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SOLUTIONS TO SINGULAR QUASILINEAR ELLIPTIC EQUATIONS ON BOUNDED DOMAINS

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Abstract. In this article we study quasilinear elliptic equations with a singular operator and at critical Sobolev growth. We prove the existence of positive solutions.

1. Introduction and statement of main results

In this article, we study the existence of solutions for the quasilinear elliptic equation

$$\begin{gathered} -\Delta u - \kappa\alpha(\Delta(|u|^{2\alpha}))|u|^{2\alpha-2}u = |u|^{q-2}u + |u|^{2^*-2}u, \quad \text{in } \Omega, \\ u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is an open bounded domain with smooth boundary $\partial\Omega$, $0 < \alpha < 1/2$, $2 \leq q < 2^*$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent.

Equation (1.1) comes from mathematical physics and was used to model some physical phenomena. Let us consider the following quasilinear Schrödinger equation introduced in [13, 14]

$$i\partial_t z = -\Delta z + w(x)z - l(|z|^2)z - \kappa\Delta h(|z|^2)h'(|z|^2)z, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $w(x)$ is a given potential, $\kappa > 0$ is a constant, $N \geq 3$. h, l are real functions of essentially pure power form.

Note that if $\kappa = 0$, then (1.2) is the standard semilinear Schrödinger equation which has been extensively studied, see [1, 2] for examples. For $\kappa > 0$, it is a quasilinear problem which has many applications in physics. The case of $h(s) = s$ was used for the superfluid film equation in plasma physics by Kurihura in [10]. It also appears in plasma physics and fluid mechanics [12], in the theory of Heisenberg ferromagnetism and magnons [9, 17] in dissipative quantum mechanics [8] and in condensed matter theory [15]. The case of $h(s) = s^\alpha, \alpha > 0$ was used to models the self-channeling of high-power ultrashort laser in matter [3].

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The study of standing waves to (1.2) of the form $z(x,t) = \exp(-iet)u(x)$ can reduce to find solutions $u(x)$ to the equation

$$-\Delta u + c(x)u - \kappa\alpha(\Delta(h(|u|^2)))h'(|u|^2)u = l(|u|^2)u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $c(x) = w(x) - e$ is a new potential function.

In recent years, problems with $h(s) = s$ have been extensively studied under different conditions imposed on the potential $c(x)$ and the perturbation $l(u)$, one can refer to [5, 6, 7, 14] and some references therein. Note that when $h(s) = s$, the main operator of the second order in (1.3) is unbounded. In order to prove the existence of solutions, Liu and Wang etc. [14] defined a change of variable $v = f^{-1}(u)$ and used it to reformulate the equation to a semilinear one, where f is defined by ODE: $f'(t) = (1 + 2f^2(t))^{-1/2}$, $t \in (0, +\infty)$ and $f(t) = -f(-t)$, $t \in (-\infty, 0)$. This method can also be found in some papers about such kind of problems thereafter, e.g. [5, 6, 7].

For problems with $h(s) = s^\alpha, \alpha > 0$, it is worthy of pointing out that when $\alpha > 1/2$, the number $2^*(2\alpha) = 2^* \times 2\alpha$ behaves like critical exponent for (1.3) (see [13]), while when $0 < \alpha \leq 1/2$, the critical number is still 2^*.

Besides the references mentioned above, there are some papers study such kind of problems with nonlinear terms at critical growth. In [19], Silva and Vieira considered the problem with $h(s) = s$, $l(|u|^2)u = K(x)u^{2(2^*)-1} + g(x,u)$, and proved the existence of solutions of (1.3). In [16], Moameni studied the problem with $h(s) = s^\alpha, \alpha > 1/2$ and $l(u)$ at critical growth under radially symmetric conditions. Recently, Li and Zhang in [11] proved the existence of a positive solution for the problem that $h(s) = s^\alpha$, $l(s) = s^{(q-2)/2} + s^{(2^*-2)/2}$, where $\alpha > 1/2$, $2(2\alpha) \leq q < 2^*(2\alpha)$.

There are two main difficulties in the study of problem (1.1). The first one is the main operator of the second order is singular in the equation provided that $0 < \alpha < 1/2$. Another one is caused by the nonlinear term $|u|^{2^*-2}u$ since the Sobolev imbedding from $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$ is not compact.

Recently, the authors in [20, 21] studied the existence of standing waves of (1.2) with $h(s) = s^\alpha, 0 < \alpha < 1/2$ in $\mathbb{R}^N$. We mention that (1.1) can be deduced from (1.3) by choosing $l(s) = s^{(q-2)/2} + s^{(2^*-2)/2}$. Inspired by [11], in this paper, we consider (1.1) on bounded domain $\Omega \subset \mathbb{R}^N$.

We denote $X := H_0^1(\Omega)$ endowed with the norm $\|u\|^2 = \langle u, u\rangle = \int_\Omega \nabla u \nabla u \, \mathrm{d}x$. Let $f(u) = |u|^{q-2}u + |u|^{2^*-2}u$. We want to find weak solutions to (1.1). By *weak solution*, we mean a function u in X satisfying that, for all $\varphi \in C_0^\infty(\Omega)$, there holds

$$\int_\Omega \nabla u \nabla \varphi \, \mathrm{d}x + \kappa\alpha \int_\Omega \nabla(|u|^{2\alpha})\nabla(|u|^{2\alpha-2}u\varphi) \, \mathrm{d}x = \int_\Omega f(u)\varphi \, \mathrm{d}x. \tag{1.4}$$

According to the variational methods, the weak solutions of (1.1) corresponds to the critical points of the functional $I : X \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_\Omega (1 + 2\kappa\alpha^2|u|^{2(2\alpha-1)})|\nabla u|^2 \, \mathrm{d}x - \int_\Omega F(u) \, \mathrm{d}x, \tag{1.5}$$

where $F(t) = \int_0^u f(s) \, \mathrm{d}s$. For $u \in X$, $I(u)$ is lower semicontinuous when $0 < \alpha < 1/2$, and not differentiable in all directions $\varphi \in X$. To overcome this difficulty, we use a change of variable to reformulate functional I. This make it possible for us to use the classical critical point theorem.

Let $g(t) = (1 + 2\kappa\alpha^2|t|^{2(2\alpha-1)})^{1/2}$, then $g(t)$ is monotone and decreasing in $t \in (0, +\infty)$. Note that for $t_0 > 0$ sufficiently small, we have

$$\int_0^{t_0} g(s)\,\mathrm{d}s \le 2\alpha\sqrt{\kappa}\int_0^{t_0} s^{2\alpha-1}\,\mathrm{d}s = \sqrt{\kappa}t_0^{2\alpha},$$

thus we can define a function $G : \mathbb{R} \to \mathbb{R}$ by

$$v = G(u) = \int_0^u g(s)\,\mathrm{d}s. \tag{1.6}$$

Then G is invertible and odd.

Let G^{-1} be the inverse function of G, then $\frac{\mathrm{d}}{\mathrm{d}v}G^{-1}(v) \in [0, 1)$. Inserting $u = G^{-1}(v)$ into (1.5), we get

$$J(v) := I(G^{-1}(v)) = \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \int_\Omega F(G^{-1}(v))\,\mathrm{d}x. \tag{1.7}$$

We can prove that (see Proposition 3.1) J is well defined on X, and is continuous in X. Moreover, it is also Gâteaux-differentiable, and for $\psi \in C_0^\infty(\Omega)$,

$$\langle J'(v), \psi\rangle = \int_\Omega \nabla v\nabla\psi\,\mathrm{d}x - \int_\Omega \frac{f(G^{-1}(v))}{g(G^{-1}(v))}\psi\,\mathrm{d}x. \tag{1.8}$$

Assume that $v \in X$ with $v > 0, x \in \Omega$ and $v = 0, x \in \partial\Omega$ be such that equality $\langle J'(v), \psi\rangle = 0$ holds for all $\psi \in C_0^\infty(\Omega)$. Let $u = G^{-1}(v)$, then by (1.6), $\nabla v = g(u)\nabla u$. Accordingly, $\nabla u = \frac{\nabla v}{g(G^{-1}(v))}$. Thus we get $u \in X$.

For $\varphi \in C_0^\infty(\Omega)$, let $\psi = g(G^{-1}(v))\varphi$, then $\nabla\psi = g(G^{-1}(v))\nabla\varphi + \frac{g'(G^{-1}(v))\varphi}{g(G^{-1}(v))}\nabla v$. Since

$$\begin{aligned}\nabla v\nabla\psi &= g(G^{-1}(v))\nabla v\nabla\varphi + \frac{g'(G^{-1}(v))\varphi}{g(G^{-1}(v))}|\nabla v|^2\\ &= g^2(u)\nabla u\nabla\varphi + g(u)g'(u)\varphi|\nabla u|^2,\end{aligned}$$

from (1.8), we obtain that

$$\int_\Omega g^2(u)\nabla u\nabla\varphi + \int_\Omega g(u)g'(u)\varphi|\nabla u|^2 - \int_\Omega f(u)\varphi = 0.$$

This implies that u such that (1.4) holds. In summary, to find a weak solution to (1.1), it suffices to find a positive weak solution to the following equation

$$-\Delta v = \frac{f(G^{-1}(v))}{g(G^{-1}(v))}, \quad x \in \Omega. \tag{1.9}$$

We assume that

(H1) assume that $q \in (2, 2^*)$ and either
  (i) $\frac{1}{4} < \alpha < \frac{1}{2}$, $q > \frac{4}{N-2} + 4\alpha$ or
  (ii) $0 < \alpha \le \frac{1}{4}$, $q > \frac{N+2}{N-2}$ holds.

Note that for $\frac{1}{4} < \alpha < \frac{1}{2}$, we have $q > \frac{4}{N-2} + 4\alpha > \frac{N+2}{N-2}$. The following theorem is the main result of this article.

Theorem 1.1. *Assume that* (H1) *holds. Then problem* (1.1) *has a positive weak solution in X.*

In Section 2, we study the properties of the function G^{-1} and show that the functional J has the mountain pass geometry. In Section 3, we first prove that every Palais-Smale sequence $\{v_n\}$ of J is bounded in X, then we employ the mountain pass theorem to prove the existence of nontrivial solution to (1.9). A crucial step is to prove that the weak limit v of $\{v_n\}$ is nonzero.

In this article, $\|\cdot\|_p$ denotes the norm of Lebesgue space $L^p(\Omega)$ and $C_k,\ k=1,2,3,\cdots$ will denote positive constants.

2. Mountain pass geometry

The following lemma gives some properties of the transformation G^{-1}.

Lemma 2.1. *The function $G^{-1}(t)$ has the following properties,*

1. *$G^{-1}(t)$ is odd, invertible, increasing and of class C^1 for $0<\alpha<1/2$, of class C^2 for $0<\alpha<1/4$;*
2. *$|\frac{\mathrm{d}}{\mathrm{d}t}G^{-1}(t)|\le 1$ for all $t\in\mathbb{R}$;*
3. *$|G^{-1}(t)|\le |t|$ for all $t\in\mathbb{R}$;*
4. *$(G^{-1}(t))^{2\alpha}/t\to\sqrt{2/\kappa}$ as $t\to 0^+$;*
5. *$2\alpha G^{-1}(t)g(G^{-1}(t))\le 2\alpha t\le G^{-1}(t)g(G^{-1}(t))$ for $t>0$;*
6. *$G^{-1}(t)/t\to 1$ as $t\to+\infty$;*

Proof. For (1) and (2), $G^{-1}(t)$ is odd and invertible by definition. Moreover, $\frac{\mathrm{d}}{\mathrm{d}t}G^{-1}(t)=[g(G^{-1}(t))]^{-1}\in[0,1]$. Thus $G^{-1}(t)$ is increasing and of class C^1 for $0<\alpha<1/2$. By direct computation, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}G^{-1}(t)=2\kappa\alpha^2(1-2\alpha)\frac{|G^{-1}(t)|^{-4\alpha}G^{-1}(t)}{\big(2\kappa\alpha^2+|G^{-1}(t)|^{2(1-2\alpha)}\big)^2}.$$

This implies that $G^{-1}(t)$ is of class C^2 provided that $0<\alpha<1/4$.

For (3), assume that $t>0$ and note that $g(G^{-1}(t))>1$, we have

$$0\le G^{-1}(t)=\int_0^{G^{-1}(t)}\mathrm{d}s\le\int_0^{G^{-1}(t)}g(s)\,\mathrm{d}s=t.$$

Then the conclusion follows since G^{-1} is odd.

For (4), note that from part (3), we have $G^{-1}(t)\to 0$ as $t\to 0$. Thus by employing L'Hôpital's Rule, we get

$$\lim_{t\to 0^+}\frac{(G^{-1}(t))^{2\alpha}}{t}=\lim_{t\to 0^+}\frac{2\alpha(G^{-1}(t))^{2\alpha-1}}{g(G^{-1}(t))}=\sqrt{\frac{2}{\kappa}}.$$

For (5), we prove the right-hand side inequality. Let $H(t)=G^{-1}(t)g(G^{-1}(t))$ and $\tilde H(t)=H(t)-2\alpha t$. Then $\tilde H(0)=0$. We prove that $\frac{\mathrm{d}}{\mathrm{d}t}\tilde H(t)\ge 0$, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}H(t)\ge 2\alpha$, and this implies the conclusion. In fact, for $t=0$, by part (4) and note that $G^{-1}(t)$ has same sign of t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}H(t)=\lim_{t\to 0}\frac{H(t)}{t}=\lim_{t\to 0}\sqrt{\frac{2}{\kappa}}\frac{|H(t)|}{|G^{-1}(t)|^{2\alpha}}=\sqrt{\frac{2}{\kappa}}\sqrt{2\kappa\alpha^2}=2\alpha.$$

For $t\neq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t)=\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{G^{-1}(t)\big(2\kappa\alpha^2+|G^{-1}(t)|^{2(1-2\alpha)}\big)^{1/2}}{|G^{-1}(t)|^{1-2\alpha}}\Big)$$

$$\geq \frac{|G^{-1}(t)|^{2(1-2\alpha)} - (1-2\alpha)|G^{-1}(t)|^{2(1-2\alpha)}}{|G^{-1}(t)|^{2(1-2\alpha)}} = 2\alpha.$$

The left-hand side inequality can be proved similarly.

For part (6), since $\frac{\mathrm{d}}{\mathrm{d}t}G^{-1}(t) > 1/2$ for $t > 0$ sufficiently large, we conclude that $G^{-1}(t) \to +\infty$ as $t \to +\infty$. Thus by employing L'Hôpital's Rule again, we have $\lim_{t\to+\infty} G^{-1}(t)/t = \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{d}t}G^{-1}(t) = 1$. $\square$

By the definition and properties of G^{-1}, we have the following imbedding results.

Lemma 2.2. *The map: $v \to G^{-1}(v)$ from X into $L^p(\Omega)$ is continuous for $2 \leq p \leq 2^*$, and is compact for $2 \leq p < 2^*$.*

The above lemma can be proved by using (2)-(3) of Lemma 2.1. In the next two lemmas, we estimate the remainder of $v - G^{-1}(v)$ at infinity. The results obtained will be used to compute the mountain pass level in the proof of the main theorem.

Lemma 2.3. *There exists $d_0 > 0$ such that*

$$\lim_{v\to+\infty}(v - G^{-1}(v)) \geq d_0.$$

Proof. Assume that $v > 0$. By Lemma 2.1, it follows that $G^{-1}(v) \leq v$ and $G^{-1}(v)g(G^{-1}(v)) \leq v$. Thus we have

$$\begin{aligned}
v - G^{-1}(v) &\geq v\Big(1 - \frac{1}{g(G^{-1}(v))}\Big) \\
&= v\frac{(2\kappa\alpha^2 + G^{-1}(v)^{2(1-2\alpha)})^{1/2} - G^{-1}(v)^{1-2\alpha}}{(2\kappa\alpha^2 + G^{-1}(v)^{2(1-2\alpha)})^{1/2}} \\
&\geq \frac{\kappa\alpha^2 v}{2\kappa\alpha^2 + G^{-1}(v)^{2(1-2\alpha)}} \\
&\geq \frac{\kappa\alpha^2 v}{2G^{-1}(v)^{2(1-2\alpha)}} \quad \text{for } v \text{ large} \\
&:= d(\alpha, v).
\end{aligned}$$

Case 1. If $\frac{1}{4} < \alpha < \frac{1}{2}$, then $0 < 1 - 2\alpha < 1$ and thus $d(\alpha, v) \to +\infty$ as $v \to +\infty$.
Case 2. If $\alpha = \frac{1}{4}$, then $1 - 2\alpha = 1$ and thus $d(\alpha, v) \to \frac{\kappa\alpha^2}{2}$ as $v \to +\infty$.
Case 3. If $0 < \alpha < \frac{1}{4}$, we claim that $v - G^{-1}(v) \to 0$ is impossible. Assume on the contrary. Note that $4\alpha < 1$ and $(G^{-1}(v))^{4\alpha-1} \to 0$ as $v \to +\infty$, by L'Hôpital's Rule, we have

$$\begin{aligned}
0 &\leq \lim_{v\to+\infty} \frac{v - G^{-1}(v)}{G^{-1}(v)^{4\alpha-1}} \\
&= \lim_{v\to+\infty} \frac{G^{-1}(v)^{1-2\alpha}}{4\alpha - 1}[(2\kappa\alpha^2 + G^{-1}(v)^{2(1-2\alpha)})^{1/2} - G^{-1}(v)^{1-2\alpha}] \\
&= \frac{\kappa\alpha^2}{4\alpha - 1} < 0,
\end{aligned}$$

a contradiction. In summary, for all $0 < \alpha < 1/2$, there exists $d_0 > 0$ such that the conclusion of the lemma holds. $\square$

Lemma 2.4. *For $G^{-1}(v)$ defined in (1.6), we have*

(i) *If* $\frac{1}{4} < \alpha < \frac{1}{2}$, *then*

$$\lim_{v\to+\infty} \frac{v - G^{-1}(v)}{v^{4\alpha-1}} = \frac{\kappa\alpha^2}{4\alpha - 1};$$

(ii) *If* $0 < \alpha \le \frac{1}{4}$, *then*

$$\lim_{v\to+\infty} \frac{v - G^{-1}(v)}{\log G^{-1}(v)} \le \begin{cases} \frac{\kappa}{16}, & \alpha = \frac{1}{4}, \\ 0, & 0 < \alpha < \frac{1}{4}. \end{cases}$$

Proof. (i) Assume that $\frac{1}{4} < \alpha < \frac{1}{2}$. By the proof of Lemma 2.3, we have $v - G^{-1}(v) \to +\infty$ as $v \to +\infty$. Then we can use L'Hopital Principle to get

$$\lim_{v\to+\infty} \frac{v - G^{-1}(v)}{v^{4\alpha-1}} = \lim_{v\to+\infty} \frac{g(G^{-1}(v)) - 1}{(4\alpha-1)v^{4\alpha-2}g(G^{-1}(v))} = \frac{\kappa\alpha^2}{4\alpha - 1}$$

(ii) Assume that $0 < \alpha \le \frac{1}{4}$. If there exists a constant $C > 0$ such that $v - G^{-1}(v) \le C$, then the conclusion holds. Otherwise, we may assume that $v - G^{-1}(v) \to +\infty$ as $v \to +\infty$. Again by L'Hopital Principle, we have

$$\begin{aligned} A &:= \lim_{v\to+\infty} \frac{v - G^{-1}(v)}{\log G^{-1}(v)} \\ &= \lim_{v\to+\infty} G^{-1}(v)\Big(\frac{1}{g(G^{-1}(v))} - 1\Big) \\ &= \lim_{v\to+\infty} \frac{2\kappa\alpha^2 G^{-1}(v)^{2\alpha}}{(2\kappa\alpha^2 + G^{-1}(v)^{2(1-2\alpha)})^{1/2} + G^{-1}(v)^{1-2\alpha}}. \end{aligned}$$

Thus $A = \frac{\kappa}{16}$ when $\alpha = \frac{1}{4}$ and $A = 0$ when $0 < \alpha < \frac{1}{4}$. This completes the proof. $\square$

3. Proof of main results

In this section, we first prove that the functional J is well defined on X, moreover, it is continuous and Gâteaux-differentiable in X; next we show that J has the mountain pass geometry, then we use mountain pass theorem to prove our main results, this include the construction of a path has level $c \in (0, S^{N/2}/N)$.

Proposition 3.1. *The functional J has the following properties:*

(1) *J is well defined on X,*
(2) *J is continuous in X,*
(3) *J is Gâteaux-differentiable.*

Proof. Conclusions (1) and (2) can be proved by using items (2)-(3) of Lemma 2.1 and Hölder's inequality, we only prove conclusion (3). Since $G^{-1} \in C^1(\mathbb{R}, \mathbb{R})$, for $v \in X$, $t > 0$ and for any $\psi \in X$, by Mean Value Theorem, there exists $\theta \in (0,1)$ such that

$$\frac{1}{t}\int_\Omega \big[F(G^{-1}(v + t\psi)) - F(G^{-1}(v))\big]\,\mathrm{d}x = \int_\Omega \frac{f(G^{-1}(v + \theta t\psi))}{g(G^{-1}(v + \theta t\psi))}\psi\,\mathrm{d}x.$$

Then by items (2),(3) of Lemma 2.1, and Lebesgue's dominated convergence theorem, we have

$$\Big|\int_\Omega \frac{f(G^{-1}(v + \theta t\psi))}{g(G^{-1}(v + \theta t\psi))}\psi\,\mathrm{d}x - \int_\Omega \frac{f(G^{-1}(v))}{g(G^{-1}(v))}\psi\,\mathrm{d}x\Big|$$

$$
\begin{aligned}
&\le \int_\Omega \Big|\frac{f(G^{-1}(v+\theta t\psi))}{g(G^{-1}(v+\theta t\psi))}\psi - \frac{f(G^{-1}(v))}{g(G^{-1}(v+\theta t\psi))}\psi\Big|\,\mathrm{d}x \\
&\quad + \int_\Omega \Big|\frac{f(G^{-1}(v))}{g(G^{-1}(v+\theta t\psi))}\psi - \frac{f(G^{-1}(v))}{g(G^{-1}(v))}\psi\Big|\,\mathrm{d}x \\
&\le \int_\Omega \big|f(G^{-1}(v+\theta t\psi)) - f(G^{-1}(v))\big||\psi|\,\mathrm{d}x \\
&\quad + \int_\Omega \big|f(G^{-1}(v))\big|\Big|\frac{1}{g(G^{-1}(v+\theta t\psi))} - \frac{1}{g(G^{-1}(v))}\Big||\psi|\,\mathrm{d}x \to 0,
\end{aligned}
$$

as $t \to 0$. Therefore,

$$
\frac{1}{t}\int_\Omega \big[F(G^{-1}(v+t\psi)) - F(G^{-1}(v))\big]\,\mathrm{d}x \to \int_\Omega \frac{f(G^{-1}(v))}{g(G^{-1}(v))}\psi\,\mathrm{d}x.
$$

This implies that J is G-differentiable. □

Remark 3.2. Let $v \in X$. Assume that $w \in X$ and $w \to v$. By using similar arguments as for Lemma 3.1, one can prove that

$$
\langle J'(w) - J'(v), \psi\rangle \to 0, \quad \forall \psi \in X.
$$

This means that J is Fréchet-differentiable.

In the following, we consider the existence of positive solutions to (1.9). From variational point of view, non-negative weak solutions of the equation correspond to the nontrivial critical points of the functional

$$
J^+(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \int_\Omega F(G^{-1}(v)^+)\,\mathrm{d}x.
$$

To avoid cumbersome notation, we denote $J^+(v)$ and $F(G^{-1}(v)^+)$ by $J(v)$ and $F(G^{-1}(v))$ respectively.

Proposition 3.3. *There exist $\rho_0, a_0 > 0$ such that $J(v) \ge a_0$ for all $\|v\| = \rho_0$.*

Proof. Note that $|G^{-1}(v)| \le v$, by Sobolev inequality, we have

$$
\begin{aligned}
J(v) &= \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \int_\Omega F(G^{-1}(v))\,\mathrm{d}x \\
&\ge \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \frac{1}{q}\int_\Omega |v|^q\,\mathrm{d}x - \frac{1}{2^*}\int_\Omega |v|^{2^*}\,\mathrm{d}x \\
&\ge C_1\|v\|^2 - C_2(\|v\|^q + \|v\|^{2^*}).
\end{aligned}
$$

Since $2^* > q > 2$, there exist $\rho > 0$ and $a_0 > 0$ such that $J(v) \ge a_0$ for all $\|v\| = \rho$. □

Proposition 3.4. *There exists $v_0 \in X$ with $\|v_0\| > \rho_0$ such that $J(v_0) < 0$.*

Proof. Let $\varepsilon > 0$ be such that $\overline{B}_{2\varepsilon} = \{x \in \mathbb{R}^N : |x| < 2\varepsilon\} \subset \Omega$. We take $\varphi \in C_0^\infty(\Omega, [0,1])$ with $\mathrm{suppt}(\varphi) = \overline{B}_{2\varepsilon}$ and $\varphi(x) = 1$ for $x \in B_\varepsilon$. Note that $\lim_{t\to+\infty} G^{-1}(t\varphi)/t\varphi = 1$, we have $F(G^{-1}(t\varphi)) \ge \frac{1}{2}F(t\varphi)$ for $t \in \mathbb{R}$ large enough. This gives

$$
J(t\varphi) \le \frac{t^2}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x - \frac{t^q}{2q}\int_{B_\varepsilon} |\varphi|^q\,\mathrm{d}x - \frac{t^{2^*}}{22^*}\int_{B_\varepsilon} |\varphi|^{2^*}\,\mathrm{d}x
$$

Choosing $t_0 > 0$ sufficient large and letting $v_0 = t_0\varphi$, we have $J(v_0) < 0$. □

As a consequence of Propositions 3.3-3.4 and the Ambrosetti-Rabinowitz Mountain Pass Theorem [18], there exists a Palais-Smale sequence $\{v_n\}$ of J at level c with

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)) > 0, \tag{3.1}$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) \neq 0, J(\gamma(1)) < 0\}$. That is, $J(v_n) \to c$, $J'(v_n) \to 0$ as $n \to \infty$.

Proposition 3.5. *Assume that $\{v_n\}$ is a Palais-Smale sequence for J, then $\{v_n\}$ and $\{G^{-1}(v_n)\}$ are bounded in X.*

Proof. Since $\{v_n\} \subset X$ is a Palais-Smale sequence, we have

$$J(v_n) = \frac{1}{2}\int_\Omega |\nabla v_n|^2 \,\mathrm{d}x - \int_\Omega F(G^{-1}(v_n)) \,\mathrm{d}x \to c, \tag{3.2}$$

and for any $\psi \in X$,

$$\langle J'(v_n), \psi\rangle = \int_\Omega \Big[\nabla v_n \nabla\psi - \frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))}\psi\Big] \,\mathrm{d}x = o(1)\|\psi\|. \tag{3.3}$$

Note that $G^{-1}(t)g(G^{-1}(t)) \to 0$ as $t \to 0$, we have $G^{-1}(v_n)g(G^{-1}(v_n)) \in X$ by direct computation. Thus we can take $\psi = G^{-1}(v_n)g(G^{-1}(v_n))$ as test functions and get

$$\begin{aligned}\langle J'(v_n), \psi\rangle &= \int_\Omega |\nabla v_n|^2 \,\mathrm{d}x - \int_\Omega f(G^{-1}(v_n))G^{-1}(v_n) \,\mathrm{d}x \\ &\quad - \int_\Omega \frac{2\kappa\alpha^2(1-2\alpha)}{2\kappa\alpha^2 + |G^{-1}(v_n)|^{2(1-2\alpha)}}|\nabla v_n|^2 \,\mathrm{d}x.\end{aligned} \tag{3.4}$$

It follows that

$$c + o(1) = J(v_n) - \frac{1}{q}\langle J'(v_n), \psi\rangle \geq \big(\frac{1}{2} - \frac{1}{q}\big)\int_\Omega |\nabla v_n|^2 \,\mathrm{d}x.$$

Since $q > 2$, we obtain that $\{v_n\}$ is bounded in X. Note that $|\nabla G^{-1}(v_n)|^2 \leq |\nabla v_n|^2$, we conclude that $\{G^{-1}(v_n)\}$ is also bounded in X. □

Since v_n is a bounded Palais-Smale sequence, there exists $v \in X$ such that $v_n \rightharpoonup v$ in X. Then by Lemma 2.1 and Lebesgue's dominated convergence theorem, for any $\psi \in X$, we have

$$\begin{aligned}&\langle J'(v_n) - J'(v), \psi\rangle \\ &= \int_\Omega (\nabla v_n - \nabla v)\nabla\psi \,\mathrm{d}x \\ &\quad - \int_\Omega \Big(\frac{|G^{-1}(v_n)|^{q-2}G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{|G^{-1}(v)|^{q-2}G^{-1}(v)}{g(G^{-1}(v))}\Big)\psi \,\mathrm{d}x \\ &\quad - \int_\Omega \Big(\frac{|G^{-1}(v_n)|^{2^*-2}G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{|G^{-1}(v)|^{2^*-2}G^{-1}(v)}{g(G^{-1}(v))}\Big)\psi \,\mathrm{d}x \to 0.\end{aligned}$$

Note that $\langle J'(v_n), \psi\rangle \to 0$, we get $J'(v) = 0$. This means that v is a weak solution of (1.1). Now we show that v is nontrivial.

Proposition 3.6. *Let $\{v_n\}$ be a Palais-Smale sequence for functional J at level $c \in (0, \frac{1}{N}S^{N/2})$, assume that $v_n \rightharpoonup v$ in X, then $v \neq 0$.*

Proof. We prove the proposition by contradiction. Assume that $v = 0$. Let $\psi = G^{-1}(v_n)g(G^{-1}(v_n))$. Reasoning as for (3.4), we get

$$\begin{aligned}
\langle J'(v_n), \psi\rangle &= \int_\Omega \frac{4\kappa\alpha^3 + |G^{-1}(v_n)|^{2(1-2\alpha)}}{2\kappa\alpha^2 + |G^{-1}(v_n)|^{2(1-2\alpha)}}|\nabla v_n|^2\,\mathrm{d}x - \int_\Omega f(G^{-1}(v_n))G^{-1}(v_n)\,\mathrm{d}x \\
&\geq \int_\Omega \frac{|G^{-1}(v_n)|^{2(1-2\alpha)}}{2\kappa\alpha^2 + |G^{-1}(v_n)|^{2(1-2\alpha)}}|\nabla v_n|^2\,\mathrm{d}x - \int_\Omega f(G^{-1}(v_n))G^{-1}(v_n)\,\mathrm{d}x \\
&= \int_\Omega |\nabla G^{-1}(v_n)|^2\,\mathrm{d}x - \int_\Omega f(G^{-1}(v_n))G^{-1}(v_n)\,\mathrm{d}x.
\end{aligned}$$

As the term $|G^{-1}(v_n)|^q$ is subcritical, we infer from $\langle J'(v_n), G^{-1}(v_n)g(G^{-1}(v_n))\rangle = o(1)$ that

$$o(1) \geq \|G^{-1}(v_n)\|^2 - \|G^{-1}(v_n)\|_{2^*}^{2^*}.$$

By Sobolev inequality, we have $\|u\|^2 \geq S\|u\|_{2^*}^2$ for all $u \in X$, where S is the best constant for the imbedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$; then we obtain

$$o(1) \geq \|G^{-1}(v_n)\|^2(1 - S^{-2^*/2}\|G^{-1}(v_n)\|^{2^*-2}).$$

Assume that $\|G^{-1}(v_n)\| \to 0$, then by Sobolev inequality, we have $\|G^{-1}(v_n)\|_r \to 0$, $\forall r \in [2, 2^*]$. Using (5) of Lemma 2.1, we conclude that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla v_n|^2\,\mathrm{d}x &= \langle J'(v_n), v_n\rangle + \int_{\mathbb{R}^N} \frac{|G^{-1}(v_n)|^{q-2}G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \frac{|G^{-1}(v_n)|^{2^*-2}G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\,\mathrm{d}x \\
&\leq \langle J'(v_n), v_n\rangle + \frac{1}{2\alpha}\int_{\mathbb{R}^N} |G^{-1}(v_n)|^q\,\mathrm{d}x + \frac{1}{2\alpha}\int_{\mathbb{R}^N} |G^{-1}(v_n)|^{2^*}\,\mathrm{d}x \\
&\to 0,
\end{aligned}$$

This contradicts $J(v_n) \to c > 0$; therefore

$$\|G^{-1}(v_n)\|_{2^*}^{2^*} \geq S^{N/2} + o(1).$$

Again by (5) of Lemma 2.1, we have

$$\begin{aligned}
c &= \lim_{n\to\infty}\Big\{J(v_n) - \frac{1}{2}\langle J'(v_n), v_n\rangle\Big\} \\
&= \lim_{n\to\infty}\Big\{\int_{\mathbb{R}^N} |G^{-1}(v_n)|^{q-2}\Big(\frac{1}{2}\frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))} - \frac{1}{q}G^{-1}(v_n)^2\Big)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{2^*-2}\Big(\frac{1}{2}\frac{G^{-1}(v_n)v_n}{g(G^{-1}(v_n))} - \frac{1}{2^*}G^{-1}(v_n)^2\Big)\,\mathrm{d}x\Big\} \\
&\geq \lim_{n\to\infty}\Big(\frac{1}{2} - \frac{1}{2^*}\Big)\int_{\mathbb{R}^N} |G^{-1}(v_n)|^{2^*}\,\mathrm{d}x \\
&\geq \frac{1}{N}S^{N/2}
\end{aligned}$$

which contradicts $c < \frac{1}{N}S^{N/2}$. Thus we conclude that $\{v_n\}$ does not vanish. $\square$

Next, we construct a path which minimax level is less than $\frac{1}{N}S^{N/2}$ and prove Theorem 1.1. We follow the strategy used in [4].

Proposition 3.7. *The minimax level c defined in* (3.1) *satisfies* $c < \frac{1}{N}S^{N/2}$.

Proof. Let
$$v^* = \frac{[N(N-2)\varepsilon^2]^{(N-2)/4}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}}$$
be the solution of $-\Delta u = u^{2^*-1}$ in $\mathbb{R}^N$. Then
$$\int_{\mathbb{R}^N} |\nabla v^*|^2 \, \mathrm{d}x = \int_{\mathbb{R}^N} |v^*|^{2^*} \, \mathrm{d}x = S^{N/2},$$
Let $\eta_\varepsilon(x) \in C_0^\infty(\Omega, [0,1])$ be a cut-off function with $\eta_\varepsilon(x) = 1$ in $B_\varepsilon = \{x \in \Omega : |x| \le \varepsilon\}$ and $\eta_\varepsilon(x) = 0$ in $B_{2\varepsilon}^c = \Omega \setminus B_{2\varepsilon}$. Let $v_\varepsilon = \eta_\varepsilon v^*$. For any $\varepsilon > 0$, there exists $t^\varepsilon > 0$ such that $J(t^\varepsilon v_\varepsilon) < 0$ for all $t > t^\varepsilon$. Define the class of paths
$$\Gamma_\varepsilon = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = t^\varepsilon v_\varepsilon\}$$
and the minimax level
$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} J(\gamma(t))$$
Let t_ε be such that
$$J(t_\varepsilon v_\varepsilon) = \max_{t \ge 0} J(t v_\varepsilon)$$
Note that the sequence $\{v_\varepsilon\}$ is uniformly bounded in X, we conclude that $\{t_\varepsilon\}$ is upper and lower bounded by two positive constants. In fact, if $t_\varepsilon \to 0$, we have $J(t_\varepsilon v_\varepsilon) \to 0$; otherwise, if $t_\varepsilon \to +\infty$, we have $J(t_\varepsilon v_\varepsilon) \to -\infty$. In both cases we get contradictions according to Proposition 3.3. This proves the conclusion.

According to [4], we have, as $\varepsilon \to 0$,
$$\|\nabla v_\varepsilon\|_2^2 = S^{N/2} + O(\varepsilon^{N-2}), \quad \|v_\varepsilon\|_{2^*}^{2^*} = S^{N/2} + O(\varepsilon^N). \tag{3.5}$$
We define
$$H(t_\varepsilon v_\varepsilon) = -\frac{1}{q} \int_\Omega G^{-1}(t_\varepsilon v_\varepsilon)^q \, \mathrm{d}x + \frac{1}{2^*} \int_\Omega [(t_\varepsilon v_\varepsilon)^{2^*} - G^{-1}(t_\varepsilon v_\varepsilon)^{2^*}] \, \mathrm{d}x.$$
By the definition of v_ε, for $x \in B_\varepsilon$, there exist two constants $c_2 \ge c_1 > 0$ such that for ε small enough,
$$c_1 \varepsilon^{-(N-2)/2} \le v_\varepsilon(x) \le c_2 \varepsilon^{-(N-2)/2}$$
and by (6) of Lemma 2.1,
$$c_1 \varepsilon^{-(N-2)/2} \le G^{-1}(v_\varepsilon(x)) \le c_2 \varepsilon^{-(N-2)/2}.$$
Note that t_ε is upper and lower bounded, there exists a constant $C_1 > 0$ such that
$$\int_{B_\varepsilon} G^{-1}(t_\varepsilon v_\varepsilon)^q \, \mathrm{d}x \ge C_1 \varepsilon^{N - q\frac{N-2}{2}} = C_1 \varepsilon^{(\frac{2^*}{2} - \frac{q}{2})(N-2)}. \tag{3.6}$$
Moreover, since $G^{-1}(t_\varepsilon v_\varepsilon) \le t_\varepsilon v_\varepsilon$ and $2^* > 2$, by Hölder inequality, we have
$$\begin{aligned}
R_\varepsilon := & \frac{1}{2^*} \int_{B_\varepsilon} [(t_\varepsilon v_\varepsilon)^{2^*} - G^{-1}(t_\varepsilon v_\varepsilon)^{2^*}] \, \mathrm{d}x \\
\le & \int_{B_\varepsilon} (t_\varepsilon v_\varepsilon)^{2^*-1} (t_\varepsilon v_\varepsilon - G^{-1}(t_\varepsilon v_\varepsilon)) \, \mathrm{d}x \\
\le & \Big(\int_{B_\varepsilon} (t_\varepsilon v_\varepsilon)^{2^*} \, \mathrm{d}x \Big)^{\frac{2^*-1}{2^*}} \Big(\int_{B_\varepsilon} (t_\varepsilon v_\varepsilon - G^{-1}(t_\varepsilon v_\varepsilon))^{2^*} \, \mathrm{d}x \Big)^{\frac{1}{2^*}}.
\end{aligned}$$

According to Lemma 2.4, there exists $C_2 > 0$ such that for $\frac{1}{4} < \alpha < \frac{1}{2}$,

$$R_\varepsilon \le C_2\Big(\int_{B_\varepsilon}(t_\varepsilon v_\varepsilon)^{2^*(4\alpha-1)}\,\mathrm{d}x\Big)^{\frac{1}{2^*}} \le C_2\varepsilon^{(1-2\alpha)(N-2)}; \tag{3.7}$$

while for $0 < \alpha \le \frac{1}{4}$, there exists a constant $\delta \in (0,1)$ such that

$$R_\varepsilon \le C_2\Big(\int_{B_\varepsilon}(t_\varepsilon v_\varepsilon)^{2^*\delta}\,\mathrm{d}x\Big)^{\frac{1}{2^*}} \le C_2\varepsilon^{\frac{1}{2}(1-\delta)(N-2)}. \tag{3.8}$$

From the above estimations (3.6)-(3.8), we get

$$H(t_\varepsilon v_\varepsilon) \le -C_1\varepsilon^{(\frac{2^*}{2}-\frac{q}{2})(N-2)} + C_2\varepsilon^{(1-2\alpha)(N-2)} \tag{3.9}$$

when $\frac{1}{4} < \alpha < \frac{1}{2}$ and

$$H(t_\varepsilon v_\varepsilon)\ leq - C_1\varepsilon^{(\frac{2^*}{2}-\frac{q}{2})(N-2)} + C_2\varepsilon^{\frac{1}{2}(1-\delta)(N-2)} \tag{3.10}$$

when $0 < \alpha \le 1/4$.

Now we have

$$J(t_\varepsilon v_\varepsilon) = \frac{t_\varepsilon^2}{2}\int_\Omega |\nabla v_\varepsilon|^2 - \frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega |v_\varepsilon|^{2^*} + H(t_\varepsilon v_\varepsilon). \tag{3.11}$$

Since the function $\xi(t) = \frac{1}{2}t^2 - \frac{1}{2^*}t^{2^*}$ achieves its maximum $\frac{1}{N}$ at point $t_0 = 1$, by using (3.5), we derive from (3.11) that

$$J(t_\varepsilon v_\varepsilon) \le \frac{1}{N}S^{N/2} + H(t_\varepsilon v_\varepsilon) + O(\varepsilon^{N-2}). \tag{3.12}$$

From assumption (H1), we conclude that

(i) for $\frac{1}{4} < \alpha < \frac{1}{2}$ and $q > \frac{4}{N-2} + 4\alpha$, we have $(\frac{2^*}{2} - \frac{q}{2})(N-2) < (1-2\alpha)(N-2)$;

(ii) for $0 < \alpha \le \frac{1}{4}$ and $q > \frac{N+2}{N-2}$, we have $(\frac{2^*}{2} - \frac{q}{2})(N-2) < \frac{1}{2}(1-\delta)(N-2)$ for $\delta > 0$ small enough.

Combining (3.9), (3.10) and (3.12) and according to conclusions (i),(ii), we get

$$c_\varepsilon = J(t_\varepsilon v_\varepsilon) < \frac{1}{N}S^{N/2}. \tag{3.13}$$

Finally, since $\Gamma_\varepsilon \subset \Gamma$, we have

$$c \le c_\varepsilon < \frac{1}{N}S^{N/2}.$$

This completes the proof. □

Proof of Theorem 1.1. Firstly, by Propositions 3.3-3.4, the functional J has the Mountain Pass Geometry. Then there exists a Palais-Smale sequence $\{v_n\}$ at level c given in (3.1). Secondly, by Proposition 3.5, the Palais-Smale sequence $\{v_n\}$ is bounded in X. By Proposition 3.6, if $c < \frac{1}{N}S^{N/2}$, then the weak limit v of $\{v_n\}$ in X is nonzero and is a critical point of J. Finally, by Proposition 3.7, there indeed exists a mountain pass which maximum level c_ε is strictly less than $\frac{1}{N}S^{N/2}$. This implies that the level $c < \frac{1}{N}S^{N/2}$ and v is a nontrivial weak solution of Eq.(1.9). By strong maximum principle, $v(x) > 0, x \in \Omega$. Let $u = G^{-1}(v)$. Since $|\nabla u| \le |\nabla v|$, we obtain that $u \in X$ and it is a positive weak solution of (1.1). □

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