Electronic Journal of Differential Equations, Vol. 2007(2007), No. 164, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A NEUMANN PROBLEM WITH THE q-LAPLACIAN ON A SOLID TORUS IN THE CRITICAL OF SUPERCRITICAL CASE

ATHANASE COTSIOLIS, NIKOS LABROPOULOS

Abstract. Following the work of Ding [21] we study the existence of a nontrivial positive solution to the nonlinear Neumann problem

$$\begin{gathered} \Delta_q u + a(x)u^{q-1} = \lambda f(x)u^{p-1}, \quad u > 0 \quad \text{on } T, \\ \nabla u|^{q-2}\frac{\partial u}{\partial \nu} + b(x)u^{q-1} = \lambda g(x)u^{\tilde{p}-1} \quad \text{on } \partial T, \\ p = \frac{2q}{2-q} > 6, \quad \tilde{p} = \frac{q}{2-q} > 4, \quad \frac{3}{2} < q < 2, \end{gathered}$$

on a solid torus of $\mathbb{R}^3$. When data are invariant under the group $G = O(2)\times I \subset O(3)$, we find solutions that exhibit no radial symmetries. First we find the best constants in the Sobolev inequalities for the supercritical case (the critical of supercritical).

1. Introduction

In this paper we study the existence of positive solutions of the Neumann boundary problem

$$\begin{gathered} \Delta_q u + a(x)u^{q-1} = \lambda f(x)u^{p-1}, \quad u > 0 \quad \text{on } T, \\ |\nabla u|^{q-2}\frac{\partial u}{\partial \nu} + b(x)u^{q-1} = \lambda g(x)u^{\tilde{p}-1} \quad \text{on } \partial T, \\ p = \frac{2q}{2-q} > 6, \quad \tilde{p} = \frac{q}{2-q} > 4, \quad \frac{3}{2} < q < 2, \end{gathered} \tag{1.1}$$

where $\frac{\partial}{\partial \nu}$ is the outer unit normal derivative, $\Delta_q u = -div(|\nabla u|^{q-2}\nabla u)$ is the q-Laplacian and for $q = 2$, $\Delta_2 = \Delta$ is the Laplace-Beltrami operator.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$ with smooth boundary $\partial\Omega$. A host of literature exists, concerning problems of the same type with (1.1), when $q = 2$; see e.g. [8, 41, 2, 3, 42, 34, 15, 31, 35, 25, 44, 37, 26, 10, 12, 11, 20, 13, 16, 9, 33, 38, 39, 36, 43] and the references therein.

2000 *Mathematics Subject Classification.* 35J65, 46E35, 58D19.
Key words and phrases. Neumann problem; q-Laplacian; solid torus; no radial symmetry; critical of supercritical exponent.

Submitted May 18, 2006. Published November 30, 2007.

In [17] the first author proved (under symmetry assumptions on Ω) the existence and the multiplicity of positive solutions and of nodal solutions for the problem

$$\begin{gathered}\Delta u + a(x)u = f(x)|u|^{p-2}u, \quad \text{on } \Omega,\\ \frac{\partial u}{\partial \nu} + b(x)u = h(x)|u|^{\tilde{p}-2}u \quad \text{on } \partial\Omega,\\ p \geq \frac{2n}{n-2}, \quad \tilde{p} \geq \frac{2(n-1)}{n-2}.\end{gathered} \tag{1.2}$$

where $a(x), f(x)$ are functions in $C^\infty(\overline{\Omega})$ and $b(x), h(x)$ are functions in $C^\infty(\partial\Omega)$.

In contrast to the case $q = 2$, the Neumann problem, for $q \neq 2$, has not been studied so extensively; see e.g. [5, 7, 19, 32, 22, 6, 24]. In all the above mentioned cases the supercritical exponent under consideration is not the highest possible one.

In problem (1.1) the main difficulty comes firstly from the dimension 3 of the domain and secondly because the exponents $p = \frac{2q}{2-q} > 6$ and $\tilde{p} = \frac{q}{2-q} > 4$, $\frac{3}{2} < q < 2$ of the equation and the boundary condition, respectively, are both the highest possible supercritital exponents (critical of supercritical). Also, the boundary condition is more complicated than the one in the above problems with $q \neq 2$. Additionally, we have to find solutions that exhibit no radial symmetries. However, since the solid torus $\overline{T} \subset \mathbb{R}^3$ is invariant under the group $G = O(2) \times I \subset O(3)$, the solutions inherit $\overline{T}$'s symmetry property.

Best constants in Sobolev inequalities are fundamental in the study of non-linear PDEs on manifolds [1, 27, 30, 23, 29, 4] and the references therein. It is also well known that Sobolev embeddings can be improved in the presence of symmetries [30, 21, 17, 28, 19] and the references therein.

In our case for any $q \in [1, 2)$ real, the embedding $H^q_{1,G}(T) \hookrightarrow L^p_G(T)$ is compact for $1 \leq p < 2q/(2-q)$, while $H^q_{1,G}(T) \hookrightarrow L^{2q/(2-q)}_{1,G}(T)$, is only continuous [18]. We will prove that for any $q \in [1, 2)$ real, the embedding $H^q_{1,G}(T) \hookrightarrow L^p_G(\partial T)$ is compact for $1 \leq p < q/(2-q)$, while $H^q_{1,G}(T) \hookrightarrow L^{q/(2-q)}_G(\partial T)$, is only continuous.

In the spirit of [1, 4] we determine the best constants of the Sobolev trace inequality

$$\|u\|^q_{L^{\tilde{p}}(\partial T)} \leq A\|\nabla u\|^q_{L^q(T)} + B\|u\|^q_{L^q(\partial T)},$$

where $\tilde{p} = q/(2-q)$, $1 \leq q < 2$, which concern the supercritical case (the critical of supercritical) and we use the above to solve the problem (1.1).

2. Notation and statement of results

Let us define the solid torus

$$\overline{T} = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - l)^2 + z^2 \leq r^2,\ l > r > 0\}$$

and $\mathcal{A} = \{(\Omega_i, \xi_i) : i = 1, 2\}$ be an atlas on T defined by

$$\begin{gathered}\Omega_1 = \{(x, y, z) \in T : (x, y, z) \notin H^+_{XZ}\},\\ \Omega_2 = \{(x, y, z) \in T : (x, y, z) \notin H^-_{XZ}\}\end{gathered}$$

where

$$\begin{gathered}H^+_{XZ} = \{(x, y, z) \in \mathbb{R}^3 : x > 0\,,\ y = 0\}\\ H^-_{XZ} = \{(x, y, z) \in \mathbb{R}^3 : x < 0\,,\ y = 0\}\end{gathered}$$

and $\xi_i : \Omega_i \to I_i \times D$, $i = 1, 2$, with

$$I_1 = (0, 2\pi), \quad I_2 = (-\pi, \pi), \quad D = \{(t, s) \in \mathbb{R}^2 : t^2 + s^2 < 1\}$$

and $\xi_i(x, y, z) = (\omega_i, t, s)$, $i = 1, 2$ with

$$\cos\omega_i = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin\omega_i = \frac{y}{\sqrt{x^2 + y^2}}, \quad i = 1, 2$$

where

$$\omega_1 = \begin{cases} \arctan\frac{y}{x}, & x \neq 0 \\ \pi/2, & x = 0, y > 0 \\ 3\pi/2, & x = 0\,,\ y < 0 \end{cases} \quad \omega_2 = \begin{cases} \arctan\frac{y}{x}, & x \neq 0 \\ \pi/2, & x = 0,\ y > 0 \\ -\pi/2, & x = 0\,,\ y < 0 \end{cases}$$

and

$$t = \frac{\sqrt{x^2 + y^2} - l}{r}\,, \quad s = \frac{z}{r}.$$

The Euclidean metric g on $(\Omega, \xi) \in \mathcal{A}$ can be expressed as

$$(\sqrt{g} \circ \xi^{-1})(\omega, t, s) = r^2(l + rt).$$

Consider the spaces of all G-invariant functions under the action of the group $G = O(2) \times I \subset O(3)$

$$H^q_{1,G} = \{u \in H^q_1(T) : u \circ \tau = u\,,\ \forall \tau \in G\},$$

where $H^q_1(T)$ is the completion of $C^\infty(T)$ with respect the to norm

$$\|u\|_{H^q_1} = \|\nabla u\|_q + \|u\|_q.$$

For all G-invariants u we define the functions $\phi(t, s) = (u \circ \xi^{-1})(\omega, t, s)$. Then we have

$$\|u\|^p_{L^p(T)} = 2\pi r^2 \int_D |\phi(t, s)|^p (l + rt)\, dt\, ds, \tag{2.1}$$

$$\|\nabla u\|^q_{L^q(T)} = 2\pi r^{2-q} \int_D |\nabla\phi(t, s)|^q (l + rt)\, dt\, ds, \tag{2.2}$$

$$\|u\|^p_{L^p(\partial T)} = 2\pi r \int_{\partial D} |\phi(t, 0)|^p (l + rt)\, dt, \tag{2.3}$$

where by ϕ we denote the extension of ϕ on ∂D.

Let $K(2, q)$ be the best constant [1] of the Sobolev inequality

$$\|\varphi\|_{L^p(\mathbb{R}^2)} \leq K(2, q)\|\nabla\varphi\|_{L^q(\mathbb{R}^2)}$$

for the Euclidean space $\mathbb{R}^2$, where $1 \leq q < 2$, $p = 2q/(2 - q)$ and $\tilde{K}(2, q)$ be the best constant [30] in the Sobolev trace embedding

$$\|\varphi\|_{L^{\tilde{p}}(\partial\mathbb{R}^2_+)} \leq \tilde{K}(2, q)\|\nabla\varphi\|_{L^q(\mathbb{R}^2_+)}$$

for the Euclidean half-space $\mathbb{R}^2_+$, where $1 \leq q < 2$, $\tilde{p} = q/(2 - q)$.

Consider a point $P_j(x_j, y_j, z_j) \in \overline{T}$, and by O_{P_j} denote the orbit of P_j under the action of the group G. Let $l_j = \sqrt{x_j^2 + y_j^2}$ be the horizontal distance of the orbit O_{P_j} from the axis $z'z$. For $\varepsilon > 0$ given and $\delta_j = l_j\varepsilon$, consider a finite covering $(T_j)_{j=1,\dots N}$ with

$$T_j = \{(x, y, z) \in \overline{T} : (\sqrt{x^2 + y^2} - l_j)^2 + (z - z_j)^2 < \delta_j^2\}$$

an open small solid torus (a tubular neighborhood of the orbit O_{P_j}).

2.1. **Best constants on the solid Torus.**

Theorem 2.1. *Let $\overline{T}$ be the solid torus and $\tilde{p}$, q be two positive real numbers such that $\tilde{p} = q/(2-q)$ with $1 < q < 2$. Then for all $\varepsilon > 0$ there exists a real number B_ε such that for all $u \in H^q_{1,G}$ the following inequality holds:*

$$\|u\|^q_{L^{\tilde{p}}(\partial T)} \leq \Big[\frac{\tilde{K}^q(2,q)}{\big[2\pi(l-r)\big]^{q-1}} + \varepsilon\Big] \|\nabla u\|^q_{L^q(T)} + B_\varepsilon \|u\|^q_{L^q(\partial T)} \tag{2.4}$$

In addition the constant $\tilde{K}^q(2,q)/[2\pi(l-r)]^{q-1}$ is the best constant for the above inequality.

2.2. **Resolution of the problem.** Consider the set

$$\Lambda = \{c = (\alpha, \beta) \in \mathbb{R}^2 : \alpha - \beta \geq \delta,\ q \leq \alpha \leq p,\ q \leq \beta \leq \tilde{p}\},$$

with $\delta \in (0, p - \tilde{p}) = \big(0, q/(2-q)\big)$, and define the functionals

$$I(u) = \int_T (|\nabla u|^q + a|u|^q) dV + \int_{\partial T} b|u|^q dS$$
$$I_c(u) = \int_T f|u|^\alpha dV + \frac{\alpha}{\beta} \int_{\partial T} g|u|^\beta dS$$

for all $u \in H^q_{1,G}$ and for any $c \in \Lambda$. $I(u)$ and $I_c(u)$ are well defined because the imbeddings of $H^q_{1,G}$ onto L^p and $L^{\tilde{p}}$ are continue according to the Sobolev theorem. Define, also

$$\Sigma_c = \{u \in H^q_{1,G} : I_c(u) = 1\},$$
$$\mu_c = \inf\{I(u) : u \in \Sigma_c\},$$
$$c_0 = (p, \tilde{p}), \quad t^+ = \sup(t, 0),$$

for all $t \in \mathbb{R}$. Consequently for the problem (1.1) we have the following theorem.

Theorem 2.2. *Let a, f, b and g be four smooth functions, G-invariant and q, p, $\tilde{p}$ be three real numbers defined as in* (1.1)*. Suppose that the function f has constant sign (e.g. $f \geq 0$). The problem* (1.1) *has a positive solution $u \in H^q_{1,G}$ if the following holds:*

$$(\sup_T f)\Big[\frac{K^q(2,q)\mu^+_{c_0}}{[\pi(l-r)]^{q/2}}\Big]^{p/2} + \frac{p}{\widetilde{p}}(\sup_{\partial T} g)^+ \Big[\frac{\widetilde{K}^q(2,q)\mu^+_{c_0}}{[2\pi(l-r)]^{q-1}}\Big]^{\widetilde{p}/2} < 1 \tag{2.5}$$

and if

(1) *$f > 0$ everywhere and g is arbitrary, or*
(2) *$f \geq 0$, $g > 0$ everywhere and $(-inf_T a)^+\kappa < 1$, where*

$$\kappa = \inf\{A > 0 : \exists B > 0\ s.t. \|\psi\|^q_{L^q(T)} \leq A\|\nabla\psi\|^q_{L^q(T)} + B\|\psi\|^q_{L^q(\partial T)}\}. \tag{2.6}$$

In the rest of this paper we denote $K = K(2,q)$, $\tilde{K} = \tilde{K}(2,q)$ and $L = 2\pi(l-r)$.

3. Proofs

Proof of Theorem 2.1. The proof is carried out in two steps.
Step 1. Suppose that there exist two real numbers A, B such that for all $u \in H^q_{1,G}$ the following inequality holds:

$$\|u\|^q_{L^{\tilde{p}}(\partial T)} \leq A\|\nabla u\|^q_{L^q(T)} + B\|u\|^q_{L^q(\partial T)}.$$

Then

$$A \geq \frac{\tilde{K}^q(2,q)}{|2\pi(l-r)|^{q-1}}$$

Consider a transformation $F : D \to \mathbb{R}^2_+$. Such a transformation, for example, is

$$F(t,s) = \Big(\frac{4t}{t^2+(1+s)^2}, \frac{2(1-t^2-s^2)}{t^2+(1+s)^2}\Big),$$

see [23]. Denote by $(\tilde{g}_{ij})$ the Euclidian metric on D, $dx\,dy$ the Euclidian metric on $\mathbb{R}^2_+$ and $d\sigma$ the induced on $\partial\mathbb{R}^2_+$. Choose a finite covering of $\bar{D}$ consisting of disks D_k, centered on p_k, such that: If $p_k \in D$, then entire D_k lies in D and if $p_k \in \partial D$, D_k is a Fermi neighborhood. In these neighborhoods we have

$$1-\varepsilon_0 \leqslant \sqrt{\det(\tilde{g}_{ij})} \leqslant 1+\varepsilon_0 \tag{3.1}$$

Suppose by contradiction that, there exists

$$A < \frac{\tilde{K}^q}{L^{q-1}} \quad \text{and} \quad B \in \mathbb{R}$$

such that the inequality

$$\|u\|^q_{L^{\tilde{p}}_G(\partial T)} \leqslant A\|\nabla u\|^q_{L^q_G(T)} + B\|u\|^q_{L^q_G(\partial T)} \tag{3.2}$$

holds for all $u \in H^q_{1,G}(T)$. Fix a point $P_0 \in \partial T$, that belongs to the orbit of minimum range $l-r$. For any $\varepsilon_0 > 0$, we can choose $\delta = \varepsilon_0(l-r) < 1$ and

$$T_\delta = \{Q \in \mathbb{R}^3 : d(Q, O_{P_0}) < \delta\}$$

such that, if $I \times U \subset I \times D$ is the image of a neighborhood of $P_0 \in \partial T$ through the chart ξ of T and $V \subset \mathbb{R}^2_+$ the image of U through F, (3.1) holds. It follows that, for any $u \in C^\infty_0(T_\delta)$, we have successively:

$$\Big(\int_{\partial T_\delta} |u|^{\tilde{p}} dS\Big)^{q/\tilde{p}} \leqslant A\int_{T_\delta} |\nabla u|^q dV + B\int_{\partial T_\delta} |u|^q dS,$$

$$\begin{aligned}
&\Big(2\pi\delta\int_{\partial D} |\phi|^{\tilde{p}}(l-r+\delta t)dt\Big)^{q/\tilde{p}} \\
&\leqslant 2\pi\delta^{2-q}A\int_D |\nabla\phi|^q(l-r+\delta t)\,dt\,ds + +2\pi\delta B\int_{\partial D}|\phi|^q(l-r+\delta t)dt,
\end{aligned}$$

$$\begin{aligned}
&\Big((1-\varepsilon_0)\delta L\int_{F(\partial D)} (|\phi|^{\tilde{p}}\sqrt{\tilde{g}})\circ F^{-1} d\sigma\Big)^{q/\tilde{p}} \\
&\leq (1+\varepsilon_0)\delta^{2-q}AL\int_{F(D)} (|\nabla\phi|^q\sqrt{\tilde{g}})\circ F^{-1}\,dx\,dy \\
&\quad + (1+\varepsilon_0)\delta LB\int_{F(\partial D)} (|\phi|^q\sqrt{\tilde{g}})\circ F^{-1} d\sigma,
\end{aligned}$$

$$\begin{aligned}
&\Big((1-\varepsilon_0)^2\delta L\int_{\partial\mathbb{R}^2_+} |\Phi|^{\tilde{p}} d\sigma\Big)^{q/\tilde{p}} \\
&\leqslant (1+\varepsilon_0)^2\delta L\Big(\delta^{1-q}A\int_{\mathbb{R}^2_+} |\nabla\Phi|^q\,dx\,dy + B\int_{\partial\mathbb{R}^2_+}|\Phi|^q d\sigma\Big),
\end{aligned}$$

$$\Big(\int_{\partial\mathbb{R}^2_+}|\Phi|^{\tilde p}d\sigma\Big)^{q/\tilde p}\leqslant f(\varepsilon_0)L^{q-1}A\int_{\mathbb{R}^2_+}|\nabla\Phi|^q\,dx\,dy+\tilde B\int_{\partial\mathbb{R}^2_+}|\Phi|^q d\sigma, \tag{3.3}$$

where $f(\varepsilon_0)=(1+\varepsilon_0)^2/(1-\varepsilon_0)^{2q/\tilde p}$, $\tilde B=f(\varepsilon_0)(\delta L)^{q-1}B$ and $\tilde p=q/(2-q)$. Because of (3.2) and since the above function $f:(0,1)\to(1,+\infty)$ with

$$f(t)=\frac{(1+t)^2}{(1-t)^{2q/\tilde p}}$$

is monotonically increasing, we can choose ε_0 small enough, such that the following inequality holds

$$A<f(\varepsilon_0)A<\frac{\tilde K^q}{L^{q-1}}$$

hence $A'<\tilde K^q$ where $A'=f(\varepsilon_0)L^{q-1}A$.

So for ε_0 small enough and for all $\Phi\in C_0^\infty(D)$ we have

$$\Big(\int_{\partial\mathbb{R}^2_+}|\Phi|^{\tilde p}d\sigma\Big)^{q/\tilde p}\leqslant A'\int_{\mathbb{R}^2_+}|\nabla\Phi|^q\,dx\,dy+\tilde B\int_{\partial\mathbb{R}^2_+}|\Phi|^q d\sigma \tag{3.4}$$

On the other hand by Hölder's inequality, for all $\Phi\in C_0^\infty(D_\delta)$, where $D_\delta\subset D$, we have

$$\int_{\partial D_\delta}|\Phi|^q d\sigma_0\leqslant[Vol(\partial D_\delta)]^{1-(q/\tilde p)}\Big(\int_{\partial D_\delta}(|\Phi|^q)^{\tilde p/q}d\sigma_0\Big)^{q/\tilde p}$$

and since $\tilde p=q/(2-q)$, that is $1-(q/\tilde p)=1-(2-q)=q-1$, we have

$$\int_{\partial D_\delta}|\Phi|^q d\sigma_0\leqslant Vol(\partial D_\delta)^{q-1}\Big(\int_{\partial D_\delta}|\Phi|^{\tilde p}d\sigma_0\Big)^{q/\tilde p} \tag{3.5}$$

Hence, choosing ε_0 small enough, by (3.4) and (3.5), we get that there exists $A''<\tilde K^q$, such that for all $\Phi\in C_0^\infty(D_\delta)$,

$$\Big(\int_{\partial\mathbb{R}^2_+}|\Phi|^{\tilde p}d\sigma\Big)^{q/\tilde p}\leqslant A''\int_{\mathbb{R}^2_+}|\nabla\Phi|^q\,dx\,dy\,. \tag{3.6}$$

Let $\Psi\in C_0^\infty(\mathbb{R}^2_+)$ and set $\Psi_\lambda(x)=\lambda^{1/\tilde p}\Psi(\lambda x)$, $\lambda>0$. For $\lambda>0$, sufficiently large, $\Psi_\lambda\in C_0^\infty(D)$ and since $\|\Psi_\lambda\|_{L^{\tilde p}(\partial\mathbb{R}^2_+)}=\|\Psi\|_{L^{\tilde p}(\partial\mathbb{R}^2_+)}$ and $\|\nabla\Psi_\lambda\|_{L^q(\mathbb{R}^2_+)}=\|\nabla\Psi\|_{L^q(\mathbb{R}^2_+)}$, by (3.6), the following inequality

$$\Big(\int_{\partial\mathbb{R}^2_+}|\Psi|^{\tilde p}d\sigma\Big)^{q/\tilde p}\leqslant A''\int_{\mathbb{R}^2_+}|\nabla\Psi|^q\,dx\,dy$$

holds for all $\Psi\in C_0^\infty(\mathbb{R}^2_+)$. This is a contradiction since $\tilde K$ is the best constant for the Sobolev inequality in $\mathbb{R}^2_+$.

Step 2. For all $\varepsilon>0$ there exists a real number B_ε such that for all $u\in H^q_{1,G}$ the following inequality holds:

$$\|u\|^q_{L^{\tilde p}(\partial T)}\leq\Big[\frac{\tilde K^q(2,q)}{[2\pi(l-r)]^{q-1}}+\varepsilon\Big]\|\nabla u\|^q_{L^q(T)}+B_\varepsilon\|u\|^q_{L^q(\partial T)}$$

Assume by contradiction that there exists $\varepsilon_0>0$ such that for all $\alpha>0$ we can find $u\in H^q_{1,G}(T)$ with

$$\|u\|^q_{L^{\tilde p}_G(\partial T)}>(\frac{\tilde K^q}{L^{q-1}}+\varepsilon_0)\|\nabla u\|^q_{L^q_G(T)}+\alpha\|u\|^q_{L^q_G(\partial T)}$$

or
$$\frac{\|\nabla u\|^q_{L^q_G(T)} + \alpha \|u\|^q_{L^q_G(\partial T)}}{\|u\|_{L^{\tilde p}_G(\partial T)}} < \big(\frac{\tilde K^q}{L^{q-1}} + \varepsilon_0\big)^{-1}$$
It follows that, the above inequality remains true for all $\varepsilon \in (0, \varepsilon_0)$ and setting
$$I_\alpha = \inf_{u\in H^{1,q}_G(T)\setminus\{0\}} \frac{\|\nabla u\|^q_{L^q_G(T)} + \alpha \|u\|^q_{L^q_G(\partial T)}}{\|u\|^q_{L^{\tilde p}_G(\partial T)}}$$
we conclude that for all $\alpha > 1$, there exists $\theta_0 > 0$ independent of α such that
$$I_\alpha < \big(\frac{\tilde K^q}{L^{q-1}} + \varepsilon_0\big)^{-1} = \frac{L^{q-1}}{\tilde K^q} - \theta_0 \tag{3.7}$$
As the quotient
$$\frac{\|\nabla u\|^q_{L^q_G(T)} + \alpha \|u\|^q_{L^q_G(\partial T)}}{\|u\|^q_{L^{\tilde p}_G(\partial T)}}$$
is homogeneous, for any fixed α we can take a minimizing sequence $(u_k) \subset H^q_{1,G}(T)$ for it satisfying $\|u_k\|^q_{L^{\tilde p}_G(\partial T)} = 1$. As
$$\|\nabla u_k\|^q_{L^q_G(T)} + \alpha \|u_k\|^q_{L^q_G(\partial T)} \to I_\alpha\,, \tag{3.8}$$
we conclude that (u_k) is bounded in $L^q_G(\partial T)$ and (∇u_k) is bounded in $L^q_G(\partial T)$. By the standard Sobolev trace inequality, we easily take, by contradiction, that there exists a constant C such that
$$\|u\|^q_{L^q(T)} \le C(\|\nabla u\|^q_{L^q(T)} + \|u\|^q_{L^q(\partial T)}). \tag{3.9}$$
These two facts together imply that $u_k \rightharpoonup u$ in $H^q_1(T)$, $u_k \to u$ in $L^q(T)$ and $u_k \to u$ in $L^q(\partial T)$. Since convergence in L^p spaces implies a.e. convergence, the function u will be G-invariant. By theorem 4 of [4] we have $u_k \to u$ in $L^{\tilde p}_G(\partial T)$, $\|u\|^q_{L^{\tilde p}_G(\partial T)} = 1$ and
$$\|\nabla u\|^q_{L^q_G(T)} + \alpha \|u\|^q_{L^q_G(\partial T)} = I_\alpha,$$
that is u is minimizing of I_α. Now, for each $\alpha > 0$, let $u_\alpha \in H^q_{1,G}(T)$ satisfy $\|u_\alpha\|_{L^{\tilde p}_G(\partial T)} = 1$ and
$$\|\nabla u_\alpha\|^q_{L^q_G(T)} + \alpha \|u_\alpha\|^q_{L^q_G(\partial T)} = I_\alpha \leqslant \frac{L^{q-1}}{\tilde K^q} - \theta_0 \tag{3.10}$$
Following arguments similar to the ones that proved u is G-invariant minimizing of I_α, we conclude that (u_α) is bounded in $H^q_{1,G}(T)$, thus we can take a subsequence of (u_α), denoted (u_α) too, such that $u_\alpha \rightharpoonup u$ in $H^q_{1,G}(T)$, $u_\alpha \to u$ in $L^q_G(T)$ and $u_\alpha \to u$ in $L^q_G(\partial T)$. Moreover, by (3.10) we obtain
$$\|u_\alpha\|^q_{L^q_G(\partial T)} < \frac{1}{\alpha}\big(\frac{L^{q-1}}{\tilde K^q} - \theta_0\big),$$
and sending α to $+\infty$ we have $u = 0$ on ∂T. Finally following the proof of theorem 4 of [4] we obtain that $\nabla u_\alpha \to \nabla u$ a.e. and so (∇u_α) is bounded in $L^q_G(T)$. Because of (2.1), (2.2) and (2.3) and since $1 < q < 2$, $\tilde p = q/\,(2-q)$ we have
$$\frac{\|\nabla u_\alpha\|^q_{L^q(T)} + \alpha \|u_\alpha\|^q_{L^q(\partial T)}}{\|u_\alpha\|^q_{L^{\tilde p}(\partial T)}}$$

$$
\begin{aligned}
&= \frac{\int_T |\nabla u_\alpha|^q dV + \alpha \int_{\partial T} |u_\alpha|^q dS}{(\int_{\partial T} |u_\alpha|^{\tilde p} dS)^{q/\tilde p}} \\
&= \frac{2\pi \left(\frac{\delta}{\lambda}\right)^{2-q} \int_D |\nabla \phi_\alpha|^q \left(l - r + \delta \frac{t}{\lambda}\right) dtds + 2\pi \frac{\delta}{\lambda} \alpha \int_{\partial D} |\phi_\alpha|^q \left(l - r + \delta \frac{t}{\lambda}\right) dt}{\left(2\pi \frac{\delta}{\lambda} \int_{\partial D} |\phi_\alpha|^{\tilde p} \left(l - r + \delta \frac{t}{\lambda}\right) dt\right)^{q/\tilde p}} \\
&= \frac{2\pi \left(\frac{\delta}{\lambda}\right)^{2-q} \int_D |\nabla \phi_\alpha|^q \left(l - r + \delta \frac{t}{\lambda}\right) dtds}{\left(2\pi \frac{\delta}{\lambda} \int_{\partial D} |\phi_\alpha|^{\tilde p} \left(l - r + \delta \frac{t}{\lambda}\right) dt\right)^{q/\tilde p}} + \frac{2\pi \frac{\delta}{\lambda} \alpha \int_{\partial D} |\phi_\alpha|^q \left(l - r + \delta \frac{t}{\lambda}\right) dt}{\left(2\pi \frac{\delta}{\lambda} \int_{\partial D} |\phi_\alpha|^{\tilde p} \left(l - r + \delta \frac{t}{\lambda}\right) dt\right)^{q/\tilde p}} \\
&= \left(\frac{1}{\lambda}\right)^{2-q-\frac{q}{\tilde p}} \frac{2\pi \delta^{2-q} \int_D |\nabla \phi_\alpha|^q \left(l - r + \delta \frac{t}{\lambda}\right) dtds}{\left(2\pi \delta \int_{\partial D} |\phi_\alpha|^{\tilde p} \left(l - r + \delta \frac{t}{\lambda}\right) dt\right)^{q/\tilde p}} \\
&\quad + \left(\frac{1}{\lambda}\right)^{1-\frac{q}{\tilde p}} \frac{2\pi \delta \alpha \int_{\partial D} |\phi_\alpha|^q \left(l - r + \delta \frac{t}{\lambda}\right) dt}{\left(2\pi \delta \int_{\partial D} |\phi_\alpha|^{\tilde p} \left(l - r + \delta \frac{t}{\lambda}\right) dt\right)^{q/\tilde p}} \\
&\le \frac{2\pi \delta^{2-q} \int_D |\nabla \phi_\alpha|^q (l - r + \delta \frac{t}{\lambda})\, dt\, ds + 2\pi \delta \alpha \int_{\partial D} |\phi_\alpha|^q (l - r + \delta \frac{t}{\lambda}) dt}{(2\pi \delta \int_{\partial D} |\phi_\alpha|^{\tilde p} (l - r + \delta \frac{t}{\lambda}) dt)^{q/\tilde p}}
\end{aligned}
$$

and as $\lambda \to +\infty$ the above inequality yields

$$
\begin{aligned}
&\frac{\|\nabla u_\alpha\|^q_{L^q(T)} + \alpha \|u_\alpha\|^q_{L^q(\partial T)}}{\|u_\alpha\|^q_{L^{\tilde p}(\partial T)}} \\
&\le \frac{L\delta^{2-q} \int_D |\nabla \phi_\alpha|^q \, dt\, ds + L\delta\alpha \int_{\partial D} |\phi_\alpha|^q dt}{(L\delta \int_{\partial D} |\phi_\alpha|^{\tilde p} dt)^{q/\tilde p}} \\
&= L^{q-1} \frac{\int_D |\nabla \phi_\alpha|^q \, dt\, ds + \delta^{1-(q/\tilde p)} \alpha \int_{\partial D} |\phi_\alpha|^q dt}{(\int_{\partial D} |\phi_\alpha|^{\tilde p} dt)^{q/\tilde p}} \\
&< L^{q-1} \frac{\int_D |\nabla \phi_\alpha|^q \, dt\, ds + \alpha \int_{\partial D} |\phi_\alpha|^q dt}{(\int_{\partial D} |\phi_\alpha|^{\tilde p} dt)^{q/\tilde p}}
\end{aligned}
$$

From the above inequality and (3.10) we obtain

$$
L^{q-1} \frac{\int_D |\nabla \phi_\alpha|^q \, dt\, ds + \alpha \int_{\partial D} |\phi_\alpha|^q dt}{(\int_{\partial D} |\phi_\alpha|^{\tilde p} dt)^{q/\tilde p}} < \frac{L^{q-1}}{\tilde K} - \theta_0
$$

or

$$
\frac{\int_D |\nabla \phi_\alpha|^q \, dt\, ds + \alpha \int_{\partial D} |\phi_\alpha|^q dt}{(\int_{\partial D} |\phi_\alpha|^{\tilde p} dt)^{q/\tilde p}} < \frac{1}{\tilde K^q} - \theta \tag{3.11}
$$

According to [4, Theorem 4] such a function satisfying inequality (3.11) does not exist, and the theorem is proved. □

3.1. **Proof of the main theorem.**

Proof of Theorem 2.2. The proof is based on ideas from [14]. We recall, in this point, some notation:

$$
\Lambda = \{c = (\alpha, \beta) \in \mathbb{R}^2 : \alpha - \beta \ge \delta,\ q \le \alpha \le p,\ q \le \beta \le \tilde p\},
$$

where $\delta \in (0, p - \tilde p) = \big(0, q/(2-q)\big)$,

$$
I(u) = \int_T (|\nabla u|^q + a|u|^q) dV + \int_{\partial T} b|u|^q dS,
$$

$$I_c(u)=\int_T f|u|^\alpha dV+\frac{\alpha}{\beta}\int_{\partial T} g|u|^\beta dS$$

where $u\in H^q_{1,G}$ and $c\in\Lambda$,

$$\begin{gathered}\Sigma_c=\{u\in H^q_{1,G}:\ I_c(u)=1\},\\ \mu_c=\inf\{I(u):u\in\Sigma_c\},\\ c_0=(p,\tilde p),\quad t^+=\sup(t,0),\ t\in\mathbb{R}.\end{gathered}$$

Because the imbeddings of $H^q_{1,G}(T)$ in $L^p_G(T)$ and $L^{\tilde p}_G(\partial T)$ are continuous but not compact, we adopt the procedure of solving an approximating equation and then we pass to the limit as in [1].

1. The proof of this part is carried out in six steps.

Step 1. A real $t_c\in\Sigma_c$. Define on $[0,+\infty)$ the continuous function h_c with

$$h_c(t)=t^\alpha\int_T fdV+\frac{\alpha}{\beta}t^\beta\int_{\partial T} gdS$$

Since $\alpha>\beta$ we have $h_c(0)=0$, $\lim_{t\to\infty}h_c(t)=+\infty$ and there exists $t_c>0$ such that $h(t_c)=1$. Hence the constant function, which in every point is equal to t_c, belongs to Σ_c, and then $\Sigma_c\neq\emptyset$.

Step 2. $\sup\{\mu_c:c\in\Lambda\}<+\infty$. We will prove that there exists $\tilde t\in\mathbb{R}$ such that $t_c\leqslant\tilde t$ for all $c\in\Lambda$ and the following holds

$$\mu_c\leqslant \mathrm{I}(t_c)=\Big(\int_T adV+\int_{\partial T} bdS\Big)t_c^q\leqslant\Big(\int_T |a|dV+\int_{\partial T}|b|dS\Big)\tilde t^q$$

• If $\int_{\partial T} gdS\geqslant 0$, since $f>0$ and $\alpha>\beta$, by equality

$$1=I_c(t_c)=t_c^\alpha\int_T fdV+\frac{\alpha}{\beta}t_c^\beta\int_{\partial T} gdS$$

arises

$$t_c=\Big(\int_T fdV+\frac{\alpha}{\beta}t_c^{\beta-\alpha}\int_{\partial T} gdS\Big)^{-1/\alpha}\geqslant\sup\{\big(\int_T fdV\big)^{-1/\alpha}:q\leqslant\alpha\leqslant p\}$$

However, if $\int_T fdV<1$, since $q\leqslant\alpha\leqslant p$, the following holds

$$\big(\int_T fdV\big)^{-1/q}\leqslant\big(\int_T fdV\big)^{-1/\alpha}<1$$

while, if $\int_T fdV\geqslant 1$, we have the inequality

$$\big(\int_T fdV\big)^{-1/q}\geqslant\big(\int_T fdV\big)^{-1/\alpha}\geqslant 1$$

Therefore, in this case, we set

$$\tilde t=\max\{1,\ \big(\int_T fdV\big)^{-1/q}\}\geqslant\sup\{\big(\int_T fdV\big)^{-1/\alpha}:q\leqslant\alpha\leqslant\frac{2q}{2-q}\}$$

• If $\int_{\partial T} gdS<0$, let $\tilde t_0\in\mathbb{R}$ such that

$$\tilde t_0\geqslant\max\big\{1,\ \big(p\frac{|\int_{\partial T} gdS|}{\int_T fdV}\big)^{1/\delta}\big\}$$

When $t \geqslant \tilde{t}_0$, because of $q \leqslant \alpha \leqslant p$, $q \leqslant \beta \leqslant \tilde{p}$ and $\beta \leqslant \alpha - \delta$, we get $(\alpha/\beta) \leqslant (p/q)$, and then

$$\begin{aligned}
h_c(t) &= t^\alpha \int_T f dV + \frac{\alpha}{\beta} t^\beta \int_{\partial T} g dS \\
&\geqslant t^\alpha \int_T f dV + \frac{p}{q} t^{\alpha-\delta} \int_{\partial T} g dS \\
&= t^\alpha \Big(1 + \frac{p}{q} \frac{|\int_{\partial T} g dS|}{\int_T f dV} t^{-\delta}\Big) \int_T f dV \\
&\geqslant t^q \Big(1 + \frac{p}{q} \frac{|\int_{\partial T} g dS|}{\int_T f dV} \tilde{t}_0^{-\delta}\Big) \int_T f dV \\
&\geqslant \frac{t^q}{q} \int_T f dV
\end{aligned}$$

Hence, for

$$\tilde{t} = \max\{q^{1/q}\big(\int_T f dV\big)^{-1/q}, \tilde{t}_0\}$$

we have $h_c(\tilde{t}) \geqslant 1$ and then $t_c \leqslant \tilde{t}$.

Step 3. $\inf\{\mu_c : c \in \Lambda\} > -\infty$. Since $f > 0$ everywhere, $m = \inf_T f > 0$ and because of $H^q_{1,G}(T) \hookrightarrow L^{\tilde{p}}_G(\partial T)$ is continuous (see [18, lemma 2.1]) there exists $C \geqslant 1$ such that for all $\psi \in H^q_{1,G}(T)$,

$$\|\psi\|_{\tilde{p},\partial T} \leqslant C(\|\nabla\psi\|_{q,T} + \|\psi\|_{q,T})$$

We set

$$\begin{gathered}
C_1 = \sup_{q \leqslant \alpha \leqslant p} \big[m^{-1/q}\big(\int_T f dV\big)^{(1/q)-(1/\alpha)}\big], \\
C_2 = \sup_{q \leqslant \beta \leqslant \tilde{p}} \big[2^{\beta-1} q^{-1} p\, C^\beta \|g\|_\infty [Vol(\partial T)]^{1-(\beta/\tilde{p})}\big], \\
C_3 = \sup_{q \leqslant \alpha \leqslant p} \big[2^{1/\alpha} C_1 (C_2+1)^{1/\alpha}\big].
\end{gathered}$$

We recall, for reals $x, y \geq 0$, the elementary inequalities

$$(x+y)^\beta \leqslant 2^{\beta-1}(x^\beta + y^\beta), \tag{3.12}$$

$$(x+y)^{1/\alpha} \leqslant 2^{1/\alpha}(x^{1/\alpha} + y^{1/\alpha}). \tag{3.13}$$

Let $u \in \Sigma_c$ with $\|u\|_{q,T} > 1$. Since $u \in \Sigma_c$ we have

$$\int_T f|u|^\alpha dV + \frac{\alpha}{\beta} \int_{\partial T} g|u|^\beta dS = 1$$

and then

$$\int_T f|u|^\alpha dV = 1 - \frac{\alpha}{\beta} \int_{\partial T} g|u|^\beta dS$$

By (3.12), (3.13) and since $q \leqslant \alpha \leqslant p$, $q \leqslant \beta \leqslant \tilde{p}$ we obtain

$$\begin{aligned}
\|u\|_{q,T} &\leqslant m^{-1/q}\Big(\int_T f|u|^q dV\Big)^{1/q} \\
&\leq m^{-1/q}\Big(\int_T f dV\Big)^{(1/q)-(1/\alpha)}\Big(\int_T f|u|^\alpha dV\Big)^{1/\alpha} \\
&\leqslant C_1\Big(1 - \frac{\alpha}{\beta} \int_{\partial T} g|u|^\beta dS\Big)^{1/\alpha}
\end{aligned}$$

$$\begin{aligned}
&\leqslant C_1(1+\frac{p}{q}\|g\|_\infty[Vol(\partial T)]^{1-(\beta/\tilde p)}\|u\|^\beta_{\tilde p,\partial T})^{1/\alpha}\\
&\leqslant C_1[1+\frac{p}{q}\|g\|_\infty[Vol(\partial T)]^{1-(\beta/\tilde p)}C^\beta(\|\nabla u\|_{q,T}+\|u\|_{q,\partial T})^\beta]^{1/\alpha}\\
&\leqslant C_1[1+2^{\beta-1}q^{-1}p\|g\|_\infty[Vol(\partial T)]^{1-(\beta/\tilde p)}C^\beta(\|\nabla u\|^\beta_{q,T}+\|u\|^\beta_{q,T})]^{1/\alpha}\\
&\leqslant C_1[C_2\|\nabla u\|^\beta_{q,T}+(C_2+1)\|u\|^\beta_{q,T}]^{1/\alpha}\\
&\leqslant 2^{1/\alpha}C_1[C_2^{1/\alpha}\|\nabla u\|^{\beta/\alpha}_{q,T}+(C_2+1)^{1/\alpha}\|u\|^{\beta/\alpha}_{q,T}]\\
&\leqslant C_3(\|\nabla u\|^{\beta/\alpha}_{q,T}+\|u\|^{\beta/\alpha}_{q,T}).
\end{aligned}$$

Since

$$\frac{\beta}{\alpha}\leqslant 1-\frac{\delta}{\alpha}\leqslant 1-\frac{\delta}{p},$$

if $\varepsilon\in(0,1)$ and $C_4=C_4(\varepsilon,p,\delta,C_3)$ is a constant such that

$$t^{1-(\delta/p)}\leqslant\frac{\varepsilon}{C_3}t+C_4,$$

for any $t\geqslant 0$, the latter inequality implies

$$\begin{aligned}
\|u\|_{q,T}&\leqslant C_3(1+\|\nabla u\|^{1-(\delta/p)}_{q,T}+\|u\|^{1-(\delta/p)}_{q,T})\\
&\leqslant\varepsilon(\|\nabla u\|_{q,T}+\|u\|_{q,T})+C_3(1+2C_4)
\end{aligned}$$

Hence if $\varepsilon_1=\varepsilon/(1-\varepsilon)$ and $C=C_3(1+2C_4)/(1-\varepsilon)$ is a constant depending on ε_1, but not on $c\in\Lambda$, by the last inequality we obtain

$$\|u\|_{q,T}\leqslant\varepsilon_1\|\nabla u\|_{q,T}+C \tag{3.14}$$

Since we can take $C\geqslant 1$, (3.14) holds for $c\in\Lambda$ and for all $u\in\Sigma_c$.

If $b\not\equiv 0$, since $H^q_{1,G}(T)\hookrightarrow L^q_G(\partial T)$ and $H^q_{1,G}(T)\hookrightarrow L^q_G(T)$ are compact, for any $\varepsilon'\in(0,1)$ we can find a constant $C'=C'(\varepsilon',b)$ such that

$$\|\psi\|^q_{q,\partial T}\leqslant\|b\|^{-1}_\infty(\varepsilon'\|\nabla\psi\|^q_{q,T}+C'\|\psi\|^q_{q,T})$$

for all $\psi\in H^q_{1,G}(T)$, and then we obtain

$$\begin{aligned}
I(\psi)&=\int_T(|\nabla\psi|^q+a|\psi|^q)dV+\int_{\partial T}b|\psi|^qdS\\
&\geq\|\nabla\psi\|^q_{q,T}-\|a\|_{\infty,T}\|\psi\|^q_{q,T}-\|b\|_{\infty,T}\|\psi\|^q_{q,\partial T}\\
&\geq(1-\varepsilon')\|\nabla\psi\|^q_{q,T}-\|a\|_{\infty,T}\|\psi\|^q_{q,T}-C'\|\psi\|^q_{q,T}\\
&=(1-\varepsilon')\|\nabla\psi\|^q_{q,T}-A(\varepsilon')\|\psi\|^q_{q,T}
\end{aligned}\tag{3.15}$$

where $A(\varepsilon')=\|a\|_{\infty,T}+C'$. An inequality of the type (3.15) is always true if $b\equiv 0$, because of inequality

$$I(\psi)\geqslant\|\nabla\psi\|^q_{q,T}-\|a\|_{\infty,T}\|\psi\|^q_{q,T}\,.$$

By (3.14) because of (3.12) we obtain

$$\|u\|^q_{q,T}\leq 2^{q-1}\varepsilon_1^q\|\nabla u\|^q_{q,T}+2^{q-1}C^q \tag{3.16}$$

Thus if we choose ε_1, small enough, such that $1-\varepsilon'-2^{q-1}A(\varepsilon')\varepsilon_1^q>0$, for all $u\in\Sigma_c$, by (3.15) and (3.16) we obtain

$$\begin{aligned}
I(u)&\geqslant\big(1-\varepsilon'-2^{q-1}A(\varepsilon')\varepsilon_1^q\big)\|\nabla u\|^q_{q,T}-2^{q-1}A(\varepsilon')C^q\\
&\geqslant-2^{q-1}A(\varepsilon')C^q
\end{aligned}$$

Hence for all $c \in \Lambda$, $\mu \geq -2^{q-1}A(\varepsilon')C^q$.

We observe that if Γ is a subset of $\cup_{c\in\Lambda}\Sigma_c$, such that $\sup\{I(u) : u \in \Gamma\} = L < +\infty$, Γ is bounded in $H^q_{1,G}(T)$. Indeed, according to the former, for any $u \in \Gamma$ we have

$$\|\nabla u\|^q_{q,T} \leqslant \frac{I(u) + 2^{q-1}A(\varepsilon')C^q}{1 - \varepsilon' - 2^{q-1}A(\varepsilon')\varepsilon_1^q} \leqslant \frac{L + 2^{q-1}A(\varepsilon')C^q}{1 - \varepsilon' - 2^{q-1}A(\varepsilon')\varepsilon_1^q} = c$$

and

$$\|u\|_{q,T} \leqslant \varepsilon_1\|\nabla u\|_{q,T} + C \leqslant c',$$

thus

$$\sup_{u\in\Gamma}(\|\nabla u\|_{q,T} + \|u\|_{q,T}) < +\infty\,.$$

Step 4. $\mu_c = \inf\{\, \mathrm{I}(u) : u \in \Sigma_c\,\}$ is attained. Suppose that $c = (\alpha,\beta) \in \Lambda$, $\alpha < p$, $\beta < \tilde{p}$. Let a sequence $(u_j) \in \Sigma_c$ such that $\lim_{j\to\infty} I(u_j) = \mu_c$. Since $|\gamma u_j| = \gamma(|u_j|)$, ($\gamma u_j$ is the trace of u_j on ∂T), a.e. on ∂T and $I(|u_j|) = I(u_j)$, $I_c(|u_j|) = I_c(u_j)$ a.e. in T , we conclude that $|u_j| \in \Sigma_c$; in a way similar to the one that employs $(|u_j|)$ in place of (u_j), $(\gamma(|u_j|)$ in place of γu_j respectively) or we can consider u_j's nonnegative a.e. (the same for γu_j's). The sequence $(I(u_j))$ is bounded in $H^q_{1,G}(T)$ implies that $\sup_{j\in\mathbb{N}}(\|u_j\|_{H^q_1(T)}) < +\infty$. Since the imbeddings of $H^q_{1,G}(T)$ in $L^q_G(T)$, $L^q_G(\partial T)$, $L^\alpha_G(T)$ and $L^\beta_G(\partial T)$ are compacts, there is a function $u_c \in H^q_{1,G}(T)$ and a subsequence (u_j) of (u_j) such that $(u_j) \rightharpoonup u_c$ in $H^q_{1,G}(T)$, $(u_j) \to u_c$ in everyone of the previous L^r spaces, $(u_j) \to u_c$ a.e. in T. (The same holds and for traces on ∂T). Hence u_c and γu_c are nonnegative. We also have $I_c(u_c) = \lim_{j\to\infty} I_c(u_j) = 1$ and then $u_c \in \Sigma_c$ and $I_c(u_c) \geqslant \mu_c$. Moreover since

$$\|\nabla u_c\|_q \leqslant \varliminf_{j\to+\infty} \|\nabla u_j\|_q$$

and

$$\int_T a u_c^q dV + \int_{\partial T} b u_c^q dS = \lim_{j\to\infty}\Big(\int_T a u_j^q dV + \int_{\partial T} b u_j^q dS\Big)$$

holds, we conclude that $I(u_c) \leqslant \lim_{j\to\infty} I(u_j) = \mu_c$ and $I(u_c) = \mu_c$.

Step 5. There exists a week solution $u_{c_0} \geq 0$. We observe that the deferential $DI_c(u)$ of I_c is $\neq 0$ for all $u \in \Sigma_c$. (Because if $DI_c(u) = 0$ for all $\psi \in C_0^\infty(T)$ then $\int_T f u|u|^{\alpha-2}\psi dV = 0$. This implies that $fu|u|^{\alpha-2}\psi = 0$ in $(C_0^\infty(T))'$ and since $f > 0$, $u = 0$. Then $u = 0$ in $H^q_{1,G}(T)$, which is impossible since $I_c(u) = 1$). After this a Lagrange multiplicand λ_c exists such that, for all $\psi \in H^q_{1,G}(T)$, it satisfies the next Euler equation

$$\begin{aligned} &\int_T (|\nabla u_c|^{q-2}\nabla u_c \nabla\psi + a u_c^{q-1}\psi) dV + \int_{\partial T} b u_c^{q-1}\psi dS \\ &= \lambda_c\Big(\int_T f u_c^{\alpha-1}\psi dV + \int_{\partial T} g u_c^{\beta-1}\psi dS\Big) \end{aligned} \tag{3.17}$$

In the following we suppose that $c = (\alpha,\beta) \to c_0 = (p,\tilde{p})$ and we will prove that there are a real λ_{c_0} and a function u_{c_0} such that

$$\begin{aligned} &\int_T (|\nabla u_{c_0}|^{q-2}\nabla u_{c_0}\nabla\psi + a u_{c_0}^{q-1}\psi) dV + \int_{\partial T} b u_{c_0}^{q-1}\psi dS \\ &= \lambda_{c_0}(\int_T f u_{c_0}^{p-1}\psi dV + \int_{\partial T} g u_{c_0}^{\tilde{p}-1}\psi dS) \end{aligned}$$

that is u_{c_0} is a week solution of (1.1). Substituting $\psi = u_c$ in (3.17) we obtain

$$\int_T (|\nabla u_c|^q + au_c^q)dV + \int_{\partial T} bu_c^q dS = \lambda_c\Big(\int_T fu_c^\alpha dV + \int_{\partial T} gu_c^\beta dS\Big)$$

or

$$\lambda_c\Big(\int_T fu_c^\alpha dV + \int_{\partial T} gu_c^\beta dS\Big) = I(u_c) = \mu_c$$

Moreover, we have

$$\begin{aligned}1 = I_c(u_c) &= \int_T fu_c^\alpha dV + \frac{\alpha}{\beta}\int_{\partial T} gu_c^\beta dS\\ &= \frac{\alpha}{\beta}(\int_T fu_c^\alpha dV + \int_{\partial T} gu_c^\beta dS) + (1-\frac{\alpha}{\beta})\int_T fu_c^\alpha dV\end{aligned}$$

and since

$$(1-\frac{\alpha}{\beta})\int_T fu_c^\alpha dV < 0,$$

we obtain

$$\int_T fu_c^\alpha dV + \int_{\partial T} gu_c^\beta dS > \frac{\beta}{\alpha} \geqslant \frac{q}{p} > 0$$

Hence λ_c and μ_c have the same sign and since the set $\{\mu_c\}_{c\in\Lambda}$ is bounded a constant C exists, such that

$$|\lambda_c| \leqslant \frac{p}{q}|\mu_c| \leqslant C$$

Since $\sup\{I(u_c) : c \in \Lambda\} < \infty$, we have (step 2)

$$\sup\{\|u_c\|_{H_1^q} : c \in \Lambda\} < \infty.$$

Moreover, because the embeddings of $H_{1,G}^q(T)$ in $L_G^p(T)$ and $L_G^{\tilde{p}}(\partial T)$ are continuous, we have

$$\begin{gathered}\sup\{\|u_c\|_{p,T} : c \in \Lambda\} < \infty,\\ \sup\{\|u_c\|_{\tilde{p},\partial T} : c \in \Lambda\} < \infty\end{gathered}$$

We observe that $\frac{\beta-1}{\tilde{p}-1} < 1$ and then

$$\begin{aligned}\|u_c^{\beta-1}\|_{\tilde{p}/(\tilde{p}-1),\partial T} &\leqslant [Vol(\partial T)]^{1-(\beta/\tilde{p})}\|u_c\|_{\tilde{p},\partial T}^{\beta-1}\\ &\leqslant [\max(1,[Vol(\partial T)]^{1-(\beta/\tilde{p})})][\max(1,\|u_c\|_{\tilde{p},\partial T}^{\tilde{p}-1})] \leqslant ct\end{aligned}$$

that is, the sets $\{u_c^{\alpha-1}\}$, (respectively $\{u_c^{\beta-1}\}$) are bounded in the Banach reflexive spaces $L^{p/(p-1)}(T)$, (respectively $L^{\tilde{p}/(\tilde{p}-1)}(\partial T)$), of which the dual $L^p(T)$ (respectively $L^{\tilde{p}}(\partial T)$) contain $H_{1,G}^q(T)$.

The above implies the existence of a sequence $c_j = (\alpha_j, \beta_j) \in \Lambda$, which converges to c_0, of a real λ_{c_0} which is the limit of (λ_{c_j}) and of a function u_{c_0} with the following properties:

(a) $u_{c_j} \rightharpoonup u_{c_0}$ on $H_1^q(T)$, (by Banach's theorem),
(b) $u_{c_j} \to u_{c_0}$ on $L^q(T)$ (resp. $L^q(\partial T)$, (by Kondrakov's theorem),
(c) $u_{c_j} \to u_{c_0}$ a.e. in T , (resp. ∂T) (by proposition 3.43 of [1]) and
(d) $u_{c_j}^{\alpha_j-1} \rightharpoonup u_{c_0}^{p-1}$ in $L^{p/(p-1)}(T)$ (resp. $u_{c_j}^{\beta_j-1} \rightharpoonup u_{c_0}^{\tilde{p}-1}$ in $L^{\tilde{p}/(\tilde{p}-1)}(\partial T)$) (by Banach theorem).

From (a) arises that

$$\int_T |\nabla u_{c_j}|^{q-2}\nabla u_{c_j}\nabla\psi dV \to \int_T |\nabla u_{c_0}|^{q-2}\nabla u_{c_0}\nabla\psi dV$$

for all $\psi \in H_1^q(T)$.
From (c) arises that u_{c_0} is G-invariant and $u_{c_0} \geqslant 0$ on T (resp. $\gamma u_{c_0} \geqslant 0$ in ∂T).

We may also assume that the sequence (μ_{c_j}) converges, with limit $\mu_0 \leqslant \mu_{c_0}$. Indeed, let $u \in \Sigma_{c_0}$ be a non-negative function such that $\mu_{c_0} \leqslant I(u) \leqslant \mu_{c_0} + \varepsilon$, with $\varepsilon > 0$. Then for all j, there exists a unique real number $t_j > 0$ such that $I_{c_j}(t_j u) = 1$ and $\lim_{j\to\infty} t_j = 1$. If this is not the case, there will exist a subsequence t_{j_k} with $\lim_{j\to\infty} t_{j_k} = l \neq 1$. But in T the following holds,

$$0 \leqslant f u^{\alpha_{j_k}} \leqslant f(1+u^p) \in L^1(T)$$

and on ∂T the following also holds,

$$0 \leqslant |g| u^{\alpha_{j_k}} \leqslant |g|(1+u^{\tilde p}) \in L^1(\partial T).$$

According to the dominated convergence theorem we have

$$I_{c_{j_k}}(t_{j_k}u) \to l^p \int_T f u^p dV + \frac{p}{\tilde p} l^{\tilde p} \int_{\partial T} g u^{\tilde p} dS$$

namely, $I_{c_0}(lu) = 1$. This is contradiction since $l \neq 1$ and $I_{c_0}(u) = 1$, whereas we know that there exists unique real number $r > 0$ such that $I_{c_0}(ru) = 1$. Since $\mu_{c_j} \leqslant I(t_j u) = t_j^q I(u)$ we conclude that $\limsup_{j\to+\infty} \mu_{c_j} \leqslant \mu_{c_0}$.

Now we can write equation (3.14) for u_{c_j}, and as $j \to +\infty$ by Lebesgue's theorem, we find that for all $\psi \in H_{1,G}^q(T)$ the following holds

$$\begin{aligned}
&\int_T (|\nabla u_{c_0}|^{q-2}\nabla u_{c_0}\nabla\psi + a u_{c_0}^{q-1}\psi) dV + \int_{\partial T} b u_{c_0}^{q-1}\psi dS \\
&= \lambda_{c_0}\Big(\int_T f u_{c_0}^{p-1}\psi dV + \int_{\partial T} g u_{c_0}^{\tilde p-1}\psi dS\Big)
\end{aligned}$$

which implies that (λ_{c_0}, u_{c_0}) is a week solution of the problem (1.1).

Step 6. $u_{c_0} > 0$ everywhere. We proved in step 5 that $u_{c_0} \geq 0$. By the maximum principle [40], the function u_{c_0} is identically equal to 0 or $u_{c_0} > 0$ everywhere in T, and finally in $\bar T$: since every point P, where u_{c_0} attains it's minimum in $\bar T$, belongs to ∂T, assume that u_{c_0} is regular and that there exists $P_0 \in \partial T$ such that $u_{c_0}(P_0) = 0$. By Hopf's lemma [40], we have that the normal derivative has strict sign, $\big(\partial u_{c_0}/\partial\nu\big)(P_0) < 0$, but the boundary condition imposes

$$|\nabla u_{c_0}|^{q-2}\frac{\partial u_{c_0}}{\partial\nu}(P_0) = (-b u_{c_0}^{q-1} + \lambda_{c_0} g u_{c_0}^{\tilde p-1})(P_0) = 0,$$

a contradiction which proves that $u_{c_0}(P) > 0$ in $\bar T$.

For the solution u_{c_0} to be strictly positive it suffices $u_{c_0} \not\equiv 0$, which implies that $\| u_{c_0}\|_{q,T} = \lim_{j\to\infty} \| u_{c_j}\|_{q,T} > 0$. By [18, theorem 3.1] and theorem 2.1 of this paper, we conclude that for any $\mathcal{K} > K/\sqrt{L/2}$ and for any $\tilde{\mathcal{K}} > \tilde K / L^{(q-1)/q}$ there exists a constant $C(\mathcal{K}, \tilde{\mathcal{K}})$ such that for all $\psi \in H_{1,G}^q(T)$ the following inequalities hold

$$\|\psi\|_{p,T}^q \leqslant \mathcal{K}^q \|\nabla\psi\|_{q,T}^q + C\|\psi\|_{q,T}^q, \tag{3.18}$$

$$\|\psi\|_{\tilde p,\partial T}^q \leqslant \tilde{\mathcal{K}}^q \|\nabla\psi\|_{q,T}^q + C\|\psi\|_{q,T}^q \tag{3.19}$$

By (3.15) we have

$$\|\nabla u_c\|_{q,T}^q \leqslant (1+\varepsilon)I(u_c) + A(\varepsilon)\|u_c\|_{q,T}^q \tag{3.20}$$

where $1+\varepsilon = 1/(1-\varepsilon')$, $A(\varepsilon) = A(\varepsilon')/(1-\varepsilon')$.

Let $\varepsilon > 0$ and $D(\varepsilon, p)$ be a constant such that for any $x, y \geqslant 0$ and $\rho \in [0, p]$ the following holds

$$(x+y)^{\rho/q} \leqslant (1+\varepsilon)x^{\rho/q} + Dy^{\rho/q} \tag{3.21}$$

By (3.18) because of (3.20), (3.21) and since $I(u_c) = \mu_c$ for $\alpha < p$, $\beta < \tilde{p}$ we can write

$$\begin{aligned}
\|u_c\|_{p,T}^\alpha &\leqslant (\mathcal{K}^q\|\nabla u_c\|_{q,T}^q + C\|u_c\|_{q,T}^q)^{\alpha/q} \\
&\leqslant [(1+\varepsilon)^{q/p}\mathcal{K}^q\mu_c^+ + (A\mathcal{K}^q + C)\|u_c\|_{q,T}^q]^{\alpha/q} \\
&\leqslant (1+\varepsilon)^q\mathcal{K}^\alpha(\mu_c^+)^{\alpha/q} + D(A\mathcal{K}^q + C)^{\alpha/q}\|u_c\|_{q,T}^\alpha
\end{aligned} \tag{3.22}$$

Similarly by (3.19) because of (3.20), (3.21) we can write,

$$\|u_c\|_{\tilde{p},\partial T}^\beta \leqslant (1+\varepsilon)^q\tilde{\mathcal{K}}^\beta(\mu_c^+)^{\beta/q} + D(A\tilde{\mathcal{K}}^q + C)^{\beta/q}\|u_c\|_{q,T}^\beta \tag{3.23}$$

So by (3.22), (3.23) and Hölder inequality we have

$$\begin{aligned}
1 = I_c(u_c) &= \int_T f u_c^\alpha dV + \frac{\alpha}{\beta}\int_{\partial T} g u_c^\beta dS \\
&\leqslant (\sup_T f)[Vol(T)]^{1-(a/p)}\|u_c\|_{p,T}^\alpha \\
&\quad + \frac{\alpha}{\beta}(\sup_{\partial T} g)^+[Vol(\partial T)]^{1-(\beta/\tilde{p})}\|u_c\|_{\tilde{p},\partial T}^\beta \\
&\leqslant (1+\varepsilon)^q[(\sup_T f)[Vol(T)]^{1-(a/p)}\mathcal{K}^\alpha(\mu_c^+)^{\alpha/q}] \\
&\quad + (1+\varepsilon)^q[\frac{\alpha}{\beta}(\sup_{\partial T} g)^+[Vol(\partial T)]^{1-(\beta/\tilde{p})}\tilde{\mathcal{K}}^\beta(\mu_c^+)^{\beta/q}] \\
&\quad + C_1\|u_c\|_{q,T}^\alpha + C_2\|u_c\|_{q,T}^\beta
\end{aligned} \tag{3.24}$$

where the constants

$$C_1 = D(A\mathcal{K}^q + C)^{\alpha/q}(\sup_T f)[Vol(T)]^{1-(a/p)},$$
$$C_2 = (\alpha/\beta)D(A\tilde{\mathcal{K}}^q + C)^{\beta/q}(\sup_{\partial T} g)^+[Vol(\partial T)]^{1-(\beta/\tilde{p})}$$

are bounded by a constant $\tilde{C}(\varepsilon, \mathcal{K}, \tilde{\mathcal{K}}) > 0$ independent of α and β. By (3.24) for $c = c_j$, since $\lim_{j\to\infty}\mu_{c_j} \leqslant \mu_{c_0}$ for $j \to +\infty$, we obtain

$$\begin{aligned}
1 &\leqslant (1+\varepsilon)^q\big[(\sup_T f)\mathcal{K}^p(\mu_{c_0}^+)^{p/q} + \frac{p}{\tilde{p}}(\sup_{\partial T} g)^+\tilde{\mathcal{K}}^{\tilde{p}}(\mu_{c_0}^+)^{\tilde{p}/q}\big] \\
&\quad + \tilde{C}(\|u_{c_0}\|_{q,T}^p + \|u_{c_0}\|_{q,T}^{\tilde{p}})
\end{aligned}$$

because the sequence $(u_{c_j}) \to u_{c_0}$ strongly into $L^q(T)$. Thus if the condition (2.5) of the theorem is satisfied and if we choose $\varepsilon > 0$ small enough and $\mathcal{K}, \tilde{\mathcal{K}}$ close enough to $K/\sqrt{L/2}$, $\tilde{K}/L^{(q-1)/q}$, respectively, we obtain $\|u_{c_0}\|_{q,T} > 0$ and hence we proved the first part of the theorem.

2. If f becomes 0, $f \geqslant 0$, we impose the condition $g > 0$, namely $\nu = \inf_{\partial T} g > 0$, the proof follows along similar lines as in case 1. Thus we have to find two positive constants $A_1, A_2 > 0$ such that if $u \in \cup_{c\in\Lambda}\Sigma_c$ the following will hold,

$$I(u) \geqslant A_1\|\nabla u\|_q^q - A_2$$

Since $g > 0$ we have $\sup_{u\in\cup_{c\in\Lambda}\Sigma_c} \|u\|_{q,\partial T} = D < +\infty$, because for such a u the following holds

$$1 = I_c(u) \geqslant \int_{\partial T} g u^\beta dS \geqslant \nu \|u\|^\beta_{\beta,\partial T} \geqslant \nu [Vol(\partial T)]^{1-(\beta/q)} \|u\|^\beta_{q,\partial T}$$

Let κ be the best constant of the inequality (2.6), namely of

$$\|\psi\|^q_{q,T} \leqslant A\|\nabla\psi\|^q_{q,T} + B\|\psi\|^q_{q,\partial T}, \psi \in H^q_{1,G}$$

If $\breve{a}\kappa < 1$ where $\breve{a} = (-\inf a)^+$ we choose A close to κ such that $A_1 = 1 - A\breve{a} > 0$ and then we have

$$\begin{aligned} I(u) &\geq \int_T (|\nabla u|^q - \breve{a}u^q)dV + \int_{\partial T} bu^q dS \\ &\geq \|\nabla u\|^q_q - \breve{a}\|u\|^q_{q,T} - \|b\|_\infty \|u\|^q_{q,\partial T} \\ &\geq \|\nabla u\|^q_q - \breve{a}A\|\nabla u\|^q_q - \breve{a}B\|u\|^q_{q,\partial T} - \|b\|_\infty\|u\|^q_{q,\partial T} \\ &= (1 - \breve{a}A)\|\nabla u\|^q_q - (\breve{a}B + \|b\|_\infty)\|u\|^q_{q,\partial T} \\ &\geq A_1\|\nabla u\|^q_q - A_2 \end{aligned}$$

where $A_1 = 1 - \breve{a}A$, $A_2 = (\breve{a}B + \|b\|_\infty)D^q$ and the theorem is proved. □

Acknowledgments. The authors would like to thank the anonymous referee for the valuable suggestions that improved this article.

References

[1] Th. Aubin, *Some non linear problems in Riemannian Geometry*, Springer, Berlin, 1998.

[2] Adimurthi and S.L. Yadava, Existence of a nonradial positive solution for the critical exponent problem with Neumann boundary condition. *J. Differential Equations*, **104**, no. 2, (1993), 298-306.

[3] Adimurthi, F. Pacella and S. L. Yadava, Interaction between the geometry of the boundary and positive solutions of a semilinear Neumann problem with critical nonlinearity, *J. Func. Anal.*, **113**, (1993), 318-350.

[4] R. J. Biezuner, Best constants in Sobolev trace inequalities, *Nonlinear Analysis.*, **54** (2003), 575-589.

[5] P. Binding, P.Drabek Y. X. Huang, On Neumann boundary value problems for some quasilinear elliptic equations, *Electronic Journal of Differential Equations*, Vol **1997**, No 05, (1997), 1-11.

[6] Gabriella Bognar, Pavel Drabek, The p-Laplacian equation with superlinear and supercritical growth, multiplicity of radial solutions *Nonlinear Analysis.***60** (2005), 719-728.

[7] J. Fernández Bonder, J. Rossi, Existence results for the $p-$Laplacian with nonlinear boundary conditions, *Journal of Mathematical Analysis and Applications* **263** (2001), 195-223.

[8] C. Budd, M. C. Knaap and L. A. Peletier, Asymptotic behavior of solutions of elliptic equations with critical exponent and Neumann boundary conditions, *Proc. Roy. Soc. Edinburgh*, **117A** (1991), 225-250.

[9] Daomin Cao and Pigong Han, A note on the Positive Energy Solutions for Elliptic Equations Involving Critical Sobolev Exponents, *Applied Mathematics Letters*, **16** (2003), 1105-1113.

[10] D. Cao, E.S. Noussair, S. Yan, Existence and nonexistence of interior-peaked solution for a nonlinear Neumann problem, *Pacific J. Math.* **200** (1) (2001), 19-41.

[11] J. Chabrowski, On the nonlinear Neumann problem involving the critical Sobolev exponent on the boundary, *J. Math. Anal. Appl.*, **290** (2004), 605-619.

[12] J. Chabrowski, P. Girao, Symmetric solutions of the Neumann problem involving a critical Sobolev exponent, *Topological Methods in Nonlinear Analysis*, **19** (2002), 1-27.

[13] J. Chabrowski, E. Tonkes, On the nonlinear Neumann problem with critical and supercritical nonlinearities, *Dissertationes Math. Rozprawy Mat.*, **417** (2003), 59pp.

[14] P. Cherrier, Probèmes de Neumann non lièaires sur les variètes riemanniennes *J Functional.*, **57** 2 (1984), 154-206.

[15] M. Chipot, I. Shafrir and Fila, On Solutions to some elliptic equations with nonlinear Neumann boundary conditions, *Advances in Differential Equations*, Vol**1**, No 1,(1996), 91-110.

[16] D. Costa, and P. Girao, Existence and nonexistence of least energy solutions of the Neumann problem for a semilinear elliptic equation with critical Sobolev exponent and a critical lower-order perturbation, *J. Differential Equations* **188**, (2003), 164202.

[17] A. Cotsiolis, Problème de Neumann avec exposant sur-critique et une non-linéarité sur-critique sur le bord, *C. R. Acad. Sci. Paris*, **322** (1996), 139-142.

[18] A. Cotsiolis and N. Labropoulos, Dirichlet problem on a solid torus in the critical of super-critical case, *Bull. Greek Math. Soc.*, To appear.

[19] F. Demengel and B. Nazaret, On some nonlinear partial differential equations involving the $p-$Laplacian and critical Sobolev traces maps, *Asymptotic Analysis*,**23**, (2000), 135-156.

[20] Yinbin Deng and Shuangjie Peng Existence of multiple positive solutions for inhomogeneous Neumann problem *J. Math. Anal. Appl.*, **271** (2002), 155-174.

[21] W. Ding, On a Conformally Invariant Elliptic Equation on R^n, *Commun. Math. Phys.*, **107** (1986), 331-335.

[22] E. Lami Dozo, O. Torne, Symmetry and Symmetry breaking for minimizers in the trace inequality, *Communications in Contemporary Mathematics*, Vol **7**, No 6 (2005), 727-746.

[23] J. Escobar, Sharp constant in a Sobolev trace inequality. *Indiana Univ. Math. J.* **37** no. 3, (1988), 687-698.

[24] Michael Filippakis, Leszek Gasinski and Nikolaos S. Papageorgiou, Multiplicity Results for Nonlinear Neumann Problems *Canad. J. Math.* Vol. **58** 1, (2006) 64-92.

[25] C. Gui and N. Ghoussoub, Multi-peak solutions for a semilinear Neumann problem involving the critical exponent, *Math. Z.* **229**, (1998), 443-474.

[26] Zheng-Chao Han and YanYan Li, The Yamabe problem on manifolds with boundary: Existence and compactness results, *Duke Mathematical Journal*, Vol **99**, No 3, (1999) 489-542.

[27] E. Hebey, *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, Courant Institute of Mathematical Sciences, Lectures Notes in Mathematics, 5, 1999.

[28] E. Hebey, M. Vaugon, Sobolev spaces in the presence of symmetries, *J.Math.Pures Appl.*, **76** (1997), 859-881.

[29] YY Li, M. Zhu., Sharp Sobolev inequalities on Riemannian manifolds with boundaries, *Comm. Pure Appl.Math.* **50** no 5 (1997), 449-487.

[30] P. L. Lions, The concentration compactness principle in the calculus of variations. The limit case, I, II. *Rev. Mat. Iberoamericana* **1** no 1,2 (1985), 145-201, 45-121.

[31] S. Maier-Paape, K. Schmitt, Zhi-Qiang Wang, On Neumann problems for semilinear elliptic equations with critical nonlinearity. Existence and symmetry of multi-peaked solutions, *Commun. in Partial Differential Equations*, Vol. **22**, (9 and 10), (1997), 1493-1527.

[32] M. Motron, Méthode des sur et sous solutions pour la résolution d'un problème de type Neumann faisant intervenir le $p-$Laplacian, *C.R.Acad. Sci. Paris*, Ser. I **335**, (2002), 341-344.

[33] Shuangjie Peng, The uniqueness of symmetric single peak solutions for a Neumann problem involving critical exponent, *Nonlinear Analysis* **56**, (2004), 19-42.

[34] D. Pierotti, S. Terracini, On a Neumann problem with critical Sobolev exponent and critical nonlinearity on the boundary, *Commun. in Partial Differential Equations* **20**(7 and 8) (1995), 1155-1187.

[35] D. Pierotti, S. Terracini, On a Neumann problem involving two critical Sobolev exponents: remarks on geometrical and topological aspects, *Calc. Var.* **5**, (1997), 271-291.

[36] Manuel del Pino, Monica Musso, Angela Pistoia, Super-critical boundary bubbling in a semi-linear Neumann problem *Ann. I. H. Poincare* AN **22** (2005), 45-82.

[37] O. Rey, An elliptic Neumann problem with critical nonlinearity in three dimensional domains, *Comm. Contemp. Math.* **1** (1999), 405-449.

[38] Olivier Rey and Juncheng Wei, Blowing up solutions for an elliptic Neumann problem with sub- or supercritical nonlinearity Part I:$N = 3$, *Journal of Functional Analysis*, **212** (2004), 472-499.

[39] Olivier Rey, Juncheng Wei, Blowing up solutions for an elliptic Neumann problem with sub- or supercritical nonlinearity. Part II: $N \geq 4$, *Ann. I. H. Poincarè*, AN **22** (2005), 459-484.

[40] J. L. Vazquez. A strong maximum principle for some quasilinear elliptic equations. *Appl. Math. Optim.*, **12**, (1984), 191-202.

[41] Xu-Jia Wang, Neumann problems os semilinear elliptic equations involving critical Sobolev exponents, *Journal of Differential equations*, **93** No 4, (1991), 283-310.
[42] Zhi-Qiang Wang, High-energy and multi-peaked solutions for a nonlinear Neumann problem with critical exponents, *Proceedings of the Royal Society of Edindurgh*, **125**A, (1995), 1003-1029.
[43] Juncheng Wei , Shusen Yan, Solutions with interior bubble and boundary layer for an elliptic Neumann problem with critical nonlinearity *C. R. Acad. Sci. Paris*, Ser. I **343**, (2006), 311-316.
[44] Meijun Zhu, Uniqueness Results through a Priori Estimates I. A Three Dimensional Neumann Problem *Journal of differential equations* **154**, (1999), 284-317.

Athanase Cotsiolis
Department of Mathematics, University of Patras, Patras 26110, Greece
E-mail address: `cotsioli@math.upatras.gr`

Nikos Labropoulos
Department of Mathematics, University of Patras, Patras 26110, Greece
E-mail address: `nal@upatras.gr`