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CONVERGENCE OF GENERALIZED EIGENFUNCTION EXPANSIONS

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Abstract. We present a simplified theory of generalized eigenfunction expansions for a commuting family of bounded operators and with finitely many unbounded operators. We also study the convergence of these expansions, giving an abstract type of uniform convergence result, and illustrate the theory by giving two examples: The Fourier transform on Hecke operators, and the Laplacian operators in hyperbolic spaces.

1. Introduction

A generalized eigenfunction expansion is a generalization of the Fourier transform. Just as the Fourier transform in higher dimensions may be regarded as an expansion for the functions in the domain of the self-adjoint operators associated with

$$\{i\frac{\partial}{\partial x_j}\}_{j=1}^n,$$

it is possible to study generalized eigenfunction expansions for families of commuting operators, not just a single operator. In this paper, we develop an alternative approach to such expansions for a commuting family of operators concentrating on questions of convergence. Instead of using the spectral projections for families of commuting operators, which appear in the spectral theorem for such families arising ultimately from the Gelfand-Naimark representation theorem, we use limits (in a topological vector sense) of such projections to produce what we call generalized eigenprojections; hence spectral properties of the operators are automatically inherited by these generalized eigenprojections.

Generalized eigenfunction expansions are widely used in mathematical physics. Also, many integral expansions, including some occurring in analytic number theory, are generalized eigenfunction expansions. There is much literature on this subject. To name a few, the classical literature is anchored by Gelfand and Vilenkin in [5], Berezhanskii in [2], and Maurin in [10]. The first modern paper on the foundations of generalized eigenfunction expansions with an application in mathematical physics was given by Simon in [14] in 1982. Also, Poerschke, Stolz, and Weidmann in [11] gave a simplified version of generalized eigenfunction expansions for a single

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self-adjoint operator with an application in mathematical physics. One difference between the classical literature and ours is that we define a generalized eigenprojection as a limit of the products of spectral projections and some real numbers in a specific space (Definition 3.2) and use it to expand the operators, thus it gives some sort of uniform convergence of the expansion in the specific space. The easiest way to illustrate is to use the Fourier series. The Fourier series of any L_2 function converges in L_2, which is the type of convergence studied by other authors. However, if the function being expanded lies in the Sobolev space W_2^1 for example, then the series converges uniformly and error estimates may be given, which apply on the entire unit ball of W_2^1. The convergence results of this paper are obtained in the same way as differentiability hypothesis are needed to guarantee uniform convergence of usual Fourier series. As a continuation and with adding more conditions, we are also able to extend some result in analytic number theory, which is given in Section 5.2 ([9]). Now a difference between a modern approach, namely Poerschke, Stolz, and Weidmann's approach, and ours is that we expand a commuting family of operators rather than a single self-adjoint operator. Although their approach also gives some asymptotic behavior of the eigenfunctions, which does not appear to have been considered by earlier authors, their approach is not extended to a family of operators. Since our approach is for a family of operators, we concentrate more on absolute convergence of the integral involved, and as a result, we obtained the same sort of uniform convergence of the integrals as by the Fourier transform on a set of functions lying in L_1. We shall compare the previous approach and ours more in Section 4.

In order to consider the convergence of the integral, and because the formalism surrounding these expansions can be confusing, we develop in this paper a formalism for generalized eigenfunction expansions which proceeds from the spectral theorem in exactly the way that the usual Fourier transform may be derived from the spectral decomposition of the operator

$$i\frac{d}{dx} \tag{1.1}$$

in $L_2(\mathbb{R})$. We obtain a theory analogue to the theory of the inverse Fourier transform. The purpose of this is to simplify the construction of the expansion and also to obtain new results about its convergence.

For the above operator (1.1), the spectral theorem gives a projection valued measure

$$\Delta \to E(\Delta)$$

in $L_2(\mathbb{R})$, where Δ is a Borel set. Here it is possible to calculate the projections $E(\Delta)$; they are the inverse images of characteristic functions under the Fourier transform. Clearly, for any λ,

$$\lim_{n\to\infty} E(\lambda + \frac{1}{n}, \lambda - \frac{1}{n}) = 0$$

in the strong operator topology on $L_2(\mathbb{R})$. However, if

$$\chi(\lambda - \frac{1}{n}, \lambda + \frac{1}{n})$$

is the characteristic function, it is clear that in the tempered distributions

$$n\chi(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}) \to 2\delta(\lambda),$$

the point measure at λ. It follows that

$$n\chi(\lambda - \frac{1}{n}, \lambda + \frac{1}{n})$$

converges in the distributions to the function $e^{-i\lambda x}$; the complex conjugate is because the embedding of functions into distributions contains a conjugation. In other words, the associated eigenfunction is actually $e^{-i\lambda x}$.

In this fashion, our approach derives the Fourier transform from a realization of the Gelfand transform of the smallest closed subalgebra of $B(\mathfrak{h})$ (the set of bounded operators from $\mathfrak{h}$ to $\mathfrak{h}$) containing the translation operators. The inverse Fourier transform is a generalized eigenfunction expansion.

One of the most important questions we study is "how does the expansion converge?". Although a variety of approaches exist for the derivation of such expansions, our results on convergence appear to be new. We need, in general, more hypothesis than in Poerschke, Stolz, and Weidmann to obtain convergent integrals and the eigenprojections. This is to be expected because, for the Fourier transform, the theory must guarantee that the transform is in L_1.

As in previous approaches, we also take the point of view that it expands operators instead of functions. As we mentioned above, the inverse Fourier transform is an example of an expansion of the identity operator (we shall explain in Section 4.1). If one takes this point of view, the obvious question is "how closely does the expansion approximate the operator?". This question is the basis for the theory of approximation numbers of operators. In the case of Sobolev spaces of functions on compact sets, the operator in question is often an embedding map from the Sobolev space into L_2. The approximation numbers are often calculated using Fourier series. For this purpose, it is necessary to have uniform estimates of the form "any element of the unit ball in the Sobolev space may be approximated in L_2 to within an accuracy of ε by $n(\varepsilon)$ terms of its Fourier series". This will be given in Corollary 3.18 in Section 3.3.

To illustrate the theory, we give two examples in the last section; one is the Fourier transform, i.e., we apply Theorem 3.16 to the Laplacian on $C_0^\infty(\mathbb{R})$, and the other is an application to analytic number theory.

2. Background

The basic idea of a generalized eigenfunction expansion follows from the next theorem.

Theorem 2.1. *Let $\mathfrak{X}$ be a compact Hausdorff space. Let μ be a positive Borel measure on $\mathfrak{X}$. Let $f \in C(\mathfrak{X})$. Let $T_f : L_2(\mathfrak{X}, \mu) \to L_2(\mathfrak{X}, \mu)$ be defined by*

$$T_f(g) = f \cdot g.$$

Then there exists a projection valued measure E on the Borel subsets of $\mathfrak{X}$ such that

$$T_f = \int_{\mathfrak{X}} f \, dE. \tag{2.1}$$

Recall that this equation is an abbreviation for

$$(T_f g, h) = \int_{\mathfrak{X}} f \, dE_{g,h}$$

where $E_{g,h}(\Delta) = (E(\Delta)g, h)$ for Borel set Δ (see [12]*). Also, if $\mathcal{B} = \{T_f : f \in C(\mathfrak{X})\}$, then, $E(\Delta)$ commutes with the set $\mathcal{B}'$ of all bounded linear transformations Q taking $L_2(\mathfrak{X}, \mu)$ into itself such that Q commutes with $\mathcal{B}$.*

Proof. Let $\hat{\Delta}$ be a Borel set in the range of f. Then let $\Delta = f^{-1}\hat{\Delta})$, and define $E(\Delta) = \chi_\Delta$ where χ_Δ is a characteristic function on Δ. Then E is a projection valued measure and

$$\int_{\mathfrak{X}} f\, dE_{g,h} = \int_{\mathfrak{X}} fg\overline{h}\, d\mu,$$

hence the theorem follows. □

The above theorem is very elementary and shows the existence of the expansion (2.1) for a multiplication operator; and thus for a family of multiplication operators (see Theorem 3.8). Our aim is to get the expansion for a commuting family of normal operators. The idea of the process is to map operators into a compact Hausdorff space (so that the operators turn into multiplication operators), get the expansion there, and pull it back to its original space.

Before we move on, let us state two corollaries follow from the above theorem.

Corollary 2.2. $\mathcal{B}' = \{T_f : f \in L_\infty\}$.

Corollary 2.3. $\{T_f : f \in L_\infty\}$ *is a von Neumann algebra.*

3. Generalized Eigenfamily

We first give some definitions and then begin the process of obtaining generalized eigenfunction expansions from Theorem 2.1.

Definition 3.1. Let W be a locally convex topological vector space and W' be its dual space with the weak*-topology. Denote $C(W, W')$ as the set of continuous conjugate lineartransforms from W into W'. We shall topologize $C(W, W')$ using sub-base open sets about 0, which are of the form $\Theta_{xV} := \{A : A(x) \subset V\}$, where $x \in W$ and V is a neighborhood in W'. With this topology, $C(W, W')$ is a locally convex topological vector space.

Definition 3.2. Let W, W', and $C(W, W')$ be defined as in Definition 3.1. Let $\mathcal{H}$ be a Hilbert space such that $W \subset \mathcal{H} \subset W'$ all dense. Denote $\mathcal{D}$ as a commuting family of normal operators $\{A\}$ on $\mathcal{H}$ such that the restriction of A to W takes W into W continuously. Suppose there is a commutative von Neumann algebra $\mathcal{A}$ with which $\mathcal{D}$ is affiliated, i.e., for every $A \in \mathcal{D}$, the spectral projections of A are in $\mathcal{A}$. Then a *generalized eigenprojection* for $\mathcal{D}$ is an operator $Q \in C(W, W')$ ($Q \not\equiv 0$), with the following properties.

(1) There exist a sequence $\{P_n\}$ of projections in $\mathcal{A}$ and a sequence $\{r_n\}$ of real numbers such that r_nP_n converges to Q in $C(W, W')$;
(2) For each $A \in \mathcal{D}$, there exists $\lambda_A \in \mathbb{C}$ with the property that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $E(A, \lambda_A, \epsilon)P_n = P_n$ for any $n > N$, where $E(A, \lambda_A, \epsilon)$ is the spectral projection for A corresponding to $\{y : |y - \lambda_A| < \epsilon\}$. Note that we fix one λ_A for each A to get a generalized eigenprojection.

Remark 3.3. The properties of a generalized eigenprojection in the above definition indicate that a generalized eigenprojection is basically the limit of spectral projections for all $A \in \mathcal{D}$.

Definition 3.4. An element of the range of a generalized eigenprojection is called a *generalized eigenfunction* of $\mathcal{D}$, and λ_A is called a *generalized eigenvalue* of A.

The above definitions of a generalized eigenfunction and eigenvalue follow from the next theorem.

Theorem 3.5. *If Q is a generalized eigenprojection for $\mathcal{D}$, then for every $A \in \mathcal{D}$ and any element ψ in the range of Q,*

$$\hat{A}^t \psi = \lambda_A \psi$$

where $\hat{A} = A|_W$ and $\hat{A}^t$ is the transpose of $\hat{A}$.

Proof. By the definition of a transpose, we have $\hat{A}^t\psi(\theta) = \psi(\hat{A}(\theta))$ where $\theta \in W$. Since $Q \in C(W, W')$ and ψ is in the range of Q, there exists $\phi \in W$ such that $Q(\phi) = \psi$. Hence $\hat{A}^t\psi(\theta) = Q(\phi)(\hat{A}(\theta))$. Since $r_n P_n \to Q$ by the definition of Q, we have

$$(\hat{A}(\theta), r_n P_n(\phi)) \to Q(\phi)(\hat{A}(\theta)). \tag{3.1}$$

Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $E(A, \lambda_A, \epsilon)P_n = P_n$ for $n > N$. Hence

$$(\hat{A}(\theta), r_n P_n(\phi)) = (\hat{A}(\theta),\, r_n E(A, \lambda_A, \epsilon) P_n(\phi)) = (\hat{A}E(A, \lambda_A, \epsilon)(\theta), r_n P_n(\phi))$$

since $\hat{A}, P_n, E(A, \lambda_A, \epsilon)$ commutes (because they are all in $\mathcal{A}$). By the spectral theorem ([12]), we have

$$(\hat{A}E(A, \lambda_A, \epsilon)(\theta), r_n P_n(\phi)) = \int t\, dF_{E(\hat{A}, \lambda_{\hat{A}}, \epsilon)(\theta), r_n P_n(\phi)} = \int_{|t-\lambda_A|<\epsilon} t\, dF_{\theta, r_n P_n(\phi)}.$$

We now use the integration by parts and then let n goes to infinity. Since ϵ is arbitrary, we get

$$(\hat{A}(\theta), r_n P_n(\phi)) = (\hat{A}E(A, \lambda_A, \epsilon)(\theta), r_n P_n(\phi)) \to \lambda_A(Q(\phi))(\theta).$$

Hence by (3.1), we get $\hat{A}^t\psi(\theta) = Q(\phi)(\hat{A}(\theta)) = \lambda_A(Q(\phi))(\theta) = \lambda_A\psi(\theta)$. $\square$

Definition 3.6. Let $\mathcal{H}, W, \mathcal{D}, \mathcal{A}$ be given as above. Let $\mathfrak{X}$ be a locally compact Hausdorff space with a positive measure μ. Denote H as an isometric isomorphism from $\mathcal{A}$ to $L_\infty(\mathfrak{X}, \mu)$ such that $H(A) = f \in C(\mathfrak{X}) \cap L_\infty(\mathfrak{X}, \mu)$ if $A \in \mathcal{D}$. Then a *generalized eigenfamily for* $W, \mathcal{D}, \mathfrak{X}$ is a function $g : \mathfrak{X} \to C(W, W')$ ($g(x) = Q_x$) such that

1. (1) g is continuous;
2. (2) for any $A \in \mathcal{D}$, $A = \int \overline{H(A)} Q_x \, d\mu(x)$; where the integral converges in $C(W, W')$;
3. (3) Q_x is a generalized eigenprojection of $\mathcal{D}$ with the generalized eigenvalue $H(A)(x)$ (note that this is for each fixed $x \in \mathfrak{X}$).

The integral expansion in 2 is called the *generalized eigenfunction expansion*.

Remark 3.7. If $A \in \mathcal{D}$ is bounded and affiliated with $\mathcal{A}$, then $A \in \mathcal{A}$ hence $\mathcal{D} \subset \mathcal{A}$, because A is a limit of spectral projections by the spectral theorem. Also, if $\mathcal{D} \subset \mathcal{A}$ and $\mathfrak{X}$ is a maximal ideal space of $\mathcal{A}$ in Definition 3.6, then H is the Gelfand transform and thus $H(A)$ is continuous.

We must now show the existence of a generalized eigenfamily in order to get some result in convergence of a generalized eigenfunction expansion. The key point is to construct the space W so that the expansion converges in $C(W, W')$. In the next section, the existence for a commuting family of bounded operators will be given, where $\mathfrak{X}$ in this case will be a certain subset of the maximal ideal space of the family of bounded operators.

3.1. **Existence of a Generalized Eigenfamily for Bounded Operators.** Let us first show the existence of a generalized eigenfamily for a family of multiplication operators (analogue to Theorem 2.1).

Theorem 3.8. *Let $\mathfrak{X}$ be a locally compact metric space and μ be a positive measure on $\mathfrak{X}$. Let $\mathcal{D} = \{T_f : f \in C(\mathfrak{X}) \cap L_\infty(\mathfrak{X}, \mu)\}$ where $T_f(g) = f \cdot g$ for any $g \in L_2(\mathfrak{X}, \mu)$. Let $W = C(\mathfrak{X}) \cap L_2(\mathfrak{X}, \mu)$. Then there exists a generalized eigenfamily for $W, \mathcal{D}, \mathfrak{X}$.*

Proof. In this case, we have a Hilbert space $\mathcal{H} := L_2(\mathfrak{X}, \mu)$ and the von Neumann algebra $\mathcal{A} := \mathcal{D}'$ by Corollary 2.3. Define $H : \mathcal{A} \to L_\infty(\mathfrak{X}, \mu)$ such that $H(T_f) = f \in L_\infty(\mathfrak{X}, \mu)$. Then H is an isometric isomorphism. Notice that, if $T_f \in \mathcal{D}$, then $H(T_f) = f \in C(\mathfrak{X}) \cap L_\infty(\mathfrak{X}, \mu)$. Now, let $x \in \mathfrak{X}$. Define $\hat{x} : W \to \mathbb{C}$ such that $\hat{x}(\phi) = \phi(x)$. Define also $g : \mathfrak{X} \to C(W, W')$ such that $g(x) = Q_x$ where $Q_x(\phi) = \overline{\phi(x)}\hat{x}$, for any $\phi \in W$. Notice $Q_x(\phi) \in W'$, i.e., for any $\psi \in W$, $(Q_x(\phi))(\psi) = \overline{\phi(x)}\hat{x}(\psi) = \overline{\phi(x)}\psi(x)$. We claim that g is a generalized eigenfamily for $W, \mathcal{D}, \mathfrak{X}$. In order to show the claim, we must show

(a) g is continuous;
(b) for $A \in \mathcal{D}$,
$$A = \int \overline{H(A)} Q_x \, d\mu; \tag{3.2}$$
(c) each Q_x is a generalized eigenprojection of $\mathcal{D}$, corresponding to the eigenvalue $H(A)(x)$.

(a) Let $\epsilon > 0$. Let V be a neighborhood of 0 in W', i.e., for $h \in W$,
$$V = \{F \in W' : |F(h)| < \epsilon\}.$$
Let $f \in W$. Let $Z := \theta_{fV}$ be the sub-base open set about 0 in $C(W, W')$, i.e.,
$$Z = \{P \in C(W, W') : P(f) \subset V\} = \{P \in C(W, W') : |P(f)(h)| < \epsilon\}.$$
Let $x \in \mathfrak{X}$. Since $f, h \in W = C(\mathfrak{X}) \cap L_2(\mathfrak{X}, \mu)$, there exists $Y \subset \mathfrak{X}$ such that, for any $y \in Y$,
$$|\overline{f(y)}h(y) - \overline{f(x)}h(x)| < \epsilon \Rightarrow |g(y)(f)(h) - g(x)(f)(h)| < \epsilon \Rightarrow g(y) - g(x) \in Z.$$
Hence g is continuous.

(b) First recall that (3.2) is an abbreviation for
$$(\psi,\, A(\phi)) = \int \overline{H(A)}(Q_x(\phi))(\psi) \, d\mu \quad \text{for } \phi, \psi \in W.$$
Since $A \in \mathcal{D} \subset \mathcal{A}$, there exists $f \in C(\mathfrak{X}) \cap L_\infty(\mathfrak{X}, \mu)$ such that $A = T_f$, i.e., $A(\phi) = T_f(\phi) = f \cdot \phi$ and $H(A) = f$. Thus we get
$$(\psi,\, A(\phi)) = \int \psi(x)\overline{A(\phi)(x)} \, d\mu = \int \psi(x)\overline{f(x)\phi(x)} \, d\mu.$$
Also,
$$\int \overline{H(A)}(Q_x(\phi))(\psi) \, d\mu = \int \overline{f(x)\phi(x)}\psi(x) \, d\mu$$

Hence the result follows.

(c) We must show, for each fixed $x \in \mathfrak{X}$,

(i) there exists $\{P_n\}$ of projections in $\mathcal{A}$ and $\{r_n\}$ of real numbers such that $r_n P_n \to Q_x$ in $C(W, W')$,

(ii) for any $\epsilon > 0$, and for each $A \in \mathcal{D}$, there is a spectral projection $E(A, \lambda_A, \epsilon)$ such that there exists $N \in \mathbb{N}$ with $E(A, \lambda_A, \epsilon)P_n = P_n$ for any $n > N$, where P_n is the projection from i and $\lambda_A = H(A)(x)$.

(i) By Corollary 2.2 and 2.3, we have $\mathcal{A} = \{T_f : f \in L_\infty(\mathfrak{X}, \mu)\}$. Let $\{\Delta_n\}$ be the sets containing x such that $\Delta_{n+1} \subset \Delta_n$ for any $n \in N$. (Such sets exist since $\mathfrak{X}$ is a metric space.) Let $P_n = T_{\chi_{\Delta_n}}$. Then $P_n \in \mathcal{A}$ and $P_n(\phi) = T_{\chi_{\Delta_n}}(\phi) = \chi_{\Delta_n} \cdot \phi$. Let $r_n = \frac{1}{\mu(\Delta_n)}$. Then we have

$$\begin{aligned}
(\psi,\, r_n P_n(\phi)) &= (\psi,\, r_n \chi_{\Delta_n}(\phi)) \\
&= \frac{1}{\mu(\Delta_n)} \int_{\Delta_n} \psi(y)\overline{\phi(y)}\, d\mu \\
&= \frac{1}{\mu(\Delta_n)} \int_{\Delta_n} \psi(x)\overline{\phi(x)}\, d\mu + \frac{1}{\mu(\Delta_n)} \int_{\Delta_n} (\psi(y)\overline{\phi(y)} - \psi(x)\overline{\phi(x)})\, d\mu.
\end{aligned}$$

The first integral is

$$\frac{\psi(x)\overline{\phi(x)}}{\mu(\Delta_n)} \int_{\Delta_n} d\mu = \frac{\psi(x)\overline{\phi(x)}}{\mu(\Delta_n)} \mu(\Delta_n) = \psi(x)\overline{\phi(x)}.$$

Since $\mathfrak{X}$ is a metric space, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|\psi(s)\overline{\phi(s)} - \psi(x)\overline{\phi(x)}| < \epsilon$ for any $s \in \Delta_n$ for $n > N$. Hence the second integral goes to zero as $n \to \infty$. Thus $(\psi, r_n P_n(\phi)) \to \psi(x)\overline{\phi(x)} = [Q_x(\phi)](\psi)$, i.e., $r_n P_n \to Q_x$ as $n \to \infty$.

(ii) Again, if $A \in \mathcal{D}$, then there exists $f \in C(\mathfrak{X}) \cap L_\infty(\mathfrak{X}, \mu)$ such that $A = T_f$ and $\lambda_A := H(A)(x) = f(x)$. Let $\epsilon > 0$ arbitrary. Let $E(A, \lambda_A, \epsilon)$ be the spectral projection of A to $\{y : |y - \lambda_A| < \epsilon\}$. We need to show $\exists N$ such that $E(A, \lambda_A, \epsilon)P_n = P_n$ for any $n > N$. We have

$$E(A, \lambda_A, \epsilon)P_n(\phi) = E(A, \lambda_A, \epsilon)T_{\chi_{\Delta_n}}(\phi) = E(A, \lambda_A, \epsilon)\chi_{\Delta_n}\phi.$$

Pick the smallest integer N such that $f(\Delta_N) \subset \{y : |y - \lambda_A| < \epsilon\}$. Then we get $E(A, \lambda_A, \epsilon)P_n = P_n$ for any $n > N$. $\square$

Remark 3.9. Instead of using a locally compact metric space $\mathfrak{X}$ in Theorem 3.8, we can use a locally compact Hausdorff space. However, in that case, we must assume some additional properties for $\mathfrak{X}$ and $\mathcal{D}$ as follows:

(1) for any $x \in \mathfrak{X}$, there exists $\{\Delta_n\}$ of sets containing x such that $\Delta_{n+1} \subset \Delta_n$ for any $n \in \mathbb{N}$;

(2) for any $\epsilon > 0$ and for any $\phi, g \in W$, there exists N such that $|\phi(s)\overline{g(s)} - \phi(\lambda)\overline{g(\lambda)}| < \epsilon$ for any $s \in \Delta_n$ for $n > N$;

(3) for the sets $\{\Delta_n\}$ from 1, $\sup_{x,y\in\Delta_n} |f(x) - f(y)| < \frac{1}{n}$ for any f such that $T_f \in \mathcal{D}$.

To show the existence for more general bounded operators, we first define the property of the locally convex topological vector space W that is required.

Definition 3.10. Let $\mathcal{H}$ and W be given as in Definition 3.2 and $\mathfrak{X}$ as in Definition 3.6. Let $\mathcal{C}$ be a commutative C^*-algebra of bounded operators on $\mathcal{H}$. Define $\mathcal{C}_e := \{Ae | A \in \mathcal{C}\}$ for $e \in \mathcal{H}$, $\mathcal{S}_e$ as the smallest closed subspace such that $\mathcal{C}_e \subset \mathcal{S}_e \subset \mathcal{H}$, and $P[\mathcal{S}_e] : \mathcal{H} \to \mathcal{S}_e$ as the projection onto $\mathcal{S}_e$. Assume there is an isometric isomorphism $\hat{G}_e : \mathcal{S}_e \to L_2(\mathfrak{X}, \mu)$ such that $\hat{G}_e|_{\mathcal{C}_e}$ is from $\mathcal{C}_e$ onto $C(\mathfrak{X})$. Then W has *the almost continuity property with respect to* $\mathcal{S}_e$ *and* $\mathfrak{X}$ if, for any $\epsilon > 0$, there exists a compact set $K \subset \mathfrak{X}$ such that $\mu(\mathfrak{X}\backslash K) < \epsilon$, $\hat{G}_e(P[\mathcal{S}_e](\phi))$ is continuous on K for any $\phi \in W$, and every open set in K has a positive measure.

We now investigate when W has the almost continuity property.

Definition 3.11. Let W_1 and W_2 be locally convex topological vector spaces. A map $E : W_1 \to W_2$ is called *k-nuclear* if there exist $\{\alpha_i\} \subset \ell_k$, $\{F_i | F_i \in W_1'\}$ equicontinuous, and $\{g_i\} \subset W_2$ uniformly bounded in W_2 such that, for any $f \in W_1$,

$$E(f) = \sum_{i=1}^{\infty} \alpha_i \cdot F_i(f) \cdot g_i.$$

Note here that, in Hilbert space setting, 1-nuclear is called the *trace class* and 2-nuclear is called the *Hilbert-Schmidt*.

Theorem 3.12. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}$ be a commutative von Neumann algebra. Suppose we have a Banach space V such that $V \subset \mathcal{H} \subset V'$, V is dense in $\mathcal{H}$, and the embedding of V into $\mathcal{H}$ is continuous. If the embedding of V into $\mathcal{H}$ is* 1*-nuclear, then there exists a constant β such that for any finite set $\{\theta_r\}_{r=1}^s$ of elements of the unit ball of V, and for any $e \in \mathcal{H}$,*

$$\sum_{r=1}^{s} |(P(\xi_r)\theta_r, e)| \leq \beta \|e\|_{\mathcal{H}}$$

for any disjoint family $\{\xi_r\}_{r=1}^s$ of Borel subsets of the maximal ideal space $\mathfrak{A}$ of $\mathcal{A}$. In case that V is a Hilbert space, the above is true with 2*-nuclear embedding of V into $\mathcal{H}$.*

For a proof of the above theorem, see [8, Theorem 230].

Theorem 3.13. *Let $\mathcal{H}$ and W be Hilbert spaces such that W is dense in $\mathcal{H}$ and the embedding of W into $\mathcal{H}$ is* 2*-nuclear. Denote $\mathcal{D}$ as a commuting family of bounded normal operators on $\mathcal{H}$ such that the restriction to W takes W into W continuously. Let $\mathcal{C}$ be a commutative C^*-algebra generated by $\mathcal{D}$. Also, let $\mathfrak{C}$ be the maximal ideal space of $\mathcal{C}$ (which is a locally compact Hausdorff space). Then W has the almost continuity property with respect to $\mathcal{S}_e$ and $\mathfrak{C}$.*

Proof. Let μ_e be a positive measure on $\mathfrak{C}$ given by Riesz Representation Theorem (RRT) for $e \in \mathcal{H}$. Let $\epsilon > 0$. Then we need to show that there exists a compact set $\mathfrak{F}$ such that $\mu_e(\mathfrak{C}\backslash\mathfrak{F}) < \epsilon$ and $\hat{G}_e(P[\mathcal{S}_e]\phi)$ is continuous on $\mathfrak{F}$ for any $\phi \in W$ (i.e., $\mathfrak{F}$ does not depend on f). Using Lemma 3.12, we construct a compact subset $\hat{K}$ such that $\mu_e(\mathfrak{C}\backslash\hat{K}) < \frac{\epsilon}{2}$ and, for any $\phi \in W$,

$$|\hat{G}_e(P[\mathcal{S}_e]\phi)(x)| < N\|\phi\| \quad \text{for } x \in \hat{K}. \tag{3.3}$$

Also, using Lusin's theorem, we construct another compact subset $\tilde{K}$ such that $\mu_e(\mathfrak{C}\backslash\tilde{K}) < \frac{\epsilon}{2}$ and $\hat{G}_e(P[\mathcal{S}_e]\phi_i)$ is continuous on $\tilde{K}$ for all ϕ_i. Let $\mathfrak{F} = \hat{K} \cap \tilde{K}$. Then $\mu_e(\mathfrak{C}\backslash\mathfrak{F}) \leq \epsilon$. Now let $\phi \in W$. Then there is a sequence $\phi_i \in S$ such that $\|\phi_i\| < \|\phi\|$ and $\phi_i \to \phi$. Then, for all $x \in \mathfrak{F}$ in the complement of a set of measure zero,

$$\hat{G}_e(P[\mathcal{S}_e]\phi_i)(x) \to \hat{G}_e(P[\mathcal{S}_e]\phi)(x) \tag{3.4}$$

On $\mathfrak{F}$, we have $|\hat{G}_e(P[\mathcal{S}_e]\phi_i)(x)| < N\|\phi_i\|$ by (3.3), hence $\{\hat{G}_e(P[\mathcal{S}_e]\phi_i)\}$ is uniformly Cauchy. Also, recall that $\{\hat{G}_e(P[\mathcal{S}_e]\phi_i)\}$ is continuous on $\mathfrak{F}$. Thus it converges to a continuous function $G(\phi)$ on $\mathfrak{F}$. However, $G(\phi)$ must agree with $\hat{G}_e(P[\mathcal{S}_e]\phi)$ almost everywhere on $\mathfrak{F}$ by (3.4). Hence the result follows. $\square$

Here is our main theorem.

Theorem 3.14. *Let $\mathcal{H}$ and W be given as in Definition 3.2 and $\mathcal{C}$ and $\mathfrak{C}$ as in Theorem 3.13. Let $\mathcal{A}$ be the smallest von Neumann algebra such that $\mathcal{C} \subset \mathcal{A}$ (cf. Remark 3.7). Also let V be a Banach space such that $W \subset V \subset \mathcal{H}$ all dense. Suppose that the embedding $E_1 : V \to \mathcal{H}$ is 1-nuclear and $E_2 : W \to V$ is 2-nuclear. Suppose also that W has the closed graph property. Then, for any cyclic vector e for $\mathcal{A}'$ (the commutant of $\mathcal{A}$) and for any $\epsilon > 0$, there exists a compact set $\mathfrak{F} \subset \mathfrak{C}$ such that $\mu_e(\mathfrak{C}\backslash\mathfrak{F}) < \epsilon$ and there exists a generalized eigenfamily for $W, \mathcal{C}, \mathfrak{F}$.*

Before we give the proof, let us note that the embedding condition on W is needed in order to get the almost continuity property (see Theorem 3.13). Also notice that the existence of a generalized eigenfamily is given for the compact set $\mathfrak{F}$ instead of the entire maximal ideal space $\mathfrak{C}$. This is because the almost continuity property of W gives a compact set in $\mathfrak{C}$ and because we need to get, for each $x \in \mathfrak{C}$, the generalized eigenprojection Q_x in $C(W, W')$, i.e., $Q_x(\phi) \in W'$ for all $x \in \mathfrak{C}$. Unfortunately, this is not true in the entire $\mathfrak{C}$, and hence we use $\mathfrak{F}$ instead of $\mathfrak{C}$.

Proof. We first note that there exists an isometric isomorphism $H : \mathcal{A} \to L_\infty(\mathfrak{C}, \mu_e)$ such that $H(A) = G(A) \in C(\mathfrak{C}) \cap L_\infty(\mathfrak{C}, \mu_e)$ for $A \in \mathcal{C}$, where G is the Gelfand transform of $\mathcal{C}$. We can construct H by observing $\hat{G}_e \mathcal{C} \hat{G}_e^{-1} = \{T_f | f \in C(\mathfrak{C})\}$. Also, by a basic property of a Hilbert space, we know there exists a set of orthonormal vectors $\{e_i\}$ in $\mathcal{H}$ such that $\mathcal{H} = \oplus \mathcal{S}_{e_i}$. Define

$$e = \sum_i (\frac{1}{n_i}) e_i$$

where $(\frac{1}{n_i})e_i$ is in ℓ_2 for any i. Then e is a cyclic vector for $\mathcal{A}'$ (i.e., $\mathcal{A}'e$ is dense in $\mathcal{H}$ where $\mathcal{A}'$ be a commutant of $\mathcal{A}$). For this e, define μ_e on $\mathfrak{C}$ using RRT as before. Since E_1 is 1-nuclear and E_2 is 2-nuclear, the embedding from W into $\mathcal{H}$ is 2-nuclear. Then by Theorem 3.13, W has almost continuity property with respect to $\mathcal{S}_{e_i}$ and $\mathfrak{C}$ for each i. That is, for $\epsilon > 0$ and for each i, there exists a compact subset $\mathfrak{F}_i \subset \mathfrak{C}$ such that $\mu_e(\mathfrak{C}\backslash\mathfrak{F}_i) < \frac{\epsilon}{2^{i+1}}$, $\hat{G}_{e_i}(P[\mathcal{S}_e](\phi))$ is continuous on $\mathfrak{F}_i$ for any $\phi \in W$, and every open set in $\mathfrak{F}_i$ has positive measure. Let $\hat{\mathfrak{F}} = \cap_i \mathfrak{F}_i$. Then $\mu_e(\mathfrak{C}\backslash\hat{\mathfrak{F}}) < \frac{\epsilon}{2}$. Also, for any i, $\hat{G}_{e_i}(P[\mathcal{S}_e](\phi))$ is continuous on $\hat{\mathfrak{F}}$ for any $\phi \in W$.

For each $x \in \hat{\mathfrak{F}}$, define Q_x on W such that

$$[Q_x(\phi)](\psi) = \sum_{i=1}^{\infty} n_i \overline{\hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\phi)(x)} \cdot \hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\psi)(x)$$

for $\phi, \psi \in W$. We would like to have $Q_x(\phi) \in W'$, however that is not true for all $x \in \hat{\mathfrak{F}}$. Hence we shall construct a set in $\mathfrak{C}$, call it $\mathfrak{F}$, such that $Q_x(\phi) \in W'$. For $x \in \mathfrak{F}$, we must show that, for a fixed $\phi \in W$,

$$\sum_{i=1}^{\infty} n_i \overline{\hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\phi)(x)} \cdot \hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\psi)(x)$$

is continuous on W. Since W has the closed graph property, it is not difficult to see that $\hat{G}_{e_i}(P[\mathcal{S}_{e_i}](\cdot))$ is continuous on W. Hence we only need to show that the sum converges uniformly. We have

$$(\psi, \phi) = \int_{\hat{\mathfrak{F}}} \sum_{i=1}^{\infty} \overline{\hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\phi)(x)} \cdot \hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\psi)(x) \cdot n_i \, d\mu_e.$$

We also observe that

$$(\psi, \phi) = \sum_i (P[\mathcal{S}_{e_i}]\phi, P[\mathcal{S}_{e_i}]\psi) = \sum_i (P[\mathcal{S}_{e_i}]\phi, \psi) \le \|\phi\| \|\psi\|.$$

Hence the integral on the right hand side of the previous equation is bounded. Now consider the set

$$\big\{x \in \mathfrak{C} : \exists \phi \text{ such that } \sum_{i=1}^{\infty} n_i \overline{\hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\phi)(x)} \cdot \hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\psi)(x) = \infty \big\}$$

Then there exists a countable dense set $\{\theta_r\}$ of all such $\{\phi\}$ from he above set. By Theorem 3.12, there exists a constant β such that

$$\sum_r (P(\Delta_r)\theta_r, \phi) \le \beta \|\phi\| \tag{3.5}$$

for any disjoint family of $\{\Delta_r\}$ of Borel subsets in $\mathfrak{C}$. Let $M \in \mathbb{N}$ such that $M \ge 4\beta\|\phi\|/\epsilon$, and define

$$\Delta_r = \big\{x \in \mathfrak{C} : \sum_{i=1}^{N} n_i \overline{\hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\phi)(x)} \cdot \hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\theta_r)(x) \ge M \text{ for some } N \big\}.$$

Then by (3.5), we get

$$\begin{aligned} \beta\|\phi\| &\ge \sum_r (P(\Delta_r)\theta_r, \phi) \\ &= \sum_r \int_{\Delta_r} \sum_{i=1}^{N} \overline{\hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\phi)(x)} \cdot \hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\psi)(x) \cdot n_i \, d\mu_e \\ &\ge M \mu_e(\cup_r \Delta_r), \end{aligned}$$

hence

$$\mu_e(\cup_r \Delta_r) \le \frac{\beta\|\phi\|}{M} \le \frac{\epsilon}{4}. \tag{3.6}$$

Let $V = \mathfrak{C} \backslash (\cup_r \Delta_r)$. Define $\tilde{\mathfrak{F}}$ as a compact set contained in V such that $\mu_e(V \backslash \tilde{\mathfrak{F}}) < \frac{\epsilon}{4}$. Then, on $\tilde{\mathfrak{F}}$, we have

$$\sum_{i=1}^{\infty} n_i \overline{\hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\phi)(x)} \cdot \hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\theta_r)(x) < M.$$

Also, by (3.6), $\mu_e(\mathfrak{C} \backslash \tilde{\mathfrak{F}}) < \frac{\epsilon}{2}$. Now let $\mathfrak{F} = \hat{\mathfrak{F}} \cap \tilde{\mathfrak{F}}$. Then $\mu_e(\mathfrak{C} \backslash \mathfrak{F}) < \epsilon$, $\hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\phi)$ is continuous on $\mathfrak{F}$ for any $\phi \in W$, and the sum is uniformly convergence for $x \in \mathfrak{F}$. Hence $Q_x \in C(W, W')$.

Define now $g : \mathfrak{F} \to C(W, W')$ such that $g(x) = Q_x$. We shall show that g is a generalized eigenfamily for $W, \mathcal{C}, \mathfrak{F}$, i.e., by Definition 3.6, we must show that

(a) g is continuous;
(b) for $A \in \mathcal{C}$, $A = \int \overline{H(A)} Q_x \, d\mu_e$;

(c) each Q_x is a generalized eigenprojection of $\mathcal{C}$, corresponding to the eigenvalue $H(A)(x)$.

(a) This follows by the similar argument as in Theorem 3.8.

(b) Let $A \in \mathcal{C}$. Since $H(A) = G(A)$, by simple substitution we get

$$(\psi, A(\phi)) = \int \overline{H(A)(x)}[Q_x(\phi)](\psi)\, d\mu_e.$$

(c) We must show that, for each fixed $x \in \mathfrak{F}$,

(i) there exists $\{P_n\}$ of projections in $\mathcal{A}$ and $\{r_n\}$ of real numbers such that $r_n P_n \to Q_x$ in $C(W, W')$;

(ii) for each $\epsilon > 0$, $A \in \mathcal{C}$, there is a spectral projection $E(G(A)(x), \lambda_{G(A)(x)}, \epsilon)$ such that there exists $N \in \mathbb{N}$ with $E(G(A)(x), \lambda_{G(A)(x)}, \epsilon)P_n = P_n$ for any $n > N$, where P_n is the projection in i.

(i) Let $x \in \mathfrak{F}$ and $\{\Omega_n\}$ be the sets containing x such that $\Omega_{n+1} \subset \Omega_n$ for all $n \in \mathbb{N}$. Define

$$P_n = \sum_i \hat{G}_{e_i}^{-1} T_{\chi\Omega_n} \hat{G}_{e_i} P[\mathcal{S}_{e_i}].$$

Since $\hat{G}_e \mathcal{C} \hat{G}_e^{-1} = \{T_f | f \in C(\mathfrak{C})\}$, we get $P_n \in \mathcal{C}$. Let $r_n = \frac{1}{\mu_e(\Omega_n)}$. Then

$$\begin{aligned}
(\psi, r_n P_n(\phi)) &= \sum_{i=1}^{\infty} \Big(\hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\psi), r_n \hat{G}_{e_i}(\hat{G}_{e_i}^{-1} T_{\chi\Omega_n} \hat{G}_{e_i} P[\mathcal{S}_{e_i}]\phi)\Big) \\
&= \sum_{i=1}^{\infty} \Big(\frac{n_i}{\mu_e(\Omega_n)} \int_{\Omega_n} \overline{\hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\phi)(x)} \cdot \hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\psi)(x)\, d\mu_e \\
&\quad + \frac{n_i}{\mu_e(\Omega_n)} \int_{\Omega_n} \Big(\overline{\hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\phi)(s)} \hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\psi)(s) \\
&\quad - \overline{\hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\phi)(x)} \hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\psi)(x)\Big)\, d\mu_e\Big).
\end{aligned}$$

The second integral approaches zero as $n \to \infty$. Hence

$$(\psi, r_n P_n(\phi)) \to \sum_{i=1}^{\infty} \frac{n_i}{\mu_e(\Omega_n)} \int_{\Omega_n} \overline{\hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\phi)(x)} \hat{G}_{e_i}(P[\mathcal{S}_{e_i}]\psi)(x)\, d\mu_e = [Q_x(\phi)](\psi).$$

Thus $r_n P_n \to Q_x$ as $n \to \infty$.

(ii) Let $A \in \mathcal{C}$. Then there exists $f \in C(\mathfrak{C})$ such that $G(A)(x) = f(x)$ for all $x \in \mathfrak{C}$. Let $\epsilon > 0$. Let $E(A, \lambda_A, \epsilon)$ be the spectral projection of A to $\{y : |y - f(x)| < \epsilon\}$. For $\phi \in W$, we have

$$\begin{aligned}
E(A, \lambda_A, \epsilon) P_n(\phi) &= E(A, \lambda_A, \epsilon) \sum_i \hat{G}_{e_i}^{-1} T_{\chi\Omega_n} \hat{G}_{e_i} P[\mathcal{S}_{e_i}](\phi) \\
&= \sum_i E(A, \lambda_A, \epsilon) \hat{G}_{e_i}^{-1} \chi_{\Omega_n} \hat{G}_{e_i} P[\mathcal{S}_{e_i}](\phi).
\end{aligned}$$

Pick the smallest integer N such that $f(\Omega_n) \subset \{y : |y - f(x)| < \epsilon\}$. Then, for $n > N$, we have

$$\begin{aligned}
E(A, \lambda_A, \epsilon) P_n(\phi) &= \sum_i E(A, \lambda_A, \epsilon) \hat{G}_{e_i}^{-1} \chi_{\Omega_n} \hat{G}_{e_i} P[\mathcal{S}_{e_i}](\phi) \\
&= \sum_i \hat{G}_{e_i}^{-1} T_{\chi\Omega_n} \hat{G}_{e_i} P[\mathcal{S}_{e_i}](\phi) = P_n(\phi).
\end{aligned}$$

□

3.2. **Existence of a Generalized Eigenfamily on the Joint Spectrum.** The expansion in the previous section was done in the maximal ideal space, although most of the applications have the expansions on the joint spectrum. In this section, we shall consider the existence of a generalized eigenfamily for a family of bounded operators with finitely many unbounded operators on the joint spectrum. Define the following:

$\{A_i\}$ is the a family of commuting normal operators in $\mathcal{H}$ which finitely many of them, say $i = 1, \dots, N$, are unbounded

$\mathcal{A}$ is the smallest von Neumann algebra which contains bounded $\{A_i\}$ and with which $\{A_i\}_{i=1}^N$ are affiliated.

e is a cyclic vector (in $\mathcal{H}$) for $\mathcal{A}'$.

$\mathfrak{A}$ is the maximal ideal space of the above $\mathcal{A}$ with measure μ_e

$G_{\mathfrak{A}} : \mathfrak{A} \to C(\mathfrak{A})$ is the Gelfand transform of $\mathfrak{A}$.

Note that the domain of $G_{\mathfrak{A}}(A_j)$ for $j = 1, \dots, N$ is the complement of the meagre set S_j and each meagre set has spectral measure zero with respect to any cyclic vector for $\mathcal{A}'$ ([7], [8]). Hence we shall use the compliment of these meagre sets instead of the entire $\mathfrak{A}$, i.e., $\mathfrak{A} := \mathfrak{A}\backslash(\cup_{j=1}^N S_j)$.

Definition 3.15. The *joint spectrum* $\mathfrak{J}$ of $\mathcal{A}$ is the closure of $\{G_{\mathfrak{A}}(A_i)(x)\}$ in the product space of the spectra of A_i. Notice here that $\mathfrak{J}$ is a locally compact metric space.

We define a measure on $\mathfrak{J}$ as follows. Let $a_i = G_{\mathfrak{A}}(A_i)$. Define $F : \mathfrak{A} \to \mathfrak{J}$ such that $F(x) = (a_1(x), a_2(x), \dots)$ and $h_i : \mathfrak{J} \to \mathbb{C}$ such that $h_i(y_1, y_2, \dots, y_i, \dots) = y_i$. Then $\sigma_e(\Delta) := \mu_e(F^{-1}(\Delta))$, where Δ is a Borel set, is the measure on $\mathfrak{J}$.

With the above set ups, we have the main theorem.

Theorem 3.16. Let $\mathcal{A}, \mathfrak{A}, G_{\mathfrak{A}}$, and $\mathfrak{J}$ be given as above. Suppose there exist a locally convex topological vector space W and a Banach space V such that $W \subset V \subset \mathcal{H}$ all dense, the embedding $E_1 : V \to \mathcal{H}$ is 1-nuclear, and the embedding $E_2 : W \to V$ is 2-nuclear. Assume also that $A \in \mathcal{A}$ takes W into W continuously. Then, for any cyclic vector e for $\mathcal{A}'$ and for any $\epsilon > 0$, there exists a compact set $\mathfrak{G} \subset \mathfrak{J}$ such that $\sigma_e(\mathfrak{J}\backslash\mathfrak{G}) < \epsilon$ and there exists a generalized eigenfamily for $W, \mathcal{A}, \mathfrak{G}$.

Proof. In order to show the existence of a generalized eigenfamily for the joint spectrum, we must construct an isometric isomorphism from $\mathcal{A}$ to $L_\infty(\mathfrak{J}, \sigma_e)$. The rest of the proof follows by the same construction as in Theorem 3.14.

Let K be a compact subset in $\mathfrak{J}$ and χ_K be the characteristic function on K. Then $\chi_K \circ F$ is a uniquely defined characteristic function of a clopen set of $\mathfrak{A}$. Hence there exists a projection $P[K]$ in $\mathcal{A}$ such that $G_{\mathfrak{A}}(P[K]) = \chi_K \circ F$. Now define $\hat{G}$ on $\mathcal{A}$ such that $\hat{G}(P[K]A_i) = \chi_K \circ h_i$ for $A_i \in \mathcal{A}$, where the domain of $\chi_K \circ h_i$ is $\{F(x) | x \in \mathfrak{A}\} \subset \mathfrak{J}$. Then $\hat{G} : \mathcal{A} \to C(\mathfrak{J})$. Let $P(y_1, y_2, \dots, y_n)$ denote any polynomial in $y_1, \dots, y_n$. Define

$$\hat{G}_{\mathfrak{J}}(P[K]P(A_1, \dots, A_n)) = \chi_K \circ P(h_1, \dots, h_n).$$

Then $\hat{G}_{\mathfrak{J}}$ is an isometry from $P_K\mathcal{A}$ into $L_\infty(K)$. Then, we can extend $\hat{G}_{\mathfrak{J}}$ to a unitary operator taking $\mathcal{H}$ onto $L_2(\mathfrak{J})$. Hence $\mathcal{A}$ is isometric to L_∞. □

The main theorem gives the generalized eigenfunction expansion in $C(W,W')$ space. We now extend the theorem to $C(Z,B)$ where Z is a Hilbert space and B is a Banach space, so that we can deal with actual uniform convergence.

Let Z be a Hilbert space such that $W \subset Z \subset \mathcal{H} = \mathcal{H}' \subset Z' \subset W'$ with each embedding to the next is continuous and each space is dense in the next. Assume that the embedding of Z into $\mathcal{H}$ is 2-nuclear. Let B be a Banach space such that $Z \subset B \subset Z'$ and the embedding of Z into B is 1-nuclear. Assume that each generalized eigenprojection $Q_x \in C(W,W')$ extends to a bounded linear transformation $\hat{Q}_x \in C(Z,Z')$. Suppose that if $\psi \in Z' \subset W'$ and $A_i^t\psi \in Z'$ for some i, where A_i^t is the transpose of $A_i \in \mathcal{A}$, then $\psi \in B$, and for the same i, $Z \subset D(A_i^2) =$ domain of A_i^2. Then:

Theorem 3.17. *With the above conditions on B and Z, and with $\mathcal{A}, \mathfrak{A}, G_{\mathfrak{A}}$, and $\mathfrak{J}$ given as above, the statement of Theorem 3.16 holds for $Z, \mathcal{A}, \mathfrak{G}$. Moreover, the generalized eigenprojections are in $C(Z,B)$ and thus the expansion converges in $C(Z,B)$.*

Corollary 3.18. *For every $\epsilon > 0$, there exists a compact subset $\mathfrak{G}$ of the joint spectrum and a positive constant δ with the following properties: for every δ-net $\{\xi_i\}$ of $\mathfrak{G}$,*

(1) *there exists a set of generalized eigenfunctions F_i (defined on Definition 3.4) such that*
$$A_n^t F_{\xi_i} = (\xi_i)_n F_{\xi_i}$$
where A_n^t is a transpose of $A_n \in \mathcal{A}$;

(2) *there exists a set of complex constants $\{c_i\}$ such that, for every θ in the unit ball of W,*
$$\Big\|\theta - \sum_{i=1}^{n} c_i F_{\xi_i}(\theta) F_{\xi_i}\Big\| < \epsilon.$$
Note here that $\{c_i\}$ works for every θ in the unit ball (by applying our theorem to the identity operator in the algebra) because our expansion converges in $C(Z,B)$ with the usual operator norm topology for Z and B.

4. Comparison to Other Approaches

One of the major differences between the previous works on a generalized eigenfunction expansion and our approach is that we use the limit of the products of spectral projections and some real numbers instead of using the elements in the dual space. To illustrate the difference, let us compare with the approach given by Maurin in [10]. First, Maurin recall the complete spectral theorem for a single hermitian operator A in a finite dimensional Hilbert space $\mathcal{H}$. In this case, he identifies the spectrum Λ of A with the eigenvalues of A (i.e., the elements in the maximal ideal space generated by A in our approach) and then defines the set $\hat{H} = \{\hat{x} : \hat{x}(\lambda) = (x, e(\lambda))\}$ where $x \in \mathcal{H}$, $\lambda \in \Lambda$, and $\{e(\lambda)\}$ is an orthonormal set of eigenvectors of A (which spans $\mathcal{H}$). The theorem states that $\{e(\lambda)\}$ determines a unitary transformation from $\mathcal{H}$ onto $\hat{H}$. An extension of the theorem to a commutative family $\mathcal{C}$ of normal operators in $\mathcal{H}$ is also given: Let Λ be the spectrum of $\mathcal{C}$. Then there exists a direct integral $\hat{H} = \int_\Lambda \hat{H}(\lambda)\, d\mu(\lambda)$ ($= \{\hat{x} : \Lambda \to \hat{H}(\lambda) : \hat{x}$ square integrable vector field $\}$) and there exists a unitary map $F : \mathcal{H} \to \hat{H}$ such that $(FAx)_k(\lambda) = \hat{A}(\lambda)\hat{x}_k(\lambda)$ for $k = 1, 2, \dots, \dim\hat{H}(\lambda)$ and

$A \in \mathcal{C}$, where $\hat{A}(\lambda)$ is the element in the maximal ideal space which identified with λ. Now he gives the fundamental theorem: Let W be a dense linear subset of $\mathcal{H}$ such that the embedding is nuclear. Then there exists a transform $F : \mathcal{H} \to \hat{H}$ such that $F(\phi) = (\phi, e_k(\lambda))$ for $k = 1, 2, \ldots, \dim \hat{H}(\lambda)$ and for $\phi \in W$, where $e_k(\lambda) \in W'$, and $F(\lambda) : W \to \hat{H}(\lambda)$ is continuous; if $A(\phi) \in W$, then $(FA\phi)_k(\lambda) = \hat{A}(\lambda)\hat{\phi}_k(\lambda)$, i.e., $(A\phi, e_k(\lambda)) = (\phi, \hat{A}(\lambda)e_k(\lambda))$ for almost all λ. From this fundamental theorem, he states that the spectral synthesis is given by

$$P\phi = \int_\lambda \sum \hat{\phi}_k(\lambda) e_k(\lambda)\, d\mu(\lambda), \tag{4.1}$$

where $\hat{\phi}_k(\lambda) = (\phi, e_k(\lambda))$ and $P : W \to W'$ is an antilinear map.

Notice that the fundamental theorem given here uses the elements $e_k(\lambda) \in W'$ (called generalized eigenelements in [10]) which are simultaneous eigenvectors for $\mathcal{C}$; hence he uses the spectral projections. In our approach, we use Q_x, the limit of the products of spectral projections and real numbers (we called a generalized eigenprojection in Definition 3.2) which is in $C(Z, B)$. Hence Maurin's convergence of the integral (4.1) is in W' and thus point-wise convergence whereas our convergence of the integral in Definition 3.6 is in $C(Z, B)$, which can not be easily deduced from Maurin's theorem since it is a different construction. Also, since we use the limit of the products, our approach can be used even when the spectral projections tend to be zero. This covers more cases than the previous approach. Another observation is that Maurin's approach does not contain asymptotics of the generalized eigenfunctions (eigenelements) thus no convergence information whereas our approach contains some information on the asymptotics of the generalized eigenfunctions because they are the elements in the range of the generalized eigenprojection. This fact (i.e., the generalized eigenfunctions are in the range of the generalized eigenprojection) is needed in order to prove some asymptotic behaviour of the eigenfunctions in a number theory application (5.2.1, [9]). As for a single self-adjoint operator, a systematic apparatus for calculating asymptotics of the generalized eigenfunctions was given by Poerschke, Stolz, and Weidmann in [11]. As for a family of operators, it does not appear to have been considered by earlier approaches.

5. Examples

5.1. **Fourier Transform.** The most well known example of a generalized eigenfunction expansion is the inverse Fourier transform. Let us illustrate how our theory works in this simple situation.

As we mentioned in the introduction, the Fourier transform may be derived from the spectral decomposition of the operator $i\dfrac{d}{dx}$ in $L_2(\mathbb{R})$. Since this operator has multiplicity one, $n\chi(\lambda - \frac{1}{n}, \lambda + \frac{1}{n})$ converges to $e^{-i\lambda x}$ for each λ. In this section, we consider the Laplacian $\mathcal{L} = \dfrac{d^2}{dx^2}$ on $W = C_0^\infty(\mathbb{R})$ with $\mathcal{H} = L_2(\mathbb{R})$. This operator has multiplicity two, and our theorem gives a generalized eigenprojection with $e^{i\lambda x}$ with two dimensional range and hence the inverse Fourier transform.

Consider commutative von Neumann algebra generated by $\mathcal{L}$ and the identity. Recall that $\mathcal{L}$ is unbounded on $\mathcal{H}$, thus the von Neumann algebra generated by $\mathcal{L}$ is the smallest von Neumann algebra with which $\mathcal{L}$ is affiliated (see [7]). Then $\mathcal{L}(f) = \lambda f$ if $f(x) = e^x$ or e^{-x} with $\lambda = 1$ and $f(x) = e^{ix}$ or e^{-ix} with $\lambda = -1$.

The Fourier transform $\hat{\phi}$ is given by

$$\hat{\phi}(t) = \int_{\mathbb{R}} \phi(x)e^{-ixt}\,dm(x)$$

where $dm(x) = \frac{1}{\sqrt{2\pi}}dx$). We know that the Fourier transform is the Gelfand transform ([12]), hence, if we let $G(\phi) = \hat{\phi}$, then G is an isometric isomorphism. Since

$$G(\mathcal{L}\phi)(t) = \int_{\mathbb{R}} \mathcal{L}\phi(x)e^{-ixt}\,dm(x) = -t^2\int_{\mathbb{R}} \phi(x)e^{-ixt}\,dm(x) = -t^2G(\phi)(t)$$

by integration by parts, let us define $H(\mathcal{L}) = -t^2$. Then H is isometric isomorphism on $\{\hat{\phi}\}$. In order to construct the spectral projection, consider $\lambda = 1$. Define $\omega_n = [1-\frac{1}{n}, 1+\frac{1}{n}]$, Borel subsets on $\mathbb{R}$. Then $H^{-1}(\omega_n) := \{x \in \mathbb{R}^+ | H(x) = -x^2 \in \omega_n\}$ is empty. Hence consider $\lambda = -1$. Define now $\omega_n = [-1-\frac{1}{n}, -1+\frac{1}{n}]$. Then $H^{-1}(\omega_n)$ is not empty this time. Hence define $\Delta_n = H^{-1}(\omega_n)$. (i.e., $\Delta_n = [\sqrt{\frac{n-1}{n}}, \sqrt{\frac{n+1}{n}}] \to 1$ if $n \to \infty$.) We now define a projection of $\mathcal{L}$ on $\{\hat{\phi}\}$ as $\tilde{E}(\Delta_n) = \chi_{\Delta_n}$ where χ_{Δ_n} is a characteristic function on Δ_n. Hence a spectral projection of $\mathcal{L}$ on L_2 can be defined by

$$E(\Delta_n) = G^{-1}\tilde{E}(\Delta_n)G. \tag{5.1}$$

One should notice here that we can not apply the spectral theorem to $\mathcal{L}$ using $E(\Delta_n)$ because $E(\Delta_n) \to 0$ as $n \to \infty$ (i.e., no expansion).

Now we shall construct the generalized eigenprojection using our theorems. Let $\epsilon > 0$. Define $\Delta = [-1-\epsilon, -1+\epsilon]$. By (5.1), $E(\Delta)$ is a spectral projection of $\mathcal{L}$ on L_2. Let $P_n = E(\Delta_n)$. Clearly, $E(\Delta)P_n = P_n$ for $n > N$ where $N := \lfloor \frac{1}{\epsilon} \rfloor$. Also, let $r_n = 1/\sqrt{\mu(\Delta_n)}$ where $\mu(\Delta_n)$ is the measure of Δ_n. Then $r_nP_n \to P_\lambda$ such that $P_\lambda(x) = e^{i\lambda x}$. Hence by Definition 3.6 and Theorem 3.16, we get

$$\mathcal{L}(\phi)(t) = \int_{\mathbb{R}} H(\mathcal{L})P_\lambda(\phi)(t)\,d\lambda = \int_{\mathbb{R}} -t^2\hat{\phi}(\lambda)e^{i\lambda t}\,d\lambda.$$

If we apply the identity in the von Neumann algebra, we get

$$\phi(t) = \int_{\mathbb{R}} \hat{\phi}(\lambda)e^{i\lambda t}\,d\lambda.$$

Hence we can consider the inverse Fourier transform as a generalized eigenfunction expansion of the Laplacian.

5.2. **Application in Number Theory.** In this section, we will apply our theorem to a family of the Hecke operators (defined below) and the Laplacian in the hyperbolic space. As a corollary, we will also get a uniform convergence of the expansion, which seems to be new. One can find details for this section in [1].

Definition 5.1. The set of all Möbius transform of the form

$$\tau' = \frac{a\tau + b}{c\tau + d},$$

where a, b, c, d are integers with $ad - bc = 1$, is called the *modular group* and denoted by Γ. The group can be represented by 2×2 integer matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\det A = 1$, provided we identify each matrix with its negative, since A and $-A$ represent the same transformation.

Definition 5.2. Let G be a subgroup of Γ. Two points τ and τ' in the upper half-plane H are said to be *equivalent* under G if $\tau' = A\tau$ for some A in G. An open subset R_G of H is called a *fundamental domain of* G if it has the following properties:

(1) No two distinct points of R_G are equivalent under G.
(2) If $\tau \in H$, there is a point τ' in the closure of R_G such that τ' is equivalent to τ under G.

Theorem 5.3. *The open set* $\mathbb{D} = \{\tau \in H : |\tau| > 1, |\tau + \overline{\tau}| < 1\}$ *is a fundamental domain of* Γ.

Definition 5.4. Let H be the upper half-plane and Γ be the modular group. We define the *Hecke operator* T_n *of order* n as

$$(T_n f)(\tau) = \frac{1}{n} \sum_{d|n} \sum_{b=1}^{d-1} f(\frac{n\tau + bd}{d^2})$$

where f is an automorphic function under Γ and $\tau \in H$.

Theorem 5.5. *Any two Hecke operators commute with each other.*

Definition 5.6. Let $\mathbb{H}$ be the hyperbolic space. The *Laplacian* $\mathcal{L}$ on $C_0^\infty(\mathbb{H})$ is defined by

$$\mathcal{L}f = -y^2(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}), \quad (x,y) \in \mathbb{H}.$$

Theorem 5.7. *The Hecke operators commute with the Laplacian. Also, there exists a commutative von Neumann algebra with which* $\mathcal{L}$ *is affiliated and in which the Hecke operators are.*

One interesting fact about the Hecke operators and the Laplacian is as follows: let T_p be the Hecke operator with a prime number p. Then we can consider T_p as a self adjoint operator from a Hilbert space $\mathcal{H}$ to $\mathcal{H}$ where $\mathcal{H}$ is the set of all modular forms with the power 2 and with

$$(f,g) = \int_{\mathbb{H}/\Gamma^0(\mathbb{N})} f(z)\overline{g(z)}\, \frac{dxdy}{y^2}.$$

Then $\mathcal{H}$ has a finite basis $\{f_1, \dots, f_r\}$, and the basis is the set of the simultaneous eigenfunctions of $\{T_p\}$ and the Laplacian.

Before we move on, let us state two theorems on how to construct 2-nuclear and 1-nuclear maps. These theorems are well known, thus we shall state without proofs ([8], [13]).

Theorem 5.8. *Let* X *be a locally compact Hausdorff space with a positive measure* μ *and* $T_0 : \mathcal{H} \to C(X)$ *be a bounded linear transformation into the set of bounded elements of* $C(X)$*, given supremum norm. Suppose* $T : \mathcal{H} \to L_2(X, \mu)$ *is given by*

$$T(g) = \theta \cdot T_0(g)$$

where θ *is a fixed element of* $L_2(X, \mu)$*. Then* T *is* 2*-nuclear.*

Theorem 5.9. *Let* D, J, M *be Hilbert spaces. Suppose* $T : D \to J$ *is* 2*-nuclear and* $S : J \to M$ *is also* 2*-nuclear. Then* $S \circ T : D \to M$ *is* 1*-nuclear.*

5.2.1. *Generalized Eigenfunction Expansion.* We shall now apply our theorem to the family of Hecke operators and the Friedrichs extension of the Laplacian (instead of the Laplacian for simplification; abusing the notation, we call the Friedrichs extension $\mathcal{L}$ from here on) to get a generalized eigenfamily with the Hilbert space $L_2(\mathbb{D})$. In order to do so, we must construct the spaces W and V with the necessary properties.

Let $\mathcal{H} = L_2(\mathbb{D})$ (then $\mathcal{H} = \mathcal{H}'$). Let us first construct the 2-nuclear map from $\mathcal{H}$ to $\mathcal{H}$.

Proposition 5.10. *Let $\beta < 1/2$. Define $T : \mathcal{H} \to \mathcal{H}$ such that*
$$T(g) = y^\beta\big(\frac{1}{y}\mathcal{L}^{-1}(g)\big).$$
Then T is 2-nuclear.

Proof. We know that $\mathcal{L}$ is positive definite and $(\mathcal{L}(f), f) > \frac{1}{4}(f, f)$. Hence $\mathcal{L}^{-1}$ is bounded by $\frac{1}{4}$. Then $\frac{1}{y}\mathcal{L}^{-1}$ is also bounded and has L_∞ norm. Since $\beta < \frac{1}{2}$, we have
$$\int |y^\beta|^2 \frac{1}{y^2}\, dy\, dx = \int y^{2\beta-2}\, dy\, dx < \int y^{-1}\, dy\, dx < \infty,$$
hence $y^\beta \in L_2(\mathbb{D})$. By Theorem 5.8, T is 2-nuclear. □

To construct a space V such that the embedding from V into $\mathcal{H}$ is 1-nuclear, recall that $S := T \circ T$ is 1-nuclear by Theorem 5.9. Notice that
$$S(g) = y^{\beta-1}\mathcal{L}^{-1}(y^{\beta-1}\mathcal{L}^{-1}(g)).$$
Also note that $S(g) = y^{\beta-1}\mathcal{L}^{-1}(y^{\beta-1}\mathcal{L}^{-1}(g))$ implies $g = \mathcal{L}(y^{1-\beta}\mathcal{L}(y^{1-\beta}S(g)))$. Hence define a space $B = C_0^\infty(\mathbb{D})$ with the norm
$$\|f|B\| = \|\mathcal{L}(y^{1-\beta}\mathcal{L}(y^{1-\beta}f))|L_2(\mathbb{D})\|$$
where $\|f|B\|$ denotes the norm of f in the space B. Let V_0 be the completion of B. Then the embedding from V_0 into $\mathcal{H}$ is 1-nuclear. In fact,
$$\begin{aligned}
\|S(f)|V_0\| &= \|\mathcal{L}(y^{1-\beta}\mathcal{L}(y^{1-\beta}S(f)))|L_2(\mathbb{D})\| \\
&= \|\mathcal{L}(y^{1-\beta}\mathcal{L}(y^{1-\beta}y^{\beta-1}\mathcal{L}^{-1}(y^{\beta-1}\mathcal{L}^{-1}(f)))|L_2(\mathbb{D})\| \\
&= \|f|L_2(\mathbb{D})\|.
\end{aligned}$$
Since S is 1-nuclear, the embedding is also 1-nuclear. Although V_0 satisfies the required embedding property, let us simplify a little. Define a space $D = C_0^\infty(\mathbb{D})$ with the norm
$$\|f|D\| = \|\mathcal{L}^2(y^{2(1-\beta)}f)|L_2(\mathbb{D})\|$$
Let V be the completion of D. Then it is not difficult to see that V_0 and V are equivalent (i.e., two norms are equivalent), hence the embedding from V into $\mathcal{H}$ is 1-nuclear. Note here that V is also invariant under $\mathcal{L}^{-1}$.

We now construct a space W such that the embedding from W into V is 2-nuclear. Let $X = C_0^\infty(\mathbb{D})$ with the norm
$$\|f|X\| = \|\mathcal{L}(y^{1-\beta}f)|V\|$$
Let W_0 be the completion of X. Then the embedding from W_0 into V is 2-nuclear by the same argument as above. As before, define another space $Y = C_0^\infty(\mathbb{D})$ with the norm
$$\|f|Y\| = \|\mathcal{L}^3(y^{3(1-\beta)}f)|L_2(\mathbb{D})\|,$$

and let W be the completion of Y. Again, W_0 and W are equivalent, hence the embedding from W into V is 2-nuclear. W is also invariant under $\mathcal{L}^{-1}$.

We apply Theorem 3.14 to get the following result.

Corollary 5.11. *Let $\mathcal{C}$ be the C^*-algebra generated by the Hecke operators, $\mathcal{L}$, and the identity. Also, let W, V, and $\mathcal{H}$ be given as above. Then, for any $A \in \mathcal{C}$, the generalized eigenfunction expansion converges in $C(W, W')$.*

Remarks for Corollary 5.11. We first note that, if $\Phi \in W'$, then by RRT there exists $\phi \in L_2(\mathbb{D})$ such that

$$\Phi(\theta) = (\phi, \mathcal{L}^3(y^\alpha \theta)) = (y^\alpha \mathcal{L}^3(\phi), \theta) \quad \text{for } \theta \in W$$

where $\alpha = 3(1-\beta) > \frac{3}{2}$ ($\beta < \frac{1}{2} \Rightarrow -\beta > -\frac{1}{2}$). Hence Φ is isometrically isomorphic to $y^\alpha \mathcal{L}^3(\phi)$. This gives us some Information on the space $C(W, W')$ in which the expansion converges. If $\Phi \in W'$ also satisfies $\mathcal{L}(\Phi) = \lambda \cdot \Phi$ for some λ, one can show that W' is isometrically isomorphic to $y^\alpha L_2(\mathbb{D})$. It is known that the eigenfunctions of the Laplacian behaves like $y^{\frac{1}{2}+\epsilon}$ for any $\epsilon > 0$ (that is $y^{\frac{1}{2}+\epsilon} L_\infty(\mathbb{D})$ instead of $y^\alpha L_2(\mathbb{D})$ in our result). If we use the fact that the multiplicity of the Hecke operators and the Laplacian is one, then it seems possible to show the same result using our theory ([9]). One should note that the multiplicity of the Laplacian itself on the cusp space is not known.

In number theory, we define the cusp space $L_{2,c}$ to be the set of automorphic functions $f \in L_2(\mathbb{D})$ such that $f_0(y) = 0$ where $f_0(y)$ is the term independent of x in the Fourier series

$$f(x, y) = \sum_{n=-\infty}^{\infty} f_n(y) \exp(2\pi i n x).$$

We also define the Eisenstein space $L_{2,E}$ to be the set of automorphic functions $f \in L_2(\mathbb{D})$ orthogonal to the cusp space, i.e., $L_2 = L_{2,c} + L_{2,E}$. Then it is known that, for $\psi \in C_0^\infty(\mathbb{D})$,

$$(\cdot, \psi) = \frac{1}{4\pi} \int_{t=-\infty,\ s=1/2+it}^{\infty} (\psi, E(z, s))(E(z,\ s), \cdot)\, dt + \sum_{i=1}^{\infty} (\psi, f_i)(f_i, \cdot) \tag{5.2}$$

where $E(z, s)$ is an Eisenstein series and f_i is an orthogonal basis (which is also an eigenfunction of $\mathcal{L}$: [6]). Applying our theory (Theorem 3.14) to the identity operator I in $\mathcal{C}$, we get

$$(\cdot, I(\psi)) = \int \overline{H(I)(x)} [Q_x(\psi)(\cdot)]\, d\mu_e. \tag{5.3}$$

Since both (5.2) and (5.3) use the same spectral decomposition, they are the same. However, there is a major difference for the convergence. In number theory, the convergence of the integral (5.2) is shown for one function at a time, i.e., one fixes a function ψ and shows the convergence. (Since we consider ψ as a "point" in W, it is like "point-wise" convergence.) In our theory, we are not expanding a function ψ but expanding the identity operator in the algebra generated by T_n and $\mathcal{L}$. Hence we know the integral converges in $C(W, W')$ by Corollary 5.11. By using the same set ups as in Theorem 3.17, we can also show the integral converges in $C(Z, B)$; that means the integral converges for every ψ in the unit ball of Z. Hence we get some sort of uniform convergence. Also, since we do not use anything specific about the fundamental domain of Γ, we believe that we can apply our method to so-called modified Hecke operators (i.e., Hecke operators defined on a set of automorphic functions under a subgroup of Γ). Nothing about expansions for those operators seems to be known.

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