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A STABILITY RESULT FOR p-HARMONIC SYSTEMS WITH DISCONTINUOUS COEFFICIENTS

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Abstract. The present paper is concerned with p-harmonic systems

$$\operatorname{div}(\langle A(x)Du(x), Du(x)\rangle^{\frac{p-2}{2}} A(x)Du(x)) = \operatorname{div}(\sqrt{A(x)}F(x)),$$

where $A(x)$ is a positive definite matrix whose entries have bounded mean oscillation (BMO), p is a real number greater than 1 and $F \in L^{\frac{r}{p-1}}$ is a given matrix field. We find a-priori estimates for a very weak solution of class $W^{1,r}$, provided r is close to 2, depending on the BMO norm of $\sqrt{A}$, and p close to r. This result is achieved using the corresponding existence and uniqueness result for linear systems with BMO coefficients [St], combined with nonlinear commutators.

0. Introduction

Consider the p-harmonic system

$$\operatorname{div}(|Du(x)|^{p-2} Du(x)) = 0 \tag{0.1}$$

in a regular domain $\Omega \subset \mathbb{R}^n$.

A vector field u in the Sobolev space $W^{1,r}_{\rm loc}(\Omega, \mathbb{R}^n)$, $r > \max\{1, p-1\}$, *is a very weak p-harmonic vector* [IS1],[L] if it satisfies

$$\int_\Omega |Du|^{p-2}\langle Du, D\phi\rangle dx = 0 \quad \forall \phi \in C^\infty_o(\Omega, \mathbb{R}^n)\,.$$

This definition was first introduced by Iwaniec and Sbordone in [IS1], they were able to prove, using commutator results, that there exists a range of exponents, close to p, $1 < r_1 < p < r_2 < \infty$, such that if $u \in W^{1,r_1}_{\rm loc}(\Omega, \mathbb{R}^n)$ is very weak p-harmonic, then u belongs to $W^{1,r_2}_{\rm loc}(\Omega, \mathbb{R}^n)$, so, in particular, is p-harmonic. J. Lewis [L], using that the maximal functions raised to a small positive power is an A_p weight in the sense of Muckenhoupt, was able to obtain similar results. Kinnunen and Zhou [KZ] gave a partial answer to a conjecture posed by Iwaniec and Sbordone; they proved that r_1 can be chosen arbitrarly close to 1, if p is close to 2. Later Greco and Verde developed the same result for p-harmonic equations with $VMO \cap L^\infty$ coefficients,

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using estimates for linear elliptic equations with VMO coefficients [D],[IS2]. Our result is concerned with p-harmonic systems with BMO coefficients:

$$\operatorname{div}(\langle A(x)Du(x), Du(x)\rangle^{\frac{p-2}{2}} A(x)Du(x)) = \operatorname{div}(\sqrt{A(x)}F(x)) \tag{0.2}$$

where $A(x) = (A_{ij}(x))$ is a symmetric, positive definite matrix with entries in BMO, F is a given matrix field in $L^{\frac{r}{p-1}}$. Our definition of very weak p-harmonic vector a priori requires that the energy functional is finite along a solution, that is:

$$\int_\Omega |\sqrt{A}Du|^r dx < \infty$$

a closed subspace of $W_0^{1,r}(\Omega, \mathbb{R}^n)$, in addition for every $\phi \in C_0^\infty(\Omega, \mathbb{R}^n)$

$$\int_\Omega |\sqrt{A}Du|^{p-2}\langle\sqrt{A}Du, \sqrt{A}D\phi\rangle dx = \int_\Omega \langle F(x), \sqrt{A}D\phi\rangle dx \tag{0.3}$$

We will use the existence and uniqueness result for linear systems with bounded mean oscillation (BMO) coefficients to derive a new Hodge decomposition for matrix fields and, then, using commutators, we will prove a continuity result for p close to 2, depending on the BMO-norm of $\sqrt{A}$.

The method of proof is different from the linear case; in fact, there we have at our disposal two commutator results: one is a powertype perturbation of the kernel of a linear bounded operator, the other is the Coifman-Rochberg-Weiss result about the linear commutator of a Calderon-Zygmund operator with a BMO matrix. In the nonlinear case, we do not know of a result for nonlinear commutators with a BMO function, so we can only use the commutator result of powertype, applied to the natural Hodge decomposition coming from the linear case. The statement is the following:

Main Theorem. *For r given in such a way that $|r-2| < \varepsilon$, determined by the BMO-norm of $\sqrt{A}$, there exists $\delta > 0$ such that if $|p-r| < \delta$ and u is a very weak p-harmonic vector, then*

$$\|\sqrt{A}Du\|_r^r \le C\|F\|_{\frac{r}{p-1}}^{\frac{r}{p-1}} \tag{0.4}$$

Further developments are presented considering some new spaces, the so-called *grand L^q spaces*, in the spirit of [GIS].

1. Definitions and preliminary results

Definition 1. Let Ω be a cube or the entire space $\mathbb{R}^n$. The John-Nirenberg space $BMO(\Omega)$ [JN] consists of all functions b which are integrable on every cube $Q \subset \Omega$ and satisfy:

$$\|b\|_* = \sup\left\{\frac{1}{|Q|}\int_Q |b - b_Q|\, dx : Q \subset \Omega\right\} < \infty$$

where $b_Q = \frac{1}{|Q|}\int_Q b(y)dy$.

Definition 2. For $1 < q < \infty$ and $0 \leq \theta < \infty$ the grand L^q-space, denoted by $L^{\theta,q)}(\Omega, \mathbb{R}^{n\times n})$, consists of matrices $F \in \bigcap_{0<\varepsilon\leq q-1} L^{q-\varepsilon}(\Omega, \mathbb{R}^{n\times n})$ such that

$$\|F\|_{\theta,q)} = \sup_{0<\varepsilon\leq q-1} \varepsilon^{\frac{\theta}{q}} \|F\|_{q-\varepsilon} < \infty$$

These spaces are Banach spaces, they were introduced for $\theta = 1$ in the study of integrability properties of the Jacobian [IS1] and were used in [GISS] to establish a degree formula for maps with non-integrable Jacobian.

Definition 3. The grand Sobolev space $W_0^{\theta,p}(\Omega, \mathbb{R}^n)$ consists of all vector fields u belonging to $\bigcap_{0<\varepsilon\leq p-1} W_0^{1,p-\varepsilon}(\Omega, \mathbb{R}^n)$ such that $Du \in L^{\theta,p}(\Omega, \mathbb{R}^{n\times n})$; a norm on this space is $\|Du\|_{\theta,p)}$.

Next, we recall a stability result for nonlinear perturbation of a kernel of a bounded linear operator; namely: $T^{-\delta} f = T(|f|^{-\delta} f)$, where

$$T : L^p(\Omega, E) \longmapsto L^p(\Omega, E)$$

is a bounded linear operator and E is a Hilbert space.

Theorem 1. *Let $T : L^p(\Omega, E) \longmapsto L^p(\Omega, E)$ be a bounded linear operator for all $p_1 \leq p \leq p_2$; then for $1 - \frac{p_2}{p} \leq \delta \leq 1 - \frac{p_1}{p}$ there is a constant $C = C(\|T\|_{p_1}, \|T\|_{p_2})$ such that if f belongs to the kernel of T, we get*

$$\|T(|f|^{-\delta} f)\|_{\frac{p}{1-\delta}} \leq C|\delta| \|f\|_p^{1-\delta} \tag{1.1}$$

A new Hogde decompostion. Consider a linear system with BMO coefficients:

$$\operatorname{div}(B(x)Du(x)) = \operatorname{div} F(x)$$

where $B(x)$ is a symmetric, positive definite matrix whose entries are in BMO, F is a given matrix field. We state the following existence and uniqueness result for the solution of the Dirichlet problem:

Theorem 2. [St] *There exists $\varepsilon > 0$, depending on the BMO-norm of B, such that for $|r - 2| < \varepsilon$ the Dirichlet problem:*

$$\begin{gathered} \operatorname{div}(BDu) = \operatorname{div} F \\ F \in L^r(\Omega, \mathbb{R}^{n\times n}), \quad u \in W_o^{1,r}(\Omega, \mathbb{R}^n) \end{gathered} \tag{1.2}$$

admits a unique solution. In particular the energy functional

$$\int_\Omega |Du|^{-\varepsilon} \langle B(x)Du, Du\rangle \, dx$$

is finite and the following a-priori estimate holds

$$\|Du\|_r \leq C\|F\|_r \tag{1.3}$$

Remark. Note that, taking into account the uniform estimate (1.3) for exponents in a range determined by the BMO-norm of B, we have actually existence and uniqueness in the grand Sobolev space $W_0^{\theta,2)}(\Omega, \mathbb{R}^n)$.

This Theorem can be rephrased in terms of a new Hodge decomposition. More precisely,

Theorem 2'. *There exists $\varepsilon > 0$, depending on the BMO-norm of B, such that for $|r-2| < \varepsilon$ a matrix field $F \in L^r(\Omega, \mathbb{R}^{n\times n})$ can be decomposed uniquely as it follows:*

$$F = BD\phi + L$$

with $\operatorname{div} L = 0$ *and* $\phi \in W_o^{1,r}(\Omega, \mathbb{R}^n)$. *Therefore, there exists a bounded linear operator*

$$S : L^r(\Omega, \mathbb{R}^{n\times n}) \to L^r(\Omega, \mathbb{R}^{n\times n})$$

given by $S(F) = BD\phi$.

It is sufficient to solve the linear system

$$\operatorname{div}(BD\phi) = \operatorname{div} F$$

We will apply Theorem 1 to the operator $T = I - S$ with $B = \sqrt{A}$. Notice that the square root operator acting on matrices with minimum eigenvalue far from zero, for example greater or equal than 1, is Lipschitz, therefore the square root of A is still in BMO. The kernel of the operator T consists of matrix fields of the form $\sqrt{A}D\phi$.

2. Proof of the Main Theorem

Consider a very weak p-harmonic vector $u \in W^{1,r}$, with r determined by Theorem 2 and with finite energy. Decompose $|\sqrt{A}Du|^{r-p}\sqrt{A}Du$ using the new Hodge decomposition:

$$|\sqrt{A}Du|^{r-p}\sqrt{A}Du = \sqrt{A}D\phi + L, \quad \operatorname{div} L = 0$$

Let us observe that $T(\sqrt{A}Du) = 0$; therefore L is a nonlinear perturbation of the kernel of a bounded linear operator; we can apply Theorem 1 with $\delta = p - r$ to get the following estimate

$$\|L\|_{\frac{r}{1-\delta}} \le C|\delta| \|\sqrt{A}Du\|_r^{1-\delta} \tag{2.1}$$

Using the above equality we find

$$\int_\Omega |Du|^r dx \le \int_\Omega |\sqrt{A}Du|^r = \int_\Omega |\sqrt{A}Du|^{p-2}\langle \sqrt{A}Du, L\rangle dx + \int_\Omega \langle F, \sqrt{A}D\phi\rangle dx\,.$$

Using Hölder's inequality on the last two terms of the above expression and (2.1),

$$\begin{aligned}\int_\Omega |\sqrt{A}Du|^r dx \le& \|\sqrt{A}Du\|_r^{p-1}\|L\|_{\frac{r}{r-p+1}} + \|F\|_{\frac{r}{p-1}}\|\sqrt{A}D\phi\|_{\frac{r}{r-p+1}}\\ \le& C|r-p|\|\sqrt{A}Du\|_r^r + C\|F\|_{\frac{r}{p-1}}\|\sqrt{A}Du\|_r^{\frac{r}{r-p+1}}\end{aligned}$$

Using Young's inequality and choosing r such that $C|r-p| < 1$, we get the assertion.

We will prove also the uniqueness of the very weak p-harmonic vector in a space larger than $W^{1,r}$, refining estimate (0.4). We begin with establishing the following Theorem, that for the p-harmonic case was established in [GIS].

Theorem 3. *For r given in such a way that $|r-2|<\varepsilon$, determined by Theorem 2, there exists δ such that if $|p-r|<\delta$ and $u,v\in W^{1,r}(\Omega,\mathbb{R}^n)$ are very weak p-harmonic vectors respectively with data $F,G\in L^{\frac{r}{p-1}}(\Omega,\mathbb{R}^{n\times n})$ with finite energy, the following estimate holds:*

$$
\begin{aligned}
&\|\sqrt{A}Du-\sqrt{A}Dv\|_r^{p-1}\\
&\le C\varepsilon^{\frac{p-1}{|p-2|}}(\|F\|_{\frac{r}{p-1}}+\|G\|_{\frac{r}{p-1}})+C\begin{cases}\|F-G\|_{\frac{r}{p-1}} & (p\ge 2)\\ \|F-G\|_{\frac{r}{p-1}}^{p-1}(\|F\|_{\frac{r}{p-1}}+\|G\|_{\frac{r}{p-1}})^{2-p} & (1<p<2)\end{cases}
\end{aligned}
\tag{2.2}
$$

Proof. Take $u\in W^{1,r}(\Omega,\mathbb{R}^n)$ with finite energy, a very weak solution of the equation:

$$
\operatorname{div}(|\sqrt{A}Du|^{p-2}ADu)=\operatorname{div}(\sqrt{A}F) \tag{2.3}
$$

and $v\in W^{1,r}(\Omega,\mathbb{R}^n)$ with finite energy, a very weak solution of

$$
\operatorname{div}(|\sqrt{A}Dv|^{p-2}ADv)=div(\sqrt{A}G) \tag{2.4}
$$

Consider the Hodge decomposition of

$$
|\sqrt{A}Du-\sqrt{A}Dv|^{r-p}(\sqrt{A}Du-\sqrt{A}Dv)=\sqrt{A}D\phi+L
$$

we have estimates:

$$
\begin{aligned}
\|\sqrt{A}D\phi\|_{\frac{r}{1-\delta}}&\le C\|\sqrt{A}Du-\sqrt{A}Dv\|_r^{1-\delta}\\
\|L\|_{\frac{r}{1-\delta}}&\le C|\delta|\|\sqrt{A}Du-\sqrt{A}Dv\|_r^{1-\delta}
\end{aligned}
$$

We can use $\sqrt{A}D\phi$ as test function in (2.3) and (2.4) and subtract the two equations, to obtain:

$$
\begin{aligned}
&\int_\Omega\langle|\sqrt{A}Du|^{p-2}\sqrt{A}Du-|\sqrt{A}Dv|^{p-2}\sqrt{A}Dv,|\sqrt{A}Du-\sqrt{A}Dv|^{r-p}(\sqrt{A}Du-\sqrt{A}Dv)\rangle\\
&\qquad=\int_\Omega\langle F-G,\sqrt{A}D\phi\rangle+\int_\Omega\langle|\sqrt{A}Du|^{p-2}\sqrt{A}Du-|\sqrt{A}Dv|^{p-2}\sqrt{A}Dv,L\rangle\,,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega(|\sqrt{A}Du|+|\sqrt{A}Dv|)^{p-2}|\sqrt{A}Du-\sqrt{A}Dv|^{2-p+r}\\
&\le C(p)\int_\Omega|F-G||\sqrt{A}D\phi|+C(p)\int_\Omega(|\sqrt{A}Du|+|\sqrt{A}Dv|)^{p-2}|\sqrt{A}Du-\sqrt{A}Dv||L|\,.
\end{aligned}
$$

Now, using Hölder's and Young's inequalities we get the assertion.

This Theorem is the key to prove uniqueness of the solution of (0.2) in the grand Sobolev space $W_0^{\theta,p}$ when the right-hand side is in a grand $L^{\theta,q)}$ space. We state the following uniqueness Theorem.

Theorem 4. *For each $F \in L^{\theta,q)}(\Omega, \mathbb{R}^{n\times n})$ with q the Hölder conjugate of p, and p in the range determined by Theorem 2, the p-harmonic system* (0.2) *may have at most one solution in the closed subspace of $W^{\theta,p}(\Omega,\mathbb{R}^n)$:*

$$\mathcal{E}^{\theta,p} = \{u \in W^{\theta,p}(\Omega,\mathbb{R}^n) : \|\sqrt{A}Du\|_{\theta,p)} < \infty\}$$

and we get the uniform estimate for the operator $\mathcal{H} : L^{\theta,q)}(\Omega,\mathbb{R}^{n\times n}) \to \mathcal{E}^{\theta,p}$ that carries F into $\sqrt{A}Du$:

$$\|\mathcal{H}F - \mathcal{H}G\|^{p-1}_{\theta,p)} \le C(n,p,\|A\|_*)\,\|F-G\|^{\alpha}_{\theta,q)}(\|F\|_{\theta,q)} + \|G\|_{\theta,q)})^{1-\alpha} \tag{2.5}$$

where

$$\alpha = \begin{cases} \frac{p-\theta(p-2)}{p} & \text{if } p \ge 2 \\ \frac{p+\theta(p-2)}{q} & \text{if } p \le 2 \end{cases}$$

If, in addition, A is in L^∞, we get existence.

In fact, given $F \in L^{\theta,q)}(\Omega,\mathbb{R}^{n\times n})$, we consider a convolution F_k with a standard mollifier; the approximations F_k converge to F in $L^{\theta',q)}(\Omega,\mathbb{R}^{n\times n})$ for every $\theta' > \theta$. Next, solve the p-harmonic system:

$$\operatorname{div}(\langle A(x)Du_k(x), Du_k(x)\rangle^{\frac{p-2}{2}} A(x)Du_k(x)) = \operatorname{div}(\sqrt{A(x)}F_k(x))$$

for $u_k \in W^{1,p}_0(\Omega,\mathbb{R}^n)$. We use estimate (2.5) with θ' in place of θ to show that u_k is a Cauchy sequence in $W^{\theta',p)}_0(\Omega,\mathbb{R}^n)$:

$$\|\sqrt{A}Du_k - \sqrt{A}Du_j\|^{p-1}_{\theta',p)} \le C(n,p,\|A\|_*)\,\|F_k - F_j\|^{\alpha}_{\theta',q)}(\|F_k\|_{\theta',q)} + \|F_j\|_{\theta',q)})^{1-\alpha}$$

Passing to the limit in the integral identities:

$$\int_\Omega |\sqrt{A}Du_k|^{p-2}\langle\sqrt{A}Du_k, \sqrt{A}D\phi\rangle dx = \int_\Omega \langle F_k(x), \sqrt{A}D\phi\rangle dx$$

we then conclude that the limit u is in $W^{\theta,p}_0(\Omega,\mathbb{R}^n)$.

References

[D] G. Di Fazio, *L^p estimates for divergence form elliptic equations with discontinuous coefficients*, Boll. Un. Mat. Ital. **7** (1996), 409–420.

[GIS] L. Greco, T. Iwaniec and C. Sbordone, *Inverting the p-Harmonic Operator*, Manuscripta Mathematica **92** (1997), 249–258.

[GISS] L. Greco, T.Iwaniec, C. Sbordone and B. Stroffolini, *Degree formulas for maps with non-integrable Jacobian*, Topological Methods in Nonlinear Analysis **6** (1995), 81–95.

[GV] L. Greco and A. Verde, *A regularity property of p-harmonic functions*, Annales Academiae Scientiarum Fennicae (to appear).

[I] T.Iwaniec, *Nonlinear Differential Forms*, Lectures in Jyväskylä,, August 1998.

[IS1] T. Iwaniec and C. Sbordone, *Weak minima of variational integrals*, J.Reine Angew.Math. **454** (1994), 143–161.

[IS2] T. Iwaniec and C. Sbordone, *Riesz transform and elliptic PDEs with VMO-coefficients*, Journal d'Analyse Math. **74** (1998), 183–212.

[KZ] J. Kinnunen and S. Zhou, *A note on very weak P-harmonic mappings*, Electronic Journal of Differential Equations, **25** (1997), 1–4.

[JN] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

[L] J. Lewis, *On very weak solutions of certain elliptic systems*, Comm. Part. Diff. Equ. **18** (1993), 1515–1537.

[S] E. M. Stein, *Harmonic Analysis*, Princeton University Press, 1993.

[St] B.Stroffolini, *Elliptic Systems of PDE with BMO coefficients*, Potential Analysis (to appear).

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