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POSITIVE SOLUTIONS OF A BOUNDARY-VALUE PROBLEM FOR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The existence of positive solutions of a two-point boundary value problem for a second order differential equation is investigated. By using indices of convergence of the nonlinearities at 0 and at $+\infty$, we provide a priori upper and lower bounds for the slope of the solutions.

1. Introduction

We show that some boundary value problems governed by a second order ordinary differential equation admits solutions with slope in a known pre-specified region of the positive axis.

Recently an increasing interest has been observed in investigating the existence of positive solutions of boundary value problems. This interest comes from situations involving nonlinear elliptic problems in annular regions; see, e.g. [2,3,7,8]. But this is not the origin. Krasnoselskii [10] already in 1964 published his book on positive solutions of abstract operator equations, where (among others) several fixed point methods were also developed.

Here, motivated mainly by the works [1,4,5,6,9] and especially from [11], we study an equation of the form

$$x'' + \operatorname{sign}(1-\alpha)q(t)f(x,x')x' = 0, \text{ a.a. on } [0,1] \tag{1.1}$$

with one of the two sets of boundary conditions (1.2) or (1.3)

$$x(0) = 0, \quad x'(1) = \alpha x'(0) \tag{1.2}$$

$$x(1) = 0, \quad x'(1) = \alpha x'(0), \tag{1.3}$$

where $\alpha > 0$ with $\alpha \neq 1$. We show that under rather mild conditions on the functions q and f the problem (1.1, 1.2) admits a solution x satisfying

$$Mt \leq x(t) \leq Nt, \ t \in I, \tag{1.4}$$

and problem (1.1, 1.3) admits a solution x satisfying

$$M(1-t) \leq x(t) \leq N(1-t), \ t \in I, \tag{1.5}$$

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where M and N are pre-specified positive constants. Our arguments for establishing the existence of solutions of these problems involve concavity properties of solutions that are used to construct a cone on which a positive integral operator is defined. Then a fixed point theorem, due to Krasnoselskii [10] mentioned above, is applied to yield the existence of positive solutions.

To organize our results in this work we introduce the meaning of the so called index of convergence of a function at a point which resembles the generalized inverse of the modulus of convergence (analogous to the modulus of continuity). By using indices of convergence of the function f at 0 and at $+\infty$ we are able to give a priori bounds for the slope of the solutions obtained for the problems (1.1, 1.2) and (1.1, 1.3).

2. The index of convergence and the main results

In the sequel we shall denote by I the interval $[0,1]$ of the real line $\mathbb{R}$. Also $C_0^1(I)$ will stand for the space of all functions $x: I \to \mathbb{R}$ such that $x(0)=0$ and x' is absolutely continuous on I. Here $x'(0)$ and $x'(1)$ mean one-sided derivatives. We furnish the set $C_0^1(I)$ with the norm

$$\|x\| := \sup\{|x'(t)| : t \in I\}.$$

Then $C_0^1(I)$ is a (real) Banach space. We shall denote by $B(0,r)$ the open ball in $C_0^1(I)$ centered at 0 and having radius $r>0$. Then $\partial B(0,r)$ and $\operatorname{cl} B(0,r)$ will denote the boundary and the closure of $B(0,r)$ respectively.

Before proceeding to our problem we want to define an auxiliary concept needed in the sequel.

Let X and Y be metric spaces with metrics ρ_X, ρ_Y respectively and let S be a nonempty set. Let also $h(\cdot,\cdot): X \times S \to Y$ be a function such that for some $e' \in X$ the limit $\lim_{e\to e'} h(e,\sigma) =: l(\sigma)$ exists for each $\sigma \in S$. This means that to any $\sigma \in S$ and $\epsilon > 0$ there corresponds a $\delta(\epsilon,\sigma) > 0$ such that $\rho_X(e,e') \le \delta(\epsilon,\sigma)$ implies $\rho_Y(h(e,\sigma), l(\sigma)) \le \epsilon$. If a $\delta > 0$ exists not depending on $\sigma \in S$, then we have uniform convergence in σ. It is clear that the set of all such $\delta's$ (for fixed ϵ) is a closed subset of the interval $(0,+\infty]$. We introduce the following simple meaning: For a given $\epsilon > 0$ the *index of uniform convergence of h at e to l* is the function defined by

$$\Delta(\epsilon; e, l) := \sup\left\{\delta > 0 : \rho_X(e', e) \le \delta \Rightarrow \rho_Y(h(e',\sigma), l(\sigma)) \le \epsilon, \text{ for all } \sigma \in S\right\}.$$

If $h(e,\sigma)$ does not depend on σ we call $\Delta(\cdot;\cdot,\cdot)$ simply *index of convergence.* It is clear that $\Delta(\cdot; e, l)$ is an increasing function taking values in the interval $(0,+\infty]$ and it has the property that whenever $\rho_X(e',e) \le \Delta(\cdot; e, l)$, then $\rho_Y(h(e',\sigma), l(\sigma)) \le \epsilon$, for all $\sigma \in S$. In the special case $X = Y = \mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$ the index of convergence can be defined via the well known pseudo-metric $\rho(\alpha,\beta)$ namely the function defined by

$$\rho(\alpha,\beta) := |\alpha - \beta| \text{ if } \alpha,\beta \in \mathbb{R},$$
$$\rho(\alpha,\pm\infty) = \rho(\pm\infty,\alpha) := \frac{1}{|\alpha|} \text{ if } \alpha \in \mathbb{R}\setminus\{0\}$$

and

$$\rho(0,\pm\infty) = \rho(\pm\infty,0) = \rho(\pm\infty,\mp\infty) = \rho(\mp\infty,\pm\infty) := +\infty.$$

To set our problem consider a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In the sequel we shall assume that the function $f(u, v)$ is continuous for $uv \neq 0$ and satisfies A1 and A2 or A3 and A4, where

A1 $\lim_{\substack{u \to 0+ \\ v \to 0+}} f(u, v) = 0$

A2 $\lim_{v \to +\infty} f(u, v) = +\infty$, uniformly for all $u \geq 0$

A3 $\lim_{\substack{u \to 0+ \\ v \to 0+}} f(u, v) = +\infty$

A4 $\lim_{v \to +\infty} f(u, v) = 0$, uniformly for all $u \geq 0$.

Let $\Delta(\cdot; (0, 0), 0)$ and $\Delta(\cdot; (0, 0), +\infty)$ be the indices of convergence of the function $f(\cdot, \cdot)$, whenever the conditions A1 and A3 are satisfied respectively (in the definition above set $e := (u, v)$ and $h(e, \sigma) = h((u, v), \sigma) := f(u, v)$ for all σ). Also, let $\Delta(\cdot; +\infty, +\infty)$ and $\Delta(\cdot; +\infty, 0)$ be the indices of uniform convergence of the function $f(u, v)$ with respect to v uniformly in u, whenever the conditions A2 and A4 are satisfied respectively (set $e := v$ and $h(e, u) = h(v, u) := f(u, v)$).

Now we return to our problems (1.1, 1.2) and (1.1, 1.3). A function x is a solution of the problem (1.1, 1.2), if x is an element of the space $C_0^1(I)$ satisfying the equation (e), for almost all $t \in I$, as well as the condition $x'(1) = \alpha x'(0)$. Similarly with the problem (1.1, 1.3). Our plans are to investigate the problem (1.1, 1.2) first and then to proceed to the other problem, which, as we shall show, is equivalent to a problem of the form (1.1, 1.2). First we notice that a function x is a solution of the problem (1.1, 1.2), if and only if it satisfies an operator equation of the form

$$x(t) = (Ax)(t), \quad t \in I, \tag{2.1}$$

for an appropriate operator A defined on $C_0^1(I)$. Fixed points of (2.1) are solutions of (1.1, 1.2). Thus we seek for the existence of fixed points of A, by following a method based on the following fixed point theorem (see, e.g., [5,10]):

Theorem 2.1. *Let $\mathcal{X}$ be a Banach space and let $\mathbb{K}$ be a cone in $\mathcal{X}$. Assume that Ω_1, Ω_2 are open subsets of $\mathcal{X}$, with $0 \in \Omega_1 \subset cl\Omega_1 \subset \Omega_2$, and let*

$$A \colon \mathbb{K} \cap (\Omega_2 \setminus cl\Omega_1) \to \mathbb{K}$$

be a completely continuous operator. If either

$$\|Au\| \leq \|u\|, \ \ u \in \mathbb{K} \cap \partial\Omega_1 \ \ \textit{and} \ \ \|Au\| \geq \|u\|, \ \ u \in \mathbb{K} \cap \partial\Omega_2$$

or

$$\|Au\| \geq \|u\|, \ \ u \in \mathbb{K} \cap \partial\Omega_1 \ \ \textit{and} \ \ \|Au\| \leq \|u\|, \ \ u \in \mathbb{K} \cap \partial\Omega_2$$

holds, then A has a fixed point.

The advantage of this theorem over other fixed point theorems is that it provides more information for the solutions, namely we can know that solutions exist in the cone and moreover they satisfy inequalities of the form (1.4) or (1.5). Next, let $\alpha > 0$ be given with $\alpha \neq 1$ and set

$$w := \min\{\alpha, \alpha^{-1}\}, \quad \beta := w(1 - w)^{-1}, \quad \gamma := \max\{1, \beta\}. \tag{2.2}$$

In the sequel we shall assume that

H1 $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function on $(\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ such that $vf(u, v) \geq 0$ for all u, v.

H2 $q : I \to \mathbb{R}^+ := [0, +\infty)$ is a Lebesgue integrable function with norm $\|q\|_1$.

Lemma 2.1. *Consider the constants* (2.2) *and the functions* f, g *satisfying H1, H2. Let*

$$\epsilon := \frac{1}{2\gamma\|q\|_1} \quad \text{and} \quad \zeta := w\beta\|q\|_1. \tag{2.3}$$

If the function f *satisfies A1, A2, then we have*

$$w\Delta(\epsilon; (0,0), 0) < \frac{1}{w\Delta(\zeta; +\infty, +\infty)} \tag{2.4}$$

and if f *satisfies A3, A4, then*

$$w\Delta(\zeta; (0,0), +\infty) < \frac{1}{w\Delta(\epsilon; +\infty, 0)} \tag{2.5}$$

Proof. Assume that A1, A2 hold but (2.4) fails. Then (since $w < 1$) we can take elements u, v such that

$$\Delta(\epsilon; (0,0), 0) > u, \quad v > \frac{1}{\Delta(\zeta; +\infty, +\infty)}.$$

From the first inequality we obtain

$$0 < |f(u,v)| \leq \epsilon \tag{2.6}$$

and from the second

$$f(u,v) \geq \frac{1}{\zeta}. \tag{2.7}$$

But observe that $\epsilon\zeta \leq \frac{w}{2} < 1$ and so (2.6), (2.7) do not agree. Thus (2.4) is true. Next, assume that $(A3)$, $(A4)$ hold, but (2.5) fails. Then, again, we find u, v such that

$$\Delta(\zeta; (0,0), +\infty) > u, \quad v > \frac{1}{\Delta(\epsilon; +\infty, 0)}.$$

From the first inequality we get (2.7) and from the second one we get (2.6), hence a contradiction. ◇

Now we are ready to state and prove our first main theorem.

Theorem 2.2. *Consider the functions* f, q *satisfying H1, H2 and let* $w, \beta, \gamma, \epsilon, \zeta$ *be the constants defined in* (2.2), (2.3)*. Then the boundary-value problem (1.1, 1.2) admits a solution* $x \in C_0^1(I)$ *such that*

$$w\Delta(\epsilon; (0,0), 0)t \leq x(t) \leq \frac{t}{w\Delta(\zeta; +\infty, +\infty)}, \quad t \in I, \tag{2.8}$$

if f *satisfies the conditions A1, A2 and*

$$w\Delta(\zeta; (0,0), +\infty)t \leq x(t) \leq \frac{t}{w\Delta(\epsilon; +\infty, 0)}, \quad t \in I, \tag{2.9}$$

if f *satisfies the conditions A3 and A4.*

Proof. We shall prove the theorem by investigating four cases depending on whether $0 < \alpha < 1$, or $\alpha > 1$ and f satisfies A1 and A2, or A3 and A4.

Case 1: Assume that $0 < \alpha < 1$ and f satisfies the conditions A1 and A2. Then $w = a$ and equation (1.1) becomes

$$x'' + q(t)f(x, x')x' = 0, \quad \text{a.a.} \quad t \in I$$

and moreover the indices $\Delta(\epsilon; (0,0), 0)$ and $\Delta(\zeta; +\infty, +\infty)$ are positive (finite) real numbers. Also it is not hard to see that the above equation with the boundary condition (1.2) is equivalent to the problem of the form (2.1), where the operator $A := A_+$ is defined by the type:

$$\begin{aligned}(A_+x)(t) :=& t\alpha(1-\alpha)^{-1}\int_0^1 q(s)f(x(s), x'(s))x'(s)\,ds \\ &+ \int_0^t\int_s^1 q(r)f(x(r), x'(r))x'(r)\,dr\,ds.\end{aligned}$$

Consider the set

$$\mathbb{K}_+ := \left\{x \in C_0^1(I) : x \geq 0, x' \text{ is non-increasing and } x'(1) = \alpha x'(0)\right\},$$

which is a cone in $C_0^1(I)$ and restrict the operator A_+ to the nonempty (because of Lemma 2.1) set

$$\mathbb{K}_+ \cap [B(0, N) \setminus \operatorname{cl} B(0, M)] \tag{2.10}$$

where

$$N := \frac{1}{\alpha\Delta(\zeta; +\infty, +\infty)} \quad \text{and} \quad M := \alpha\Delta(\epsilon; (0,0), 0).$$

It is easy to show that A_+ is completely continuous with range in $\mathbb{K}_+$. (Recall that $0 \leq vf(u, v)$, $0 < \alpha < 1$ and $0 \leq q(t)$, $t \in I$.)

Let $x \in \mathbb{K}_+$ be such that $\|x\| = M$. Then, for each $t \in I$, we have $|x'(t)| < \Delta(\epsilon; (0,0), 0)$ and $0 \leq x(t) < t\Delta(\epsilon; (0,0), 0) \leq \Delta(\epsilon; (0,0), 0)$. Hence

$$|f(x(t), x'(t))| \leq \epsilon, \;\; t \in I$$

and therefore,

$$\begin{aligned}0 \leq (A_+x)'(t) =& \alpha(1-\alpha)^{-1}\int_0^1 q(s)f(x(s), x'(s))x'(s)ds \\ &+ \int_t^1 q(s)f(x(s), x'(s))x'(s)ds \\ \leq& \alpha(1-\alpha)^{-1}\int_0^1 q(s)|f(x(s), x'(s))||x'(s)|ds \\ &+ \int_t^1 q(s)|f(x(s), x'(s))||x'(s)|ds \\ \leq& \epsilon\beta\|x\|\|q\|_1 + \epsilon\|x\|\|q\|_1 \leq 2\epsilon\gamma\|q\|_1\|x\|,\end{aligned}$$

which, by the choice of ϵ (see (2.3)) gives that

$$x \in \mathbb{K}_+ \cap \partial B(0, M) \Rightarrow \|A_+x\| \leq \|x\|. \tag{2.11}$$

Next let $x \in \mathbb{K}_+$ be a function such that $\|x\| = N$. Since x' is non-increasing we have $x'(0) \geq x'(t) \geq x'(1) = \alpha x'(0), \quad t \in [0,1]$. This chain of inequalities implies that

$$N = x'(0) \geq x'(t) > 0, \quad t \in [0,1]$$

and

$$x'(t) \geq x'(1) = \alpha x'(0) = \alpha N = \frac{1}{\Delta(\zeta; +\infty, +\infty)}, \quad t \in I.$$

Then we have

$$f(x(s), x'(s)) \geq \frac{1}{\zeta}, \;\; s \in I$$

(see A2) and so

$$\begin{aligned}(A_+x)'(1) &= \alpha(1-\alpha)^{-1} \int_0^1 q(s) f(x(s), x'(s)) x'(s) ds \\ &\geq \alpha(1-\alpha)^{-1} \frac{1}{\zeta} x'(1) \|q\|_1 = \alpha^2 (1-\alpha)^{-1} \frac{1}{\zeta} x'(0) \|q\|_1 \\ &= x'(0) = N.\end{aligned}$$

This means that, if x is in $\mathbb{K}_+ \cap \partial B(0, N)$, then

$$\|A_+x\| \geq \|x\|. \tag{2.12}$$

Apply now Theorem 2.1 by taking into account (2.11), (2.12) and Lemma 2.1. So, we conclude that there is a solution x of the problem (1.1, 1.2) satisfying $M \leq x'(t) \leq N$, for all $t \in I$. The latter implies (2.8), since $x(0) = 0$.

Case 2: Assume that $0 < \alpha < 1$ and f satisfies the conditions A3, A4. Then $w = a$ and following the same lines as above we obtain that if $x \in \mathbb{K}_+ \cap \partial B(0, \alpha \Delta(\zeta; (0,0), +\infty))$, then $\|A_+x\| \geq \|x\|$, and if

$$x \in \mathbb{K}_+ \cap \partial B(0, \frac{1}{\alpha \Delta(\epsilon; +\infty, 0)}),$$

then $\|A_+x\| \leq \|x\|$. Then, Lemma 2.1 and Theorem 2.1 imply the desired result.

Case 3: Assume that $\alpha > 1$ and f satisfies the conditions A1 and A2. Then $w = \alpha^{-1}$, $\beta = (\alpha - 1)^{-1}$ and equation (1.1) becomes

$$x'' - q(t) f(x, x') x' = 0, \;\; \text{a.a. } t \in I. \tag{2.13}$$

We transform the problem (2.13, 1.2) into the functional equation (2.1), where the operator $A := A_-$ is now defined by

$$\begin{aligned}(A_-x)(t) :=& t(\alpha-1)^{-1} \int_0^1 q(s) f(x(s), x'(s)) x'(s) ds \\ &+ \int_0^t \int_0^s q(r) f(x(r), x'(r)) x'(r) \, dr \, ds\,.\end{aligned}$$

Here we consider the cone

$$\mathbb{K}_- := \{x \in C_0^1(I) : x \geq 0, x' \text{ is non-decreasing and } x'(1) = \alpha x'(0)\}$$

and restrict the operator A_- to the set

$$\mathbb{K}_- \cap [B(0,N_1) \setminus \mathrm{cl}\, B(0,M_1)]$$

where

$$N_1 := \frac{\alpha}{\Delta(\zeta;+\infty,+\infty)} \quad \text{and} \quad M_1 := \frac{1}{\alpha}\Delta(\epsilon;(0,0),0).$$

Observe that, if $x \in \mathbb{K}_- \cap \partial B(0,N_1)$, then $0 \le x'(0) \le x'(t) \le x'(1) = N_1$, $t \in I$ and so

$$x'(t) \ge x'(0) = \frac{1}{\alpha}x'(1) = \frac{1}{\Delta(\zeta;+\infty,+\infty)}.$$

This means that $f(x(t),x'(t)) \ge 1/\zeta$ for $t \in I$; therefore,

$$\begin{aligned}(A_-x)'(0) =&(\alpha-1)^{-1}\int_0^1 q(s)f(x(s),x'(s))x'(s)ds\\ \ge&(\alpha-1)^{-1}\frac{1}{\zeta}\|q\|_1 x'(0) = (\alpha(\alpha-1))^{-1}\frac{1}{\zeta}\|q\|_1 N_1 = N_1,\end{aligned}$$

which implies that

$$\|A_-x\| \ge \|x\|. \tag{2.14}$$

Similarly, if $x \in \mathbb{K}_- \cap \partial B(0,M_1)$, then $\|x\| = M_1$. Thus $|x'(t)| \le \Delta(\epsilon;(0,0),0)$, $t \in I$ and so $0 \le x(t) \le \Delta(\epsilon;(0,0),0)$. Finally we obtain

$$\begin{aligned}0 \le (A_-x)'(t) =&(\alpha-1)^{-1}\int_0^1 q(s)f(x(s),x'(s))x'(s)ds\\ &+\int_0^t q(s)f(x(s),x'(s))x'(s)ds\\ \le&(\alpha-1)^{-1}\epsilon\|q\|_1 M_1 + \epsilon\|q\|_1 M_1 \le M_1,\end{aligned}$$

and so $\|A_-x\| \le \|x\|$. Taking into account this inequality, (2.14) and Lemma 2.1 we apply Theorem 2.1 and get the result.

Case 4: Assume that $\alpha > 1$ and f satisfies the conditions A3 and A4. Then, as in Case 2, we obtain that if

$$x \in \mathbb{K}_- \cap \partial B(0, \frac{1}{\alpha}\Delta(\zeta;(0,0),+\infty))\,,$$

then $\|A_-x\| \ge \|x\|$, and if

$$x \in \mathbb{K}_- \cap \partial B(0, \frac{\alpha}{\Delta(\epsilon;+\infty,0)})\,,$$

then $\|A_-x\| \le \|x\|$. These facts together with Lemma 2.1 and Theorem 2.1 imply the result and the proof is complete. $\diamondsuit$

Now consider the problem (1.1, 1.3). Assume for the moment that x is a solution of it and let

$$y(t) := x(1-t), \;\; t \in I.$$

Then observe that y satisfies the boundary-value problem

$$\begin{gathered}y'' + \mathrm{sign}(1-\hat\alpha)\hat q(t)\hat f(y(t),y'(t))y'(t) = 0\\ y(0) = 0, \quad y'(1) = \hat\alpha y'(0)\,,\end{gathered}$$

where $\hat\alpha := \alpha^{-1}$, $\hat q(t) := q(1-t)$ and $\hat f(u,v) := f(u,-v)$. Clearly this problem is the same with (1.1, 1.2) discussed above. So, by using this transformation and Theorem 2.2 we conclude the following.

Theorem 2.3. *Consider the boundary-value problem (1.1, 1.3), where $\alpha > 0$ with $\alpha \neq 1$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function continuous on $(\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ and such that $vf(u,v) \leq 0$ for all u, v. Also, let $q : I \to \mathbb{R}^+$ be a Lebesgue integrable function with norm $\|q\|_1$. Let $w, \beta, \gamma, \epsilon, \zeta$ be the constants defined in (2.2) and (2.3).*

If the function f satisfies

$$\lim_{\substack{u\to 0+\\ v\to 0-}} f(u,v) = 0 \ \ and \ \lim_{v\to -\infty} f(u,v) = +\infty, \ \ uniformly \ for \ all \ u \geq 0,$$

then the boundary-value problem (1.1, 1.3) admits a solution $x(t)$, $t \in I$ satisfying

$$w\Delta(\epsilon; (0,0), 0)(1-t) \leq x(t) \leq \frac{1-t}{w\Delta(\zeta; +\infty, +\infty)}, \quad t \in I,$$

while, if the function f satisfies

$$\lim_{\substack{u\to 0+\\ v\to 0-}} f(u,v) = +\infty \ \ and \ \lim_{v\to -\infty} f(u,v) = 0, \ \ uniformly \ for \ all \ u > 0,$$

then the boundary-value problem (1.1, 1.3) admits a solution $x(t)$, $t \in I$ satisfying

$$w\Delta(\zeta; (0,0), +\infty)(1-t) \leq x(t) \leq \frac{1-t}{w\Delta(\epsilon; +\infty, 0)}, \quad t \in I.$$

3. Two applications

(i) Consider the boundary-value problem

$$x'' + \frac{1}{2\sqrt{t}}(ax^{2\mu} + b{x'}^{2\mu}){x'}^2 = 0, \quad t \in [0,1] \tag{3.1}$$

$$x(0) = 0, \ \ x'(1) = \frac{1}{2}x'(0) \tag{3.2}$$

where $a \geq 0$, $b > 0$ and μ is any positive integer. Observe that for the function $f(u,v) := (au^{2\mu} + bv^{2\mu})v$ the assumptions A1 and A2 are satisfied. Here we have $w = 1/2$, $\beta = 1$, $\gamma = 1$, $\epsilon = \zeta = 1/2$ and

$$\Delta(\frac{1}{2}; (0,0), 0) := (2(a+b))^{-\frac{1}{2\mu+1}},$$

$$\Delta(\frac{1}{2}; +\infty, +\infty) := \Big(\frac{b}{2}\Big)^{\frac{1}{2\mu+1}}.$$

Hence, there is a solution x of the boundary-value problem (3.1)-(3.2) such that

$$\frac{1}{2^{2(\mu+1)}(a+b)} t^{2\mu+1} \leq x(t)^{2\mu+1} \leq \frac{2^{2(\mu+1)}}{b} t^{2\mu+1}, \ \ t \in [0,1].$$

(ii) Consider the one-parameter differential equation

$$x'' + \lambda {x'}^2 = 0, \ \ \text{on } I,$$

associated with the conditions (1.2) with $0 < \alpha < 1$ and $\lambda > 0$. Applying Theorem 2.2 (Case 1) we conclude that there is a solution x satisfying

$$\frac{1}{2\lambda}\min\{\alpha, 1-\alpha\}t \leq x(t) \leq \frac{(1-\alpha)t}{\lambda\alpha^3}, \quad t \in [0,1].$$

Indeed, such a solution (and only this) is given by $x(t) := \frac{1}{\lambda}\ln\big(1 + (1-\alpha)\alpha^{-1}t\big)$, $t \in [0,1]$.

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