ON THE BOUNDS OF THE DOMINATION NUMBER OF PERMUTATION GRAPHS

THESIS

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CHAPTER 1

INTRODUCTION AND BACKGROUND

Within the last thirty five years, the area of graph theory in mathematics has seen extensive growth in both research and in the classroom. The importance of the subject is directly due to its applications in fields such as computer science, operations research, and communications networking. Furthermore, within this vast subject has grown an increasing interest in the study of domination in graphs. If we define a simple, finite graph $G = (V(G), E(G))$, we abbreviate $V(G)$ and $E(G)$ to V and E, respectively. Thus, for any graph G, a set S of vertices in G is called a dominating set if every vertex $v \in G$ is either an element of S or is adjacent to an element of S. The domination number of G, denoted by $\gamma(G)$, is then the minimum cardinality of a dominating set of G.

In 1998, Haynes, Hedetniemi and Slater [4] published the first comprehensive book on the domination of graphs. In this book are listed a few great examples regarding the applications of dominating sets. One such example is to model a computer network by a graph G where the vertices represent the computers and the edges represent the communication lines between different pairs of the computers. Therefore, if a time delay became an issue when collecting data for the entire network, a dominating set of computers could be chosen so as to minimize the number of computers needed to collect data from the entire network. Another example used is to model a street map of a city by a graph G where each edge represents one city block. Thus, a dominating set for G would include the street intersections on a bus route so that each child that lived in the neighborhood would be required to walk no more than 2 blocks to a bus stop. This example reflects the fact that domination has applications in fields outside of science. Also, because the dominating set of a graph is of minimum cardinality, when applied to areas such as computer or social networking, a dominating set corresponds to minimizing spending or time consumption. Therefore, bounds on the domination number has become a primary focus in graph theory.

Another area of particular interest in graph theory is permutation graphs. Let G be a simple graph with labeled vertices and α be a permutation from the vertex set of G to a copy of the vertex set of G, denoted by G'. Then the permutation graph, denoted by $P_\alpha(G)$, can be obtained by connecting all vertices of G to all vertices of G' according to α. For example, let $G = P_3$ be a path of length 3 and let α be the identity permutation. That is, for all $v \in V(G)$, $\alpha(v) = v'$. Then, $P_\alpha(G)$ is shown below in Figure 1.1.

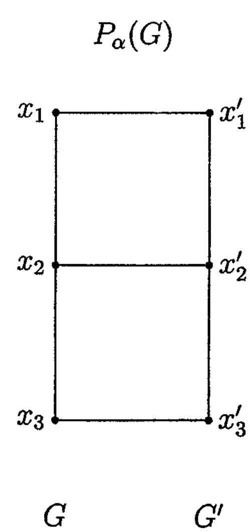

Figure 1.1: $P_\alpha(G)$: Graph of a path of length 3 with the identity permutation

Finding a permutation between two identical graphs impacts networking problems significantly. Take the computer network example from above, only consider having two identical graphs representing networks in different areas. It would be much more efficient if for every computer in the first graph, there was a communication line to a computer in the second graph. Ideally, we would be able to find a specific permutation so as to reduce the dominating set for the permutation graph, rather than treating the two networks separately, which would directly reduce time delays/spending. Therefore, the focus of this thesis is on the bounds of the domination number of permutation graphs.

It has been shown in [5] before that for any permutation α on the vertex set of any graph G,

$$\gamma(G) \leq \gamma\big(P_\alpha(G)\big) \leq 2\gamma(G).$$

However, we find it to be beneficial if we could tighten these bounds for specific types of graphs. For example, it was posed in 1999 by Gu, [3], that $\gamma(G) = \gamma\big(P_\alpha(G)\big)$ for any permutation α on $V(G)$ if and only if G consists of isolated vertices. Since then, several papers have shown that $\gamma(G) \neq \gamma\big(P_\alpha(G)\big)$ for some types of special graphs. For example, it was shown by Gibson [2] that if G is a connected bipartite graph, then the above statement is true and Burger [1] later proved that this is also true for regular graphs. However, the conjecture itself remains open. Thus, our primary goal is to prove the validity of the conjecture, consequently classifying the only graphs that meet the lower bound on the domination number for any permutation imposed on the vertex set.

With regards to the upper bound on the domination number, it was shown in [3] that complete graphs, K_n, satisfy the upper bound, or for any permutation α,

$$\gamma(P_\alpha(K_n)) = 2\gamma(K_n).$$

However, we intend to research other such graphs for which the above equality is true.

Furthermore, notice that $\gamma(P_\alpha(G))$ can take on various values within its bounds. Thus, in Chapter 3, we shall investigate two different scenarios. First, if we are given positive integers a, b such that $a \leq b \leq 2a$, is it possible to find a graph with a permutation α on $V(G)$ such that $\gamma(G) = a$ and $\gamma(P_\alpha(G)) = b$? Secondly, if we are given a specific graph where $\gamma(G) = a$, is it possible to find a permutation on $V(G)$ such that $\gamma(P_\alpha(G)) = b$ for all values of b such that $a \leq b \leq 2a$?

Definitions and Notations

Given a simple graph G, for any vertex $v \in G$, the degree of v in G, or $\deg_G(v)$, is the number of vertices adjacent to v and $\Delta(G)$ is the maximum degree of any vertex in G. Notice that the degree of any vertex $v \in V(G)$ in $P_\alpha(G)$, denoted by $\deg_{P_\alpha(G)}(v)$, is $\deg_G(v) + 1$. Therefore, for all $v \in V(G)$, it is imperative to distinguish the degree of v with respect to G from the degree of v with respect to $P_\alpha(G)$. Also, for $S \subset V(G)$, the number of vertices of S is denoted $|S|$ and the number of vertices in G, $|V(G)|$, is called the order of G.

For any vertex $v \in G$, the neighborhood set of v, $N(v)$, is the set of vertices adjacent to v. We shall use the notation $N[v]$ to denote the closed neighborhood of v, $N(v) \cup \{v\}$. Also, it is important to distinguish the neighborhood set of v with respect to G, denoted

$N_G(v)$, from the neighborhood set of v with respect to $P_\alpha(G)$, denoted $N_{P_\alpha(G)}(v)$. Note that $|N_{P_\alpha(G)}(v)| = |N_G(v)| + 1$.

The complement of a graph G, denoted by $\overline{G}$, is a graph on the same vertices such that two vertices of $\overline{G}$ are adjácent if and only if they are not adjacent in G. By this definition, the complement of a complete graph, $\overline{K_n}$, is a graph consisting of n isolated vertices.

Let $H = \{a_1, a_2, ..., a_n\}$. We shall use (1) to denote the identity permutation on a finite set H and use $\alpha = (a_1 a_2 ... a_k)$ to denote a cyclic permutation, that is,

$$\alpha(v) = \begin{cases} a'_{i+1} & \text{if } v = a_i, 1 \leq i \leq k-1 \\ a'_1 & \text{if } v = a_k \\ v' & \text{otherwise} \end{cases}$$

Additionally, we shall use $\alpha = (a_1 a_2 ... a_k)(b_1 b_2 .. b_l)$ to denote the product of two cycles,

$$\alpha(v) = \begin{cases} a'_{i+1} & \text{if } v = a_i, 1 \leq i \leq k-1 \\ a'_1 & \text{if } v = a_k \\ b'_{i+1} & \text{if } v = b_i, 1 \leq i \leq l-1 \\ b'_1 & \text{if } v = b_k \\ v' & \text{otherwise} \end{cases}$$

CHAPTER 2

BOUNDS ON THE DOMINATION NUMBER

In this chapter, we shall investigate the bounds on $\gamma\big(P_\alpha(G)\big)$ for all graphs and all possible permutations on $V(G)$. It has been shown informally that

$$\gamma(G) \leq \gamma\big(P_\alpha(G)\big) \leq 2\gamma(G).$$

Therefore, we shall first show that this inequality is in fact true.

Theorem 1. *Let G be a graph. Then for any permutation α on $V(G)$,*

$$\gamma(G) \leq \gamma\big(P_\alpha(G)\big) \leq 2\gamma(G).$$

Proof. We shall prove the lower bound by contradiction. Suppose that $\gamma(G) = n$ and there exists a permutation α such that $\gamma\big(P_\alpha(G)\big) \leq n-1$. Let S be a dominating set for $P_\alpha(G)$ where $|S| \leq n-1$. Notice that $S = A \cup B$ where $A \subseteq V(G)$ and $B \subseteq V(G')$. Thus, the number of vertices in B must dominate the number of vertices in G not dominated by A. Since for all $v' \in V(G')$, v' is a copy of $v \in V(G)$, let

$B^* = \{v \in V(G) | \alpha(v) \in B\}$. Then, $A \cup B^*$ is a dominating set for $P_\alpha(G)$ and

$$\gamma(G) \leq |A \cup B^*| = |A \cup B| \leq n - 1,$$

contradicting the assumption that $\gamma(G) = n$. Thus, $\gamma(G) \leq \gamma\big(P_\alpha(G)\big)$. To prove the upper bound, let $\gamma(G) = n$. Then there exists a dominating set S for G where $|S| = n$. Notice that $S \cup S'$ completely dominates $V(G)$ and $V(G')$ and $|S \cup S'| = 2n$. Thus, $S \cup S'$ is a dominating set for $P_\alpha(G)$ and hence $\gamma\big(P_\alpha(G)\big) \leq 2\gamma(G)$. □

Theorem 1 provides bounds for the domination number of $P_\alpha(G)$. However, it is important to know whether both lower and upper bounds are achievable. The following lemma shows the lower bound is sharp.

Lemma 1. *Let* $G = \overline{K_n}$. *Then* $\gamma(G) = \gamma\big(P_\alpha(G)\big) = n$, *for any permutation* α *on* $V(G)$.

Proof. It is clear that $\gamma(\overline{K_n}) = n$. Since for any permutation α, $P_\alpha(\overline{K_n})$ consists of n nonadjacent edges, $\gamma\big(P_\alpha(\overline{K_n})\big) = n$. Thus, $\gamma(\overline{K_n}) = \gamma\big(P_\alpha(\overline{K_n})\big) = n$. □

In 1999, Dr. Weizhen Gu posited the following conjecture:

Conjecture: *Let G be a graph. Then* $\gamma(G) = \gamma\big(P_\alpha(G)\big)$ *for all permutations on* $V(G)$ *if and only if* $G = \overline{K_m}$.

As we have seen from Lemma 1, if G is a graph consisting only of isolated vertices, then $\gamma(G) = \gamma\big(P_\alpha(G)\big)$ for all α. Thus, one direction of the conjecture is very obvious. However, proving the converse is much more difficult because not only must all possible graphs be considered, but all possible permutations on the graph's vertex set as well. In order to avoid this, we have chosen to prove the following equivalent statement: If

$G \neq \overline{K_m}$, then there exists a permutation α on $V(G)$ such that $\gamma(P_\alpha(G)) > \gamma(G)$. We first look at this statement when $\gamma(G) = 1$ or 2.

Lemma 2. *Let G be a graph with $\gamma(G) = 1$. If $|V(G)| \geq 2$, then $\gamma(P_\alpha(G)) = 2$ for any permutation α on $V(G)$.*

Proof. Suppose there exists a permutation α such that $\gamma(P_\alpha(G)) = 1$. Then, there must exist a vertex $a \in V(P_\alpha(G))$ such that $|N_{P_\alpha(G)}[a]| = |V(P_\alpha(G))|$. However, for any vertex $v \in V(P_\alpha(G))$,

$$|N_{P_\alpha(G)}[v]| \leq |N_G[v]| + 1 \leq |V(G)| + 1 < |V(G)| + |V(G)| = 2|V(G)|.$$

Therefore,

$$2|V(G)| = |V(P_\alpha(G))| = |N_{P_\alpha(G)}[a]| < 2|V(G)|,$$

which is a contradiction. Hence, $\gamma(P_\alpha(G)) \geq 2$ for any α on $V(G)$. On the other hand, by Theorem 1,

$$\gamma(P_\alpha(G)) \leq 2\gamma(G) = 2.$$

Thus, $\gamma(P_\alpha(G)) = 2$ for any permutation. □

Notice, when $\gamma(G) = 1$ and $|V(G)| \geq 2$, G must be connected. But when $\gamma(G) = 2$, G can be either connected or disconnected. These two cases will be investigated in Lemma 3 and Lemma 5, respectively.

Lemma 3. *Let G be a disconnected graph with at least 3 vertices. If $\gamma(G) = 2$, then there exists a permutation α on V(G) such that $\gamma(P_\alpha(G)) \geq 3$.*

Proof. Since $\gamma(G) = 2$ and G is disconnected, then G must consist of 2 disjoint components, namely X and Y. Also, because G contains at least 3 vertices, without loss of generality, we may assume that $|X| \geq 2$.

Assume for the sake of contradiction that $\gamma(P_\alpha(G)) < 3$ for any α on $V(G)$. Since $\gamma(G) = 2$, by Theorem 1, $\gamma(P_\alpha(G)) = 2$ for any α on $V(G)$. Let α be the identity permutation, that is $\alpha(v) = v'$ for all $v \in V(G)$, represented below by Figure 2.1. Let $S = \{a, b\}$ be a dominating set of $P_\alpha(G)$. Because of the choice of α, $P_\alpha(G)$ contains 2 disjoint components, $X \cup X'$ and $Y \cup Y'$, where X' and Y' are copies of X and Y, respectively. Therefore, without loss of generality, we may assume that $a \in X \cup X'$ and $b \in Y \cup Y'$. Since S is a dominating set of $P_\alpha(G)$, both a and b must completely dominate their respective components. However, if $a \in X$, then a cannot completely dominate X' since $|X'| \geq 2$. Thus $a \notin X$. A similar argument shows that $a \notin X'$. This contradiction shows that $\gamma(P_\alpha(G)) \geq 3$ for some permutation α on $V(G)$. $\square$

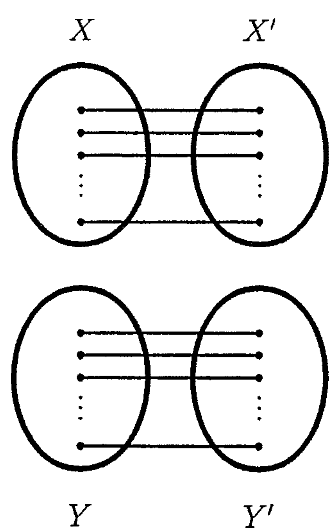

Figure 2.1: Identity Permutation graph $P_\alpha(G)$,G consists of two components X, Y

Remark: Although Lemma 3 shows the existence of a permutation for which $\gamma(P_\alpha(G)) \geq 3$, it does not provide any information about the existence of a permutation α so that $\gamma(P_\alpha(G)) = 2$. To characterize such a disconnected graph, consider the following:

Lemma 4. *Let G be a disconnected graph with $\gamma(G) = 2$. Then $\gamma(P_\alpha(G)) \geq 3$ for all α on $V(G)$ if and only if each component of G contains at least 2 vertices.*

Proof. Suppose $\gamma(P_\alpha(G)) \geq 3$ for all α on $V(G)$ and $|V(G)| = k$. Since G is disconnected and $\gamma(G) = 2$, then $G = A \cup B$ where A and B are disjoint sets and $\gamma(A) = \gamma(B) = 1$. Also, since $|V(G)| > 2$, without loss of generality, $|V(A)| = n > 2$. However, because $\gamma(A) = 1$, there must exist a vertex $v \in A$ such that $deg_G(v) = k - 2$. Now, for the sake of contradiction, assume B consists of one vertex, namely x, and define α on $V(G)$ as follows:

$$\alpha(u) = \begin{cases} x' & \text{if } u = v \in A \\ v' & \text{if } u = x \in B \\ u' & \text{otherwise} \end{cases}$$

Thus, $S = \{v, v'\}$ is a dominating set for $P_\alpha(G)$ since v dominates A and B' and v' dominates A' and B. However, this contradicts the assumption that $\gamma(P_\alpha(G)) \geq 3$ for all α on $V(G)$. Therefore, both components of G must consist of at least 2 vertices.

Conversely, suppose each component of G contains at least 2 vertices and for the sake of contradiction, assume there exists a permutation α on $V(G)$ such that $\gamma(P_\alpha(G)) = 2$. Let $S = \{a, b\}$ be a minimum dominating set for $P_\alpha(G)$. If $a, b \in V(G)$, then S does not dominate G', since $|V(G')| > 2$. Therefore, without loss of generality, assume $a \in V(G)$ and $b \in V(G')$. Since a can only dominate its respective component, then b must dominate the other component. However, each component contains at least 2 vertices. Thus, S cannot be a dominating set for $P_\alpha(G)$. Therefore, for all α on $V(G)$, $\gamma(P_\alpha(G)) \geq 3$. $\square$

Although Lemma 4 classifies all the graphs for which $\gamma(G) = 2$ and $\gamma(P_\alpha(G)) \geq 3$ for all α on $V(G)$, it also proves that the only disconnected graph for which $\gamma(G) = \gamma(P_\alpha(G)) = 2$ is when $V(G) = V(A) \cup \{x\}$ where $\gamma(A) = 1$. Later, we shall prove the more general case when $\gamma(G) = n$ and G consists of n disjoint components, each containing at least n vertices. For now, we shall consider the case when G is a connected graph where $\gamma(G) = 2$.

Lemma 5. *Let G be a connected graph with at least 3 vertices. If $\gamma(G) = 2$, then there exists a permutation α on $V(G)$ such that $\gamma(P_\alpha(G)) \geq 3$.*

Proof. For the sake of contradiction, assume $\gamma(P_\alpha(G)) = 2$ for any α on V(G). Since $\gamma(G) = 2$, $\Delta(G) \leq n - 2$, where $|V(G)| = n$.

Case I: Suppose $\Delta(G) < n - 2$. Let $S = \{a, b\}$ be a dominating set for $P_\alpha(G)$. Thus, $\deg_G(a) \leq n - 3$ and $|N_G[a]| \leq n - 3 + 1 = n - 2$. Therefore,

$$|N_{P_\alpha(G)}[a]| \leq n - 2 + 1 = n - 1$$

since a shares an edge with exactly one more vertex in the permutation graph. Similarly, $|N_{P_\alpha(G)}[b]| \leq n - 1$. Since S is a dominating set of $P_\alpha(G)$,

$$|V(P_\alpha(G))| \leq |N_{P_\alpha(G)}[a]| + |N_{P_\alpha(G)}[b]| \leq 2n - 2 < 2n = |V(P_\alpha(G))|,$$

which is a contradiction. Hence, for all α, $\gamma(P_\alpha(G)) \geq 3$.

Case II: Suppose $\Delta(G) = n - 2$. Let $v \in V(G)$ be such that $\deg_G(v) = n - 2$. Let $w \in V(G)$ be such that $vw \notin E(G)$. Let

$$\begin{aligned} A &= \{v \mid \deg_G(v) = n-2, \deg_G(w) < n-2, vw \notin E(G)\} \\ &= \{a_1, a_2, \ldots, a_k\}, \end{aligned}$$

where $k < n$, and let

$$\begin{aligned} B &= \{v \mid \deg_G(v) = \deg_G(w) = n-2, vw \notin E(G)\} \\ &= \{b_i, x_i \mid \deg_G(b_i) = \deg_G(x_i) = n-2, b_i x_i \notin E(G), 1 \leq i \leq l\} \\ &= \{b_1, b_2, \ldots, b_l, x_1, x_2, \ldots, x_l\}. \end{aligned}$$

Let $S = \{c, d\}$ be a dominating set of $P_\alpha(G)$. Without loss of generality, let $c \in V(G)$. It is clear from Case 1 that $\deg_G(c) = n - 2$. Note that $c \in A \cup B$ and $A \cap B = \emptyset$.

(i) Assume $|B| = 0$ and thus $c \in A$, so $c = a_j$ for some $j \leq k$. Define $\alpha_1 = (a_1 a_2 \ldots a_k)$. Let w be the vertex of G so that $cw \notin E(G)$. It follows that w must be dominated by d. Since $|V(G)| \geq 3$, $d \in V(G')$. Because of our choice of α_1 and $\deg_G(w) < n - 2$, $d = w'$. However, $w'a'_j \notin E(G')$. Therefore, a'_j is not dominated by d. But a'_j is not dominated by c either. Thus, S is not a dominating set for $P_{\alpha_1}(G)$.

(ii) Assume $c \in B$ and $|B| = 2$, thus $B = \{x, b\}$. Since $|B| = 2$, there exists a vertex

$y \in N_G[x]$ such that $deg_G(y) < n - 2$. Define $\alpha_2 = (xy)$. Since $\deg_G(c) = n - 2$, either $c = x$ or $c = b$. If $c = x$, then c does not dominate b. Therefore, $d = b'$. However, b' does not dominate $x' \in V(G')$. But neither does c since $\alpha(c) = y'$. Thus, S is not a dominating set for $P_{\alpha_2}(G)$. If $c = b$, then c does not dominate x. Therefore, $d = y'$. However, $\deg'_G(y') < n - 2$, therefore S is not a dominating set for $P_{\alpha_2}(G)$.

(iii) Assume $c \in B$, $|B| > 2$ and $c = b_m$ for some $m \leq l$. Define $\alpha_3 = (a_1 a_2 ... a_k)(x_1 x_2 ... x_l)$. Note that x_m is not adjacent to b_m. So x_m must be dominated by d. From Case II (1), it is clear that $d \in V(G')$. Also, by our choice of α_3, $d = x'_{m+1}$. Since $b'_{m+1} x'_{m+1} \notin E(G')$, d does not dominate b'_{m+1}. However, c does not dominate b'_{m+1} either. Thus, S is not a dominating set for $P_{\alpha_3}(G)$.

Thus, there exists an α such that $\gamma\big(P_\alpha(G)\big) \geq 3$. □

Notice that Figure 2.2 represents Lemma 5, Case II(ii), where $G = C_4$. Thus, for every $x \in V(G)$, $deg_G(x) = 4 - 2 = 2$. We invite the reader to find a dominating set S for $P_\alpha(G)$ such that S contains only 2 vertices. Lemma 3 and Lemma 5 show that if $\gamma(G) = \gamma\big(P_\alpha(G)\big) = m$ for any permutation α on $V(G)$, then $G = \overline{K_m}$, where $m = 1, 2$. We shall next show that this is true when given a graph G such that $\Delta(G) = n - m$, where $|V(G)| = n$.

Theorem 2. *Let G be a connected graph with $\gamma(G) = k \geq 3$, and $|V(G)| = n$. If $\Delta(G) = n - k$, then there exists a permutation α such that $\gamma\big(P_\alpha(G)\big) > k$.*

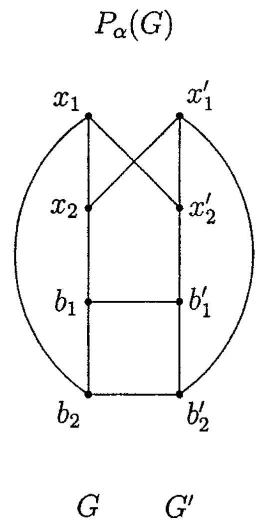

Figure 2.2: Permutation Graph of a cycle of length 4

Proof. Since $\Delta(G) = n - k$, there must exist a vertex $v \in V(G)$ such that $\deg(v) = n - k$. Therefore, consider the subgraph of G induced by A, where $A = N[v]$. Notice that there are exactly $k - 1$ vertices in $G \setminus A$. Let$\overline{A} = \{a_1, a_2, ..., a_{k-1}\}$, where $a_i v \notin E(G)$, for all $i \in \{1, ..., k-1\}$. Since $\gamma(G) = k$, for all $i, j \in \{1, ..., k-1\}$, $i \neq j$, $a_i a_j \notin E(G)$. That is, a_i and a_j are nonadjacent. Furthermore, since G is connected, for all $i \in \{1, ..., k-1\}$, a_i must have a neighbor in A and if $a_i x \in E(G)$, for some $x \in A$, then for any $j \neq i$, $a_j x \notin E(G)$, otherwise $\gamma(G) < k$. Now suppose we define α such that every vertex in A maps to a vertex in A', the copy of A in G', and for every $a_i \in \overline{A}$, $\alpha(a_i) = a_i'$, $i \in \{1, ..., k-1\}$. Next, let S be a dominating set for $P_\alpha(G)$ such that $S = C \cup D$ where $C \subset V(G)$ and $D \subset V(G')$. If $C \subset \overline{A}$ and $D \subset \overline{A'}$, then v and v' are not dominated. If $C \subset A$ and $D \subset A'$, since no vertex in A is adjacent to more than one vertex in $\overline{A}$ and by choice of α, then for S to dominate $P_\alpha(G)$, $|C| \geq k - 1$ and $|D| \geq k - 1$. Thus, $|S| > k$ since $k \geq 3$. Therefore, C must contain a vertex $x \in A$ and since $|A'| > k$, D must contain a vertex $y \in A'$. Next, in order to find a minimum dominating set S of $P_\alpha(G)$ such that $|S| = k$ with the restrictions placed on α, $\{x, y\}$ must dominate some $a_i a_i'$ pair in $\overline{A} \cup \overline{A'}$. Otherwise, S must contain x, y, and $k-1$ vertices in $\overline{A} \cup \overline{A'}$ to completely dominate $P_\alpha(G)$. Therefore, S must contain x, y, and $k-2$ vertices from $\overline{A} \cup \overline{A'}$. Also, $x \neq v$ and $y \neq v'$, since they dominate no vertex in $\overline{A} \cup \overline{A'}$.

Case I: Suppose $\deg_G(x) < n-k$, x dominates a_j and y dominates a_j' for some $j \in \{1, ..., k-1\}$. Since each $a_i \in \overline{A}$ has a neighbor in A, for all $i \in \{1, ..., k-1\}$, pick one such neighbor of a_i and call it z_i. Define $\alpha = (z_1 ... z_{k-1})$. Since S was assumed to be a dominating set of $P_\alpha(G)$, if there exists a vertex $p \in N_G(v) - N_G(\overline{A})$, then x must dominate $p \in V(G)$ and y must dominate $p' \in V(G')$. Additionally, since x dominates a_j and y dominates a_j', then neither a_j nor a_j' are in S. Thus, x or y must dominate $N(a_1) \in G$. If x dominates $N(a_1)$, then $\{x, a_1, ..a_{j-1}, a_{j+1}, .., a_{k-1}\}$ would have been a dominating set of the original graph, G, contradicting $\gamma(G) = k$. Therefore, there must exist a vertex $m \in N(a_1)$ not adjacent to x. Note that since S was assumed to be a dominating set of $P_\alpha(G)$, that y must dominate m. The only way for y to dominate m is if the original copy of y in G is m itself. Therefore, x and y are nonadjacent in $N(a_j) = \{x, y\}$. However, by definition of α, then $x = z_j$ and therefore neither x nor y dominates $x' \in V(G')$. Thus, this case cannot occur and $\deg_G(x) = n-k$.

Since the $\deg_G(x) = n-k$, then there must exist at least one other vertex $u \in V(G)$ such that $\deg_G(u) = n-k$. Since x must dominate a_j for some $j \in \{1, ..., k-1\}$, then consider all $u \in V(G)$ such that $\deg_G(u) = n-k$ and u dominates a_j. Thus, there must exist exactly one vertex $w \in A$ such that $vw \notin E(G)$.

Let

$$\begin{aligned} Q &= \{u \mid \deg_G(u) = n-k, \deg_G(w) < n-k, uw \notin E(G)\} \\ &= \{q_1, q_2, ..., q_t\}, \end{aligned}$$

where $t < n$, and let

$$\begin{aligned} R &= \{u \mid \deg_G(u) = \deg_G(w) = n-k, uw \notin E(G)\} \\ &= \{b_i, c_i \mid \deg_G(b_i) = \deg_G(c_i) = n-k, b_i c_i \notin E(G), 1 \leq i \leq l\} \\ &= \{b_1, b_2, ..., b_l, c_1, c_2, ..., c_l\}. \end{aligned}$$

Case II: If $x \in Q$, then $x = q_i$, for some $i \in \{1, .., t\}$. Note that q_i is not adjacent to some $s \in A$. Define $\alpha = (vq_1...q_t)$. Assume again that q_i dominates $a_j \in \overline{A}$. Notice that if s is adjacent to some other vertex in $\overline{A}$, say a_2, then $\{q_i, s, a_3, ..., a_{k-1}\}$ would have been an original dominating set for G. Therefore, s cannot be dominated by a vertex in $\overline{A}$. By definition of α, y must be s' since $a'_j \notin S$. However, q'_i is not dominated in A'. Therefore, this case cannot occur.

Case III: If $x \in R$, then without loss of generality, we may assume that $x = c_i$ for some $i \in \{1, ..., l\}$. Note that c_i is not adjacent to b_i. Define $\alpha = (vc_1...c_l)$. Assume again that c_i dominates $a_j \in \overline{A}$. Notice that if b_i is adjacent to some other vertex in $\overline{A}$, say a_2, then $\{c_i, b_i, a_3, ..., a_{k-1}\}$ would have been an original dominating set of $P_\alpha(G)$. Therefore, b_i cannot be dominated by a vertex in $\overline{A}$. By definition of α, y must be b'_i since $a'_j \notin S$.

However, c_i' is not dominated in A'. Therefore, this case cannot occur.

Therefore, in every case, S cannot dominate all the vertices in $P_\alpha(G)$. Thus, there exists a permutation where $\gamma\big(P_\alpha(G)\big) > k$.

□

Lastly, we need to show that the upper bound on $\gamma\big(P_\alpha(G)\big)$ is sharp and try to classify some graphs for which $\gamma\big(P_\alpha(G)\big) = 2\gamma(G)$ for all α. As mentioned in Chapter 1, it has been shown that all complete graphs meet the upper bound for all α. Therefore, we next determine for which disconnected graphs the above is true.

Theorem 3. *Let G be a disconnected graph where $\gamma(G) = n$ and G consists of n disjoint components, X_i for $i \in \{1, 2, \ldots, n\}$. If $|X_i| > n$ for all $i \in \{1, 2, \ldots, n\}$, then $\gamma\big(P_\alpha(G)\big) = 2n$ for all permutations α from $V(G)$ to $V(G)$.*

Proof. Since $\gamma(G) = n$ and there are n disjoint components, then there exists a vertex $x_i \in X_i$ for all $i \in \{1, 2, \ldots, n\}$ such that x_i is adjacent to every other vertex in X_i. Consider the graph $P_\alpha(G)$ for any α, and for the sake of contradiction, assume that there exists a dominating set S for $P_\alpha(G)$ such that $|S| < 2n$. Since S contains vertices from $V(G)$ and vertices from $V(G')$, then consider $S = A \cup B$, where $A \in V(G)$ and $B \in V(G')$. Since $S < 2n$, one of A or B has at most $n - 1$ vertices. Without loss of generality, let $|B| = k \leq n - 1$. Then, at least $n - k$ components of $V(G')$ do not contain any vertices of B, namely $C_1, C_2, \ldots, C_{n-k}$. Therefore, since S dominates all vertices of $P_\alpha(G)$ and B does not dominate the vertices in components $C_1, C_2, \ldots, C_{n-k}$, A has to dominate these components. Note that $|A| = |S| - |B| = 2n - 1 - k$. However, the number of vertices in the components C_i, $i \in \{1, .., n - k\}$, is at least $(n - k)(n + 1)$.

Therefore, $2n - k - 1 = |A| \geq (n-k)(n+1)$. This inequality holds if and only if

$$2n - k - 1 \geq n^2 + n - kn - k$$

which implies

$$n - 1 \geq n^2 - kn. \tag{2.1}$$

Since $k \leq n - 1$, we have

$$n^2 - kn \geq n^2 - (n-1)n = n. \tag{2.2}$$

From (2.1) and (2.2), we have

$$n - 1 \geq n$$

This contradiction shows that $|A| \geq (n-k)(n+1)$ is impossible, which implies that A does not consist of enough vertices to dominate all vertices in the components C_i, $i \in \{1, \ldots, n-k\}$. Therefore, S is not a dominating set of $P_\alpha(G)$, and $\gamma\big(P_\alpha(G)\big) = 2n$ for all permutations α. □

Notice that if we relax the conditions in Theorem 3 so that there exists a component, X_j, in G that contains exactly n vertices, then we can find a permutation on $V(G)$ such that $\gamma\big(P_\alpha(G)\big) < 2n$. To see this, consider the graph G as just described where $X_j = \{y_1, y_2, \ldots, y_n\}$. Assume that $x_j = y_j$ for all $j \in \{1, 2, \ldots, n\}$, otherwise we can relabel y's in X_j. Define α in the following way:

$$\alpha(u) = \begin{cases} y_i' & \text{if } u = x_i \in X_i, 1 \leq i \leq n \\ u' & \text{otherwise} \end{cases}$$

Therefore, the set $S = \{x_1, x_2, \ldots, x_n, x_1', \ldots, x_{j-1}', x_{j+1}', \ldots, x_n'\}$ is a dominating set for $P_\alpha(G)$ where $|S| = 2n - 1$. Thus, the condition that each component in G must contain more than n vertices is strict. To see this, consider the graph $G = P_3 \cup P_2$ and $P_\alpha(G)$ shown in Figure 2.3. It is easy to see that $\gamma(G) = 2$ and $\gamma\big(P_\alpha(G)\big) = 3$.

$P_\alpha(G)$

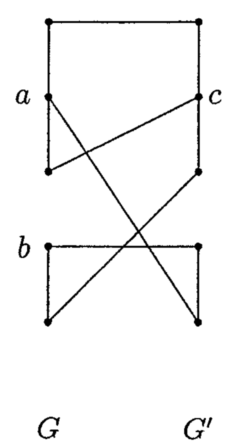

Figure 2.3: Permutation Graph of graph G where $\gamma(G) = 2$, $\gamma\big(P_\alpha(G)\big) = 3$

Although Theorem 3 only classifies one type of disconnected graph for which $\gamma\big(P_\alpha(G)\big) = 2\gamma(G)$ for all α, a similar argument could potentially classify all connected graphs for which the above is true. However, further research is needed to show such a claim.

CHAPTER 3

RESTRICTING DOMINATION NUMBER

In the last section, we focused on finding graphs whose permutation graph achieved either the lower or upper bounds for the domination number. However, as $\gamma(G)$ increases, the range of the bounds for $\gamma\big(P_\alpha(G)\big)$ also increases. Therefore, the next logical question is to find out if there are limitations on graphs given specific values for $\gamma(G)$ and $\gamma\big(P_\alpha(G)\big)$.

First, we shall investigate that if given two positive integers, a and b where $a \leq b \leq 2a$, is it possible to find a graph and a permutation on $V(G)$ such that $\gamma(G) = a$ and $\gamma\big(P_\alpha(G)\big) = b$ for all $a, b \in N$? If no restrictions are placed on G, it is relatively easy to find a graph and permutation on $V(G)$ that meet the above criteria, as seen in the following theorem.

Theorem 4. *For any positive integers a, b where $a \leq b \leq 2a$, there exists a graph G and permutation α on $V(G)$ so that $\gamma(G) = a$ and $\gamma\big(P_\alpha(G)\big) = b$.*

Proof. Let G be a graph consisting of $b - a$ isolated edges and $2a - b$ isolated vertices. Note that any minimum dominating set, S, for G must contain at least $b - a$ vertices from the isolated edges and all of the isolated vertices, thus $|S| = 2a - b + b - a = a$. Therefore, $\gamma(G) = a$. Next, impose the permutation $\alpha = (1)$ on $V(G)$, represented below with $k = 4$ and $l = 6$. Therefore, $P_\alpha(G)$ consists of $b - a$ disjoint cycles of length 4 and

$2a - b$ disjoint edges. Thus, any minimum dominating set, R, for $P_\alpha(G)$ must contain exactly $2a - b$ vertices from the disjoint edges and $2(b - a) = 2b - 2a$ vertices from the disjoint cycles. Therefore, $|R| = 2a - b + 2b - 2a = b$. Hence, $\gamma(P_\alpha(G)) = b$. □

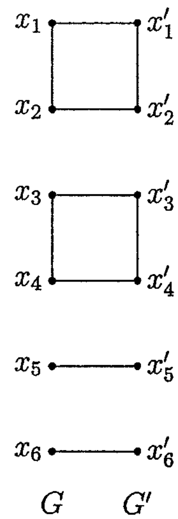

Figure 3.1: Permutation Graph where $k = 4$, $l = 6$

It is proved in Theorem 4 that a disconnected graph can always be found given any values for a and b. However, a better result would be to find a connected graph with the given parameters. Further research is needed to prove such a result.

Now, suppose we are given any graph G where $\gamma(G) = a$. If G does not meet the criteria in Theorem 4, would it be possible to find a permutation α on $V(G)$ such that $\gamma(P_\alpha(G)) = b$ for all $b \in N$, where $a \leq b \leq 2a$? As mentioned above, the distance from a to $2a$ increases as the value of a increases. Therefore, we shall focus on just a few cases.

When $\gamma(G) = 1$, clearly $\gamma(P_\alpha(G)) = 1, 2$. In Chapter 2, it was proven that $\gamma(P_\alpha(G)) = 1$ only when G consists of one isolated vertex. Thus, for any $G \neq K_1$,

$\gamma(P_\alpha(G)) = 2$ for all permutations on $V(G)$.

When $\gamma(G) = 2$, then $\gamma(P_\alpha(G)) = 2, 3, 4$. Rather than trying to find permutations for every possible graph such that $\gamma(P_\alpha(G))$ attains the given values, we wish to classify the graphs for which a permutation would actually exist for a given value. First, let us consider classifying the graphs for which there exists a permutation on $V(G)$ such that $\gamma(P_\alpha(G)) = 2$.

Theorem 5. *Let G be a graph with n vertices and $\gamma(G) = 2$. Then there exists a permutation α on $V(G)$ such that $\gamma(P_\alpha(G)) = 2$ if and only if $\Delta(G) = n - 2$.*

Proof. ($\Longrightarrow$) Since $\gamma(G) = 2$, then $\Delta(G) \leq n - 2$. Assume that $\Delta(G) \neq n - 2$. Then, $\Delta(G) \leq n - 3$. That is, for every vertex $v \in V(G)$, $\deg_G(v) \leq n - 3$. Since there exists a permutation α such that $\gamma(P_\alpha(G)) = 2$, let $S = \{a, b\}$ be a dominating set of $P_\alpha(G)$. Therefore,

$$|N_{P_\alpha(G)}[a]| \leq n - 3 + 1 + 1 = n - 1.$$

However,

$$|N_{P_\alpha(G)}[a]| + |N_{P_\alpha(G)}[b]| \leq n - 1 + n - 1 = 2n - 2 < 2n.$$

Thus, S is not a dominating set of $P_\alpha(G)$, and $\Delta(G) = n - 2$.

($\Longleftarrow$) Suppose $\Delta(G) = n - 2$. Then, there must exist a vertex $v \in V(G)$ such that $\deg_G(v) = n - 2$. Let x be the only vertex in $V(G)$ that is not adjacent to v. Define $\alpha = (xv)$. Then, $S = \{v, v'\}$ is a dominating set of $P_\alpha(G)$ and

$$\gamma(P_\alpha(G)) \leq 2. \tag{3.1}$$

Since $\gamma(G) = 2$, by Theorem 1,

$$\gamma\big(P_\alpha(G)\big) \geq 2. \tag{3.2}$$

From (3.1) and (3.2), $\gamma\big(P_\alpha(G)\big) = 2$. $\square$

Theorem 5 restricts the structure of a graph where there exists a permutation on $V(G)$ such that $\gamma\big(P_\alpha(G)\big) = 2$. However, when we consider classifying the graphs for which there exists a permutation such that $\gamma\big(P_\alpha(G)\big) = 3$, the restrictions on G are relaxed. To see this, consider Lemma 6.

Lemma 6. *Let G be a graph with n vertices where $\gamma(G) = 2$. If there exists a permutation α on $V(G)$ such that $\gamma\big(P_\alpha(G)\big) = 3$, then*

$$n - 3 \leq \Delta(G) \leq n - 2.$$

Proof. Since $\gamma(G) = 2$ and there exists a permutation α on $V(G)$ such that $\gamma\big(P_\alpha)\big) = 3 > \gamma(G)$, then $G \neq \overline{K_2}, \overline{K_3}$. Therefore, $G = K_2 \cup K_1$ or G contains at least 4 vertices. In the first case,

$$\Delta(G) = 1 \geq 3 - 3 = 0,$$

thus the the statement is true. In the second case, we shall prove $\Delta(G) \geq n - 3$ by contradiction. Thus, suppose that $\Delta(G) \leq n - 4$. Then, for every vertex $v \in V(G)$, $\deg_G(v) \leq n - 4$. Since there exists a permutation α such that $\gamma\big(P_\alpha(G)\big) = 3$, let $S = \{a, b, c\}$ be a dominating set of $P_\alpha(G)$. Therefore,

$$|N_{P_\alpha(G)}[a]| \leq n - 4 + 1 + 1 = n - 2.$$

However,

$$|N_{P_\alpha(G)}[a]| + |N_{P_\alpha(G)}[b]| \leq n-2+n-2 = 2n-4$$

Thus, there exist at least 4 vertices in $P_\alpha(G)$ not dominated by a or b. Notice that if 2 vertices of S, say a, b are in $V(G)$ then only 2 vertices in $V(G')$ are dominated by a and b. However, $\deg_{G'}(c) \leq n-4$. Thus, c cannot dominate the remaining $n-2$ vertices in $V(G')$. Thus, a, b and c must lie in the same copy of G. Without loss of generality, we may assume that a, b and $c \in V(G)$. Thus, only 3 vertices of G' can be dominated by S. However, $|V(G')| \geq 4$, implying that S is not a dominating set of $P_\alpha(G)$. Therefore, $\Delta(G) \geq n-3$. □

Lemma 7. *Let G be a graph with n vertices where $\gamma(G) = 2$ and $\Delta(G) = n-2$. Then there exists a permutation α on $V(G)$ such that $\gamma(P_\alpha(G))= 3$.*

Proof. **Case 1.** Suppose G is a disconnected graph. Since $\Delta(G) = n-2$, then there must exist a vertex $v \in V(G)$ such that $\deg_G(v) = n-2$. Let x be the only vertex in $V(G)$ not adjacent to v and define $\alpha = (1)$. Notice that $S = \{v, v', x\}$ is a dominating set of $P_\alpha(G)$. Thus,

$$\gamma(P_\alpha(G)) \leq 3. \tag{3.3}$$

However, by the same argument used in Lemma 3,

$$\gamma(P_\alpha(G)) \geq 3. \tag{3.4}$$

Thus, by (3.3) and (3.4), $\gamma(P_\alpha(G) = 3$.

Case 2. Suppose G is a connected graph. Since $\Delta(G) = n-2$, then there must exist a vertex $v \in V(G)$ such that $\deg_G(v) = n-2$. Let $w \in V(G)$ be such that $vw \notin E(G)$. Define

$$\begin{aligned} A &= \{v \mid \deg_G(v) = n-2, \deg_G(w) < n-2, vw \notin E(G)\} \\ &= \{a_i, y_i \mid \deg_G(a_i) = n-2, \deg_G(y_i) < n-2, a_i y_i \notin E(G), 1 \leq i \leq k\} \\ &= \{a_1, a_2, \ldots, a_k, y_1, y_2, \ldots, y_k\}, \end{aligned}$$

where $k < n$, and let

$$\begin{aligned} B &= \{v \mid \deg_G(v) = \deg_G(w) = n-2, vw \notin E(G)\} \\ &= \{b_i, x_i \mid \deg_G(b_i) = \deg_G(x_i) = n-2, b_i x_i \notin E(G), 1 \leq i \leq l\} \\ &= \{b_1, b_2, \ldots, b_l, x_1, x_2, \ldots, x_l\}. \end{aligned}$$

(i) If $|B| = 0$, then define $\alpha_1 = (a_1 a_2 ... a_k)$. Notice that $S = \{a_1, a_1', y_1\}$ is a dominating set of $P_{\alpha_1}(G)$ and therefore $\gamma(P_{\alpha_1}(G)) \leq 3$. However, since this is the same α_1 defined in Lemma 5, $\gamma(P_{\alpha_1}(G)) \geq 3$. Thus, $\gamma(P_{\alpha_1}(G)) = 3$.

(ii) If $|B| = 2$, then there must exist a vertex $m \in V(G)$ such that $\deg_G(m) < n-2$. Therefore, define $\alpha_2 = (wm)(a_1 a_2 ... a_k)$. Notice that $S = \{b, w', m\}$ is a dominating set of $P_{\alpha_2}(G)$ and therefore $\gamma(P_{\alpha_2}(G)) \leq 3$. However, this is the same β defined in Lemma 5 and thus $\gamma(P_\beta(G)) \geq 3$. Thus, $\gamma(P_{\alpha_2}(G)) = 3$.

(iii) If $|B| > 2$, then define $\alpha_3 = (w_1 w_2 ... w_l)(a_1 a_2 ... a_k)$. Notice that $S = \{w_1, w_1', b_1'\}$ is a dominating set of $P_{\alpha_3}(G)$ and therefore $\gamma(P_{\alpha_3}(G)) \leq 3$. However, this is the same

α_3 defined in Lemma 5 and thus $\gamma(P_{\alpha_3}(G)) \geq 3$. Thus, $\gamma(P_{\alpha_3}(G)) = 3$.

□

Lemma 8. *Let G be a graph with n vertices where $\gamma(G) = 2$ and $\Delta(G) = n - 3$. Then there exists a permutation α on $V(G)$ such that $\gamma(P_\alpha(G)) = 3$.*

Proof. Since $\Delta(G) = n - 3$, there must exist a vertex $v \in V(G)$ such that $\deg_G(v) = n - 3$. Therefore, let x and y be the only two vertices in $V(G)$ not adjacent to v and define $\alpha = (vx)$. Notice that $S = \{v, v', y\}$ is a dominating set of $P_\alpha(G)$. Therefore,

$$\gamma(P_\alpha(G)) \leq 3. \tag{3.5}$$

However, since $\Delta(G) < n - 2$, by Theorem 5

$$\gamma(P_\alpha(G)) \geq 3. \tag{3.6}$$

Thus, by (3.5) and (3.6), $\gamma(P_\alpha(G)) = 3$. □

Theorem 6. *Let G be a graph with n vertices and $\gamma(G) = 2$. Then there exists a permutation α on $V(G)$ such that $\gamma(P_\alpha(G)) = 3$ if and only if*

$$n - 3 \leq \Delta(G) \leq n - 2.$$

Proof. One direction follows from Lemma 6 and the other direction follows from Lemmas 7 and 8. □

CHAPTER 4

CONCLUSION

The primary goal of this thesis was to prove the statement that if $\gamma(G) = \gamma\big(P_\alpha(G)\big)$, for all permutations α, then $G = \overline{K_m}$. However, we were only able to show that the above is true if $\Delta(G) = n - k$, where $|V(G)| = n$ and $\gamma(G) = k$.

Therefore, we have yet to consider the case when $\Delta(G) < n - k$. Be that as it may, the results we have found further the progress of solving this open conjecture.

Additionally, we were able to determine a specific permutation on disconnected graphs for which the upper bound on the domination number was met and classify the graphs G where $\gamma(G) = 2$ and $\gamma\big(P_\alpha(G)\big) = 2, 3$. Although these results only give us some insight for specific cases of graphs or domination number, we have discovered other areas that require further research. Overall, this thesis is more of an introduction to the type of logic needed to prove conjectures in this area of graph theory and research that has yet to come.

Areas of Further Research

Below are listed but a few of the questions that will require additional research:

1. Given a, b such that $a \leq b \leq 2a$, is it possible to find a connected graph G and permutation on $V(G)$ so that $\gamma(G) = a$ and $\gamma\big(P_\alpha(G)\big) = b$?

2. For which graphs will the $P_\alpha(G)$ induced by the identity permutation meet the lower bound in terms of domination?

3. If G is a connected graph where $\gamma(G) = k$, $|V(G)| = n$, and $\Delta(G) < n - k$, does there exists a permutation α on $V(G)$ such that $\gamma\big(P_\alpha(G)\big) > k$.

4. For which graphs G, where $\gamma(G) = 3$, will there exist a permutation α on $V(G)$ such that $\gamma\big(P_\alpha(G)\big) = 3, 4, 5, 6$?

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VITA

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