here.

4 Quasi-orthogonal 2×2 Systems

Next, we consider the system of equations ([9]),

$$u_t + vu_x = 0, \tag{52}$$
$$v_t + uv_x = 0. \tag{53}$$

In the following, we let Γ denote the v-characteristic, defined by

$$\frac{dx}{dt}(t,\alpha) = v(t,x(t,\alpha)), \tag{54}$$

where α is a Lagrangian coordinate, and

$$x(0,\alpha) = \alpha. \tag{55}$$

Theorem 4.1 *Let $(u,v)(t,x) \in C^1([0,t^*);C^1(\mathbb{R}))$ be a solution to (52), (53), for some maximal t^*. Then $(u,v)(t,.) : \mathbb{R}^2 - \Sigma \to \Sigma$ as $t \to t^* < \infty$ whenever $u_0' < 0$ or $v_0' < 0$.*

Proof Equation (52) implies that on Γ,

$$u(t,x(t,\alpha)) = u_0(\alpha). \tag{56}$$

Now, from (53),

$$v_t + vv_x = (v-u)v_x, \tag{57}$$

and differentiating (52),

$$u_{tx} + vu_{xx} = -u_xv_x. \tag{58}$$

So (57) and (58) together give

$$(v-u)(u_{tx} + vu_{xx}) + (v_t + vv_x)u_x = 0, \tag{59}$$

which reduces to

$$\frac{d}{dt}((v-u)u_x) = 0, \tag{60}$$

where

$$\frac{d}{dt} \equiv D_1 = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} \tag{61}$$

and we have used (56). As a result of (60), then

$$(v(t,x(t,\alpha)) - u_0(\alpha))u_x(t,x(t,\alpha)) = (v_0(\alpha) - u_0(\alpha))u_0'(\alpha). \tag{62}$$

Now by (54)

$$\frac{dx}{dt} = v \Rightarrow \frac{dx_\alpha}{dt} = v_x x_\alpha \Rightarrow \frac{d\ln|x_\alpha|}{dt} = v_x \tag{63}$$

and by (58),

$$u_{tx} + vu_{xx} = -u_x v_x \Rightarrow \frac{d\ln|u_x|}{dt} = -v_x. \tag{64}$$

(63), (64) therefore show

$$\frac{d\ln|u_x|}{d\ln|x_\alpha|} = -1, \ (t,x) \in \Gamma, \tag{65}$$

from which it follows easily that

$$|u_x| \to \infty \text{ as } |x_\alpha| \to 0 \tag{66}$$

since (65) implies

$$\int_{u_0'(\alpha)}^{u_x(t,x(t,\alpha))} d\ln|u_x| = -\int_\alpha^{x(t,\alpha)} d\ln|x_\alpha|, \tag{67}$$

and so

$$u_x(t,x(t,\alpha)) = u_0'(\alpha)x_\alpha^{-1}(t,\alpha). \tag{68}$$

Here we have used continuity in time of the *local* initial value problem and (55) to remove the absolute value signs. Together with (62), (68) also gives

$$v(t,x(t,\alpha)) - u_0(\alpha) = (v_0(\alpha) - u_0(\alpha))x_\alpha(t,\alpha). \tag{69}$$

Next, using (54), (56) and (69), we obtain

$$x_t + (u_0 - v_0)x_\alpha = u_0, \tag{70}$$

a linear, non-constant coefficient equation for $x(t,\alpha)$. Introducing a second coordinate, a, for (t,α) space, such that

$$\frac{d\alpha}{dt}(t,a) = u_0(\alpha(t,a)) - v_0(\alpha(t,a)) \equiv w_0(\alpha(t,a)), \tag{71}$$

with $\alpha(0,a) \equiv \alpha_0(a)$, and denoting

$$\mathcal{D} = \frac{\partial}{\partial t} + w_0 \frac{\partial}{\partial \alpha}, \tag{72}$$

(70) then implies that

$$\mathcal{D}x(t, \alpha(t,a)) = u_0(\alpha(t,a)), \tag{73}$$

where $x(0, \alpha(0,a)) = \alpha_0(a)$. Since initial data lie in $\mathbb{R}^2 - \Sigma$, therefore $w_0(\alpha_0(a)) \neq 0$ and (71) gives

$$\mathcal{Q}(\alpha(t,a)) - \mathcal{Q}(\alpha_0(a)) \equiv \int_{\alpha_0(a)}^{\alpha(t,a)} \frac{d\alpha}{w_0(\alpha)} = t \tag{74}$$

where $\mathcal{Q}'(\alpha) \equiv 1/w_0(\alpha)$. So provided $w_0(\alpha(t,a)) \neq 0$,

$$\alpha(t,a) = \mathcal{Q}^{-1}(\mathcal{Q}(\alpha_0(a)) + t). \tag{75}$$

By (73), then

$$\mathcal{D}x(t, \alpha(t,a)) = u_0(\alpha(t,a)) = u_0(\mathcal{Q}^{-1}(\mathcal{Q}(\alpha_0(a)) + t)). \tag{76}$$

If we now define a Lagrangian variable $X(t,a)$ by

$$X(t,a) = x(t, \alpha(t,a)),\ X(0,a) = \alpha_0(a), \tag{77}$$

then $X_t = \mathcal{D}x$ by (72), and

$$X_t(t,a) = u_0(\alpha(t,a)) = u_0(\mathcal{Q}^{-1}(\mathcal{Q}(\alpha_0(a)) + t)) \tag{78}$$

implies

$$X(t,a) = \alpha_0(a) + \mathcal{S}_0(\mathcal{Q}(\alpha_0(a)) + t) - \mathcal{S}_0(\mathcal{Q}(\alpha_0(a))) \tag{79}$$

where $\mathcal{S}_0' = u_0 \circ \mathcal{Q}^{-1}$. As a result, using (75), (77) and (79),

$$x(t, \mathcal{Q}^{-1}(\mathcal{Q}(\alpha_0(a)) + t)) = \alpha_0(a) + \mathcal{S}_0(\mathcal{Q}(\alpha_0(a)) + t) - \mathcal{S}_0(\mathcal{Q}(\alpha_0(a))), \tag{80}$$

or, since (75) implies $\mathcal{Q}(\alpha_0(a)) = \mathcal{Q}(\alpha(t,a)) - t$, then (80) reads

$$x(t, \alpha) = \mathcal{Q}^{-1}(\mathcal{Q}(\alpha) - t) + \mathcal{S}_0(\mathcal{Q}(\alpha)) - \mathcal{S}_0(\mathcal{Q}(\alpha) - t). \tag{81}$$

In particular, on differentiating (81),

$$x_\alpha(t, \alpha) = \frac{\mathcal{Q}'(\alpha)}{\mathcal{Q}'(\mathcal{Q}^{-1}(\mathcal{Q}(\alpha) - t))} + \mathcal{S}_0'(\mathcal{Q}(\alpha))\mathcal{Q}'(\alpha) - \mathcal{S}_0'(\mathcal{Q}(\alpha) - t)\mathcal{Q}'(\alpha), \tag{82}$$

and so, since $\mathcal{S}_0' = u_0 \circ \mathcal{Q}^{-1},\ \mathcal{Q}' = 1/w_0$, by means of (71)

$$\begin{aligned} x_\alpha(t,\alpha) &= \frac{1}{w_0(\alpha)}(w_0(\mathcal{Q}^{-1}(\mathcal{Q}(\alpha)-t)) + u_0(\alpha) - u_0(\mathcal{Q}^{-1}(\mathcal{Q}(\alpha)-t))) \\ &= \frac{u_0(\alpha) - v_0(\mathcal{Q}^{-1}(\mathcal{Q}(\alpha)-t))}{u_0(\alpha)-v_0(\alpha)}. \end{aligned} \tag{83}$$

This then implies breakdown, by (68), provided there exists some positive time, t, at which $u_0(\alpha) = v_0(\mathcal{Q}^{-1}(\mathcal{Q}(\alpha)-t))$, *ie.* provided $t = \mathcal{Q}(\alpha) - \mathcal{Q}(v_0^{-1}(u_0(\alpha))) > 0$, if v_0 possesses a local inverse. Since $\mathcal{Q}' = 1/w_0$, then $\mathcal{Q}(\alpha)$ is locally increasing if $u_0(\alpha) > v_0(\alpha)$ and locally decreasing if $u_0(\alpha) < v_0(\alpha)$. It is an elementary exercise to show that this is consistent with $t > 0$ only if $v_0'(\alpha) < 0$. Then $t^* = inf_\alpha\, t$. Interchanging u and v in the above proof gives the result stated in the Theorem, with t^* the infimum, over α, of all $t > 0$ constructed as above. □

Remark. Recalling (69), which can be written

$$x_\alpha(t,\alpha) = \frac{u_0(\alpha) - v(t, x(t,\alpha))}{u_0(\alpha)-v_0(\alpha)}, \tag{84}$$

and comparing (83) with (84) shows that v evolves along Γ as

$$v(t, x(t,\alpha)) = v_0(\mathcal{Q}^{-1}(\mathcal{Q}(\alpha)-t)). \tag{85}$$

5 Numerical Results

In order to examine the onset of singularity formation for the system

$$\begin{cases} u_t + vu_x &= 0 \\ v_t + uv_x &= 0 \end{cases}$$

numerically, the graphics shown in Figure 1 were obtained using a simple finite difference scheme

$$u_i^{n+1} = u_i^n - 0.02 v_i^n (u_{i+1}^n - u_{i-1}^n), \tag{86}$$
$$v_i^{n+1} = v_i^n - 0.02 u_i^n (v_{i+1}^n - v_{i-1}^n). \tag{87}$$

Step sizes are $\Delta t = 0.01$ and $\Delta x = 1$, and initial data takes the form

$$u_0 = 0.0095 j(150-j)\sin(0.06(j-37.5)),\ 0 \le j \le 150\,,$$

and

$$v_0 = .01k(150-k),\ 0 \le k \le 150\,.$$

The singularity forms immediately the u and v curves touch, which takes place at $t = 0.11$.

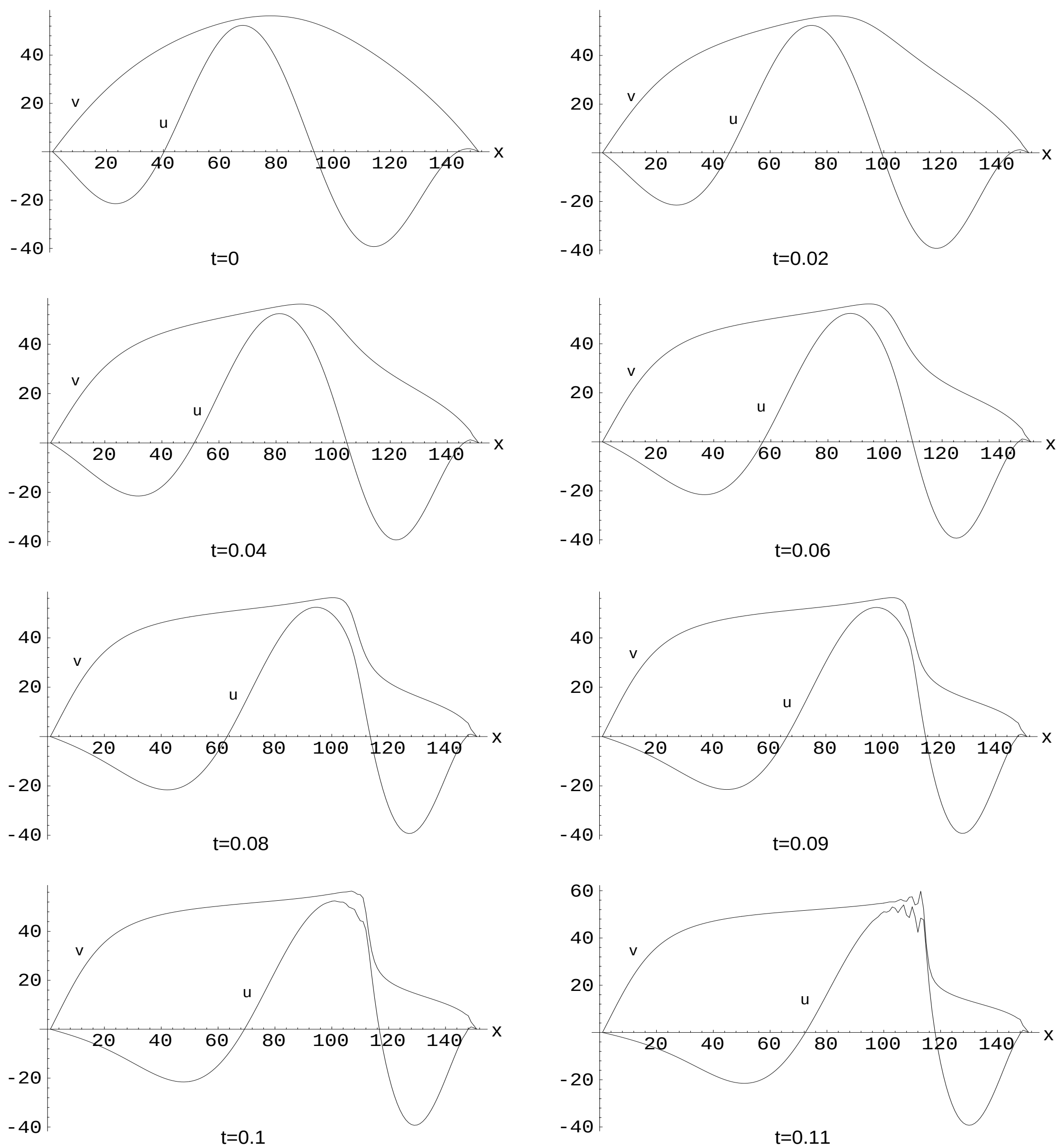

Figure 1: Singularity formation for smooth initial data.

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