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ATTRACTORS FOR DISSIPATIVE LATTICE DIFFERENTIAL EQUATIONS WITH LOCAL AND NONLOCAL NONLINEARITIES

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Abstract. We study the dynamics of some dissipative lattice differential equations with local and nonlocal nonlinearities. Using a difference inequality due to Nakao [14] and suitable estimates, we prove the existence of global attractors. In addition, we briefly discuss the dynamics of two periodic lattice differential equations.

1. Introduction

Lattice differential equations (LDEs) are used to model various problems that occur in important areas of science and technology, for instance, in biophysics [13], electrical engineering [4], image processing [5], chemical reaction theory [10], etc.. They also naturally arise as spatial discretization of continuous models. The dynamics of nonlinear LDEs is a wide-ranging theme that includes as subjects the existence of solutions, existence and stability of traveling waves, asymptotic behavior, attractors and their properties, etc., see e.g. [1, 2, 3, 8, 20, 25] and the references therein. In particular, the existence of attractors for LDEs is a subject that attracts a great deal of attention. In this article, we study the dynamics of some dissipative LDEs with local and nonlocal nonlinearities in arbitrary spatial dimensions. Our main objective is to prove the existence of global attractors. Firstly, we consider the following class of second order LDEs

$$\begin{gathered}\ddot{u}_n(t) + (-1)^p \Delta_d^p u_n(t) + \alpha u_n(t) + F(n, u_n(t), \nabla^+ u_n(t)) + g(n, \dot{u}_n(t)) = f_n, \\ u_n(0) = u_{0,n}, \quad \dot{u}_n(0) = u_{1,n}\,,\end{gathered} \tag{1.1}$$

where $n \in \mathbb{Z}^d$ and $t \in \mathbb{R}^+$. In (1.1), α is a positive constant, p is any positive integer, $\Delta_d^p = \Delta_d \circ \cdots \circ \Delta_d$, p times, and Δ_d denotes the d-dimensional discrete Laplacian operator defined by $\Delta_d u_n = \sum_{i=1}^d (u_{n+e_i} + u_{n-e_i} - 2u_n)$, where $\{e_i\}_{i=1}^d$ is the canonical basis of $\mathbb{R}^n$. We assume that the nonlinear term $F(n, u_n(t), \nabla^+ u_n(t))$ has the form

$$F(n, u_n(t), \nabla^+ u_n(t)) = h_0(n, u_n(t)) - \sum_{i=1}^d \partial_i^- h_i(\partial_i^+ u_n(t)), \tag{1.2}$$

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where $h_0 : \mathbb{Z}^d \times \mathbb{R} \to \mathbb{R}$, $h_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, d$, are functions satisfying appropriate assumptions stated in Section 2 and $\partial_i^+ u_n = u_{n+e_i} - u_n$ and $\partial_i^- u_n = u_n - u_{n-e_i}$, with $i = 1, \dots, d$. Moreover, $g(n, \dot{u}_n(t))$ is a nonlinear dissipation.

Many papers in the literature deal with the existence of attractors for LDEs. Here, we just mention some works that are related to our model (1.1). The existence of attractor and finite-dimensional approximations for the case $p = 1$, $h_i \equiv 0$ for $i = 1, \dots, d$, was studied by Zhou [33]. The existence of a global attractor for the class (1.1) when $d = 1$ and $h_i \equiv 0$ for $i = 1, \dots, d$, was first investigated by Oliveira and Pereira [18]. Later, for this last class with a suitable delay term, the existence of a pullback attractor was established by Wang and Bai [29]. Another paper related to (1.1) is [32] in which the authors studied the existence and the upper semi-continuity of attractors for second order lattices with retarded terms in dimension one ($d = 1$) when $p = 1$. In addition, the investigation of the existence of attractors for second order LDEs in weighted spaces was considered in [7] by Han and some contributions concerning non-autonomous and stochastic LDEs are provided in [6] and [30], respectively.

An important special class of LDEs included in (1.1) is

$$\ddot{u}_n(t) + (-1)^p \Delta_d^p u_n(t) + \alpha u_n(t) + h_0(n, u_n(t)) + g_0(\dot{u}_n(t)) = f_n. \tag{1.3}$$

Models of type (1.3) when $p = 1$ are known as discrete nonlinear Klein-Gordon models and appear in different physical contexts, see e.g. [21, 23]. When $p = 2$, equation (1.3) can be regarded as discrete versions of beam equations. An example of dissipative term for which our results apply is $g_0 \in C^1(\mathbb{R};\mathbb{R})$, $g_0(0) = 0$, and $g_0'(s) \geq c_0 > 0$ for all $s \in \mathbb{R}$. Observe that we do not demand any relation on the parameters α and c_0, and $g_0'(s)$ need not be bounded above.

The inclusion of the term $-\sum_{i=1}^d \partial_i^- h_i(\partial_i^+ u_n)$ in (1.2) was motivated by some studies on the dynamics of nonlinear LDEs in periodic spaces [16, 17, 19], and continuous models of beam equations studied in the literature, see e.g. [12, 24, 31]. Note that if we choose $p = 2$, $h_i(s) = |s|^{q-2}s$, $q \geq 2$, $i = 1, \dots, d$, in (1.1), then we obtain a discrete version of the beam equation

$$u_{tt} + \Delta^2 u + \alpha u + h_0(x, u) - \Delta_q u + g(x, u_t) = f(x),$$

where $x \in \mathbb{R}^d, u = u(x,t)$ and $\Delta_q u = \sum_{i=1}^d \frac{\partial}{\partial x_i}\big(|\frac{\partial u}{\partial x_i}|^{q-2}\frac{\partial u}{\partial x_i}\big)$ is the usual q-Laplacian operator.

Secondly, we study the existence of global attractors for the following class of dissipative LDEs with a nonlocal nonlinearity

$$\begin{gathered} \ddot{u}_n(t) + (-1)^p \Delta_d^p u_n(t) + \alpha u_n(t) + F(u_n(t)) + g(n, \dot{u}_n(t)) = f_n, \\ u_n(0) = u_{0,n}, \quad \dot{u}_n(0) = u_{1,n}, \end{gathered} \tag{1.4}$$

where

$$F(u_n(t)) = h'(u_n(t)) \sum_{m \in \mathbb{Z}^d} V(n-m) h(u_m(t)). \tag{1.5}$$

In (1.4) and (1.5), $g : \mathbb{Z}^d \times \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}^+$, and $V : \mathbb{Z}^d \to \mathbb{R}^+$ are functions satisfying suitable conditions stated in Section 2. To the best of our knowledge, the existence of attractor for LDEs of type (1.4) with a nonlocal nonlinearity as in (1.5) has not been considered before. The continuous convolution term corresponding to (1.5) is known as Hartree-type nonlinearity and appears in Schrödinger and Klein-Gordon equations, see [22, 27].

In addition to the problems described above we also briefly discuss the existence of global attractors for two periodic LDEs. We first consider a periodic problem for the LDE in (1.1) in arbitrary spatial dimension, then we consider the following one-dimensional periodic LDE with $\alpha = 0$ and nonlinear dissipation

$$\ddot{u}_n(t) + (-1)^p \Delta_1^p u_n(t) - \partial_1^- h(\partial_1^+ u_n(t)) - \partial_1^- g(\partial_1^+ \dot{u}_n(t)) = f_n, \tag{1.6}$$

where $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfy appropriate conditions stated in Section 4. Note that when $g(s) = \mu\, s$, with $\mu > 0$, we obtain the strong damping $\mu\, \Delta_1 \dot{u}_n(t)$. When $p = 2$, $g(s) = \mu s$, and $h(s) = \alpha_1 s^5 - \alpha_2 s^3 - \alpha_3 s$, with $\alpha_i > 0, i = 1, 2, 3$, model (1.6) can be regarded as a discrete version of the beam equation studied by Racke and Shang in [24]. However, unlike the continuous model treated in [24], here, the global attractor we obtain exists in the whole phase space where the dynamics is considered.

Our main purpose in this paper is to prove the existence of global attractors for the LDEs (1.1) and (1.4) by combining the use of a difference inequality due to Nakao [14] with a method to derive "tail estimates of solutions" introduced by Wang in [28]. As far as we know, in the context of the dynamics of discrete models, this approach was only used for LDEs of type (1.1) in dimension one in [18] when $h_i(s) \equiv 0$, $i = 1, \dots, d$, under conditions on the nonlinear terms $h_0(n, s)$ and $g(n, s)$ more restrictive than those used in this paper. We note that our assumptions do not require any growth condition on the dissipative term $g(n, s)$. In particular, our results apply to cases with nonlinear dissipative terms such as $g(n, s) = a_n(s^3 + s)$ or $g(n, s) = b_n \sinh s$, when the real constants a_n and b_n are suitably chosen.

This paper is organized as follows. In Section 2, we state the assumptions on the functions h_0, h_i, $i = 1, \dots, d$, in (1.2), g in (1.1), and h and V in (1.5) that we need to prove the existence of solutions and global attractors. Then, after introducing some notation, we briefly discuss the global well-posedness of problems (1.1) and (1.4). In Section 3, we establish the existence of global attractors for the semigroups generated by the solutions of (1.1) and (1.4). We first prove the existence of absorbing sets, then we prove the asymptotic compactness of the semigroups. The proofs are based on Nakao's method [14] and suitable estimates. In Section 4, we show how some arguments used in Section 3 can be adapted to prove the existence of global attractors for the periodic problems described above. In the case of model (1.6), we also used a Poincaré inequality valid for the periodic space where the problem is considered. Our results, in particular, generalize and complement the studies of [16, 17, 18]. Finally, in appendix A, we present examples of functions that satisfy some assumptions used in this paper.

2. Existence of solutions

In this section, we briefly discuss the existence of solutions for the initial value problems (1.1) and (1.4). We begin establishing the appropriate assumptions on the functions in (1.1), (1.2), and (1.5) and introducing some notation. We denote by ℓ^p the space of real sequences $u = (u_n)_{n \in \mathbb{Z}^d}$ such that $\|u\|_{\ell^p} < \infty$, where

$$\|u\|_{\ell^p} = \Big(\sum_{n \in \mathbb{Z}^d} |u_n|^p \Big)^{1/p}, \quad \text{if } 1 \le p < \infty,$$
$$\|u\|_{\ell^\infty} = \sup_{n \in \mathbb{Z}^d} |u_n|, \quad \text{if } p = \infty.$$

When $p=2$, ℓ^2 is a Hilbert space with the inner product

$$(u,v)_{\ell^2}=\sum_{n\in\mathbb{Z}^d} u_n\, v_n, \quad u,v\in\ell^2.$$

In this case, we denote by $\|\cdot\|$ the corresponding norm. Also, to simplify notation, we denote a sequence $(u_n)_{n\in\mathbb{Z}^d}$ by (u_n).

We recall that for the ℓ^p spaces the following embedding relation holds:

$$\ell^q\subset\ell^p, \quad \|u\|_{\ell^p}\le\|u\|_{\ell^q}, \quad 1\le q\le p\le\infty.$$

For the functions $h_0:\mathbb{Z}^d\times\mathbb{R}\to\mathbb{R}$, $h_i:\mathbb{R}\to\mathbb{R}, i=1,\dots,d$, and $g:\mathbb{Z}^d\times\mathbb{R}\to\mathbb{R}$, we assume that

(A1) For each $s_0>0$, there exist positive constants $L_j=L_j(s_0)$, $j=1,2,3$, such that
  - (i) $|h_0(n,s_1)-h_0(n,s_2)|\le L_1|s_1-s_2|$,
  - (ii) $|g(n,s_1)-g(n,s_2)|\le L_2|s_1-s_2|$,
  - (iii) $|h_i(s_1)-h_i(s_2)|\le L_3|s_1-s_2|$,

  for all s_1,s_2 in $\mathbb{R}$, $|s_1|\le s_0$, $|s_2|\le s_0$, for all $n\in\mathbb{Z}^d$ and $i=1,\dots,d$. In addition, $h_0(n,0)=0$, $g(n,0)=0$, $h_i(0)=0$, for all $n\in\mathbb{Z}^d$ and $i=1,\dots,d$.

(A2) There exist sequences of nonnegative real numbers $b_1=(b_{1,n})\in\ell^1$, $b_2=(b_{2,n})\in\ell^1$, and a positive constant k_1 such that

$$sh_0(n,s)+b_{1,n}\ge k_1(\tilde{h}_0(n,s)+b_{2,n})\ge 0, \quad \forall s\in\mathbb{R} \text{ and } n\in\mathbb{Z}^d,$$

  where $\tilde{h}_0(n,s)=\int_0^s h_0(n,\sigma)d\sigma$.

(A3) There exist positive constants $k_{0,i}$ such that

$$sh_i(s)\ge k_{0,i}\tilde{h}_i(s), \quad \forall s\in\mathbb{R} \text{ and } i=1,\dots,d, \text{ where } \tilde{h}_i(s)=\int_0^s h_i(\sigma)d\sigma\ge 0.$$

(A4) There exist constants $k_2>0$, $r\ge 0$ and a positive integer n_0 satisfying

$$sg(n,s)\ge k_2|s|^{r+2}, \quad \text{if } |n|_0\le n_0,$$
$$sg(n,s)\ge k_2|s|^{2}, \quad \text{if } |n|_0> n_0,$$

  for all $s\in\mathbb{R}$, where $|n|_0=\max_{1\le i\le d}|n_i|$, if $n=(n_1,\dots,n_d)$.

Regarding the LDE (1.1) with the nonlocal term (1.4), we assume that g satisfies the same hypotheses above and that $h:\mathbb{R}\to\mathbb{R}^+$ and $V:\mathbb{Z}^d\to\mathbb{R}^+$ satisfy

(A5) $h\in C^2(\mathbb{R};\mathbb{R}^+)$, $h(0)=h'(0)=0$ and there exists a positive constant c_1 such that $sh'(s)\ge c_1h(s)\ge 0$ for all $s\in\mathbb{R}$.

(A6) $V=(V(m))\in\ell^2$ and $V(m)=V(-m)$ for all $m\in\mathbb{Z}^d$.

The following notation will be useful.

$$\nabla^+u_n=(\partial_1^+u_n,\dots,\partial_d^+u_n), \quad \nabla^-u_n=(\partial_1^-u_n,\dots,\partial_d^-u_n),$$
$$\nabla^+u_n\cdot\nabla^+v_n=\sum_{i=1}^d\partial_i^+u_n\partial_i^+v_n, \quad |\nabla^+u_n|^2=\nabla^+u_n\cdot\nabla^+u_n,$$
$$D^pu_n=\begin{cases}\Delta_d^{p/2}u_n, & \text{if } p \text{ is even}\\ \nabla^+\big(\Delta_d^{\frac{p-1}{2}}u_n\big), & \text{if } p \text{ is odd,}\end{cases} \tag{2.1}$$

where $\Delta_d^0=I$.

Lemma 2.1. *If $u=(u_n)\in\ell^2$ then $\sum_{n\in\mathbb{Z}^d}|D^pu_n|^2\le(4d)^p\|u\|^2$.*

Proof. Using the elementary inequality $\big(\sum_{i=1}^d a_i\big)^2 \le d\sum_{i=1}^d a_i^2$, $a_i \in \mathbb{R}$, and arguing by induction we find that

$$\sum_{n\in\mathbb{Z}^d} |\Delta_d^k u_n|^2 \le (4d)^{2k} \sum_{n\in\mathbb{Z}^d} |u_n|^2,\ \forall k \in \mathbb{N}. \tag{2.2}$$

In (2.2) and hereafter $\mathbb{N}$ denotes the set of all positive integers. If p is odd then, from (2.1) and (2.2), we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}^d} |D^p u_n|^2 &= \sum_{n\in\mathbb{Z}^d} |\nabla^+(\Delta_d^{\frac{p-1}{2}} u_n)|^2 = \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d |\partial_i^+ \Delta_d^{\frac{p-1}{2}} u_n|^2 \\
&\le 4\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d |\Delta_d^{\frac{p-1}{2}} u_n|^2 = 4d\sum_{n\in\mathbb{Z}^d} |\Delta_d^{\frac{p-1}{2}} u_n|^2 \\
&\le 4d(4d)^{p-1}\|u\|^2 = (4d)^p\|u\|^2.
\end{aligned}$$

Similarly, using (2.1) and (2.2) we can treat the case when p is even. □

Lemma 2.2. *For any $u = (u_n)$ and $v = (v_n)$ in ℓ^2 we have*

$$(-1)^p \sum_{n\in\mathbb{Z}^d} (\Delta_d^p u_n) v_n = \begin{cases} \sum_{n\in\mathbb{Z}^d} \Delta_d^{p/2} u_n \Delta_d^{p/2} v_n, & \text{if } p \text{ is even} \\ \sum_{n\in\mathbb{Z}^d} \nabla^+(\Delta_d^{\frac{p-1}{2}} u_n)\cdot\nabla^+(\Delta_d^{\frac{p-1}{2}} v_n), & \text{if } p \text{ is odd.} \end{cases}$$

Proof. Since $u = (u_n)$ and $v = (v_n)$ belong to ℓ^2, we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}^d} (\Delta_d u_n) v_n &= \sum_{i=1}^d \sum_{n\in\mathbb{Z}^d} (\partial_i^+ u_n) v_n - \sum_{i=1}^d \sum_{n\in\mathbb{Z}^d} (\partial_i^- u_n) v_n \\
&= \sum_{i=1}^d \sum_{n\in\mathbb{Z}^d} (\partial_i^+ u_n) v_n - \sum_{i=1}^d \sum_{n\in\mathbb{Z}^d} (\partial_i^+ u_n) v_{n+e_i} \\
&= -\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d \partial_i^+ u_n \partial_i^+ v_n = -\sum_{n\in\mathbb{Z}^d} \nabla^+ u_n \cdot \nabla^+ v_n.
\end{aligned}$$

This proves Lemma 2.2 if $p = 1$. The general case follows by induction on p. □

In what follows, given a sequence $u = (u_n)$, we will write

$$\begin{gathered}
h_0(u) = (h_0(n, u_n)), \quad g(u) = (g(n, u_n)), \\
Au = ((-1)^p \Delta_d^p u_n), \quad B(u) = \Big(-\sum_{i=1}^d \partial_i^- h_i(\partial_i^+ u_n)\Big).
\end{gathered} \tag{2.3}$$

Also, we will use the Hilbert space $H = \ell^2 \times \ell^2$ equipped with the usual inner product and norm,

$$((u,v),(w,z))_H = (u,w)_{\ell^2} + (v,z)_{\ell^2} \quad \text{and} \quad \|(u,v)\|_H = \big(\|u\|^2 + \|v\|^2\big)^{1/2},$$

for any (u,v) and (w,z) in H.

Lemma 2.3. *Under assumption (A1), we have*

(i) *h_0, B, and g are locally Lipschitz continuous maps from ℓ^2 into itself.*
(ii) *$A : \ell^2 \to \ell^2$ is a bounded operator and $\|Au\| \le (4d)^{p/2}\|u\|$ for all $u \in \ell^2$.*

Proof. (i) Given $u = (u_n)$ in ℓ^2, using (A1), we have

$$\begin{aligned}
\|B(u)\|^2 &= \sum_{n\in\mathbb{Z}^d} \Big| \sum_{i=1}^d \partial_i^- h_i(\partial_i^+ u_n) \Big|^2 \le d \sum_{n\in\mathbb{Z}^d} \sum_{i=1}^d \Big|\partial_i^- h_i(\partial_i^+ u_n)\Big|^2 \\
&\le d \sum_{n\in\mathbb{Z}^d} \sum_{i=1}^d |h_i(\partial_i^+ u_n) - h_i(\partial_i^- u_n)|^2 \\
&\le dL_3(2\|u\|)^2 \sum_{n\in\mathbb{Z}^d} \sum_{i=1}^d \big(|\partial_i^+ u_n - \partial_i^- u_n|^2\big) \\
&\le (4d)^2 L_3(2\|u\|)^2 \|u\|^2 < \infty.
\end{aligned}$$

If $u = (u_n)$ and $v = (v_n)$ belong to ℓ^2, with $\|u\| \le R$ and $\|v\| \le R$, then, using (A1) again, we have

$$\begin{aligned}
\|B(u) - B(v)\|^2 &\le d \sum_{n\in\mathbb{Z}^d} \sum_{i=1}^d |\partial_i^- h_i(\partial_i^+ u_n) - \partial_i^- h_i(\partial_i^+ v_n)|^2 \\
&= \sum_{n\in\mathbb{Z}^d} \sum_{i=1}^d |h_i(\partial_i^+ u_n) - h_i(\partial_i^+ v_n) - [h_i(\partial_i^- u_n) - h_i(\partial_i^- v_n)]|^2 \\
&\le 2dL_3(2R)^2 \sum_{n\in\mathbb{Z}^d} \sum_{i=1}^d \big(|\partial_i^+ (u_n - v_n)|^2 + |\partial_i^- (u_n - v_n)|^2\big) \\
&\le (4d)^2\, L_3(2R)^2 \|u - v\|^2.
\end{aligned}$$

This shows that B is a locally Lipschitz continuous map from ℓ^2 into itself. Similarly, we prove that $h_0, g : \ell^2 \to \ell^2$ are locally Lipschitz continuous maps.

(ii) It follows immediately from (2.2). □

Using Lemma 2.3 we can write the initial value problem (1.1) in ℓ^2 as

$$\begin{gathered}
\ddot{u}(t) + Au(t) + \alpha u(t) + h_0(u(t)) + B(u(t)) + g(\dot{u}(t)) = f,\ t > 0, \\
u(0) = u_0, \quad \dot{u}(0) = u_1,
\end{gathered} \tag{2.4}$$

where $u_0 = (u_{0,n})$, $u_1 = (u_{1,n})$, $f = (f_n)$, $u(t) = (u_n(t))$, $\dot{u}(t) = (\dot{u}_n(t))$ and $\ddot{u}(t) = (\ddot{u}_n(t))$.

Theorem 2.4. *Assume that* (A1)–(A3) *hold and let u_0, u_1 and f belong to ℓ^2. Assume also that $sg(n, s) \ge 0$ for all $n \in \mathbb{Z}^d$ and $s \in \mathbb{R}$. Then the initial value problem* (2.4) *has a unique solution $u \in C^2(\mathbb{R}^+; \ell^2)$. Moreover, for each $\tau > 0$, the map $\mathfrak{I} : H \to C([0, \tau]; H)$, defined by $\mathfrak{I}(u_0, u_1)(t) = (u(t), \dot{u}(t))$, $0 \le t \le \tau$, is continuous.*

Proof. Introducing the change of variable $\dot{u} = v$ we can rewrite problem (2.4) in the space H as

$$\begin{gathered}
\frac{dw}{dt}(t) + \mathcal{B}w(t) = 0, \quad t > 0, \\
w(0) = w_0,
\end{gathered}$$

where $w = (u, v)$, $\frac{dw}{dt} = (\dot{u}, \dot{v})$, $w_0 = (u_0, u_1)$, and

$$\mathcal{B}(w) = (-v, Au + \alpha u + h_0(u) + B(u) + g(v) - f). \tag{2.5}$$

Using Lemma 2.3 we can easily prove that the map $\mathcal{B}$ defined by (2.5) is a locally Lipschitz continuous map from H into itself. Then, an application of the Theory of Ordinary Differential Equations in Banach Spaces shows that the initial value problem (2.4) has a unique solution $u \in C^2([0,\tau_{\max});\ell^2)$ such that either $\tau_{\max} = \infty$ or $\tau_{\max} < \infty$ and $\lim_{t\to\tau_{\max}^-} \|(u(t),\dot u(t))\|_H = \infty$.

To extend the solution globally we proceed as follows. Taking the inner product of equation (2.4) with $\dot u(t)$ in ℓ^2 and using Lemma 2.2 we find

$$\frac{d}{dt}E(t) = -(g(\dot u(t)),\dot u(t))_{\ell^2} \le 0, \quad \forall 0 \le t < \tau_{\max}, \tag{2.6}$$

where $E(t)$ is the energy associated with the initial value problem (2.4) given by

$$\begin{aligned} E(t) &= \frac12\|\dot u(t)\|^2 + \frac12\|D^p u(t)\|^2 + \frac{\alpha}{2}\|u(t)\|^2 + \sum_{n\in\mathbb{Z}^d}\tilde h_0(n,u_n(t)) \\ &\quad + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d \tilde h_i(\partial_i^+ u_n) - (f,u(t))_{\ell^2}. \end{aligned} \tag{2.7}$$

In (2.7) and hereafter $\|D^p u\|^2 = \sum_{n\in\mathbb{Z}^d}|D^p u_n|^2$. Thus, $E(t) \le E(0)$, for all $0 \le t < \tau_{\max}$. Since $|(f,u)_{\ell^2}| \le \frac{\alpha}{4}\|u\|^2 + \frac{4}{\alpha}\|f\|^2$, using (A2), (A3) and (2.7) we deduce that

$$\|(u(t),\dot u(t))\|_H^2 \le \alpha_0^{-1}\tilde E(t), \quad \forall 0 \le t < \tau_{\max}, \tag{2.8}$$

where $\alpha_0 = \min\{\frac12, \frac{\alpha}{4}\}$ and

$$\tilde E(t) = E(t) + \frac{4}{\alpha}\|f\|^2 + \|b_2\|_{\ell^1}, \quad \forall 0 \le t < \tau_{\max}. \tag{2.9}$$

From (2.8) and (2.9) we conclude that $\tau_{\max} = \infty$. Finally, under the assumptions of Theorem 2.4, the continuity of $\mathfrak{I}$ can be proved using (2.8) and the Gronwall inequality. Since the arguments are well known, we omit the details here. $\square$

Now, let us consider the initial value problem (1.4) with the nonlocal term (1.5). By assumptions (A5) and (A6) we can define the map $F : \ell^2 \to \ell^2$ by

$$F(u) = \Big(h'(u_n)\sum_{m\in\mathbb{Z}^d} V(n-m)h(u_m)\Big), \quad \forall u = (u_n) \in \ell^2.$$

Then, using the above notation, we can write (1.4) in the space ℓ^2 as

$$\begin{gathered} \ddot u(t) + Au(t) + \alpha u(t) + F(u(t)) + g(\dot u(t)) = f, \quad t > 0, \\ u(0) = u_0, \quad \dot u(0) = u_1, \end{gathered} \tag{2.10}$$

To see that F is a locally Lipschitz continuous map from ℓ^2 into iself, let $u = (u_n)$ and $v = (v_n)$ in ℓ^2 such that $\|u\| \le R$ and $\|v\| \le R$. Since $h \in C^2(\mathbb{R};\mathbb{R}^+)$, $h(0) = h'(0) = 0$ by (A5) and $V = (V(m)) \in \ell^2$ by (A6) and $|u_n| \le R$ and $|v_n| \le R$, for all $n \in \mathbb{Z}^d$, then

$$\begin{aligned} \|F(u) - F(v)\|^2 &\le 2\sum_{n\in\mathbb{Z}^d}|h'(u_n) - h'(v_n)|^2\Big(\sum_{m\in\mathbb{Z}^d} V(n-m)|h(u_m)|\Big)^2 \\ &\quad + 2\sum_{n\in\mathbb{Z}^d}|h'(v_n)|^2\Big(\sum_{m\in\mathbb{Z}^d} V(n-m)|h(u_m) - h(v_m)|\Big)^2 \\ &\le 2M_1^2M_2^2\|V\|^2\|u\|^2\|u-v\|^2 + 2M_1^2M_2^2\|V\|^2\|v\|^2\|u-v\|^2 \\ &\le 4R^2M_1^2M_2^2\|V\|^2\|u-v\|^2, \end{aligned}$$

where $M_1 = \max_{|s|\leq 2R} |h'(s)|$ and $M_2 = \max_{|s|\leq 2R} |h''(s)|$.

Proceeding as we did for the initial value problem (1.1), we can prove the following results.

Theorem 2.5. *Assume* (A1)-(ii)*, with* $g(n,0) = 0$ *for all* $n \in \mathbb{Z}^d$*,* (A5), (A6) *and let* u_0*,* u_1*, and* f *belong to* ℓ^2*. Assume also that* $sg(n,s) \geq 0$ *for all* $n \in \mathbb{Z}^d$ *and* $s \in \mathbb{R}$*. Then the initial value problem* (2.10) *has a unique solution* $u \in C^2(\mathbb{R}^+; \ell^2)$*. Moreover, for each* $\tau > 0$*, the map* $\mathfrak{I} : H \to C([0,\tau]; H)$*, defined by* $\mathfrak{I}(u_0, u_1)(t) = (u(t), \dot{u}(t))$*,* $0 \leq t \leq \tau$*, is continuous.*

In this case, as before, we obtain the identity (2.6) with the energy function

$$\begin{aligned} E(t) &= \frac{1}{2}\|\dot{u}(t)\|^2 + \frac{1}{2}\|D^p u(t)\|^2 + \frac{\alpha}{2}\|u(t)\|^2 \\ &\quad + \frac{1}{2}\sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d} V(n-m)h(u_m(t))h(u_n(t)) - (f, u(t))_{\ell^2}. \end{aligned} \tag{2.11}$$

The same inequality in (2.8) can be derived with

$$\tilde{E}(t) = E(t) + \frac{4}{\alpha}\|f\|^2, \quad \forall 0 \leq t < \infty. \tag{2.12}$$

3. Existence of global attractors

Our aim in this section is to prove the existence of global attractors for the semigroups generated by the solutions of the initial value problems (2.4) and (2.10). Let us first consider the initial value problem (2.4). Using Theorem 2.4 we can define a semigroup of continuous operators $\{S(t)\}_{t\geq 0}$ on H as follows

$$S(t)(u_0, u_1) = (u(t), \dot{u}(t)), \quad \forall (u_0, u_1) \in H. \tag{3.1}$$

To prove the existence of a global attractor for $\{S(t)\}_{t\geq 0}$ in H it is sufficient to prove that $\{S(t)\}_{t\geq 0}$ has an absorbing set in H and that it is asymptotically compact in H, see e.g. [26]. Our proofs are based on a difference inequality by Nakao as stated in Lemma 3.1 below. This difference inequality was introduced in [14] to study the existence of attractors for some nonlinear wave equations with nonlinear dissipation. Some other applications to the study of the dynamics of continuous models can be seen in [9, 11, 15]. In the context of LDEs, it was used in [16, 17, 18].

Lemma 3.1 (Nakao [14])**.** *Let* $\psi(t)$ *be a nonnegative continuous function on* $[0, T)$*,* $T > 1$*, possibly* $T = \infty$*, satisfying*

$$\sup_{t\leq s\leq t+1} \psi(s)^{1+\gamma} \leq C[\psi(t) - \psi(t+1)] + K, \quad \forall 0 \leq t < T - 1, \tag{3.2}$$

with some $C > 0$*,* $K > 0$ *and* $\gamma > 0$*. Then*

$$\psi(t) \leq \big[C^{-1}\gamma(t-1)^+ + \big(\sup_{0\leq s\leq 1}\psi(s)\big)^{-\gamma}\big]^{-\frac{1}{\gamma}} + K^{\frac{1}{\gamma+1}}, \quad 0 \leq t < T.$$

If (3.2) *holds with* $\gamma = 0$*, then*

$$\psi(t) \leq \sup_{0\leq s\leq 1}\psi(s)\Big(\frac{C}{C+1}\Big)^{[t]} + K, \quad 0 \leq t < T,$$

where $[t]$ *is the largest integer less than or equal to* t *and* $\beta^+ = \max\{\beta, 0\}$*.*

Let us now introduce some notation that will be used in this section. We first observe that if (A4) holds then, in particular, we have

$$sg(n,s) \geq 0, \forall n \in \mathbb{Z}^d \text{ and } s \in \mathbb{R}.$$

Let

$$P(t)^2 = E(t) - E(t+1) \quad \text{and} \quad Q(t) = \sum_{n\in\mathbb{Z}^d} \dot{u}_n(t) g(n, \dot{u}_n(t)), \quad \forall t \geq 0.$$

According to (2.6), $E(t)$ is a non-increasing function on $[0,\infty)$ and

$$P(t)^2 = \int_t^{t+1} Q(s)\,ds, \quad \forall t \geq 0. \tag{3.3}$$

Also, by the Mean Value Theorem for integrals, there exist real numbers $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\int_t^{t+\frac{1}{4}} Q(s)\,ds = \frac{1}{4} Q(t_1) \quad \text{and} \quad \int_{t+\frac{3}{4}}^{t+1} Q(s)\,ds = \frac{1}{4} Q(t_2). \tag{3.4}$$

To simplify notation we will write $\hat{h}(u) = h_0(u) + B(u)$ for all $u \in \ell^2$.

Lemma 3.2. *Assume that* (A1)–(A4) *hold and let* u_0*,* u_1*, and* f *belong to* ℓ^2*. Then there exists a positive constant* C_0 *such that*

$$\sup_{t\leq s\leq t+1} \tilde{E}(s) \leq C_0\big(P(t)^{\frac{4}{r+2}} + P(t)^4 + P(t)^2 + \|f\|^2 + \|b_1\|_{\ell^1}\big), \forall t \geq 0. \tag{3.5}$$

Proof. Taking the inner product of 2.4 with $u = u(t)$ in ℓ^2, using Lemma 2.2, and integrating the result over $[t_1, t_2]$ we obtain

$$\begin{aligned}
&\int_{t_1}^{t_2} \|D^p u(s)\|^2 ds + \alpha \int_{t_1}^{t_2} \|u(s)\|^2 ds + \int_{t_1}^{t_2} (\hat{h}(u(s)), u(s))_{\ell^2} ds \\
&= (\dot{u}(t_1), u(t_1))_{\ell^2} - (\dot{u}(t_2), u(t_2))_{\ell^2} + \int_{t_1}^{t_2} \|\dot{u}(s)\|^2 ds \\
&\quad - \int_{t_1}^{t_2} (g(\dot{u}(s)), u(s))_{\ell^2}\,ds + \int_{t_1}^{t_2} (f, u(s))_{\ell^2}\,ds.
\end{aligned} \tag{3.6}$$

Let us estimate the terms in the right hand side of (3.6). We initially write

$$(\dot{u}(t_j), u(t_j))_{\ell^2} = \sum_{|n|_0 \leq n_0} \dot{u}_n(t_j) u_n(t_j) + \sum_{|n|_0 > n_0} \dot{u}_n(t_j) u_n(t_j), \quad j = 1, 2, \tag{3.7}$$

with n_0 as in (A4). Using (2.8) we have

$$\Big(\sum_{|n|_0 \leq n_0} |u_n(t_j)|^2\Big)^{1/2} \leq \alpha_0^{-1/2} \sup_{t\leq s\leq t+1} \tilde{E}(s)^{1/2}. \tag{3.8}$$

Using Hölder's inequality, (A4), (3.3), and (3.4) we have

$$\begin{aligned}
\sum_{|n|_0 \leq n_0} |\dot{u}_n(t_j)|^2 &\leq (2n_0+1)^{\frac{rd}{r+2}} \Big(\sum_{|n|_0 \leq n_0} |\dot{u}_n(t_j)|^{r+2}\Big)^{\frac{2}{r+2}} \\
&\leq (2n_0+1)^{\frac{rd}{r+2}} \Big(k_2^{-1} \sum_{|n|_0 \leq n_0} g(n, \dot{u}_n(t_j)) \dot{u}_n(t_j)\Big)^{\frac{2}{r+2}} \\
&\leq (2n_0+1)^{\frac{rd}{r+2}} k_2^{-\frac{2}{r+2}} 4^{\frac{2}{r+2}} P(t)^{\frac{4}{r+2}}.
\end{aligned} \tag{3.9}$$

Similarly, we can estimate the second term in (3.7) to find

$$\sum_{|n|_0>n_0} |\dot{u}_n(t_j)u_n(t_j)| \le 2\alpha_0^{-1/2}k_2^{-1/2}P(t) \sup_{t\le s\le t+1} \tilde{E}(s)^{1/2}. \tag{3.10}$$

From (3.7)-(3.10) we conclude that there is a positive constant $C_{0,1}$ depending only on n_0, α, k_2, r, and d such that

$$|\,(u(t_1),\dot{u}(t_1))_{\ell^2} - (u(t_2),\dot{u}(t_2))_{\ell^2}\,| \le C_{0,1}\Big(P(t)^{\frac{2}{r+2}} + P(t)\Big) \sup_{t\le s\le t+1} \tilde{E}(s)^{1/2}. \tag{3.11}$$

Proceeding as in (3.9), replacing t_j by $s\in[t_1,t_2]$, and noticing that

$$\sum_{|n|_0>n_0} |\dot{u}_n(s)|^2 \le k_2^{-1}Q(s),$$

for any $s\in[t_1,t_2]$ by (A4), we obtain

$$\begin{aligned}\int_{t_1}^{t_2} \|\dot{u}(s)\|^2 ds &\le (2n_0+1)^{\frac{rd}{r+2}} k_2^{-\frac{2}{r+2}} \int_{t_1}^{t_2} Q(s)^{\frac{2}{r+2}} ds + k_2^{-1}\int_{t_1}^{t_2} Q(s)ds \\ &\le (2n_0+1)^{\frac{rd}{r+2}} k_2^{-\frac{2}{r+2}} P(t)^{\frac{4}{r+2}} + k_2^{-1}P(t)^2.\end{aligned} \tag{3.12}$$

Next, for each $t\ge 0$ fixed, we define the sets

$$I_1(t) = \left\{n\in\mathbb{Z}^d;\ |\dot{u}_n(t)|\le 1\right\}, \quad I_2(t) = \mathbb{Z}^d\backslash I_1(t).$$

Note that by (A1), $|g(n,\dot{u}_n(s))| \le L_2(1)|\dot{u}_n(s)|$, whenever $n\in I_1(s)$. Then

$$\begin{aligned}\sum_{n\in I_1(s)} u_n(s)g(n,\dot{u}_n(s)) &\le \frac{\alpha}{2}\|u(s)\|^2 + \frac{1}{2\alpha}\sum_{n\in I_1(s)} |g(n,\dot{u}_n(s))|^2 \\ &\le \frac{\alpha}{2}\|u(s)\|^2 + \frac{1}{2\alpha}L_2(1)\sum_{n\in I_1(s)} |\dot{u}_n(s)|\,|g(n,\dot{u}_n(s))| \\ &= \frac{\alpha}{2}\|u(s)\|^2 + \frac{1}{2\alpha}L_2(1)\sum_{n\in I_1(s)} \dot{u}_n(s)g(n,\dot{u}_n(s)) \\ &\le \frac{\alpha}{2}\|u(s)\|^2 + \frac{1}{2\alpha}L_2(1)Q(s).\end{aligned} \tag{3.13}$$

In addition, using (A4) and (2.8) we have

$$\begin{aligned}\sum_{n\in I_2(s)} u_n(s)g(n,\dot{u}_n(s)) &\le \sum_{n\in I_2(s)} |u_n(s)|\,|\dot{u}_n(s)|\,|g(n,\dot{u}_n(s))| \\ &\le \|u(s)\|\sum_{n\in I_2(s)} |\dot{u}_n(s)|\,|g(n,\dot{u}_n(s))| \\ &= \|u(s)\|\sum_{n\in I_2(s)} \dot{u}_n(s)g(n,\dot{u}_n(s)) \\ &\le \|u(s)\|Q(s) \le \alpha_0^{-1/2}Q(s)\sup_{t\le s\le t+1}\tilde{E}(s)^{1/2}.\end{aligned} \tag{3.14}$$

It follows from (3.13), (3.14), and (3.3) that

$$\begin{aligned}\int_{t_1}^{t_2} (g(\dot{u}(s),u(s))_{\ell^2} ds &\le \frac{\alpha}{2}\int_{t_1}^{t_2}\|u(s)\|^2 ds + \frac{1}{2\alpha}L_2(1)P(t)^2 \\ &\quad + \alpha_0^{-1/2}P(t)^2 \sup_{t\le s\le t+1}\tilde{E}(s)^{1/2}.\end{aligned} \tag{3.15}$$

Finally, in view of (2.8), we easily see that

$$\Big|\int_{t_1}^{t_2}(f,u(s))_{\ell^2}\,ds\Big| \le \alpha_0^{-1/2}\|f\|\sup_{t\le s\le t+1}\tilde{E}(s)^{1/2}. \tag{3.16}$$

Substituting (3.11), (3.12), (3.15), and (3.16) into (3.6) we obtain the estimate

$$\begin{aligned}
&\int_{t_1}^{t_2}\|D^p u(s)\|^2 ds + \frac{\alpha}{2}\int_{t_1}^{t_2}\|u(s)\|^2 ds + \int_{t_1}^{t_2} J_1(s)ds \\
&\le C_{0,2}\Big[\big(P(t)^{\frac{2}{r+2}} + P(t)^2 + P(t) + \|f\|\big) \\
&\quad \times \sup_{t\le s\le t+1}\tilde{E}(s)^{1/2} + P(t)^{\frac{4}{r+2}} + P(t)^2\Big] + \|b_1\|_{\ell^1},
\end{aligned} \tag{3.17}$$

where $C_{0,2}$ is a positive constant depending only on $n_0,\alpha,k_2,r,d,L_2(1)$, and

$$J_1(t) = \sum_{n\in\mathbb{Z}^d}\Big(u_n(t)h_0(n,u_n(t)) + b_{1,n}\Big) + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d \partial_i^+ u_n(t)h_i(\partial_i^+ u_n(t)) \ge 0,$$

for all $t\ge 0$, because of (A2) and (A3).

On the other hand, using hypotheses (A2) and (A3) we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^d}\tilde{h}_0(n,u_n(t)) + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d \tilde{h}_i(\partial_i^+ u_n(t)) \\
&\le k_1^{-1}\sum_{n\in\mathbb{Z}^d}\Big(u_n(t)h_0(n,u_n(t)) + b_{1,n}\Big) - \|b_2\|_{\ell^1} \\
&\quad + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d k_{0,i}^{-1}\Big(\partial_i^+ u_n(t)h_i(\partial_i^+ u_n(t))\Big) \\
&\le k_0\Big[\sum_{n\in\mathbb{Z}^d}\Big(u_n(t)h_0(n,u_n(t)) + b_{1,n}\Big) + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d \partial_i^+ u_n(t)h_i(\partial_i^+ u_n(t))\Big] \\
&\quad - \|b_2\|_{\ell^1},
\end{aligned} \tag{3.18}$$

where $k_0 = \max\{k_1^{-1}, k_{0,i}^{-1}, i=1,\dots,d\}$.

Then, integrating (2.9) over $[t_1,t_2]$ and using (2.7), (3.17), (3.12), and (3.18) we deduce that

$$\begin{aligned}
&\int_{t_1}^{t_2}\tilde{E}(s)ds \\
&= \int_{t_1}^{t_2}E(s)ds + \frac{4}{\alpha}\int_{t_1}^{t_2}\|f\|^2 ds + \int_{t_1}^{t_2}\|b_2\|_{\ell^1}ds \\
&\le \max\{k_0,1\}\Big[\int_{t_1}^{t_2}\|D^p u(s)\|^2 ds + \frac{\alpha}{2}\int_{t_1}^{t_2}\|u(s)\|^2 ds + \int_{t_1}^{t_2}J_1(s)ds\Big] \\
&\quad + \frac{1}{2}\int_{t_1}^{t_2}\|\dot{u}(s)\|^2 ds + \frac{4}{\alpha}\int_{t_1}^{t_2}\|f\|^2 ds \\
&\le C_{0,3}\big[\big(P(t)^{\frac{2}{r+2}} + P(t)^2 + P(t) + \|f\|\big)\sup_{t\le s\le t+1}\tilde{E}(s)^{1/2} + P(t)^{\frac{4}{r+2}} + P(t)^2\big] \\
&\quad + \frac{4}{\alpha}\|f\|^2 + \max\{k_0,1\}\|b_1\|_{\ell^1},
\end{aligned} \tag{3.19}$$

for some positive constant $C_{0,3}$ depending only on $n_0, \alpha, k_0, k_2, r, d$, and $L_2(1)$.

Furthermore, by the Mean Value Theorem for integrals, there exists $t^* \in [t_1, t_2]$ such that

$$\frac{1}{2}\tilde{E}(t^*) \le (t_2 - t_1)\ \tilde{E}(t^*) = \int_{t_1}^{t_2} \tilde{E}(s)\,ds. \tag{3.20}$$

If $\tilde{t} \in [t, t^*]$ then using (2.6) and (3.3) we obtain

$$\tilde{E}(\tilde{t}) = \tilde{E}(t^*) + \int_{\tilde{t}}^{t^*} Q(s)\,ds \le \tilde{E}(t^*) + P(t)^2.$$

Analogously, we obtain the same estimate if $\tilde{t} \in [t^*, t+1]$. Therefore,

$$\tilde{E}(s) \le \tilde{E}(t^*) + P(t)^2,\ \forall s \in [t, t+1].$$

Using this fact, (3.19) and (3.20), we complete the proof. $\square$

In what follows we will write $K_0 = \|f\|^2 + \|b_1\|_{\ell^1}$ and assume that $K_0 > 0$.

Lemma 3.3. *Under the assumptions of Lemma 3.2 there exists $\rho_0 > 0$ such that $B[0;\rho_0] = \{(w,z) \in H;\ \|(w,z)\|_H \le \rho_0\}$ is an absorbing set for $\{S(t)\}_{t\ge 0}$ in H.*

Proof. Let $\mathcal{O}$ be any bounded subset of H and let $\rho = \rho(\mathcal{O})$ be a positive constant such that $\|(w,z)\|_H \le \rho, \forall (w,z) \in \mathcal{O}$. Assume that $(u_0, u_1) \in \mathcal{O}$. Let us first consider the case $r > 0$. Since $\tilde{E}(t)$ is a non-increasing function, then using (A1)-(A3) and Lemma 2.1 we can find a positive constant μ_0 depending only on $p, d, \alpha, \rho, k_1, k_{0,i}$, $i = 1, \dots, d$, $\|f\|$, $\|b_1\|_{\ell^1}$, and $\|b_2\|_{\ell^1}$ such that

$$P(t)^2 = E(t) - E(t+1) = \tilde{E}(t) - \tilde{E}(t+1) \le 2\tilde{E}(0) \le \mu_0, \quad \forall t \ge 0. \tag{3.21}$$

It follows from (3.5) and (3.21) that

$$\sup_{t\le s\le t+1} \tilde{E}(s)^{1+\frac{r}{2}} \le C_{1,1}[\tilde{E}(t) - \tilde{E}(t+1)] + \big(2C_0K_0\big)^{1+\frac{r}{2}}, \tag{3.22}$$

where $C_{1,1}$ is a positive constant depending only on r, C_0, and μ_0. Applying the first part of Lemma 3.1 to (3.22) with $\psi(t) = \tilde{E}(t)$, $\gamma = \frac{r}{2}$, $C = C_{1,1}$, and $K = \big(2C_0K_0\big)^{1+\frac{r}{2}}$ we obtain

$$\tilde{E}(t) \le \Big[C_{1,1}^{-1}\frac{r}{2}(t-1)^+ + \Big(\sup_{0\le s\le 1} \tilde{E}(s)\Big)^{-r/2}\Big]^{-2/r} + 2C_0K_0. \tag{3.23}$$

From (3.23), using (3.21), we deduce that

$$\tilde{E}(t) \le C_1(1+t)^{-2/r} + 2C_0K_0, \tag{3.24}$$

where C_1 is a positive constant depending only on r, C_0, and μ_0. Combining (3.24) with (2.8) we obtain

$$\|S(t)(u_0,u_1)\|_H^2 \le \alpha_0^{-1}C_1(1+t)^{-\frac{2}{r}} + 2\alpha_0^{-1}C_0K_0 \quad \forall t \ge 0. \tag{3.25}$$

Consequently, $\|S(t)(u_0,u_1)\|_H \le \rho_0$, for all $t \ge \tau$ if we take

$$\rho_0 = 2\Big(\frac{C_0K_0}{\alpha_0}\Big)^{1/2}, \quad \tau = \tau(\mathcal{O}) = \max\Big\{0, \Big(\frac{C_1}{2C_0K_0}\Big)^{r/2} - 1\Big\}.$$

This completes the proof of Lemma 3.3 if $r > 0$. The proof for the case $r = 0$ is analogous. Indeed, using (3.5) and (3.21) we have

$$\sup_{t\le s\le t+1} \tilde{E}(s) \le C_0(2+\mu_0)[\tilde{E}(t) - \tilde{E}(t+1)] + C_0K_0,\ \forall t \ge 0,$$

with C_0 and μ_0 as before. Then, applying the second part of Lemma 3.1 to this inequality and using (2.8) again, we deduce that

$$\begin{aligned}\|S(t)(u_0,u_1)\|_H^2 &\le \alpha_0^{-1}\frac{\mu_0}{2}\Big(\frac{C_2}{C_2+1}\Big)^{[t]} + \alpha_0^{-1}C_0K_0 \\ &\le \alpha_0^{-1}\frac{\mu_0}{2}\Big(\frac{C_2+1}{C_2}\Big)e^{-\nu t} + \alpha_0^{-1}C_0K_0, \quad \forall t\ge 0,\end{aligned} \tag{3.26}$$

where $C_2 = C_0(2+\mu_0)$ and $\nu = \ln\left(\frac{C_2+1}{C_2}\right)$. This implies Lemma 3.3 in the case $r=0$ with $\rho_0 = \left(\frac{2C_0K_0}{\alpha_0}\right)^{1/2}$ and $\tau = \tau(\mathcal{O}) = \max\left\{0, \frac{1}{\nu}\ln\left(\frac{\mu_0(C_2+1)}{2C_2C_0K_0}\right)\right\}$. □

Next, we prove that the semigroup $\{S(t)\}_{t\ge 0}$ is asymptotically compact in H. Here, the main step consists in using a method introduced by B. Wang in [28] combined with Lemma 3.1 to derive an appropriate estimate for the "tail" of the solution of (2.4). More precisely, we will show that, for all $\epsilon > 0$, there exist $\tau(\epsilon) > 0$ and a positive integer $k(\epsilon)$ such that

$$\sum_{|n|_0\ge k(\epsilon)} [(\dot u_n(t))^2 + (u_n(t))^2] < \epsilon, \quad \text{for all } t\ge \tau(\epsilon),$$

whenever the initial data (u_0,u_1) belongs to the absorbing set $B[0;\rho_0]$. To do this, we will need the following auxiliary lemma whose proof relies on the following elementary identities valid for any sequences $w=(w_n)$ and $z=(z_n)$ and $i = 1,\dots,d$.

$$\partial_i^+(w_nz_n) = (\partial_i^+w_n)z_{n+e_i} + w_n\partial_i^+z_n, \tag{3.27}$$

$$\partial_i^+(w_nz_n) = (\partial_i^+w_n)z_n + w_n\partial_i^+z_n + \partial_i^+w_n\partial_i^+z_n, \tag{3.28}$$

$$partial_i^-(w_nz_n) = (\partial_i^-w_n)z_n + w_n\partial_i^-z_n - \partial_i^-w_n\partial_i^-z_n, \tag{3.29}$$

$$\Delta_d(w_nz_n) = (\Delta_d w_n)z_n + w_n\Delta_d z_n + \nabla^+w_n\cdot\nabla^+z_n + \nabla^-w_n\cdot\nabla^-z_n. \tag{3.30}$$

Note that (3.30) follows from (3.28) and (3.29).

Lemma 3.4. *Let $u=(u_n(t))$ belong to $C^1(\mathbb{R}^+;\ell^2)$, and (θ_n) belong to ℓ^2. Then*

$$(-1)^p\sum_{n\in\mathbb{Z}^d}\Delta_d^p u_n(t)\,(\theta_n\dot u_n(t)) = \frac12\frac{d}{dt}\sum_{n\in\mathbb{Z}^d}\theta_n|D^pu_n(t)|^2 + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d(\partial_i^+\theta_n)z_{p,n}^{(i)}(t),$$

where

$$\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d|z_{p,n}^{(i)}(t)| \le C(p,d)\|(u(t),\dot u(t))\|_H^2, \quad \forall t\ge 0,$$

for some positive constant $C(p,d)$ depending on p and d.

Proof. A proof of this lemma when $d=1$ was first presented in [18]. We will argue by induction. Consider first p odd. If $p=1$ then using Lemma 2.2 and (3.27) we have

$$\begin{aligned}-\sum_{n\in\mathbb{Z}^d}\Delta_d u_n(\theta_n\dot u_n) &= \sum_{n\in\mathbb{Z}^d}\nabla^+u_n\cdot\nabla^+(\theta_n\dot u_n) \\ &= \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d\partial_i^+u_n\big(\theta_n\partial_i^+\dot u_n + \partial_i^+\theta_n\dot u_{n+e_i}\big)\end{aligned}$$

$$= \frac{1}{2}\frac{d}{dt}\sum_{n\in\mathbb{Z}^d}\theta_n|\nabla^+u_n|^2 + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d \partial_i^+\theta_n z_{1,n}^{(i)},$$

where $z_{1,n}^{(i)} = (\partial_i^+u_n)\dot{u}_{n+e_i}$ satisfies $\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d |z_{1,n}^{(i)}| \le 2d\|(u,\dot{u})\|_H^2$ by Lemma 2.1.

Assume now that Lemma 3.4 holds for $p = 2k-1$ with $k \in \mathbb{N}$. Using Lemma 2.2 again and (3.30) we have

$$\begin{aligned}
&(-1)^{2k+1}\sum_{n\in\mathbb{Z}^d}\Delta_d^{2k+1}u_n(\theta_n\dot{u}_n)\\
&= -\sum_{n\in\mathbb{Z}^d}\Delta_d^{2k}u_n\Delta_d(\theta_n\dot{u}_n)\\
&= -\sum_{n\in\mathbb{Z}^d}\Delta_d^{2k}u_n\big[(\Delta_d\theta_n)\dot{u}_n + \theta_n\Delta_d\dot{u}_n + \nabla^+\theta_n\cdot\nabla^+\dot{u}_n + \nabla^-\theta_n\cdot\nabla^-\dot{u}_n\big].
\end{aligned} \tag{3.31}$$

By Lemma 2.2 and (3.28) we also see that

$$\begin{aligned}
&-\sum_{n\in\mathbb{Z}^d}\Delta_d^{2k}u_n(\Delta_d\theta_n)\dot{u}_n\\
&= \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d\partial_i^+\theta_n(\Delta_d^{2k}u_n)\partial_i^+\dot{u}_n + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d\partial_i^+\theta_n(\partial_i^+\Delta_d^{2k}u_n)\dot{u}_n\\
&\quad + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d\partial_i^+\theta_n(\partial_i^+\Delta_d^{2k}u_n)\partial_i^+\dot{u}_n.
\end{aligned} \tag{3.32}$$

In addition,

$$\begin{aligned}
&-\sum_{n\in\mathbb{Z}^d}\Delta_d^{2k}u_n\nabla^-\theta_n\cdot\nabla^-\dot{u}_n\\
&= -\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d\Delta_d^{2k}u_{n+e_i}\partial_i^+\theta_n\partial_i^+\dot{u}_n\\
&= -\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d\partial_i^+\Delta_d^{2k}u_n\partial_i^+\theta_n\,\partial_i^+\dot{u}_n - \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d\Delta_d^{2k}u_n\partial_i^+\theta_n\,\partial_i^+\dot{u}_n.
\end{aligned} \tag{3.33}$$

Substituting (3.32) and (3.33) into (3.31) results

$$\begin{aligned}
&(-1)^{2k+1}\sum_{n\in\mathbb{Z}^d}\Delta_d^{2k+1}u_n(\theta_n\dot{u}_n)\\
&= -\sum_{n\in\mathbb{Z}^d}\Delta_d^{2k}u_n(\theta_n\Delta_d\dot{u}_n)\\
&\quad + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d\partial_i^+\theta_n\big[(\partial_i^+\Delta_d^{2k}u_n)\dot{u}_n - \Delta_d^{2k}u_n(\partial_i^+\dot{u}_n)\big].
\end{aligned} \tag{3.34}$$

Then, using the induction hypothesis with $v(t) = (\Delta_d u_n(t)) \in C^1(\mathbb{R}^+;\ell^2)$, from (3.34), we obtain

$$\begin{aligned} &(-1)^{2k+1} \sum_{n\in\mathbb{Z}^d} \Delta_d^{2k+1} u_n(\theta_n \dot{u}_n) \\ &= \frac{1}{2}\frac{d}{dt}\sum_{n\in\mathbb{Z}^d} \theta_n |D^{2k-1} v_n|^2 + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d (\partial_i^+\theta_n) z_{2k-1,n}^{(i)} \\ &\quad + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d \partial_i^+\theta_n \big[(\partial_i^+\Delta_d^{2k} u_n)\dot{u}_n - \Delta_d^{2k} u_n(\partial_i^+\dot{u}_n)\big], \end{aligned} \tag{3.35}$$

where, in view of Lemma 2.1,

$$\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d |z_{2k-1,n}^{(i)}| \le C(2k-1,d)\|(v,\dot{v})\|_H^2 \le 16d^2 C(2k-1,d)\|(u,\dot{u})\|_H^2,$$

$$\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d |(\partial_i^+\Delta_d^{2k} u_n)\dot{u}_n - \Delta_d^{2k} u_n(\partial_i^+\dot{u}_n)| \le (4d)^{4k+1}\|(u,\dot{u})\|_H^2.$$

Therefore, the proof of Lemma 3.4 for $p = 2k+1$ can be concluded from (3.35) with $z_{2k+1,n}^{(i)} = z_{2k-1,n}^{(i)} + (\partial_i^+\Delta_d^{2k} u_n)\dot{u}_n - \Delta_d^{2k} u_n(\partial_i^+\dot{u}_n)$.

Consider now p even. Since the proof is similar to the one in the previous case, we summarize it as follows. In the case $p = 2$, using Lemma 2.2 and working with the identity (3.30) we can prove that

$$\sum_{n\in\mathbb{Z}^d} \Delta_d^2(\theta_n \dot{u}_n) = \frac{1}{2}\frac{d}{dt}\sum_{n\in\mathbb{Z}^d} \theta_n(\Delta_d u_n)^2 + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d (\partial_i^+\theta_n) z_{2,n}^{(i)},$$

where, by Lemma 2.1, $z_{2,n}^{(i)} = (\Delta_d u_n)\partial_i^+\dot{u}_n - (\partial_i^+\Delta_d u_n)\dot{u}_n$ satisfies

$$\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d |z_{2,n}^{(i)}| \le (4d)^3\|(u,\dot{u})\|_H^2.$$

Next, assuming Lemma 3.4 valid for $p = 2k$, with $k \in \mathbb{N}$ and proceeding as in (3.33)-(3.35) we can prove that

$$\begin{aligned} \sum_{n\in\mathbb{Z}^d} \Delta_d^{2k+2} u_n(\theta_n \dot{u}_n) &= \frac{1}{2}\frac{d}{dt}\sum_{n\in\mathbb{Z}^d} \theta_n |D^{2k+2} v_n|^2 + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d \partial_i^+\theta_n z_{2k,n}^{(i)} \\ &\quad + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d \partial_i^+\theta_n \big[\Delta_d^{2k+1} u_n(\partial_i^+\dot{u}_n) - (\partial_i^+\Delta_d^{2k+1} u_n)\dot{u}_n\big], \end{aligned}$$

with $z_{2k+2,n}^{(i)} = z_{2k,n}^{(i)} + \Delta_d^{2k+1} u_n(\partial_i^+\dot{u}_n) - (\partial_i^+\Delta_d^{2k+1} u_n)\dot{u}_n$ satisfying

$$\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d |z_{2k+2,n}^{(i)}| \le [16d^2 C(2k,d) + (4d)^{4k+3}]\|(u,\dot{u})\|_H^2,$$

by Lemma 2.1. This completes the proof. □

Remark 3.5. Along the proof of Lemma 3.4 if we replace $\theta_n\dot{u}_n(t)$ by $\theta_n u_n(t)$ then we can easily check that

$$(-1)^p\sum_{n\in\mathbb{Z}^d}\Delta_d^p u_n(t)\,(\theta_n u_n(t))=\sum_{n\in\mathbb{Z}^d}\theta_n|D^p u_n(t)|^2+\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d(\partial_i^+\theta_n)\tilde{z}_{p,n}^{(i)}(t), \tag{3.36}$$

with $\tilde{z}_{p,n}^{(i)}(t)$ satisfying the same estimate in Lemma 3.4.

Let us now introduce some notation analogous to those used in the proof of Lemma 3.2. Let $u=(u_n(t))$ be the global solution of (2.4) obtained by Theorem 2.4. We consider a function $\theta\in C^1(\mathbb{R}^+;\mathbb{R})$ satisfying $\theta\equiv 0$ in $[0,1]$, $\theta\equiv 1$ in $[2,\infty)$, $0\le\theta(t)\le 1$ and $|\theta'(t)|\le 2$ for all $t\ge 0$. Let $w=(w_n(t))$, where $w_n(t)=\theta_n u_n(t)$, with $\theta_n=\theta\Big(\frac{|n|_0}{k}\Big)$ and k fixed in $\mathbb{N}$. We define the modified energy

$$\begin{aligned}E_\theta(t)&=\frac{1}{2}\sum_{n\in\mathbb{Z}^d}\theta_n(\dot{u}_n(t))^2+\frac{1}{2}\sum_{n\in\mathbb{Z}^d}\theta_n|D^p u_n(t)|^2+\frac{\alpha}{2}\sum_{n\in\mathbb{Z}^d}\theta_n(u_n(t))^2\\&\quad+\sum_{n\in\mathbb{Z}^d}\theta_n\tilde{h}_0(n,u_n(t))+\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d\theta_n\tilde{h}_i(\partial_i^+u_n(t))-\sum_{n\in\mathbb{Z}^d}\theta_n f_n u_n(t),\end{aligned}\tag{3.37}$$

for $t\ge 0$. Differentiating $E_\theta(t)$ with respect to t, using (2.4) and Lemma 3.4, we have

$$\frac{d}{dt}E_\theta(t)=-\sum_{n\in\mathbb{Z}^d}\theta_n g(n,\dot{u}_n(t))\dot{u}_n(t)-\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d(\partial_i^+\theta_n)z_{p,n}^{(i)}(t),\ \forall t\ge 0, \tag{3.38}$$

with $z_{p,n}^{(i)}(t)$ as in Lemma 3.4. From (3.38), for all $t\ge 0$, we can write

$$E_\theta(t)-E_\theta(t+1)=\int_t^{t+1}Q_\theta(s)ds+\int_t^{t+1}R_\theta(s)ds,$$

where

$$Q_\theta(t)=\sum_{n\in\mathbb{Z}^d}\theta_n g(n,\dot{u}_n(t))\dot{u}_n(t),\quad R_\theta(t)=\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d(\partial_i^+\theta_n)z_{p,n}^{(i)}(t).$$

Also, observe that from (3.37) and the hypotheses (A2) and (A3) we have

$$E_\theta(t)\ge\alpha_0\sum_{n\in\mathbb{Z}^d}\theta_n[\dot{u}_n(t)^2+u_n(t)^2]-\frac{4}{\alpha}\sum_{n\in\mathbb{Z}^d}\theta_n f_n^2-\sum_{n\in\mathbb{Z}^d}\theta_n b_{2,n},\quad\forall t\ge 0, \tag{3.39}$$

where, as before, $\alpha_0=\min\{\frac{1}{2},\frac{\alpha}{4}\}$.

The above considerations motivate us to define on $[0,\infty)$ the following nonnegative functions

$$P_\theta(t)^2=E_\theta(t)-E_\theta(t+1)-\int_t^{t+1}R_\theta(s)ds=\int_t^{t+1}Q_\theta(s)ds, \tag{3.40}$$

$$\tilde{E}_\theta(t)=E_\theta(t)+\frac{4}{\alpha}\sum_{n\in\mathbb{Z}^d}\theta_n f_n^2+\sum_{n\in\mathbb{Z}^d}\theta_n b_{2,n}. \tag{3.41}$$

Finally, to simplify notation, for any sequences $w=(w_n)$ and $z=(z_n)$, we will write

$$(w,z)_\theta=\sum_{n\in\mathbb{Z}^d}\theta_n w_n z_n,\quad\|w\|_\theta^2=\sum_{n\in\mathbb{Z}^d}\theta_n|w_n|^2,$$

$$\|w\|_{\ell^1_\theta} = \sum_{n\in\mathbb{Z}^d} \theta_n |w_n|, \quad \|D^p w\|^2_\theta = \sum_{n\in\mathbb{Z}^d} \theta_n |D^p w_n|^2,$$

$$(\hat{h}(w), w)_\theta = \sum_{n\in\mathbb{Z}^d} \theta_n w_n h_0(n, w_n) - \sum_{n\in\mathbb{Z}^d} \sum_{i=1}^d \theta_n w_n \partial_i^- h_i(\partial_i^+ w_n).$$

Lemma 3.6. *Under the assumptions of Lemma 3.2 and $(u_0, u_1) \in B[0; \rho_0]$, there exist positive constants $C_{2,i}$, $i = 1, 2, 3$, with $C_{2,1}$, $C_{2,2}$ depending on ρ_0 and $C_{2,3}$ depending only on $k_1, k_{0,i}$, $i = 1, \dots, d$, and α, such that*

$$\sup_{t\le s\le t+1} \tilde{E}_\theta(s) \le C_{2,1}\big(P_\theta(t)^{\frac{4}{r+2}} + P_\theta(t)^2\big) + \frac{1}{k} C_{2,2} + C_{2,3}(\|f\|^2_\theta + \|b_1\|_{\ell^1_\theta}),\ \forall t \ge 0.$$

Proof. We first observe that $(u_0, u_1) \in B[0; \rho_0]$ implies the existence of a positive constant r_0 depending on ρ_0 and some parameters of the problem (see (3.24) and (3.26)) such that

$$\|(u(t), \dot{u}(t))\| \le r_0, \quad \forall t \ge 0. \tag{3.42}$$

By the Mean Value Theorem for integrals there exist $t_1 \in [t, t + \frac{1}{4}]$ and $t_2 \in [t + \frac{3}{4}, t + 1]$ such that

$$\int_t^{t+\frac{1}{4}} Q_\theta(s)\, ds = \frac{1}{4} Q_\theta(t_1) \quad \text{and} \quad \int_{t+\frac{3}{4}}^{t+1} Q_\theta(s)\, ds = \frac{1}{4} Q_\theta(t_2). \tag{3.43}$$

Taking the inner product of equation (2.4) with $w(t)$ in ℓ^2, using (3.36) and integrating the result over $[t_1, t_2]$ we find

$$\begin{aligned} &\int_{t_1}^{t_2} \|D^p u(s)\|^2_\theta ds + \alpha \int_{t_1}^{t_2} \|u(s)\|^2_\theta ds + \int_{t_1}^{t_2} (\hat{h}(u(s)), u(s))_\theta ds \\ &= (u(t_1), \dot{u}(t_1))_\theta - (u(t_2), \dot{u}(t_2))_\theta + \int_{t_1}^{t_2} \|\dot{u}(s)\|^2_\theta ds \\ &\quad - \int_{t_1}^{t_2} (g(\dot{u}(s)), u(s))_\theta ds + \int_{t_1}^{t_2} (f, u(s))_\theta ds - \int_{t_1}^{t_2} \tilde{R}_\theta(s) ds, \end{aligned} \tag{3.44}$$

where

$$\tilde{R}_\theta(t) = \sum_{n\in\mathbb{Z}^d} \sum_{i=1}^d \partial_i^+ \theta_n \tilde{z}^{(i)}_{p,n}(t), \tag{3.45}$$

with $\tilde{z}^{(i)}_{p,n}(t)$ satisfying

$$\sum_{n\in\mathbb{Z}^d} \sum_{i=1}^d |\tilde{z}^{(i)}_{p,n}(t)| \le C(p, d) r_0^2, \quad \forall t \ge 0. \tag{3.46}$$

Observe that by the definition of θ_n we have that $|\partial_i^+ \theta_n| \le \frac{2}{k}$, $i = 1, \dots, d$. Since $(u_0, u_1) \in B[0; \rho_0]$, from (3.46) we see that

$$\Big| \int_{t_1}^{t_2} \tilde{R}_\theta(s) ds \Big| \le \sum_{n\in\mathbb{Z}^d} \sum_{i=1}^d \int_{t_1}^{t_2} |\partial_i^+ \theta_n|\, |\tilde{z}^{(i)}_{p,n}(s)| ds \le \frac{2}{k} C(p, d) r_0^2. \tag{3.47}$$

Next, we follow the same steps of the proof of Lemma 3.2. We first write

$$(u(t_j), \dot{u}(t_j))_\theta = \sum_{|n|_0 \le n_0} \theta_n u_n(t_j) \dot{u}_n(t_j) + \sum_{|n|_0 > n_0} \theta_n u_n(t_j) \dot{u}_n(t_j), \quad j = 1, 2, \tag{3.48}$$

with n_0 as in (A4). Using (3.39) and (3.41) we have

$$\Big(\sum_{|n|_0\le n_0}\theta_n|u_n(t_j)|^2\Big)^{1/2}\le \alpha_0^{-1/2}\sup_{t\le s\le t+1}\tilde{E}_\theta(s)^{1/2}. \tag{3.49}$$

Using Hölder's inequality, (A4), (3.43), and (3.40), we also have

$$\begin{aligned}
\sum_{|n|_0\le n_0}\theta_n|\dot{u}_n(t_j)|^2 &\le (2n_0+1)^{\frac{rd}{r+2}}\Big(\sum_{|n|_0\le n_0}\theta_n^{\frac{r+2}{2}}|\dot{u}_n(t_j)|^{r+2}\Big)^{\frac{2}{r+2}}\\
&\le (2n_0+1)^{\frac{rd}{r+2}}\Big(k_2^{-1}\sum_{|n|_0\le n_0}\theta_n g(n,\dot{u}_n(t_j))\dot{u}_n(t_j)\Big)^{\frac{2}{r+2}}\\
&\le (2n_0+1)^{\frac{rd}{r+2}}k_2^{-\frac{2}{r+2}}4^{\frac{2}{r+2}}P_\theta(t)^{\frac{4}{r+2}}.
\end{aligned}\tag{3.50}$$

Similarly, we can estimate the second term in (3.48) to find that

$$\sum_{|n|_0>n_0}|\dot{u}_n(t_j)u_n(t_j)|\le 2\alpha_0^{-1/2}k_2^{-1/2}P_\theta(t)\sup_{t\le s\le t+1}\tilde{E}_\theta(s)^{1/2}. \tag{3.51}$$

Thus, from (3.48)-(3.51) we conclude that there is a positive constant $C_{1,1}$ depending only on n_0,α,k_2,r, and d such that

$$|(u(t_1),\dot{u}(t_1))_\theta-(u(t_2),\dot{u}(t_2))_\theta|\le C_{1,1}\Big(P_\theta(t)^{\frac{2}{r+2}}+P_\theta(t)\Big)\sup_{t\le s\le t+1}\tilde{E}_\theta(s)^{1/2}. \tag{3.52}$$

In the same way we obtain the estimate

$$\int_{t_1}^{t_2}\|\dot{u}(s)\|_\theta^2ds\le (2n_0+1)^{\frac{rd}{r+2}}k_2^{-\frac{2}{r+2}}P_\theta(t)^{\frac{4}{r+2}}+4k_2^{-1}P_\theta(t)^2. \tag{3.53}$$

Now, to estimate the third term in right hand side of (3.44), we define the sets $I_1(t)=\big\{n\in\mathbb{Z}^d;|\dot{u}_n(t)|\le 1\big\}$ and $I_2(t)=\mathbb{Z}^d\backslash I_1(t)$, with $t\ge 0$ fixed. Using the assumptions (A1) and (A4) we see that

$$\begin{aligned}
&\sum_{n\in I_1(s)}\theta_n u_n(s)g(n,\dot{u}_n(s))\\
&\le \frac{\alpha}{2}\sum_{n\in I_1(s)}\theta_n|u_n(s)|^2+\frac{1}{2\alpha}\sum_{n\in I_1(s)}\theta_n|g(n,\dot{u}_n(s))|^2\\
&\le \frac{\alpha}{2}\|u(s)\|_\theta^2+\frac{L_2(1)}{2\alpha}\sum_{n\in I_1(s)}\theta_n|\dot{u}_n(s)|\,|g(n,\dot{u}_n(s))|\\
&= \frac{\alpha}{2}\|u(s)\|_\theta^2+\frac{L_2(1)}{2\alpha}\sum_{n\in I_1(s)}\theta_n\dot{u}_n(s)g(n,\dot{u}_n(s))\\
&\le \frac{\alpha}{2}\|u(s)\|_\theta^2+\frac{L_2(1)}{2\alpha}Q_\theta(s).
\end{aligned}\tag{3.54}$$

Moreover, using (A4) and (3.42) we have

$$\begin{aligned}\sum_{n\in I_2(s)}\theta_n u_n(s)g(n,\dot u_n(s)) &\le \sum_{n\in I_2(s)}\theta_n|u_n(s)|\,|\dot u_n(s)|\,|g(n,\dot u_n(s))|\\ &\le \|u(s)\|\sum_{n\in I_2(s)}\theta_n|\dot u_n(s)|\,|g(n,\dot u_n(s))|\\ &= \|u(s)\|\sum_{n\in I_2(s)}\theta_n\dot u_n(s)g(n,\dot u_n(s))\\ &\le r_0Q_\theta(s).\end{aligned}\tag{3.55}$$

From (3.54), (3.55), and (3.40) it follows that

$$\int_{t_1}^{t_2}(g(\dot u(s),u(s))_\theta ds \le \frac{\alpha}{2}\int_{t_1}^{t_2}\|u(s)\|_\theta^2ds + \Big(\frac{1}{2\alpha}L_2(1)+r_0^2\Big)P_\theta(t)^2.\tag{3.56}$$

Finally, using (3.39) and (3.41) we also have

$$\Big|\int_{t_1}^{t_2}(f,u(s))_\theta ds\Big| \le \int_{t_1}^{t_2}\|f\|_\theta\,\|u(s)\|_\theta ds \le \alpha_0^{-1/2}\|f\|_\theta\sup_{t\le s\le t+1}\tilde E_\theta(s)^{1/2}.\tag{3.57}$$

Therefore, using (3.47), (3.52), (3.53), (3.56), and (3.57) in (3.44) we conclude that there exists a positive constant $C_{1,2}$, which depends only on $n_0,\alpha,k_2,r,d,L_2(1)$, and r_0, such that

$$\begin{aligned}&\int_{t_1}^{t_2}\|D^pu(s)\|_\theta^2ds + \frac{\alpha}{2}\int_{t_1}^{t_2}\|u(s)\|_\theta^2ds + \int_{t_1}^{t_2}(\hat h(u(s)),u(s))_\theta ds\\ &\le C_{1,2}\Big[\big(P_\theta(t)^{\frac{2}{r+2}}+P_\theta(t)\big)\sup_{t\le s\le t+1}\tilde E_\theta(s)^{1/2} + P_\theta(t)^{\frac{4}{r+2}} + P_\theta(t)^2\Big]\\ &+\alpha_0^{-1/2}\|f\|_\theta\sup_{t\le s\le t+1}\tilde E_\theta(s)^{1/2} + \frac{2}{k}C(p,d)r_0^2.\end{aligned}\tag{3.58}$$

On the other hand, integrating (3.37) over $[t_1,t_2]$ we obtain

$$\begin{aligned}\int_{t_1}^{t_2}E_\theta(s)ds &= \frac12\int_{t_1}^{t_2}\|D^pu(s)\|_\theta^2ds + \frac{\alpha}{2}\int_{t_1}^{t_2}\|u(s)\|_\theta^2ds + \int_{t_1}^{t_2}J_{1,\theta}(s)ds\\ &\quad+\frac12\int_{t_1}^{t_2}\|\dot u(s)\|_\theta^2ds - \int_{t_1}^{t_2}(f,u(s))_\theta ds,\end{aligned}\tag{3.59}$$

where

$$J_{1,\theta}(s) = \sum_{n\in\mathbb{Z}^d}\theta_n\tilde h_0(n,u_n(s)) + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d\theta_n\tilde h_i(\partial_i^+u_n(s)).$$

Let us compare (3.58) and (3.59). Using the identity (3.27) we can write

$$\begin{aligned}&(\hat h(u(t)),u(t))_\theta\\ &=\sum_{n\in\mathbb{Z}^d}\theta_nu_n(t)h_0(n,u_n(t)) + \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^d\theta_n\partial_i^+u_n(s)h_i(\partial_i^+u_n(t))\\ &\quad+R_{1,\theta}(t) = J_{2,\theta}(t)+R_{1,\theta}(t)-\|b_1\|_{\ell_\theta^1},\end{aligned}$$

where

$$R_{1,\theta}(t)=\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^{d}(\partial_i^+\theta_n)u_{n+e_i}(t)h_i(\partial_i^+u_n(t)),$$

$$J_{2,\theta}(t)=\sum_{n\in\mathbb{Z}^d}\theta_n[u_n(t)h_0(n,u_n(t))+b_{1,n}]+\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^{d}\theta_n\partial_i^+u_n(t)h_i(\partial_i^+u_n(t))\geq 0,$$

for all $t\geq 0$ because of (A2) and (A3).

Substituting the above expression into (3.58) yields

$$\begin{aligned}&\int_{t_1}^{t_2}\|D^pu(s)\|_\theta^2ds+\frac{\alpha}{2}\int_{t_1}^{t_2}\|u(s)\|_\theta^2ds+\int_{t_1}^{t_2}J_{2,\theta}(s)ds\\&\leq C_{1,2}\Big[\big(P_\theta(t)^{\frac{2}{r+2}}+P_\theta(t)\big)\sup_{t\leq s\leq t+1}\tilde{E}_\theta(s)^{1/2}+P_\theta(t)^{\frac{4}{r+2}}+P_\theta(t)^2\Big]\\&\quad+\alpha_0^{-1/2}\|f\|_\theta\sup_{t\leq s\leq t+1}\tilde{E}_\theta(s)^{1/2}+\frac{2}{k}C(p,d)r_0^2+\int_{t_1}^{t_2}\|b_1\|_{\ell_\theta^1}ds\\&\quad-\int_{t_1}^{t_2}R_{1,\theta}(s)ds.\end{aligned}\tag{3.60}$$

Since $(u_0,u_1)\in B[0;\rho_0]$ and $|\partial_i^+\theta_n|\leq\frac{2}{k}$, $i=1,\dots,d$, then using (3.42) and (A1) we have

$$\Big|\int_{t_1}^{t_2}R_{1,\theta}(s)ds\Big|\leq\frac{4d}{k}L_3(2r_0)r_0^2.\tag{3.61}$$

In addition, by (A2) and (A3) we see that

$$\begin{aligned}J_{1,\theta}(s)&\leq k_1^{-1}\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^{d}\theta_n[u_n(s)h_0(n,u_n(s))+b_{1,n}]-\|b_2\|_{\ell_\theta^1}\\&\quad+\sum_{n\in\mathbb{Z}^d}\sum_{i=1}^{d}k_{0,i}^{-1}\theta_n\partial_i^+u_n(s)h_i(\partial_i^+u_n(s))\leq k_0J_{2,\theta}(s)-\|b_2\|_{\ell_\theta^1},\end{aligned}\tag{3.62}$$

where $k_0=\max\{k_1^{-1},k_{0,i}^{-1},i=1,\dots,d\}$. Therefore, from (3.41), (3.53), (3.57), (3.59), (3.60), (3.61), and (3.62) we deduce that

$$\begin{aligned}&\int_{t_1}^{t_2}\tilde{E}_\theta(s)ds\\&\leq C_{1,3}\big[\big(P_\theta(t)^{\frac{2}{r+2}}+P_\theta(t)\big)\sup_{t\leq s\leq t+1}\tilde{E}_\theta(s)^{1/2}+P_\theta(t)^{\frac{4}{r+2}}+P_\theta(t)^2\big]\\&\quad+2\alpha_0^{-1/2}\|f\|_\theta\sup_{t\leq s\leq t+1}\tilde{E}_\theta(s)^{1/2}+\frac{1}{k}C_{1,4}+\max\{k_0,1\}\|b_1\|_{\ell_\theta^1}+\frac{4}{\alpha}\|f\|_\theta^2,\end{aligned}\tag{3.63}$$

where $C_{1,3}$ and $C_{1,4}$ are positive constants depending on r_0.

Finally, as in the end of the proof of Lemma 3.2, we can find $t^*\in[t_1,t_2]$ so that

$$\frac{1}{2}\tilde{E}_\theta(t^*)\leq(t_2-t_1)\tilde{E}_\theta(t^*)=\int_{t_1}^{t_2}\tilde{E}_\theta(s)ds.\tag{3.64}$$

If $\tilde{t}\in[t,t^*]$, then, using (3.41), (3.38), and (3.40), we have

$$\tilde{E}_\theta(\tilde{t})-\tilde{E}_\theta(t^*)=E_\theta(\tilde{t})-E_\theta(t^*)$$

$$
\begin{aligned}
&= \int_{t^*}^{\tilde{t}} Q_\theta(s)ds + \int_{\tilde{t}}^{t^*} R_\theta(s)ds \\
&\le P_\theta(t)^2 + \int_t^{t+1} |R_\theta(s)|ds.
\end{aligned}
$$

Similarly, if $\tilde{t} \in [t^*, t+1]$ then

$$
\tilde{E}_\theta(\tilde{t}) - \tilde{E}_\theta(t^*) = -\int_{t^*}^{\tilde{t}} Q_\theta(s)ds - \int_{\tilde{t}}^{t^*} R_\theta(s)ds \le \int_t^{t+1} |R_\theta(s)|ds.
$$

Therefore,

$$
\sup_{t \le s \le t+1} \tilde{E}_\theta(s) \le \tilde{E}_\theta(t^*) + P_\theta(t)^2 + \int_t^{t+1} |R_\theta(s)|ds, \tag{3.65}
$$

where, as in (3.47), we have

$$
\int_t^{t+1} |R_\theta(s)|ds \le \frac{2}{k} C(p,d) r_0^2. \tag{3.66}
$$

The conclusion of the proof of Lemma 3.6 now follows from (3.63)-(3.66). □

Lemma 3.7. *Under the assumptions of Lemma 3.6, for each $\epsilon > 0$, there exist $\tau(\epsilon) > 0$ and a positive integer $k(\epsilon)$ such that*

$$
\sum_{|n|_0 \ge k(\epsilon)} [(\dot{u}_n(t))^2 + (u_n(t))^2] < \epsilon, \quad \forall t \ge \tau(\epsilon).
$$

Proof. We will denote by $C_{3,i}$, $i = 1, \dots, 6$, the positive constants depending on ρ_0 that appear along the proof. Let us first assume that $r > 0$. Since (3.42) holds, using (A1), we have

$$
\begin{aligned}
P_\theta(t)^2 = \int_t^{t+1} Q_\theta(s)ds &\le \int_t^{t+1} \sum_{n \in \mathbb{Z}^d} \theta_n |g(n, \dot{u}_n(s))||\dot{u}_n(s)|ds \\
&\le L_2(r_0) \int_t^{t+1} \|\dot{u}(s)\|^2 ds \le L_2(r_0) r_0^2.
\end{aligned} \tag{3.67}
$$

By (3.66) we also have

$$
\begin{aligned}
P_\theta(t)^2 &= \tilde{E}_\theta(t) - \tilde{E}_\theta(t+1) - \int_t^{t+1} R_\theta(s)ds \\
&\le \tilde{E}_\theta(t) - \tilde{E}_\theta(t+1) + \frac{2}{k} C(p,d) r_0^2.
\end{aligned} \tag{3.68}
$$

Using Lemma 3.6, (3.67), and (3.68) we obtain

$$
\begin{aligned}
\sup_{t \le s \le t+1} \tilde{E}_\theta(s)^{1+\frac{r}{2}} &\le 2^{1+\frac{r}{2}} C_{2,1}^{1+\frac{r}{2}} P_\theta(t)^2 \Big(1 + P_\theta(t)^{\frac{2r}{r+2}}\Big)^{1+\frac{r}{2}} \\
&\quad + 2^{1+\frac{r}{2}} \Big[\frac{1}{k} C_{2,2} + C_{2,3}(\|f\|_\theta^2 + \|b_1\|_{\ell^1_\theta})\Big]^{1+\frac{r}{2}} \\
&\le C_{3,1}[\tilde{E}_\theta(t) - \tilde{E}_\theta(t+1)] + 2^{1+\frac{r}{2}} \frac{1}{k} C_{3,2} \\
&\quad + \Big[\frac{2}{k} C_{2,2} + 2C_{2,3}(\|f\|_\theta^2 + \|b_1\|_{\ell^1_\theta})\Big]^{1+\frac{r}{2}}.
\end{aligned} \tag{3.69}
$$

Applying the first part of Lemma 3.1 to inequality (3.69) we deduce that

$$\begin{aligned}\tilde{E}_\theta(t) \le \big[C_{3,1}^{-1}\frac{r}{2}(t-1)^+ + \big(\sup_{0\le s\le 1}\tilde{E}_\theta(s)\big)^{-\frac{r}{2}}\big]^{-\frac{2}{r}} + C_{3,3}k^{-\frac{2}{k+2}} \\ + 2C_{2,3}(\|f\|_\theta^2 + \|b_1\|_{\ell_\theta^1}), \quad \forall t\ge 0.\end{aligned} \tag{3.70}$$

Since $(u_0,u_1)\in B[0;\rho_0]$ then proceeding as in (3.21) we can bound the term $\sup_{0\le s\le 1}\tilde{E}_\theta(s)$ in (3.70) by a positive constant that depends on ρ_0. Thus, from (3.70), (3.39), and (3.41) we obtain

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d}\theta_n[\dot{u}_n(t)^2+u_n(t)^2] \le \alpha_0^{-1}C_{3,4}(1+t)^{-\frac{2}{r}} + \alpha_0^{-1}C_{3,3}k^{-\frac{2}{r+2}} \\ + 2\alpha_0^{-1}C_{2,3}(\|f\|_\theta^2 + \|b_1\|_{\ell_\theta^1}), \quad \forall t\ge 0.\end{aligned} \tag{3.71}$$

Since $f\in\ell^2$ and $b_1\in\ell^1$ then, given $\epsilon>0$, there exists a positive integer $k(\epsilon)$ such that

$$\alpha_0^{-1}C_{3,3}k^{-\frac{2}{r+2}} + 2\alpha_0^{-1}C_{2,3}\Big(\sum_{|n|_0>k}|f_n|^2 + \sum_{|n|_0>k}b_{1,n}\Big) < \frac{\epsilon}{2}, \quad \forall k\ge\frac{k(\epsilon)}{2}.$$

Then, from (3.71) it follows that

$$\sum_{n\in\mathbb{Z}^d}\theta_n[\dot{u}_n(t)^2+u_n(t)^2] \le \alpha_0^{-1}C_{3,4}(1+t)^{-\frac{2}{r}} + \frac{\epsilon}{2},$$

for all $t\ge 0$ and $k\ge\frac{k(\epsilon)}{2}$. Choosing $\tau(\epsilon)=\max\{0,\tau_1(\epsilon)\}$, with $\tau_1(\epsilon)=\Big(\frac{2C_{3,4}}{\alpha_0\epsilon}\Big)^{\frac{r}{2}}-1$ we obtain

$$\sum_{|n|_0\ge 2k}[\dot{u}_n(t)^2+u_n(t)^2] < \epsilon, \quad \forall t\ge\tau(\epsilon) \text{ and } \forall k\le\frac{k(\epsilon)}{2}.$$

This estimate implies Lemma 3.6 if $r>0$.

Suppose now that $r=0$. By Lemma 3.6 and (3.68) we have

$$\begin{aligned}\sup_{t\le s\le t+1}\tilde{E}_\theta(s) &\le 2C_{2,1}P_\theta(t)^2 + \frac{1}{k}C_{2,2} + C_{2,3}(\|f\|_\theta^2 + \|b_1\|_{\ell_\theta^1}) \\ &\le 2C_{2,1}[\tilde{E}_\theta(t) - \tilde{E}_\theta(t+1)] + C_{3,5}\frac{1}{k} + C_{2,3}(\|f\|_\theta^2 + \|b_1\|_{\ell_\theta^1}).\end{aligned} \tag{3.72}$$

Applying the second part of Lemma 3.1 to inequality (3.72) yields

$$\tilde{E}_\theta(t) \le \sup_{0\le s\le 1}\tilde{E}_\theta(s)\Big(\frac{2C_{2,1}}{2C_{2,1}+1}\Big)^{[t]} + C_{3,5}\frac{1}{k} + C_{2,3}(\|f\|_\theta^2 + \|b_1\|_{\ell_\theta^1}), \quad \forall t\ge 0. \tag{3.73}$$

Then, estimating the term $\sup_{0\le s\le 1}\tilde{E}_\theta(s)$ in (3.73) by a positive constant depending on ρ_0 and using (3.39) and (3.41) again we obtain

$$\begin{aligned}&\sum_{n\in\mathbb{Z}^d}\theta_n[\dot{u}_n(t)^2+u_n(t)^2] \\ &\le \alpha_0^{-1}C_{3,6}e^{-\nu t} + \alpha_0^{-1}C_{3,5}\frac{1}{k} + \alpha_0^{-1}C_{2,3}(\|f\|_\theta^2 + \|b_1\|_{\ell_\theta^1}), \quad \forall t\ge 0,\end{aligned}$$

with $\nu=\ln\big(\frac{2C_{2,1}+1}{2C_{2,1}}\big)$, which implies Lemma 3.7 if $r=0$. □

Lemma 3.8. *Under the assumptions of Lemma 3.6, the semigroup $\{S(t)\}_{t\ge 0}$ is asymptotically compact in H.*

In view of Lemmas 3.3 and 3.7, the proof of the above lemma is similar to that of Lemma 3.9 in [18], so we omit its proof here. Using the notation of Lemma 3.3 and denoting by

$$\operatorname{dist}(A,B)=\sup_{a\in A}\inf_{b\in B}\|a-b\|_H$$

the Hausdorff semi-distance between two subsets A and B of H we can state our first main result in this section.

Theorem 3.9. *Assume that* (A1)–(A4) *hold and let* u_0, u_1*, and* f *belong to* ℓ^2*. Then the semigroup* $\{S(t)\}_{t\geq 0}$ *associated with the initial value problem* (2.4) *possesses a unique global attractor* $\mathcal{A}$ *in* H*. Moreover,* $\mathcal{A}\subset B[0;\rho_0]$ *and for any bounded set* B_0 *in* H *there exist positive constants* $C(B_0)$ *and* ν *depending on* B_0 *such that*

$$\operatorname{dist}\big(S(t)B_0, B[0;\rho_0]\big)\leq C(B_0)\begin{cases}(1+t)^{-\frac{1}{r}}, & \text{if } r>0\\ e^{-\nu t}, & \text{if } r=0\end{cases}. \tag{3.74}$$

Proof. In view of Lemmas 3.3, 3.7 and [26, Theorem 1.1], the ω-limit set $\mathcal{A}=\omega(B[0;\rho_0])$ is the unique global attractor for $\{S(t)\}_{t\geq 0}$ in H. Clearly, $\mathcal{A}\subset B[0;\rho_0]$ because $\mathcal{A}$ is a bounded set of H and invariant under $\{S(t)\}_{t\geq 0}$. Let us prove the absorbing rate in (3.74) when $r>0$. By (3.25) for any $(u_0,u_1)\in B_0$ we can find a positive constant $C(B_0)$ depending on B_0 such that

$$\|S(t)(u_0,u_1)\|_H\leq C(B_0)(1+t)^{-\frac{1}{r}}+2^{-1/2}\rho_0,\quad \forall t\geq 0. \tag{3.75}$$

Now, for any $(u_0,u_1)\in B_0$ and $t\geq 0$ fixed, from (3.75) we easily see that

$$\inf_{(w_0,w_1)\in B[0;\rho_0]}\|S(t)(u_0,u_1)-(w_0,w_1)\|_H\leq C(B_0)(1+t)^{-1/r}.$$

Therefore,

$$\operatorname{dist}\big(S(t)B_0, B[0;\rho_0]\big)\leq C(B_0)(1+t)^{-1/r}.$$

The exponential absorbing rate in (3.74), i.e., when $r=0$, is proved in the same manner using (3.26). □

Next, we will prove the existence of a global attractor for the semigroup $\{S(t)\}_{t\geq 0}$ associated with the initial value problem (2.10). Thus, in the rest of this section, $u=u(t)$ denotes the global solution of (2.10) obtained by Theorem 2.5 and the semigroup $\{S(t)\}_{t\geq 0}$ is defined as in (3.1). We also consider $E(t)$ and $\tilde{E}(t)$ defined by (2.11) and (2.12), respectively, and the functions $P(t)^2$ and $Q(t)$ as before satisfying (3.3) and (3.4).

Lemma 3.10. *Assume that* (A1), (A4)–(A6) *hold and let* u_0*,* u_1*, and* f *belong to* ℓ^2*. Then there exists* $\rho_1>0$ *such that*

$$B[0;\rho_1]=\{(w,z)\in H;\ \|(w,z)\|_H\leq\rho_1\}$$

is an absorbing set for $\{S(t)\}_{t\geq 0}$ *in* H*.*

Proof. The proof of this lemma is similar to the proofs of Lemmas 3.2 and 3.3. We first proceed as in the proof of Lemma 3.2 to obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\|D^pu(s)\|^2ds+\frac{\alpha}{2}\int_{t_1}^{t_2}\|u(s)\|^2ds+\int_{t_1}^{t_2}\tilde{J}_2(s)ds\\&\le \tilde{C}_{0,2}\Big[\big(P(t)^{\frac{2}{r+2}}+P(t)^2+P(t)+\|f\|\big)\sup_{t\le s\le t+1}\tilde{E}(s)^{1/2}\\&\quad+P(t)^{\frac{4}{r+2}}+P(t)^2\Big],\end{aligned}\tag{3.76}$$

where $\tilde{C}_{0,2}$ is a positive constant that depends only on n_0,α,k_2,r,d, $L_2(1)$ and

$$\tilde{J}_2(t)=\sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d}V(n-m)h(u_m(t))h'(u_n(t))u_n(t),\quad \forall t\ge 0.$$

Using hypotheses (A5) and (A6) we have that

$$\sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d}V(n-m)h(u_m(t))h(u_n(t))\le c_1^{-1}\tilde{J}_2(t),\quad \forall t\ge 0.\tag{3.77}$$

Integrating (2.12) over $[t_1,t_2]$ and using (2.11), (3.76), and (3.77) we obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\tilde{E}(s)ds\\&\le \max\{c_1^{-1},1\}\tilde{C}_{0,2}\Big[\big(P(t)^{\frac{2}{r+2}}+P(t)^2+P(t)+\|f\|\big)\sup_{t\le s\le t+1}\tilde{E}(s)^{1/2}\\&\quad+P(t)^{\frac{4}{r+2}}+P(t)^2\Big]+\frac{1}{2}\int_{t_1}^{t_2}\|\dot{u}(s)\|^2ds+\Big|\int_{t_1}^{t_2}(f,u(s))_{\ell^2}ds\Big|+\frac{4}{\alpha}\|f\|^2.\end{aligned}$$

Then, still arguing as in the proof of Lemma 3.2, we prove the following inequality analogous to (3.5),

$$\sup_{t\le s\le t+1}\tilde{E}(s)\le \tilde{C}_0\big[P(t)^{\frac{4}{r+2}}+P(t)^4+P(t)^2+\|f\|^2\big],\ \forall t\ge 0,\tag{3.78}$$

for some positive constant $\tilde{C}_0$.

Now, let $\mathcal{O}$ be a bounded subset of H and choose $\rho=\rho(\mathcal{O})>0$ so that $\|(w,z)\|_H\le\rho$, for all $(w,z)\in\mathcal{O}$. If $(u_0,u_1)\in\mathcal{O}$, then, as in (3.21), now using Lemma 2.1, (A5), and (A6), we can bound $P(t)^2$ by a positive constant $\tilde{\mu}_0$ depending on p,d,α,ρ, $\|V\|$, and $\|f\|$. Therefore, the proof of Lemma 3.10 can be concluded as in the proof o Lemma 3.3, using (3.78) and Lemma 3.1. □

Next, let us modify the proofs of Lemmas 3.6 and 3.7. We will keep the same notation used in Lemma 3.6. Under the assumptions of Lemma 3.10, we define

$$\begin{aligned}E_\theta(t)&=\frac{1}{2}\sum_{n\in\mathbb{Z}^d}\theta_n(\dot{u}_n(t))^2+\frac{1}{2}\sum_{n\in\mathbb{Z}^d}\theta_n|D^pu_n(t)|^2+\frac{\alpha}{2}\sum_{n\in\mathbb{Z}^d}\theta_n(u_n(t))^2\\&\quad+\tilde{J}_{1,\theta}(t)-\sum_{n\in\mathbb{Z}^d}\theta_nf_nu_n(t),\ \forall t\ge 0,\end{aligned}\tag{3.79}$$

where

$$\tilde{J}_{1,\theta}(t)=\frac{1}{2}\sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d}\theta_nV(n-m)h(u_m(t))h(u_n(t)).\tag{3.80}$$

Differentiating $E_\theta(t)$ with respect to t, using (2.10), (A5), (A6) and Lemma 3.4, we find that

$$\frac{d}{dt}E_\theta(t) = -\sum_{n\in\mathbb{Z}^d}\theta_n g(n,\dot{u}_n(t))\dot{u}_n(t) - R_\theta(t), \quad \forall t\ge 0, \tag{3.81}$$

with $R_\theta(t) = R_{1,\theta}(t) + R_{2,\theta}(t)$, where

$$R_{1,\theta}(t) = \sum_{n\in\mathbb{Z}^d}\sum_{i=1}^{d}(\partial_i^+\theta_n)z_{p,n}^{(i)}(t), \tag{3.82}$$

$$R_{2,\theta}(t) = -\frac{1}{2}\sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d}(\theta_n-\theta_m)V(n-m)h(u_n(t))h'(u_m(t))\dot{u}_m(t), \tag{3.83}$$

and the functions $z_{p,n}^{(i)}(t)$, $i=1,\dots,d$ are as in Lemma 3.4. Thus, defining on $[0,\infty)$ the functions

$$P_\theta(t)^2 = E_\theta(t) - E_\theta(t+1) - \int_t^{t+1}R_\theta(s)ds, \tag{3.84}$$

$$\tilde{E}_\theta(t) = E_\theta(t) + \frac{4}{\alpha}\sum_{n\in\mathbb{Z}^d}\theta_n f_n^2, \tag{3.85}$$

from (3.79) and (3.81) we obtain

$$P_\theta(t)^2 = \int_t^{t+1}Q_\theta(s)ds \ge 0, \quad \forall t\ge 0, \tag{3.86}$$

$$\tilde{E}_\theta(t) \ge \alpha_0\sum_{n\in\mathbb{Z}^d}\theta_n[\dot{u}_n(t)^2 + u_n(t)^2], \quad \forall t\ge 0, \tag{3.87}$$

where, as before, in (3.86), $Q_\theta(t) = \sum_{n\in\mathbb{Z}^d}\theta_n g(n,\dot{u}_n(t)\dot{u}_n(t))$ for all $t\ge 0$.

Lemma 3.11. *Under the assumptions of Lemma 3.10 and also $(u_0,u_1)\in B[0;\rho_1]$, there exist positive constants $\tilde{C}_{2,i}$, $i=1,2,3$, with $\tilde{C}_{2,1},\tilde{C}_{2,2}$ depending on ρ_1 and $\tilde{C}_{2,3}$ depending only on α, such that*

$$\begin{aligned}\sup_{t\le s\le t+1}\tilde{E}_\theta(s) &\le \tilde{C}_{2,1}\Big(P_\theta(t)^{\frac{4}{r+2}} + P_\theta(t)^2\Big)\\ &\quad + \tilde{C}_{2,2}\Big[\frac{1}{k^2}\Big(1+\sum_{|j|_0\le l}|j|_0^2|V(j)|^2\Big) + \sum_{|j|_0>l}|V(j)|^2\Big]^{1/2} + \tilde{C}_{2,3}\|f\|_\theta^2,\end{aligned}$$

for any positive integer l and any $t\ge 0$.

Proof. Since $(u_0,u_1)\in B[0;\rho_1]$, as in Lemma 3.6, there exists a positive constant r_1 depending on ρ_1 such that

$$\|(u(t),\dot{u}(t))\|_H \le r_1, \quad \forall t\ge 0. \tag{3.88}$$

Let t_1, t_2 satisfy (3.43) with $Q_\theta(t)$ as above. Taking the inner product of equation (2.10) with $w(t)$ in ℓ^2 and integrating the result over $[t_1, t_2]$ we find that

$$\begin{aligned} &\int_{t_1}^{t_2} \|D^p u(s)\|_\theta^2 ds + \alpha \int_{t_1}^{t_2} \|u(s)\|_\theta^2 ds + \int_{t_1}^{t_2} \tilde{J}_{2,\theta}(s) ds \\ &= (\dot{u}(t_1), u(t_1))_\theta - (\dot{u}(t_2), u(t_2))_\theta + \int_{t_1}^{t_2} \|\dot{u}(s)\|_\theta^2 ds \\ &\quad - \int_{t_1}^{t_2} (g(\dot{u}(s)), u(s))_\theta ds + \int_{t_1}^{t_2} (f, u(s))_\theta ds - \int_{t_1}^{t_2} \tilde{R}_\theta(s) ds, \end{aligned} \tag{3.89}$$

where

$$\tilde{J}_{2,\theta}(t) = \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} \theta_n V(n-m) h(u_m(t)) h'(u_n(t)) u_n(t), \tag{3.90}$$

$$\tilde{R}_\theta(t) = \sum_{n \in \mathbb{Z}^d} \sum_{i=1}^{d} (\partial_i^+ \theta_n) \tilde{z}_{p,n}^{(i)}(t),$$

and the functions $\tilde{z}_{p,n}^{(i)}(t)$, $i = 1, \dots, d$ satisfy the estimate (3.46) with $r_0 = r_1$.

Substituting the estimates obtained in the proof of Lemma 3.6 into (3.89) yields

$$\begin{aligned} &\int_{t_1}^{t_2} \|D^p u(s)\|_\theta^2 ds + \frac{\alpha}{2} \int_{t_1}^{t_2} \|u(s)\|_\theta^2 ds + \int_{t_1}^{t_2} \tilde{J}_{2,\theta}(s) ds \\ &\le \tilde{C}_{1,2} \Big[\big(P_\theta(t)^{\frac{2}{r+2}} + P_\theta(t) \big) \sup_{t \le s \le t+1} \tilde{E}_\theta(s)^{1/2} + P_\theta(t)^{\frac{4}{r+2}} + P_\theta(t)^2 \Big] \\ &\quad + \alpha_0^{-1/2} \|f\|_\theta \sup_{t \le s \le t+1} \tilde{E}_\theta(s)^{1/2} + \frac{2}{k} C(p,d) r_1^2, \end{aligned} \tag{3.91}$$

where $\tilde{C}_{1,2}$ is a positive constant that depends on $n_0, \alpha, k_2, r, d, L_2(1)$, and r_1.

By (A5) and (A6) we know that $\tilde{J}_{2,\theta}(t) \ge 2c_1 \tilde{J}_{1,\theta}(t)$, $\forall t \ge 0$. Then, integrating (3.79) over $[t_1, t_2]$, using (3.53), (3.57), and (3.91), we obtain

$$\begin{aligned} &\int_{t_1}^{t_2} \tilde{E}_\theta(s) ds \\ &\le \tilde{C}_{1,3} \big[\big(P_\theta(t)^{\frac{2}{r+2}} + P_\theta(t) \big) \sup_{t \le s \le t+1} \tilde{E}_\theta(s)^{1/2} + P_\theta(t)^{\frac{4}{r+2}} + P_\theta(t)^2 \big] \\ &\quad + \alpha_0^{-1/2} \|f\|_\theta \sup_{t \le s \le t+1} \tilde{E}_\theta(s)^{1/2} + \frac{1}{k} \tilde{C}_{1,4} + \frac{4}{\alpha} \|f\|_\theta^2, \end{aligned} \tag{3.92}$$

with some positive constants $\tilde{C}_{1,3}$ and $\tilde{C}_{1,4}$ depending on r_1.

To complete the proof we need to estimate the term $R_\theta(t)$. Observe that for any $m, j \in \mathbb{Z}^d$, by the definition of θ_n, we know that

$$|\theta_{m+j} - \theta_m| \le \frac{2}{k} |j|_0 \quad \text{and} \quad |\theta_{m+j} - \theta_m| \le 2.$$

Since $(u_0, u_1) \in B[0;\rho_1]$, for any integer $l > 1$, using (3.88), (A5), and (A6) we can estimate $R_{2,\theta}(t)$ as follows

$$
\begin{aligned}
&|R_{2,\theta}(t)| \\
&\le \frac{1}{2}\sum_{m\in\mathbb{Z}^d}\Big(|h'(u_m(t))|\,|\dot u_m(t)|\sum_{n\in\mathbb{Z}^d}|\theta_n-\theta_m|V(n-m)|h(u_n(t))|\Big) \\
&\le \frac{1}{2}\beta_1\Big(\sum_{m\in\mathbb{Z}^d}|h'(u_m(t))|^2\,|\dot u_m(t)|^2\Big)^{1/2}\Big(\sum_{j\in\mathbb{Z}^d}|\theta_{m+j}-\theta_m|^2|V(j)|^2\Big)^{1/2} \\
&\le \beta_1^2 r_1\Big(\frac{1}{k^2}\sum_{|j|_0\le l}|j|_0^2|V(j)|^2+\sum_{|j|_0>l}|V(j)|^2\Big)^{1/2},
\end{aligned}
\tag{3.93}
$$

where $\beta_1 = \max_{|s|\le r_1}\{|h(s)|, |h'(s)|\}$.

Estimating the term (3.82) as in (3.47) we obtain

$$
\int_t^{t+1}|R_{1,\theta}(s)|ds \le \frac{2}{k}C(p,d)r_1^2. \tag{3.94}
$$

Now, as in the proof of Lemma 3.6, by the Mean-Value Theorem for integrals there exists $t^* \in [t_1, t_2]$ such that

$$
\frac{1}{2}\,\tilde E_\theta(t^*) \le (t_2-t_1)\tilde E_\theta(t^*) = \int_{t_1}^{t_2}\tilde E_\theta(s)ds \tag{3.95}
$$

and, as in (3.65), we have

$$
\sup_{t\le s\le t+1}\tilde E_\theta(s) \le \tilde E_\theta(t^*) + P_\theta(t)^2 + \int_t^{t+1}|R_\theta(s)|ds. \tag{3.96}
$$

Finally, from (3.92)-(3.96) we conclude the proof. $\square$

Lemma 3.12. *Under the assumptions of Lemma 3.10, for each $\epsilon > 0$, there exist $\tau(\epsilon) > 0$ and a positive integer $k(\epsilon)$ such that*

$$
\sum_{|n|_0\ge k(\epsilon)}[(\dot u_n(t))^2+(u_n(t))^2] < \epsilon, \quad \forall t \ge \tau(\epsilon).
$$

Proof. Let us consider the case $r > 0$. We will denote by $\tilde C_{3,i}$, $i = 1,\dots,5$, the positive constants depending on ρ_1 which will appear along the proof. By Lemma 3.11 we have

$$
\begin{aligned}
\sup_{t\le s\le t+1}\tilde E_\theta(s)^{1+\frac{r}{2}} &\le 2^{1+\frac{r}{2}}P_\theta(t)^2\big[\tilde C_{2,1}\big(1+P_\theta(t)^{\frac{2r}{r+2}}\big)\big]^{1+\frac{r}{2}} \\
&\quad + 2^{1+\frac{r}{2}}\big[\tilde C_{2,2}\,\vartheta(k,l)+\tilde C_{2,3}\|f\|_\theta^2\big]^{1+\frac{r}{2}}, \quad \forall t\ge 0,
\end{aligned}
\tag{3.97}
$$

where

$$
\vartheta(k,l) = \frac{1}{k^2}\Big(1+\sum_{|j|_0\le l}|j|_0^2|V(j)|^2\Big)+\sum_{|j|_0>l}|V(j)|^2.
$$

In (3.97) we can bound the term $P_\theta(t)^{\frac{2r}{r+2}}$ by a positive constant depending on r_1 using (3.86), (A1) and (3.88) (see (3.67)). Then, using (3.84) and the estimates (3.93) and (3.94) again, from (3.97) it follows that

$$
\begin{aligned}
\sup_{t\le s\le t+1}\tilde E_\theta(s)^{1+\frac{r}{2}} &\le \tilde C_{3,1}[\tilde E_\theta(t)-\tilde E_\theta(t+1)] + 2^{1+\frac{r}{2}}\tilde C_{3,2}\vartheta(k,l) \\
&\quad + 2^{1+\frac{r}{2}}\big[\tilde C_{2,2}\vartheta(k,l)+\tilde C_{2,3}\|f\|_\theta^2\big]^{1+\frac{r}{2}}.
\end{aligned}
$$

Applying the first part of Lemma 3.1 to this inequality and using the elementary inequality $(a+b)^s \leq a^s + b^s$, valid for all real numbers $a, b > 0$ and $0 < s \leq 1$, we deduce that

$$\begin{aligned} \tilde{E}_\theta(t) \leq \big[\tilde{C}_{3,1}^{-1}\frac{r}{2}(t-1)^+ + \big(\sup_{0\leq s\leq 1} \tilde{E}_\theta(s)\big)^{-\frac{r}{2}}\big]^{-\frac{2}{r}} + \tilde{C}_{3,3}\big[\vartheta_1(k,l) + \vartheta_2(l)\big] \\ + 2\tilde{C}_{2,3}\|f\|_\theta^2, \quad \forall t \geq 0, \end{aligned} \tag{3.98}$$

where

$$\vartheta_1(k,l) = \frac{1}{k^2}\Big(1 + \sum_{|j|_0\leq l} |j|_0^2|V(j)|^2\Big) + \Big[\frac{1}{k^2}\Big(1 + \sum_{|j|_0\leq l} |j|_0^2|V(j)|^2\Big)\Big]^{\frac{2}{r+2}},$$

$$\vartheta_2(l) = \sum_{|j|_0>l} |V(j)|^2 + \Big(\sum_{|j|_0>l} |V(j)|^2\Big)^{\frac{2}{r+2}}.$$

Since $(u_0, u_1) \in B[0;\rho_1]$, using (3.79), (3.85), Lemma 2.1, and (A5) we can bound the term $\sup_{0\leq s\leq 1} \tilde{E}_\theta(s)$ in (3.98) by a positive constant depending on $\rho_1, p, d, \alpha, \|V\|$, and $\|f\|$. Therefore, from (3.98) and (3.87), we obtain the inequality

$$\begin{aligned} \sum_{n\in\mathbb{Z}^d} \theta_n[\dot{u}_n(t)^2 + u_n(t)^2] \leq \alpha_0^{-1}\tilde{C}_{3,4}(1+t)^{-\frac{2}{r}} + \alpha_0^{-1}\tilde{C_{3,3}}[\vartheta_1(k,l) + \vartheta_2(l)] \\ + 2\alpha_0^{-1}\tilde{C}_{2,3}\|f\|_\theta^2, \quad \forall t \geq 0, \end{aligned}$$

Now, given $\epsilon > 0$, using (A6) we choose $l = l(\epsilon) > 1$ such that $\alpha_0^{-1}\tilde{C}_{3,3}\vartheta_2(l) < \varepsilon/4$. Thus,

$$\begin{aligned} \sum_{n\in\mathbb{Z}^d} \theta_n[\dot{u}_n(t)^2 + u_n(t)^2] \leq \alpha_0^{-1}\tilde{C}_{3,4}(1+t)^{-\frac{2}{r}} + \alpha_0^{-1}\tilde{C}_{3,3}\vartheta_1(k, l(\epsilon)) \\ + 2\alpha_0^{-1}\tilde{C}_{2,3}\|f\|_\theta^2 + \frac{\epsilon}{4}, \quad \forall t \geq 0. \end{aligned}$$

Since $f \in \ell^2$, then there exists a positive integer $k(\epsilon)$ such that

$$\alpha_0^{-1}\tilde{C}_{3,3}\vartheta_1(k, l(\epsilon)) + 2\alpha_0^{-1}\tilde{C}_{2,3}\|f\|_\theta^2 < \frac{\epsilon}{4},$$

for all $k \geq \frac{k(\epsilon)}{2}$. Consequently, the above inequality reduces to

$$\sum_{n\in\mathbb{Z}^d} \theta\Big(\frac{|n|_0}{k}\Big)\big[\dot{u}_n(t)^2 + u_n(t)^2\big] \leq \alpha_0^{-1}\tilde{C}_{3,4}(1+t)^{-\frac{2}{r}} + \frac{\epsilon}{2}, \tag{3.99}$$

for all $t \geq 0$ and $k \geq \frac{k(\epsilon)}{2}$. Now, choosing $\tau(\epsilon) = \max\big\{0, \big(\frac{2\tilde{C}_{3,4}}{\alpha_0\epsilon}\big)^{r/2} - 1\big\}$, from (3.99) we obtain

$$\sum_{|n|\geq 2k} [\dot{u}_n(t)^2 + u_n(t)^2] \leq \epsilon, \quad \forall t \geq \tau(\epsilon),\ \forall k \geq \frac{k(\epsilon)}{2},$$

which implies Lemma 3.12 if $r > 0$.

If $r = 0$ then by Lemma 3.11, (3.93) and (3.94) we have

$$\sup_{t\leq s\leq t+1} \tilde{E}_\theta(s) \leq 2\tilde{C}_{2,1}[\tilde{E}_\theta(t) - \tilde{E}_\theta(t+1)] + \tilde{C}_{3,5}\vartheta(k,l) + \tilde{C}_{2,3}\|f\|_\theta^2,$$

with $\vartheta(k,l)$ as before. Applying the second part of Lemma 3.1 to this inequality we can conclude the proof as we did in the case $r > 0$. $\square$

Lemma 3.13. *Under the same assumptions of Lemma 3.10 the semigroup* $\{S(t)\}_{t\geq 0}$ *associated with the initial value problem* (2.10) *is asymptotically compact in* H.

The above lemma is a consequence of Lemma 3.12 and well known arguments (see [18, Lemma 3.9]). As a direct consequence of Lemmas 3.10, 3.13 and [26, Theorem 1.1] we obtain our second main result in this section.

Theorem 3.14. *Assume that* (A1), (A4)–(A6) *hold and let* u_0, u_1, *and* f *belong to* ℓ^2. *Then the semigroup* $\{S(t)\}_{t\geq 0}$ *associated with the initial value problem* (2.10) *possesses a unique global attractor* $\mathcal{A}$ *em* H. *Moreover,* $\mathcal{A} \subset B[0;\rho_1]$ *and the absorbing rates* (3.74) *hold with* ρ_0 *replaced by* ρ_1.

4. Periodic lattice differential equations

Let N be a fixed positive integer (the period). We denote by $\ell^2_{\rm per}$ the Hilbert space of all real sequences $u = (u_n)_{n\in\mathbb{Z}^d}$ such that

$$u_{n+Ne_i} = u_n, \quad i = 1, \dots, d,\ n \in \mathbb{Z}^d, \tag{4.1}$$

equipped with the inner product and norm given by

$$(u, v)_{\ell^2_{\rm per}} = \sum_{n=1}^{N} u_n v_n, \quad \|u\| = \Big(\sum_{n=1}^{N} |u_n|^2\Big)^{1/2},$$

for all $u = (u_n)_{n\in\mathbb{Z}^d}$ and $v = (v_n)_{n\in\mathbb{Z}^d}$. Throughout this section we donote by $\sum_{n=1}^N u_n$ the sum $\sum_{n_1=1}^N \cdots \sum_{n_d=1}^N u_{(n_1,\dots,n_d)}$, whenever $u = (u_n)_{n\in\mathbb{Z}^d}$ satisfies (4.1). We also consider the Hilbert space $\dot{\ell}^2_{\rm per}$ of all sequences $u = (u_n)_{n\in\mathbb{Z}^d}$ in $\ell^2_{\rm per}$ such that $\sum_{n=1}^N u_n = 0$, equipped with the inner product and norm above.

As in the previous sections we will denote a sequence $(u_n)_{n\in\mathbb{Z}^d}$ by (u_n). Also, we will refer to a sequence $u = (u_n)_{n\in\mathbb{Z}^d}$ satisfying (4.1) as a periodic sequence.

Using (4.1) it is easy to check that for any $u = (u_n)$ and $v = (v_n)$ in $\ell^2_{\rm per}$, Lemma 2.1, inequality (2.2), and Lemma 2.2 are still valid with $\sum_{n\in\mathbb{Z}^d}$ replaced by $\sum_{n=1}^N$ and $|D^p u_n|^2$ defined by (2.1).

We assume that the functions $h_0 : \mathbb{Z}^d \times \mathbb{R} \to \mathbb{R}$, $h_i : \mathbb{R} \to \mathbb{R}, i = 1, \dots, d$, and $g : \mathbb{Z}^d \times \mathbb{R} \to \mathbb{R}$ satisfy (A1)–(A3) stated in Section 2, where in (A2), $b_1 = (b_{1,n})$, and $b_2 = (b_{2,n})$ are nonnegative periodic sequences. In addition, we assume that

- (A4') For each $s \in \mathbb{R}$, $(h_0(n, s))$, $(g(n, s))$ are periodic sequences and there exist real constants $k_2 > 0, r \geq 0$ such that $sg(n, s) \geq k_2 |s|^{r+2}$, for all $n \in \mathbb{Z}^d$ and $s \in \mathbb{R}$.

Let us first consider the periodic problem

$$\begin{gathered} \ddot{u}_n(t) + (-1)^p \Delta_d^p u_n(t) + \alpha u_n(t) + F(n, u_n(t), \nabla^+ u_n(t)) + g(n, \dot{u}_n(t)) = f_n, \\ u_n(0) = u_{0,n}, \quad \dot{u}_n(0) = u_{1,n}, \\ u_{n+Ne_i}(t) = u_n(t), \end{gathered} \tag{4.2}$$

where $n \in \mathbb{Z}^d$, $i = 1, \dots, d$, $t \geq 0$, $(u_{0,n})$, $(u_{1,n})$, (f_n) belong to $\ell^2_{\rm per}$, and the term $F(n, u_n(t), \nabla^+ u_n(t))$ is as in (1.2).

We introduce the Hilbert space $H = \ell^2_{\rm per} \times \ell^2_{\rm per}$ equipped with the natural inner product given by

$$\big((u, v), (w, z)\big)_H = (u, w)_{\ell^2_{\rm per}} + (v, z)_{\ell^2_{\rm per}}, \tag{4.3}$$

for all (u, v) and (w, z) in H.

Proceeding as in Sections 2 and 3 we can study the dynamics of the problem (4.2) in the space H and extend the results of Theorems (2.4) and (3.9) to this case. Since the proofs are entirely analogous, here, we will just indicate what should be done.

- We first observe that, under the above assumptions, we can prove that the maps h_0, B, g defined in (2.3) are locally Lipschitz continuous from $\ell^2_{\rm per}$ into itself and that the linear operator A, also defined in (2.3), is bounded in $\ell^2_{\rm per}$. Then we can write the problem (4.2) in $\ell^2_{\rm per}$ as the initial value problem (2.4). As a result, we can use the Theory Ordinary Differential Equations in Banach Spaces and the energy equation associated with the problem (4.2), which is given by (2.7), with $\sum_{n\in\mathbb{Z}^d}$ replaced by $\sum_{n=1}^{N}$ and ℓ^2 by $\ell^2_{\rm per}$, to obtain the same results of Theorem 2.4 for problem (4.2) in the space H above.
- The second step consists in defining the semigroup $\{S(t)\}_{t\geq 0}$ of continuous operator associate with problem (4.2) as in (3.1). Since the dimension of $H = \ell^2_{\rm per} \times \ell^2_{\rm per}$ is finite, in order to prove the existence of a global attractor in H we only need to prove the existence of an absorbing set for the semigroup $\{S(t)\}_{t\geq 0}$ in H. This can be done as in Lemmas 3.2 and 3.3 using Lemma 3.1.

Next, let us discuss the existence of a global attractor for the one-dimensional periodic problem

$$\begin{gathered}
\ddot{u}_n(t) + (-1)^p \Delta_1^p u_n(t) - \partial_1^- h(\partial_1^+ u_n(t)) - \partial_1^- g(\partial_1^+ \dot{u}_n(t)) = f_n, \\
u_n(0) = u_{0,n}, \quad \dot{u}_n(0) = u_{1,n}, \\
u_{n+N}(t) = u_n(t),
\end{gathered} \tag{4.4}$$

where $n \in \mathbb{Z}$ and $t \geq 0$. In (4.4) $(u_{0,n})$, $(u_{1,n})$, (f_n) belong to $\dot{\ell}^2_{\rm per}$, and we assume that the functions $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfy the assumptions

(A7) For each $s_0 > 0$, there exist positive constants $L_j = L_j(s_0)$, $j = 1, 2$, such that
  (i) $|h(s_1) - h(s_2)| \leq L_1 |s_1 - s_2|$,
  (ii) $|g(s_1) - g(s_2)| \leq L_2 |s_1 - s_2|$,
  for all s_1, s_2 in $\mathbb{R}$, $|s_1| \leq s_0$, $|s_2| \leq s_0$ and $h(0) = 0$, $g(0) = 0$.

(A8) There exist constants $k_0 > 0$, $k_1 \geq 0$ and $k_2 \geq 0$ such that

$$sh(s) + k_1 \geq k_0(\tilde{h}(s) + k_2) \geq 0, \quad \forall s \in \mathbb{R}, \text{ where } \tilde{h}(s) = \int_0^s h(\sigma)d\sigma.$$

(A9) There exist constants $k_3 > 0, r \geq 0$ such that $sg(s) \geq k_3 |s|^{r+2}$ for all $s \in \mathbb{R}$.

Examples of functions satisfying the above assumptions are given in Appendix A.

We want to study the dynamics of problem (4.4) in the Hilbert space $\dot{H} = \dot{\ell}^2_{\rm per} \times \dot{\ell}^2_{\rm per}$, equipped with the inner product $(\cdot, \cdot)_{\dot{H}}$ defined as in (4.3), with H replaced by $\dot{H}$. We denote by $\|\cdot\|_{\dot{H}}$ the corresponding norm. Under assumptions (A7)–(A9), the well-posedness of problem (4.4) can be proved following the same steps of Section 2. Indeed, introducing the notation

$$\begin{gathered}
Au = ((-1)^p \Delta_1^p u_n), \quad B(u) = \big(-\partial_1^- h(\partial_1^+ u_n)\big), \\
G(u) = \big(-\partial_1^- g(\partial_1^+ u_n)\big),
\end{gathered}$$

for any $u = (u_n)$ in $\dot{\ell}^2_{\rm per}$, we can write (4.4) in $\dot{\ell}^2_{\rm per}$ as

$$\begin{gathered} \ddot{u}(t) + Au(t) + B(u(t)) + G(\dot{u}(t)) = f, \quad t > 0, \\ u(0) = u_0, \quad \dot{u}(0) = u_1, \end{gathered} \tag{4.5}$$

where $u_0 = (u_{0,n})$, $u_1 = (u_{1,n})$, $f = (f_n)$, $u(t) = (u_n(t))$, $\dot{u}(t) = (\dot{u}_n(t))$, and $\ddot{u}(t) = (\ddot{u}_n(t))$.

Therefore, we can prove the existence of a unique solution $u = u(t)$ of (4.5) belonging to $C^2([0, \tau_{\max}); \dot{\ell}_{\rm per})$ with the property that either $\tau_{\max} = \infty$ or

$$\tau_{\max} < \infty \quad \text{and} \quad \lim_{t \to \tau_{\max}^-} \|(u(t), \dot{u}(t))\|_{\dot{H}} = \infty.$$

To extend the solution $u = u(t)$ globally we will need the following consequence of Poincaré inequality valid in $\dot{\ell}^2_{\rm per}$.

Lemma 4.1. *Let $u = (u_n)$ belong to $\dot{\ell}^2_{\rm per}$. Then*

$$\sum_{n=1}^N u_n^2 \le C_0^p \sum_{n=1}^N |D^p u_n|^2,$$

for all $p \in \mathbb{N}$, where $C_0 = 4N^2$.

Proof. By Poincaré inequality (see [17]), we know that

$$\sum_{n=1}^N u_n^2 \le C_0 \sum_{n=1}^N |\partial_1^+ u_n|^2, \tag{4.6}$$

where $C_0 = 4N^2$. Since $\partial_1^+ u_n \in \dot{\ell}^2_{\rm per}$, using (4.6) and observing that

$$\sum_{n=1}^N |\partial_1^+ v_n|^2 = \sum_{n=1}^N |\partial_1^- v_n|^2$$

for any $v = (v_n) \in \dot{\ell}^2_{\rm per}$ because of the periodicity, we obtain

$$\sum_{n=1}^N u_n^2 \le C_0^2 \sum_{n=1}^N |\Delta_1 u_n|^2.$$

Then by induction we can conclude the proof. □

Theorem 4.2. *Assume that* (A7)–(A9) *hold and let u_0, u_1, and f belong to $\dot{\ell}^2_{\rm per}$. Then the initial value problem* (4.5) *has a unique solution $u \in C^2(\mathbb{R}^+; \dot{\ell}^2_{\rm per})$. Moreover, for each $\tau > 0$, the map $\mathfrak{I} : \dot{H} \to C([0, \tau]; \dot{H})$, defined by $\mathfrak{I}(u_0, u_1)(t) = (u(t), \dot{u}(t))$, $0 \le t \le \tau$, is continuous.*

Proof. Taking the inner product of equation 4.5 with $\dot{u} = \dot{u}(t)$ in $\dot{\ell}^2_{\rm per}$ we find

$$\frac{d}{dt} E(t) = -\sum_{n=1}^N \partial_1^+ \dot{u}_n(t) g(\partial_1^+ \dot{u}_n(t)) \le 0, \quad \forall 0 \le t < \tau_{\max}, \tag{4.7}$$

because $sg(s) \ge 0$ for all $s \in \mathbb{R}$ by (A9), where

$$E(t) = \frac{1}{2}\|\dot{u}(t)\|^2 + \frac{1}{2}\|D^p u(t)\|^2 + \sum_{n=1}^N \tilde{h}(\partial_1^+ u_n(t)) - (f, u(t))_{\ell^2_{\rm per}}. \tag{4.8}$$

Recall that $\|D^p u(t)\|^2 = \sum_{n=1}^N |D^p u_n(t)|^2$. From (4.7) and (4.8), using Lemma 4.1 and (A8) we obtain

$$\|(u(t),\dot u(t))\|_{\dot H}^2 \le \alpha_0^{-1}\tilde E(t) \le \alpha_0^{-1}\tilde E(0) < \infty, \forall 0 \le t < \tau_{\max}, \tag{4.9}$$

where

$$\tilde E(t) = E(t) + k_2 N + 4C_0^P\|f\|^2, \forall 0 \le t < \tau_{\max}, \tag{4.10}$$

and $\alpha_0 = \min\{\frac12, \frac{1}{C_0^p}\}$. Hence $\tau_{\max} = \infty$.

Finally, the proof of the continuity of $\mathfrak{I}$ is standard using Gronwall inequality and so we omit it here. □

Let us denote by $\{S(t)\}_{t\ge0}$ the semigroup of continuous operator in $\dot H$ associated with problem (4.5), which is defined in the same manner as in (3.1).

Lemma 4.3. *Under the assumptions of Theorem 4.2 there exists $\rho_0 > 0$ such that $B[0;\rho_0] = \{(w,z) \in \dot H : \|(w,z)\|_{\dot H} \le \rho_0\}$ is an absorbing set for $\{S(t)\}_{t\ge0}$ in $\dot H$.*

Proof. The proof is similar to that of Lemma 3.2. Let us indicate the appropriate modifications. Define

$$P(t)^2 = E(t) - E(t+1) \quad\text{and}\quad Q(t) = \sum_{n=1}^N \partial_1^+\dot u_n(t) g(\partial_1^+\dot u_n(t)),$$

for all $t \ge 0$ and let t_1, t_2 be chosen as in (3.4). Also note that by (4.7)

$$P(t)^2 = \int_t^{t+1} Q(s)ds \ge 0, \quad \forall t \ge 0.$$

Using (4.6), Hölder's inequality, (4.9), and (A9) we have

$$\begin{aligned}
|(\dot u(t_j), u(t_j))| &\le \alpha_0^{-1/2} C_0^{1/2} \sup_{t\le s\le t+1} \tilde E(s)^{1/2}\Big(\sum_{n=1}^N |\partial_1^+\dot u_n(t_j)|^2\Big)^{1/2} \\
&\le N^{\frac{r}{2(r+2)}}\alpha_0^{-1/2} C_0^{1/2} \sup_{t\le s\le t+1} \tilde E(s)^{1/2}\Big(\sum_{n=1}^N |\partial_1^+\dot u_n(t_j)|^{r+2}\Big)^{\frac{1}{r+2}} \\
&\le C_{0,1} P(t)^{\frac{2}{r+2}} \sup_{t\le s\le t+1} \tilde E(s)^{1/2}, \quad j = 1,2,
\end{aligned}$$

for some positive constant $C_{0,1}$. By Lemma 4.1,

$$\sum_{n=1}^N |\partial_1^+ u_n(t)|^2 \le 4\sum_{n=1}^N |u_n(t)|^2 \le 4C_0^p\|D^p u(t)\|^2, \quad \forall t \ge 0. \tag{4.11}$$

For each t fixed, define the sets $I_1(t) = \{n \in \{1,\dots,N\}; |\partial_1^+\dot u_n(t)| \le 1\}$ and $I_2(t) = \{1,\dots,N\}\backslash I_1(t)$. Using Lemma 4.1, (4.9), (4.11), and (A8) we see that, for any $t \le s \le t+1$,

$$\sum_{n\in I_1(s)} \partial_1^+ u_n(s) g(\partial_1^+\dot u_n(s)) \le \frac12\|D^p u(s)\|^2 + 2C_0^p L_2(1) Q(s)$$

and

$$\begin{aligned}
\sum_{n\in I_2(s)} \partial_1^+ u_n(s) g(\partial_1^+\dot u_n(s)) &\le \sum_{n\in I_2(s)} |\partial_1^+ u_n(s)|\,|\partial_1^+\dot u_n(s)|\,|g(\partial_1^+\dot u_n(s))| \\
&\le 2\|u(s)\| \sum_{n\in I_2(s)} |\partial_1^+\dot u_n(s)|\,|g(\partial_1^+\dot u_n(s))|
\end{aligned}$$

$$\le 2\alpha_0^{-1/2} Q(s) \sup_{t\le s\le t+1} \tilde{E}(s)^{1/2}.$$

Therefore,

$$\begin{aligned}
&\int_{t_1}^{t_2} \sum_{n=1}^{N} \partial_1^+ u_n(s) g(\partial_1^+ \dot{u}_n(s)) ds \\
&\le \frac{1}{2}\int_{t_1}^{t_2} \|D^p u(s)\|^2 ds + \big[2C_0^p L_2(1) + 2\alpha_0^{-1/2} \sup_{t\le s\le t+1} \tilde{E}(s)^{1/2}\big] P(t)^2.
\end{aligned}$$

The other estimates are analogous to those of Lemma 3.2 and we can prove the inequality

$$\sup_{t\le s\le t+1} \tilde{E}(s) \le C_0\big(P(t)^{\frac{4}{r+2}} + P(t)^4 + P(t)^2 + \|f\|^2 + k_1\big), \quad \forall t \ge 0,$$

for some positive constant C_0, depending on $N, p, r, L_2(1), k_0$, and k_3. Then, arguing as in the proof of Lemma 3.3 and using Lemma 3.1 we can conclude the proof. □

Since $\dot{H}$ has finite dimension, it follows from Lemma 4.3 that the semigroup $\{S(t)\}_{t\ge 0}$ is asymptotically compact in $\dot{H}$. Thus, we obtain the following result.

Theorem 4.4. *Assume that* (A7)–(A9) *hold and let* u_0*,* u_1*, and* f *belong to* $\dot{\ell}^2_{\rm per}$*. Then the semigroup* $\{S(t)\}_{t\ge 0}$ *associated with problem* (4.5) *possesses a unique global attractor* $\mathcal{A}$ *in* $\dot{H}$*. Moreover,* $\mathcal{A} \subset B[0;\rho_0]$ *and absorbing rates analogous to those in* (3.74) *are valid.*

Remark 4.5. Theorem 4.4 remains valid if we replace the dissipative term $-\partial_1^- g(\partial_1^+ \dot{u}_n(t))$ in (4.4) by $+\mu \dot{u}_n(t)$, with $\mu > 0$.

Remark 4.6. The restriction on the dimension in (4.4) is due to the fact that we used Lemma 4.1 which we do not know if it is valid when the operator Δ_1^p is replaced by Δ_d^p, with $p > 1$. The case with Δ_d when d is arbitrary was considered in [17].

5. Appendix A

In this appendix we give examples of functions satisfying the assumptions (A1)–(A9).

Example 5.1. We start with functions satisfying (A1)–(A4). The following functions $h_0 : \mathbb{Z}^d \times \mathbb{R} \to \mathbb{R}$ satisfy (A1) and (A2).

(a) $h_0(n,s) = a_n|s|^\gamma s - \lambda_n |s|^\sigma s$, with $\gamma > \sigma \ge 0$, $\lambda = (\lambda_n) \in \ell^1$, $a = (a_n) \in \ell^\infty$, and $a_n \ge \alpha > 0$ for all $n \in \mathbb{Z}^d$.

(b) $h_0(n,s) = a_n \sinh s$, with $a = (a_n) \in \ell^\infty$ and $a_n \ge 0, \forall n \in \mathbb{Z}^d$.

(c)
$$h_0(n,s) = \begin{cases} a_n|s|s, & |s| \le 1 \\ a_n \frac{s}{|s|}, & |s| > 1 \end{cases},$$
where $a = (a_n)$ is as in b).

Note that in (a) assumption (A2) is satisfied with

$$\begin{gathered}
k_1 = \gamma + 2, \quad b_{1,n} = (\gamma+2) b_{2,n}, \quad b_{2,n} = \mu \lambda_n^{q+1}, \quad \text{if } \lambda_n \ge 0 \\
k_1 = \sigma + 2, \quad b_{1,n} = 0, \quad b_{2,n} = 0 \quad \text{if } \lambda_n \le 0,
\end{gathered}$$

where

$$\mu = \alpha^{\frac{\sigma+2}{\sigma-\gamma}}\Big(\frac{1}{\sigma+2} - \frac{1}{\gamma+2}\Big), \quad q = \frac{\sigma+2}{\gamma-\sigma}.$$

Also (b) and (c) fulfill (A2) with $k_1 = 1$, $b_{1,n} = 0$, and $b_{2,n} = 0$.

The following functions $h_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, d$, satisfy (A1) and (A3).

(d) $h_i(s) = \mu_i |s|^{q-2} s$, $\mu_i > 0$, and $q \geq 2$.
(e) $h_i(s) = \mu_i s\, e^{\lambda_i s^2}$, μ_i, and $\lambda_i > 0$.
(f) $h_i(s) = \sum_{j=0}^{m_i} a_{i,m_i-j} s^{2m_i+1-2j}$, where m_i is a positive integer and $a_{i,m_i-j} > 0$, $i = 1, \dots, d$, $j = 0, \dots, m_i$.

In addition, we observe that assumptions (A1) and (A3) are fulfilled if the functions $h_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, d$, are non-decreasing and locally Lipschitz continuous and satisfy $sh_i(s) \geq 0$ for all $s \in \mathbb{R}$.

Some examples of functions $g(n, s)$ satisfying (A1) and (A4) are as follows.

(g) $g(n, s) = g_0(s)$, with $g_0 \in C^1(\mathbb{R}, \mathbb{R})$, $g_0(0) = 0$ and $g_0'(s) \geq c_0 > 0$.
(h)
$$g(n, s) = \begin{cases} a_n |s|^r s + b_n g_1(s), & |n|_0 \leq n_0, s \in \mathbb{R} \\ c_n g_0(s), & |n|_0 > n_0, s \in \mathbb{R} \end{cases},$$
where n_0 is a positive integer, g_0, g_1 are locally Lipschitz functions satisfying $g_0(0) = g_1(0) = 0$, $sg_0(s) \geq c_0 s^2$, $sg_1(s) \geq 0$, with $c_0 > 0$, for all $s \in \mathbb{R}$, $(a_n), (b_n)$, and (c_n) belong to ℓ^∞, $a_n > 0, b_n \geq 0$, $c_n > 0$, for all $n \in \mathbb{Z}^d$ and $c_n \geq c_* > 0$ if $|n|_0 > n_0$.

Example 5.2. We now give examples of functions $h : \mathbb{R} \to \mathbb{R}^+$ and $V : \mathbb{Z}^d \to \mathbb{R}^+$ that satisfy (A5) and (A6).

(a) $h(s) = \mu |s|^\gamma$, $\mu > 0$, $\gamma \geq 2$.
(b) $h(s) = \mu(\cosh s - 1)$, $\mu > 0$.
(c) $V(n) = \frac{e^{-\lambda |n|}}{|n|^\alpha}$, if $n \neq 0$ and $V(0) = 0$, where $\lambda > 0, \alpha \geq 0$ and $|n| = \big(\sum_{i=1}^d |n_i|^2\big)^{1/2}$.

Example 5.3. Finally, we present functions $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying (A7)–(A9). Other examples can be built using some functions from the preceding examples.

(a) $h(s) = \mu \arctan s$, $\mu > 0$.
(b) $h(s) = se^{-s^2}$.
(c) $h(s) = \sum_{j=0}^m a_{m-j} s^{2m+1-2j}$, where $m \geq 1$ is a positive integer, $a_m > 0$ and $a_{m-j} < 0$, $j = 1, \dots, m$.

Concerning example (c), clearly $k_2 = -\min_{s\in\mathbb{R}} \tilde{h}(s) \geq 0$. Note that $sh(s) = \tilde{h}(s) + h_1(s)$, where

$$h_1(s) = \Big(1 - \frac{1}{2m+2}\Big) a_{m-1} s^{2m+2} + \Big(1 - \frac{1}{2m}\Big) a_{m-1} s^{2m} + \dots + \frac{1}{2} a_0 s^2.$$

Let $s_0 > 0$ be such that $h_1(s) \geq k_2$ for $|s| > s_0$ and set

$$k_1 = k_2 - \Big(1 - \frac{1}{2m}\Big) a_{m-1} s_0^{2m} - \dots - \frac{1}{2} a_0 s_0^2.$$

Then it is easy to check that assumption (A8) is satisfied with $k_0 = 1$.

(d) $g(s) = g_0(s)|s|^r s + g_1(s)$, where $r \geq 0$, g_0, g_1 are locally Lipschitz functions satisfying $g_0(s) \geq c_0 > 0$, $g_1(0) = 0$ and $sg_1(s) \geq 0$, for all $s \in \mathbb{R}$. A function of this type is $g(s) = s^3 + s$.

(e) $g(s) = \sinh s$.

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