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EVEN ORDER SELF ADJOINT TIME SCALE PROBLEMS

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Abstract. Even order self adjoint differential time scale expressions are introduced, together with associated self adjoint boundary conditions; the result is established by induction. Several fourth-order nabla-delta delta-nabla examples are given for select self adjoint boundary conditions, together with the specific corresponding Green's functions over common time scales. One derived Green's function is shown directly to be symmetric.

1. Introduction

Some self adjoint boundary value problems (BVPs) for second order differential equations on time scales were constructed and studied earlier in [1] by making use of both delta and nabla derivatives. Next, certain BVPs for higher order equations on time scales were investigated in [2, 3, 4] where, however, the considered BVPs turned out, in general, nonself adjoint because their Green's function were found nonsymmetric. Therefore it remained unclear as to how to place the successive delta and nabla derivatives for higher order to get self adjoint differential expressions that can yield symmetric Green's functions. Guseinov [5] offered a possible resolution of this problem; in this paper we offer a direct proof by mathematical induction of his conjecture, in the case where we stack nabla derivatives and one delta derivative on the inside first, followed by stacked deltas and one nabla on the outside (see below). In a subsequent, closely related sequel [6], a more abstract but comprehensive approach is used to establish self adjoint delta-nabla equations and boundary conditions, using quasi-derivative notation to consolidate (though unfortunately also obscure) notationally all of the stacking and alternating of delta and nabla derivatives. The proofs there are given in an indirect way using a Lagrange bracket scheme. In both papers specific fourth-order examples are given, this using nabla-delta equations, [6] using delta-nabla equations.

Let $\mathbb{T}$ be a time scale, $p_0(t), p_1(t), \dots, p_n(t)$ real-valued smooth functions defined on $\mathbb{T}$, and $a \in \mathbb{T}^{\kappa^n}$, $b \in \mathbb{T}_{\kappa^n}$, with $\sigma^n(a) < \rho^n(b)$. Consider the $2n$th order differential expression

$$L_{2n}y(t) = \sum_{i=0}^{n-1} \left(p_{i+1}y^{\nabla^i\Delta}\right)^{\Delta^i\nabla}(t) + p_0^\rho(t)y(t). \tag{1.1}$$

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We show that this expression is self adjoint with respect to the inner product

$$\langle y, z\rangle = \int_a^b y(t)z(t)\nabla t,$$

that is, the identity

$$\langle L_{2n}y, z\rangle = \langle y, L_{2n}z\rangle$$

holds provided that y and z satisfy some appropriate boundary conditions at a and b. In what follows such boundary conditions, self adjoint boundary conditions, will be presented. For the convenience of the reader a section on time scale essentials is included. This operator was considered in [5], however the proof of Theorem 2.2 was only given for the cases $n = 1$ and $n = 2$. We extend this proof to the general case, and include examples of fourth order Green's functions.

Basic Time Scale Notions

Any arbitrary nonempty closed subset of the reals $\mathbb{R}$ can serve as a time scale $\mathbb{T}$; see [7], [8].

Definition 1.1. For $t \in \mathbb{T}$ define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

Define the graininess operators $\mu_\sigma, \mu_\rho : \mathbb{T} \to [0,\infty)$ via $\mu_\sigma(t) = \sigma(t) - t$ and $\mu_\rho(t) = \rho(t) - t$.

Definition 1.2. A function $f : \mathbb{T} \to \mathbb{R}$ is right dense continuous (rd-continuous) provided it is continuous at all right dense points of $\mathbb{T}$ and its left sided limit exists (finite) at left dense points of $\mathbb{T}$. The set of all right dense continuous functions on $\mathbb{T}$ is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

Similarly, a function $f : \mathbb{T} \to \mathbb{R}$ is left dense continuous (ld-continuous) provided it is continuous at all left dense points of $\mathbb{T}$, and its right sided limit exists (finite) at right dense points of $\mathbb{T}$. The set of all left dense continuous functions is denoted

$$C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R}).$$

Take $\mathbb{T}_\kappa$ to be $\mathbb{T} - \{m_1\}$ if $\mathbb{T}$ has a right scattered minimum m_1, or to be $\mathbb{T}$ otherwise. In the same way, $\mathbb{T}^\kappa$ is $\mathbb{T} - \{m_2\}$ if $\mathbb{T}$ has a left scattered maximum m_2, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. In addition use the notation $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$, et cetera.

Definition 1.3 (Delta Derivative)**.** Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

The function $f^\Delta(t)$ is the delta derivative of f at t.

Definition 1.4 (Nabla Derivative)**.** For $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}_\kappa$, define $f^\nabla(t)$ to be the number (provided it exists) with the property that given any $\epsilon>0$, there is a neighborhood U of t such that

$$|f(\rho(t))-f(s)-f^\nabla(t)[\rho(t)-s]|\le\epsilon|\rho(t)-s|\quad\text{for all } s\in U.$$

The function $f^\nabla(t)$ is the nabla derivative of f at t.

In the case $\mathbb{T}=\mathbb{R}$, $f^\Delta(t)=f'(t)=f^\nabla(t)$. When $\mathbb{T}=\mathbb{Z}$, $f^\Delta(t)=f(t+1)-f(t)$ and $f^\nabla(t)=f(t)-f(t-1)$. By $f^{\Delta^2}(t)$ we mean $(f^\Delta)^\Delta(t)$, and similarly for higher order delta and nabla derivatives.

Definition 1.5 (Delta Integral)**.** Let $f:\mathbb{T}\to\mathbb{R}$ be a function, and $a,b\in\mathbb{T}$. If there exists a function $F:\mathbb{T}\to\mathbb{R}$ such that $F^\Delta(t)=f(t)$ for all $t\in\mathbb{T}$, then F is a delta antiderivative of f. In this case the integral is given by the formula

$$\int_a^b f(\tau)\Delta\tau=F(b)-F(a)\text{ for } a,b\in\mathbb{T}.$$

Definition 1.6 (Nabla Integral)**.** Let $f:\mathbb{T}\to\mathbb{R}$ be a function, and $a,b\in\mathbb{T}$. If there exists a function $F:\mathbb{T}\to\mathbb{R}$ such that $F^\nabla(t)=f(t)$ for all $t\in\mathbb{T}$, then F is a nabla antiderivative of f. In this case the integral is given by the formula

$$\int_a^b f(\tau)\nabla\tau=F(b)-F(a)\text{ for } a,b\in\mathbb{T}.$$

Remark 1.7. All right dense continuous functions are delta integrable, and all left dense continuous functions are nabla integrable.

Theorem 1.8. *If $f,g:\mathbb{T}\to\mathbb{R}$ are left dense continuous then*

$$\int_a^b f(t)g^\nabla(t)\nabla t=(fg)(b)-(fg)(a)-\int_a^b f^\nabla(t)g(\rho(t))\nabla t.$$

The following statement (Theorems 2.5 and 2.6 in [1]) will be used:

Theorem 1.9.

(i) *If $f:\mathbb{T}\to\mathbb{R}$ is $\Delta-$ differentiable on $\mathbb{T}^\kappa$ and if f^Δ is continuous on $\mathbb{T}^\kappa$, then f is $\nabla-$ differentiable on $\mathbb{T}_\kappa$ and*
$$f^\nabla(t)=f^\Delta(\rho(t))\quad\text{for all } t\in\mathbb{T}_\kappa.$$

(ii) *If $f:\mathbb{T}\to\mathbb{R}$ is $\nabla-$ differentiable on $\mathbb{T}_\kappa$ and if f^∇ is continuous on $\mathbb{T}_\kappa$, then f is $\Delta-$ differentiable on $\mathbb{T}^\kappa$ and*
$$f^\Delta(t)=f^\nabla(\sigma(t))\quad\text{for all } t\in\mathbb{T}^\kappa.$$

2. Self adjoint Differential Expressions and Boundary Conditions

Throughout we assume that the leading coefficient $p_n(t)$ is such that $p_n(t)\neq 0$ for all $t\in\mathbb{T}$. The following lemma is easily shown using induction and Theorem 1.9.

Lemma 2.1. *Assume that $f^{\Delta^n\nabla}$ for $n\in\mathbb{N}_0$ and g satisfy the conditions of Theorems 1.8 and 1.9. Then*

$$\int_a^b f^{\Delta^n\nabla}(t)g(t)\nabla t=\sum_{i=0}^n(-1)^i f^{\Delta^{n-i}}(t)g^{\nabla^i}(t)\big|_a^b-(-1)^n\int_a^b f(\rho(t))g^{\nabla^{n+1}}(t)\nabla t.$$

Theorem 2.2. *$\langle L_{2n}y, z\rangle = \langle L_{2n}z, y\rangle$ if and only if*

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^j\big(p_{i+1}y^{\nabla^i\Delta}\big)^{\Delta^{i-j}}(t)z^{\nabla^j}(t)\big|_a^b = \sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^j\big(p_{i+1}z^{\nabla^i\Delta}\big)^{\Delta^{i-j}}(t)y^{\nabla^j}(t)\big|_a^b.$$

Proof. By definition,

$$\begin{aligned}\langle L_{2n}y, z\rangle &= \int_a^b\Big(\sum_{i=0}^{n-1}\big(p_{i+1}y^{\nabla^i\Delta}\big)^{\Delta^i\nabla}(t) + p_0^\rho(t)y(t)\Big)z(t)\nabla t\\ &= \sum_{i=0}^{n-1}\int_a^b\big(p_{i+1}y^{\nabla^i\Delta}\big)^{\Delta^i\nabla}(t)z(t)\nabla t + \int_a^b p_0^\rho(t)y(t)z(t)\nabla t.\end{aligned}$$

Consider

$$\int_a^b\Big(p_{i+1}y^{\nabla^i\Delta}\Big)^{\Delta^i\nabla}(t)z(t)\nabla t.$$

Using Lemma 2.1 and Theorem 1.9, we have

$$\begin{aligned}&\int_a^b\big(p_{i+1}y^{\nabla^i\Delta}\big)^{\Delta^i\nabla}(t)z(t)\nabla t\\ &= \sum_{j=0}^{i}(-1)^j\big(p_{i+1}y^{\nabla^i\Delta}\big)^{\Delta^{i-j}}(t)z^{\nabla^j}(t)\big|_a^b - (-1)^i\int_a^b\big(p_{i+1}y^{\nabla^i\Delta}\big)^{\nabla}(t)z^{\nabla^{i+1}}(t)\nabla t\\ &= \sum_{j=0}^{i}(-1)^j\big(p_{i+1}y^{\nabla^i\Delta}\big)^{\Delta^{i-j}}(t)z^{\nabla^j}(t)\big|_a^b - (-1)^i\int_a^b\big(p_{i+1}y^{\nabla^i\Delta}\big)(\rho(t))z^{\nabla^{i+1}}(t)\nabla t\\ &= \sum_{j=0}^{i}(-1)^j\big(p_{i+1}y^{\nabla^i\Delta}\big)^{\Delta^{i-j}}(t)z^{\nabla^j}(t)\big|_a^b - (-1)^i\int_a^b p_{i+1}(\rho(t))y^{\nabla^{i+1}}(t)z^{\nabla^{i+1}}(t)\nabla t.\end{aligned}$$

Thus

$$\begin{aligned}\langle L_{2n}y, z\rangle = \sum_{i=0}^{n-1}\sum_{j=0}^{i}\Big[&(-1)^j\big(p_{i+1}y^{\nabla^i\Delta}\big)^{\Delta^{i-j}}(t)z^{\nabla^j}(t)\big|_a^b\\ &- (-1)^i\int_a^b p_{i+1}(\rho(t))y^{\nabla^{i+1}}(t)z^{\nabla^{i+1}}(t)\nabla t\Big] + \int_a^b p_0^\rho(t)y(t)z(t)\nabla t.\end{aligned}$$

Similarly

$$\begin{aligned}\langle L_{2n}z, y\rangle = \sum_{i=0}^{n-1}\sum_{j=0}^{i}\Big[&(-1)^j\big(p_{i+1}z^{\nabla^i\Delta}\big)^{\Delta^{i-j}}(t)y^{\nabla^j}(t)\big|_a^b\\ &- (-1)^i\int_a^b p_{i+1}(\rho(t))z^{\nabla^{i+1}}(t)y^{\nabla^{i+1}}(t)\nabla t\Big] + \int_a^b p_0^\rho(t)z(t)y(t)\nabla t.\end{aligned}$$

Therefore $\langle L_{2n}y, z\rangle = \langle L_{2n}z, y\rangle$ if and only if

$$\sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^j\big(p_{i+1}y^{\nabla^i\Delta}\big)^{\Delta^{i-j}}(t)z^{\nabla^j}(t)\big|_a^b = \sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^j\big(p_{i+1}z^{\nabla^i\Delta}\big)^{\Delta^{i-j}}(t)y^{\nabla^j}(t)\big|_a^b.$$

$\square$

If $n=1$, then $\langle L_2y,z\rangle=\langle y,L_2z\rangle$ if and only if

$$p_1(t)[y^{\Delta}(t)z(t)-y(t)z^{\Delta}(t)]|_a^b=0. \tag{2.1}$$

The requirement (2.1) will give a way for finding all self adjoint boundary conditions associated with L_2. If, for example, y and z both satisfy the Sturm-Liouville boundary conditions of the form

$$\alpha u(a)+\beta u^{\Delta}(a)=0,\quad \gamma u(b)+\delta u^{\Delta}(b)=0\quad (|\alpha|+|\beta|\neq 0, |\gamma|+|\delta|\neq 0),$$

then (2.1) is satisfied. Another set of boundary conditions that guarantee (2.1) are the "periodic" boundary conditions

$$u(a)=u(b),\quad p_1(a)u^{\Delta}(a)=p_1(b)u^{\Delta}(b).$$

Note that the self adjoint expression (1.1) solely is not enough for the symmetry of the Green's function of L_{2n} subject to some boundary conditions at a and b. In addition, the boundary conditions must also be chosen self adjoint, that is, so that to have $\langle L_{2n}y,z\rangle=\langle y,L_{2n}z\rangle$ for y,z satisfying those boundary conditions.

3. Fourth Order Self Adjoint Boundary Value Problems

If $n=2$, then $\langle L_4y,z\rangle=\langle y,L_4z\rangle$ if and only if

$$\begin{aligned}&\big\{[p_2(t)y^{\nabla\Delta}(t)]^{\Delta}+p_1(t)y^{\Delta}(t)\}z(t)|_a^b-y(t)\{[p_2(t)z^{\nabla\Delta}(t)]^{\Delta}+p_1(t)z^{\Delta}(t)]\big\}\big|_a^b\\&-p_2(t)[y^{\nabla\Delta}(t)z^{\nabla}(t)-y^{\nabla}(t)z^{\nabla\Delta}(t)]|_a^b=0.\end{aligned} \tag{3.1}$$

The requirement (3.1) will give a way for finding all self adjoint boundary conditions associated with L_4. If, for example, y and z both satisfy the boundary conditions of the form

$$u(a)=0,\quad u^{\nabla}(a)=0,\quad u^{\nabla\Delta}(b)=0,\quad p_2(\sigma(b))u^{\nabla\Delta^2}(b)+p_1(b)u^{\Delta}(b)=0,$$

then (3.1) is satisfied. (Note that $[p_2(t)u^{\nabla\Delta}(t)]^{\Delta}=p_2^{\Delta}(t)u^{\nabla\Delta}(t)+p_2(\sigma(t))u^{\nabla\Delta^2}(t)$). Consider the differential expression $L_4y(t)$ with $p_0^{\rho}(t)=p_1(t)\equiv 0$, rewritten here as

$$Ly(t)=(py^{\nabla\Delta})^{\Delta\nabla}(t) \tag{3.2}$$

subject to the boundary conditions

$$y(a)=0,\quad y^{\nabla}(a)=0,\quad p(b)y^{\nabla\Delta}(b)=0,\quad (py^{\nabla\Delta})^{\Delta}(b)=0 \tag{3.3}$$

on an arbitrary time scale. Without calculating the Green's function $G(t,s)$ of (3.2), (3.3) we can state that it must be symmetric: $G(t,s)=G(s,t)$. Indeed, as it was noted above the operator L generated by (3.2), (3.3) is self adjoint: $\langle Ly,z\rangle=\langle y,Lz\rangle$. It is easily seen that the inverse of a self adjoint operator also is self adjoint. Thus we have

$$\langle L^{-1}f,g\rangle=\langle f,L^{-1}g\rangle,\quad \text{for all } f,g. \tag{3.4}$$

On the other hand, L^{-1} is given by

$$L^{-1}f(t)=\int_a^b G(t,s)f(s)\nabla s. \tag{3.5}$$

From (3.4) and (3.5) it follows that $G(t,s)=G(s,t)$. In practice it can be more difficult to construct the Green's function and show directly that it is symmetric.

In this case it is relatively straightforward, and we demonstrate the technique. Here the Green's function $G(t,s)$ is given by

$$G(t,s)=\begin{cases}\int_a^t\big(\int_a^\tau \frac{s-x}{p(x)}\Delta x\big)\nabla\tau & t\le s\\ \int_a^t\big(\int_a^\tau \frac{s-x}{p(x)}\Delta x\big)\nabla\tau+\int_s^t\big(\int_s^\tau \frac{x-s}{p(x)}\Delta x\big)\nabla\tau & t\ge s.\end{cases}\tag{3.6}$$

We show that

$$G(t,s)=\begin{cases}\int_a^t\big(\int_a^\tau \frac{s-x}{p(x)}\Delta x\big)\nabla\tau & t\le s\\ \int_a^s\big(\int_a^\tau \frac{t-x}{p(x)}\Delta x\big)\nabla\tau & t\ge s.\end{cases}$$

Let

$$v_1(t,s):=\int_a^t\Big(\int_a^\tau \frac{s-x}{p(x)}\Delta x\Big)\nabla\tau+\int_s^t\Big(\int_s^\tau \frac{x-s}{p(x)}\Delta x\Big)\nabla\tau$$

and

$$v_2(t,s):=\int_a^s\Big(\int_a^\tau \frac{t-x}{p(x)}\Delta x\Big)\nabla\tau.$$

Then

$$w_1(s):=v_1^{\nabla_t}(t,s)=\int_a^s \frac{s-x}{p(x)}\Delta x$$

and

$$w_2(s):=v_2^{\nabla_t}(t,s)=\int_a^s\Big(\int_a^\tau \frac{1}{p(x)}\Delta x\Big)\nabla\tau.$$

Taking the nabla derivative with respect to s,

$$w_1^\nabla(s)=\int_a^s\frac{1}{p(x)}\Delta x+\frac{\rho(s)-\rho(s)}{p(\rho(s))}=\int_a^s\frac{1}{p(x)}\Delta x=w_2^\nabla(s);$$

since $w_1(a)=w_2(a)$, $w_1(s)=w_2(s)$, or $v_1^{\nabla_t}(t,s)=v_2^{\nabla_t}(t,s)$. Again, since $v_1(s,s)=v_2(s,s)$, $v_1(t,s)=v_2(t,s)$. Therefore $G(t,s)=G(s,t)$.

Example 3.1. Let $\mathbb{E}=\{1-q^{\mathbf{N}_0}\}\cup\{1\}$. Taking $a=0$ and $b=1$ with $p(t)\equiv 1$ we have the following:

$$\mathbb{T}=\mathbb{R}:\quad G(t,s)=\begin{cases}\frac{t^2[3s-t]}{6} & t\le s\\ \frac{s^2[3t-s]}{6} & t\ge s\end{cases}$$

$$\mathbb{T}=h\mathbb{Z}:\quad G(t,s)=\begin{cases}\frac{t\sigma(t)[3s-\rho(t)]}{6} & t\le s\\ \frac{s\sigma(s)[3t-\rho(s)]}{6} & t\ge s\end{cases}$$

$$\mathbb{T}=\mathbb{E}:\quad G(t,s)=\begin{cases}\frac{t\sigma(t)[(q^2+q+1)s-q^2\rho(t)]}{(q+1)(q^2+q+1)} & t\le s\\ \frac{s\sigma(s)[(q^2+q+1)t-q^2\rho(s)]}{(q+1)(q^2+q+1)} & t\ge s\end{cases}$$

Note that as $h\to 0$, the Green's function for $h\mathbb{Z}$ becomes the Green's function for $\mathbb{R}$, as one would expect. Allowing q to take on the value of 1, one can see that the Green's function for $\mathbb{E}$ also becomes the Green's function for $\mathbb{R}$ In addition, as predicted the Green's functions are symmetric.

Remark 3.2. Some of the other self adjoint boundary conditions associated with (3.2) include

$$
\begin{array}{ll}
y(a) = y^{\nabla}(a) = 0, & y(b) = y^{\nabla}(b) = 0; \\
y(a) = y^{\nabla}(a) = 0, & y(b) = p(b)y^{\nabla\Delta}(b) = 0; \\
y(a) = y^{\nabla}(a) = 0, & y^{\nabla}(b) = (py^{\nabla\Delta})^{\Delta}(b) = 0; \\
y(a) = y^{\nabla}(a) = 0, & p(b)y^{\nabla\Delta}(b) = (py^{\nabla\Delta})^{\Delta}(b) = 0; \\
y(a) = p(a)y^{\nabla\Delta}(a) = 0, & y(b) = y^{\nabla}(b) = 0; \\
y(a) = p(a)y^{\nabla\Delta}(a) = 0, & y(b) = p(b)y^{\nabla\Delta}(b) = 0; \\
y(a) = p(a)y^{\nabla\Delta}(a) = 0, & y^{\nabla}(b) = (py^{\nabla\Delta})^{\Delta}(b) = 0; \\
y(a) = p(a)y^{\nabla\Delta}(a) = 0, & p(b)y^{\nabla\Delta}(b) = (py^{\nabla\Delta})^{\Delta}(b) = 0; \\
y^{\nabla}(a) = (py^{\nabla\Delta})^{\Delta}(a) = 0, & y(b) = y^{\nabla}(b) = 0; \\
y^{\nabla}(a) = (py^{\nabla\Delta})^{\Delta}(a) = 0, & y(b) = p(b)y^{\nabla\Delta}(b) = 0; \\
y^{\nabla}(a) = (py^{\nabla\Delta})^{\Delta}(a) = 0, & y^{\nabla}(b) = (py^{\nabla\Delta})^{\Delta}(b) = 0; \\
y^{\nabla}(a) = (py^{\nabla\Delta})^{\Delta}(a) = 0, & p(b)y^{\nabla\Delta}(b) = (py^{\nabla\Delta})^{\Delta}(b) = 0; \\
p(a)y^{\nabla\Delta}(a) = (py^{\nabla\Delta})^{\Delta}(a) = 0, & y(b) = y^{\nabla}(b) = 0; \\
p(a)y^{\nabla\Delta}(a) = (py^{\nabla\Delta})^{\Delta}(a) = 0, & y(b) = p(b)y^{\nabla\Delta}(b) = 0; \\
p(a)y^{\nabla\Delta}(a) = (py^{\nabla\Delta})^{\Delta}(a) = 0, & y^{\nabla}(b) = (py^{\nabla\Delta})^{\Delta}(b) = 0; \\
p(a)y^{\nabla\Delta}(a) = (py^{\nabla\Delta})^{\Delta}(a) = 0, & p(b)y^{\nabla\Delta}(b) = (py^{\nabla\Delta})^{\Delta}(b) = 0;
\end{array}
$$

and the periodic conditions

$$
\begin{gathered}
y(a) = y(b), \quad (py^{\nabla\Delta})^{\Delta}(a) = (py^{\nabla\Delta})^{\Delta}(b), \\
y^{\nabla}(a) = y^{\nabla}(b), \quad p(a)y^{\nabla\Delta}(a) = p(b)y^{\nabla\Delta}(b).
\end{gathered}
$$

Example 3.3. Consider (3.2) with the boundary conditions

$$
p(a)y^{\nabla\Delta}(a) = \left(py^{\nabla\Delta}\right)^{\Delta}(a) = 0, \quad y(b) = y^{\nabla}(b) = 0.
$$

The Green's function here is given by

$$
G(t,s) = \begin{cases} \int_s^b \left(\int_\tau^b \frac{x-t}{p(x)} \Delta x \right) \nabla \tau & t \le s \\ \int_t^b \left(\int_\tau^b \frac{x-s}{p(x)} \Delta x \right) \nabla \tau & t \ge s. \end{cases}
$$

If $p(t) \equiv 1$ we have

$$
\mathbb{T} = \mathbb{R}: \quad G(t,s) = \begin{cases} \frac{(b-s)^2(2b+s-3t)}{6} & t \le s \\ \frac{(b-t)^2(2b+t-3s)}{6} & t \ge s. \end{cases}
$$

Example 3.4. Again consider (3.2) with the boundary conditions

$$
y(a) = p(a)y^{\nabla\Delta}(a) = 0 \quad y^{\nabla}(b) = \left(py^{\nabla\Delta}\right)^{\Delta}(b) = 0.
$$

Then the Green's function is

$$
G(t,s) = \begin{cases} (t-a)\int_a^s \int_\tau^b \frac{\Delta x}{p(x)} \nabla\tau - \int_a^t \int_a^\tau \frac{x-a}{p(x)} \Delta x \nabla\tau & t \le s \\ (s-a)\int_a^t \int_\tau^b \frac{\Delta x}{p(x)} \nabla\tau - \int_a^s \int_a^\tau \frac{x-a}{p(x)} \Delta x \nabla\tau & t \ge s. \end{cases}
$$

If $p(t) \equiv 1$ we have

$$\mathbb{T} = \mathbb{R}: \quad G(t,s) = \begin{cases} \frac{(t-a)(s-a)(2b-s-a)}{2} + \frac{(a-t)^3}{6} & t \le s \\ \frac{(s-a)(t-a)(2b-t-a)}{2} + \frac{(a-s)^3}{6} & t \ge s. \end{cases}$$

For boundary conditions

$$y^{\nabla}(a) = \left(py^{\nabla\Delta}\right)^{\Delta}(a) = 0 \quad y(b) = p(b)y^{\nabla\Delta}(b) = 0,$$

the Green's function is

$$G(t,s) = \begin{cases} (b-t)\int_s^b \int_a^\tau \frac{\Delta x}{p(x)} \nabla\tau - \int_s^b \int_t^\tau \frac{x-t}{p(x)} \Delta x \nabla\tau & t \le s \\ (b-s)\int_t^b \int_a^\tau \frac{\Delta x}{p(x)} \nabla\tau - \int_t^b \int_s^\tau \frac{x-s}{p(x)} \Delta x \nabla\tau & t \ge s. \end{cases}$$

Remark 3.5. It can be similarly seen by using Theorem 1.9 (ii) that the differential expression

$$Q_{2n}y(t) = \sum_{i=0}^{n-1} \left(p_{i+1}y^{\Delta^i\nabla}\right)^{\nabla^i\Delta}(t) + p_0^{\sigma}(t)y(t) \tag{3.7}$$

is a self adjoint expression with respect to the inner product

$$\langle y, z\rangle = \int_a^b y(t)z(t)\Delta t.$$

Remark 3.6. In [3] it is shown (Example 18) that in the case $\mathbb{T} = \mathbb{Z}$ the Green's function of

$$Ly(t) = \left(y^{\Delta^2}\right)^{\nabla^2} \tag{3.8}$$

with the boundary conditions

$$y(a) = 0, \quad y^{\Delta}(a) = 0, \quad y^{\Delta^2}(b) = 0, \quad y^{\Delta^2\nabla}(b) = 0 \tag{3.9}$$

is not symmetric. Note that the expression (3.8) is of the form (3.2) and (3.7), since in the case $\mathbb{T} = \mathbb{Z}$ the operations Δ and ∇ commute, and so the expression (3.8) is self adjoint in the case $\mathbb{T} = \mathbb{Z}$. However, the boundary conditions (3.9), in contrast to the boundary conditions (3.3), are not self adjoint. This is why the Green's function turned out nonsymmetric. (Note that if we replace in the self adjoint boundary conditions for usual differential equations the usual derivative by delta or nabla derivative, the obtained boundary conditions need not be self adjoint on time scales.)

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