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MINIMAL WAVE SPEED ON A DIFFUSIVE SIR MODEL WITH NONLOCAL DELAYS

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Abstract. This article concerns the minimal wave speed of a diffusive SIR model with nonlocal delays, in which the dynamics of disease has no positive outbreak threshold. By constructing a pair of super and sub-solutions, we establish the existence of traveling wave solutions with the minimal wave speed.

1. Introduction

The geographic spread of epidemics is less well understood and much less well studied than the temporal development and control of diseases and epidemics [16, Chapter 13]. Since Kermack and McKendrick [10], many epidemic systems have been proposed to model the evolutionary process of disease, which includes the so-called SIS model, SIR model, SEIR model and so on. Moreover, there are also some models involving spatial migration of individuals, see Rass and Radcliffe [18] and references cited therein. In particular, the threshold dynamics of these models has been widely studied, we refer to Anderson and May [1], Anderson et al. [2], Brauer and Castillo-Chavez [3], Draief and Massoulie [6], Hethcote [8].

In the literature, the traveling wave solutions of epidemic models have been studied since they can characterize several important features of spatial propagation of the epidemic. For example, constant wave speeds of traveling wave solutions could model the almost fixed spreading speeds of the epidemic, see Murray [16, pp. 668, pp. 675] for two cases. Moreover, the minimal wave speed could reflect the speed at which the epidemic spreads (see Diekmann [4, 5]). Partly because of the fact that many epidemic models can not generate monotone semiflows, their dynamical behavior is very plentiful, we may refer to the books mentioned above.

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In this article, we study the minimal wave speed of traveling wave solutions of the following diffusive SIR model with nonlocal delays [11, 21, 22],

$$\begin{aligned}
\frac{\partial S(x,t)}{\partial t} &= d_1\Delta S(x,t) - \frac{\beta S(x,t)\int_0^\infty\int_{\mathbb{R}} J(y,s)I(x-y,t-s)\,dy\,ds}{S(x,t)+\int_0^\infty\int_{\mathbb{R}} J(y,s)I(x-y,t-s)\,dy\,ds},\\
\frac{\partial I(x,t)}{\partial t} &= d_2\Delta I(x,t) + \frac{\beta S(x,t)\int_0^\infty\int_{\mathbb{R}} J(y,s)I(x-y,t-s)\,dy\,ds}{S(x,t)+\int_0^\infty\int_{\mathbb{R}} J(y,s)I(x-y,t-s)\,dy\,ds}\\
&\quad - \gamma I(x,t),\\
\frac{\partial R(x,t)}{\partial t} &= d_3\Delta R(x,t) + \gamma I(x,t),
\end{aligned} \tag{1.1}$$

in which $x\in\mathbb{R}, t>0$. Here $d_i>0, i=1,2,3$, denote diffusion rates of the susceptible individuals S, the infected individuals I and the removed individuals R, respectively. In addition, $\beta>0$ is the transmission coefficient, $\gamma>0$ is the recovery/remove rate and $J(y,s)$ satisfies proper integrable and measurable conditions describing the interaction between the infected individuals at an earlier time $t-s$ at location y and susceptible individuals at location x at the present time t (see Ruan [19]).

Observing that $R(x,t)$ does not appear in the equations of $S(x,t), I(x,t)$, and Li et al. [11, Section 5] have discussed the properties of $R(x,t)$ by $S(x,t), I(x,t)$, then it suffices to investigate the equations on S, I in (1.1); that is,

$$\begin{aligned}
\frac{\partial S(x,t)}{\partial t} &= d_1\Delta S(x,t) - \frac{\beta S(x,t)\int_0^\infty\int_{\mathbb{R}} J(y,s)I(x-y,t-s)\,dy\,ds}{S(x,t)+\int_0^\infty\int_{\mathbb{R}} J(y,s)I(x-y,t-s)\,dy\,ds},\\
\frac{\partial I(x,t)}{\partial t} &= d_2\Delta I(x,t) + \frac{\beta S(x,t)\int_0^\infty\int_{\mathbb{R}} J(y,s)I(x-y,t-s)\,dy\,ds}{S(x,t)+\int_0^\infty\int_{\mathbb{R}} J(y,s)I(x-y,t-s)\,dy\,ds}\\
&\quad - \gamma I(x,t).
\end{aligned} \tag{1.2}$$

Hereafter, a traveling wave solution of (1.2) is a special translation invariant solution taking the form

$$S(x,t)=S(\xi),\quad I(x,t)=I(\xi),\quad \xi=x+ct\in\mathbb{R},$$

in which $c>0$ is the wave speed at which the wave profile (S,I) propagates in the whole $\mathbb{R}$. If we consider the traveling wave solution of (1.2), then for all $\xi\in\mathbb{R}$, one has

$$\begin{aligned}
cS'(\xi) &= d_1S''(\xi) - \frac{\beta S(\xi)(J*I)(\xi)}{S(\xi)+(J*I)(\xi)},\\
cI'(\xi) &= d_2I''(\xi) + \frac{\beta S(\xi)(J*I)(\xi)}{S(\xi)+(J*I)(\xi)} - \gamma I(\xi)
\end{aligned} \tag{1.3}$$

with

$$(J*I)(\xi)=\int_0^\infty\int_{\mathbb{R}} J(y,s)I(\xi-y-cs)\,dy\,ds.$$

Moreover, to describe the evolutionary phenomenon that the initial susceptible group admits a constant density $S_0>0$ while all individuals eventually become the removed, we shall investigate (1.3) with the following asymptotic behavior

$$\begin{aligned}
\lim_{\xi\to-\infty} S(\xi) &=: S(-\infty)=S_0, \quad \lim_{\xi\to\infty} S(\xi)=:S(\infty)=0,\\
\lim_{\xi\to-\infty} I(\xi) &=: I(-\infty)=0, \quad \lim_{\xi\to\infty} I(\xi)=:I(\infty)=0.
\end{aligned} \tag{1.4}$$

Under proper convergence conditions clarified later, let c^* be the smallest constant such that $c \geq c^*$ implies

$$d_2\lambda^2 - c\lambda + \beta \int_0^\infty \int_{\mathbb{R}} J(y,s)e^{\lambda(y-cs)}\,dy\,ds - \gamma = 0$$

admitting a positive root. In Li et al. [11], it has been proven that (1.3) has a nontrivial positive solution satisfying (1.4) if $c > c^*$ and $\beta/\gamma > 1$, while $0 < c < c^*$ and $\beta/\gamma > 1$ or $\beta/\gamma < 1$ implies the nonexistence of such a solution. Wang et al. [21] obtained a similar conclusion if the nonlocal delays vanish. Very recently, Li and Yang [12] studied the model with nonlocal dispersal version in [22, 21]. However, these results do not answer the existence or nonexistence of traveling wave solutions if $c = c^*$. The purpose of this paper is to complete these results on the minimal wave speed $c = c^*$.

In light of the ideas in [7, 14, 23], by constructing super and sub-solutions and applying Schauder fixed point theorem, we confirm the existence of nontrivial positive solutions of (1.3) with (1.4) if $c = c^*$. This extends the results in [11, 21], and indicates that c^* is the true minimal wave speed. Thus, we can obtain some evident control strategies of diseases and epidemics, e.g., reducing the movement ability of infected individuals and improving the recovery ratio. Furthermore, we also find different decay estimations, namely, $I(\xi)$ decays exponentially as $\xi \to -\infty$ if $c > c^*$ [11], while $c = c^*$ implies different decay behavior.

2. Preliminaries

In this article, we discuss the existence of traveling wave solutions of (1.2) when the kernel function satisfies the following assumptions:

- (A1) $J(y,s) = J(-y,s) \geq 0$, $y \in \mathbb{R}$, $s \geq 0$, $\int_0^\infty \int_{\mathbb{R}} J(y,s)\,dy\,ds = 1$;
- (A2) for each $c > 0$, there exists $\lambda_c \leq \infty$ such that
$$\int_0^\infty \int_{\mathbb{R}} J(y,s)e^{\lambda(y-cs)}\,dy\,ds < \infty \forall \lambda \in (0,\lambda_c),$$
$$d_2\lambda^2 - c\lambda + \beta \int_0^\infty \int_{\mathbb{R}} J(y,s)e^{\lambda(y-cs)}\,dy\,ds \to \infty, \quad \lambda \to \lambda_c-;$$
- (A3) for each $c > 0$, there exists $\mu > 0$ such that
$$\int_0^\infty \int_{\mathbb{R}} J(y,s)e^{\mu|y-cs|}\,dy\,ds < \infty, \quad \int_0^\infty \int_{\mathbb{R}} J(y,s)e^{\mu s}\,dy\,ds < \infty;$$
- (J4) $\int_0^\infty \int_{\mathbb{R}} sJ(y,s)\,dy\,ds < \infty$;
- (A5) $J(y,s)$ admits non-empty compact support with respect to y, namely, there exists a positive number $K > 0$ such that $J(y,s) \equiv 0$ for all $|y| \geq K$ and $s \in (0,+\infty)$.

For any $\lambda > 0, c > 0$, define

$$\Lambda(\lambda,c) := d_2\lambda^2 - c\lambda + \beta \int_0^\infty \int_{\mathbb{R}} J(y,s)e^{\lambda(y-cs)}\,dy\,ds - \gamma.$$

The properties of $\Lambda(\lambda,c)$ have been analyzed by Tian and Weng [20, Lemma 3.1], which can be described by the following lemma.

Lemma 2.1. *Assume that $\beta > \gamma$. Then there exists $c^* > 0$ such that*

(1) $\Lambda(\lambda,c) = 0$ *has no real roots if $c < c^*$;*

(2) *if* $c > c^*$, *then* $\Lambda(\lambda, c) = 0$ *has two positive real roots* $\lambda_1(c), \lambda_2(c)$ *such that* $\Lambda(\lambda, c) < 0, \lambda \in (\lambda_1(c), \lambda_2(c))$;
(3) *if* $c = c^*$, *then* $\Lambda(\lambda, c) = 0$ *only admits a unique positive real root* λ^* *and* $\Lambda(\lambda, c^*) > 0$ *for all* $\lambda > 0$ *and* $\lambda \neq \lambda^*$. *In addition,*

$$\Lambda_\lambda(\lambda^*, c^*) := 2d_2\lambda^* - c^* + \beta \int_0^\infty \int_{\mathbb{R}} J(y,s)(y - c^* s) e^{\lambda^*(y - c^* s)}\, dy\, ds = 0.$$

Lemma 2.2. *Assume that* $\beta > \gamma$. *Further suppose that* $S_+(\xi), S_-(\xi), I_+(\xi), I_-(\xi)$ *are continuous functions such that*

(i) $0 \leq S_-(\xi) \leq S_+(\xi) \leq S_0$, $0 \leq I_-(\xi) \leq I_+(\xi) \leq (\frac{\beta}{\gamma} - 1)S_0$, $\xi \in \mathbb{R}$;
(ii) *they are twice differentiable except finite points* $\mathbb{T} \subset \mathbb{R}$, *and* $S'_+(\xi)$, $S'_-(\xi)$, $I'_+(\xi), I'_-(\xi)$, $S''_+(\xi)$, $S''_-(\xi)$, $I''_+(\xi)$, $I''_-(\xi)$ *are bounded for* $\xi \in \mathbb{R} \backslash \mathbb{T}$;
(iii) *if* $\xi \in \mathbb{T}$, *then the left and right derivatives satisfy* $S'_+(\xi^-) \geq S'_+(\xi^+)$, $I'_+(\xi^-) \geq I'_+(\xi^+)$, $S'_-(\xi^-) \leq S'_-(\xi^+), I'_-(\xi^-) \leq I'_-(\xi^+)$;
(iv) *if* $\xi \in \mathbb{R} \backslash \mathbb{T}$, *then*

$$cS'_+(\xi) \geq d_1 S''_+(\xi) - \frac{\beta S_+(\xi)(J * I_-)(\xi)}{S_+(\xi) + (J * I_-)(\xi)}, \tag{2.1}$$

$$cI'_+(\xi) \geq d_2 I''_+(\xi) + \frac{\beta S_+(\xi)(J * I_+)(\xi)}{S_+(\xi) + (J * I_+)(\xi)} - \gamma I_+(\xi), \tag{2.2}$$

$$cS'_-(\xi) \leq d_1 S''_-(\xi) - \frac{\beta S_-(\xi)(J * I_+)(\xi)}{S_-(\xi) + (J * I_+)(\xi)}, \tag{2.3}$$

$$cI'_-(\xi) \leq d_2 I''_-(\xi) + \frac{\beta S_-(\xi)(J * I_-)(\xi)}{S_-(\xi) + (J * I_-)(\xi)} - \gamma I_-(\xi). \tag{2.4}$$

Then (1.3) *admits a positive solution* (S, I) *such that*

$$S_-(\xi) \leq S(\xi) \leq S_+(\xi), \quad I_-(\xi) \leq I(\xi) \leq I_+(\xi), \quad \xi \in \mathbb{R}.$$

Remark 2.3. In Lemma 2.2, $(S_+(\xi), I_+(\xi)), (S_-(\xi), I_-(\xi))$ are a pair of super and sub-solutions of (1.3), see Li et al. [11].

Lemma 2.2 can be proved using the Schauder fixed point theorem, as done in [11]. The same method was used earlier by Ma [15] for delayed quasimonotone systems, and by Huang and Zou [9] for delayed predator-prey systems (the monotone conditions are similar to those in (1.3)). So we omit that proof here.

3. Main Results

In this section, we establish the existence of nontrivial positive solutions of (1.3)-(1.4) with $c = c^*$ by Lemma 2.2. To this end, we first construct a pair of proper super and sub-solutions of (1.3) with $c = c^*$ under assumptions (A1)–(A5). We define the continuous functions

$$S_+(\xi) = S_0,$$

$$S_-(\xi) = \begin{cases} S_0 - pe^{\lambda_3 \xi}, & \xi < \xi_1, \\ \epsilon e^{-\lambda_4 \xi}, & \xi \geq \xi_1, \end{cases}$$

$$I_+(\xi) = \begin{cases} -\rho \xi e^{\lambda^* \xi}, & \xi < \xi_2, \\ (\frac{\beta}{\gamma} - 1) S_0, & \xi \geq \xi_2, \end{cases}$$

$$I_-(\xi)=\begin{cases}-\rho\xi e^{\lambda^*\xi}-L(-\xi)^{1/2}e^{\lambda^*\xi}, & \xi<\xi_3,\\ 0, & \xi\ge\xi_3,\end{cases}$$

where

$$\lambda_3=\min\Big\{\frac{\lambda^*}{2},\frac{c^*}{2d_1}\Big\},\quad \lambda_4=\sqrt{\beta/d_1},$$

and $\xi_1,\xi_2,\xi_3\in\mathbb{R}$, $p>S_0$, $\epsilon>0$, $\rho>0$, $L>0$ will be clarified later.

Lemma 3.1. *Assume that $\beta>\gamma$ and (A1)–(A5) hold. Then (1.3) with $c=c^*$ admits a solution satisfying*

$$S_-(\xi)\le S(\xi)\le S_+(\xi),\quad I_-(\xi)\le I(\xi)\le I_+(\xi),\quad \xi\in\mathbb{R}.$$

Proof. Clearly, from Lemma 2.2 and the definitions of $S_+(\xi),S_-(\xi),I_+(\xi),I_-(\xi)$, it suffices to verify (2.1)-(2.4) by selecting proper parameters. Next we define

$$m:=\int_0^\infty\int_{\mathbb{R}}J(y,s)e^{\lambda^*(y-c^*s)}\,dy\,ds,$$
$$n:=\int_0^\infty\int_{\mathbb{R}}J(y,s)(y-c^*s)e^{\lambda^*(y-c^*s)}\,dy\,ds\,.$$

Then (A1)–(A5) indicate that m,n are bounded and

$$\xi m+n\le(\xi+K)m-c^*\int_0^\infty\int_{\mathbb{R}}J(y,s)se^{\lambda^*(y-c^*s)}\,dy\,ds,$$

and so $\xi m+n<0$ for all $\xi<-K$. In addition, Lemma 2.2 indicates that

$$d_2\lambda^{*2}-c^*\lambda^*+\beta m-\gamma=0,\quad 2d_2\lambda^*-c^*+\beta n=0.$$

Note that if $u\ge0$, $v\ge0$ with $u+v>0$, then

$$\frac{uv}{u+v}\le\min\{u,v\}.$$

Now, we verify (2.1)-(2.4) one by one.

(1) $S_+(\xi)=S_0$. Since $S_+(\xi)$ is positive and $I_-(\xi)$ is nonnegative, then (2.1) is straightforward.

(2) Let $\rho>0$ be a positive constant such that

$$\sup_{\xi\in\mathbb{R}}\{-\rho\xi e^{\lambda^*\xi}\}>\big(\frac{\beta}{\gamma}-1\big)S_0,$$

and ξ_2,ξ^* be the only two negative real roots of $-\rho\xi e^{\lambda^*\xi}=\big(\frac{\beta}{\gamma}-1\big)S_0$. Denote by ξ_2 the smaller one, then there exists $\rho>0$ large enough such that $\xi^*-\xi_2>K$. To illustrate that the parameters are admissible, we give Figure 1.

If $\xi<\xi_2$, then $I_+(\xi)=-\rho\xi e^{\lambda^*\xi}$, and it suffices to prove that

$$c^*I_+'(\xi)\ge d_2I_+''(\xi)+\beta(J*I_+)(\xi)-\gamma I_+(\xi). \tag{3.1}$$

Note that

$$\begin{aligned}(J*I_+)(\xi)&=\int_0^\infty\int_{\mathbb{R}}J(y,s)I_+(\xi-y-c^*s)\,dy\,ds\\ &\le\int_0^\infty\int_{\xi-\xi^*-c^*s}^{+\infty}J(y,s)I_+(\xi-y-c^*s)\,dy\,ds\\ &=-\rho\int_0^\infty\int_{\mathbb{R}}J(y,s)(\xi-y-c^*s)e^{\lambda^*(\xi-y-c^*s)}\,dy\,ds\end{aligned}$$

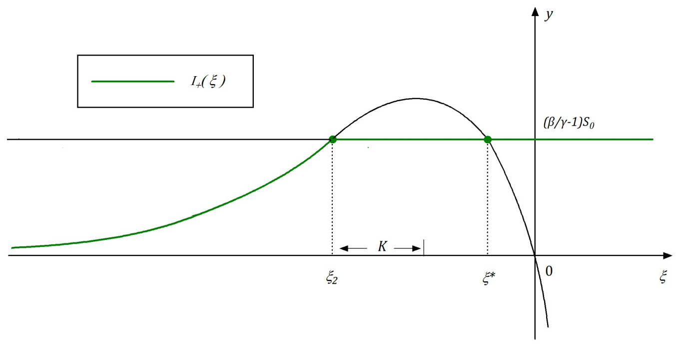

FIGURE 1. $I_+(\xi)$.

$$
\begin{aligned}
&= -\rho \int_0^\infty \int_{\mathbb{R}} J(y,s)(\xi + y - c^*s) e^{\lambda^*(\xi+y-c^*s)}\,dy\,ds \\
&= -\rho\xi e^{\lambda^*\xi} \int_0^\infty \int_{\mathbb{R}} J(y,s) e^{\lambda^*(y-c^*s)}\,dy\,ds \\
&\quad - \rho e^{\lambda^*\xi} \int_0^\infty \int_{\mathbb{R}} J(y,s)(y - c^*s) e^{\lambda^*(y-c^*s)}\,dy\,ds \\
&= -\rho\xi e^{\lambda^*\xi} m - \rho e^{\lambda^*\xi} n\,.
\end{aligned}
$$

Then (3.1) holds if

$$
c^* I_+'(\xi) \ge d_2 I_+''(\xi) - \beta\rho\xi e^{\lambda^*\xi} m - \beta\rho e^{\lambda^*\xi} n - \gamma I_+(\xi).
$$

From the definition of $I_+(\xi)$, for any $\xi < \xi_2$, direct calculations yield

$$
\begin{gathered}
I_+'(\xi) = -\rho e^{\lambda^*\xi}(1 + \lambda^*\xi), \\
I_+''(\xi) = -\rho e^{\lambda^*\xi}(2\lambda^* + (\lambda^*)^2\xi).
\end{gathered}
$$

It follows that

$$
\begin{aligned}
&c^* I_+'(\xi) - d_2 I_+''(\xi) + \beta\rho\xi e^{\lambda^*\xi} m + \beta\rho e^{\lambda^*\xi} n + \gamma I_+(\xi) \\
&= \Lambda_\lambda(\lambda^*, c^*)\rho e^{\lambda^*\xi} + \Lambda(\lambda^*, c^*)\rho\xi e^{\lambda^*\xi} = 0,
\end{aligned}
$$

which implies (3.1). Furthermore, if $\xi > \xi_2$, then

$$
\frac{\beta S_+(\xi)(J * I_+)(\xi)}{S_+(\xi) + (J * I_+)(\xi)} \le \frac{\beta S_0(\frac{\beta}{\gamma} - 1)S_0}{S_0 + (\frac{\beta}{\gamma} - 1)S_0} = \gamma(\frac{\beta}{\gamma} - 1)S_0
$$

such that (2.2) is also evident.

(3) For any $\rho > 0$ and $\xi^* < 0$ given in (2), denote

$$
p = S_0 e^{\lambda_3(K-\xi^*)} + \sup_{\xi<0} \frac{-\beta\rho e^{(\lambda^*-\lambda_3)\xi}(\xi m + n)}{c^*\lambda_3 - d_1\lambda_3^2}.
$$

Let $\epsilon > 0$ such that

$$
S_0 - p e^{\lambda_3\xi} = \epsilon e^{-\lambda_4\xi}
$$

admits two negative real roots and we choose the larger one as ξ_1, then ξ_1 is admissible and $\xi_1 \le \xi^* - K$, see Figure 2.

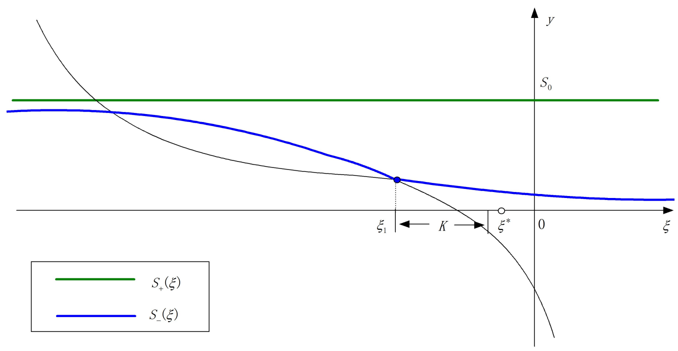

FIGURE 2. $S_-(\xi)$ and $S_+(\xi)$.

If $\xi < \xi_1$, then $S_-(\xi) = S_0 - pe^{\lambda_3 \xi} > 0$, and (2.3) is true once

$$c^* S_-'(\xi) \le d_1 S_-''(\xi) - \beta (J * I_+)(\xi), \quad \xi < \xi_1.$$

Since $\xi < \xi_1 \le \xi^* - K$, it follows that

$$\begin{aligned} (J * I_+)(\xi) &= \int_0^\infty \int_{\mathbb{R}} J(y,s) I_+(\xi - y - c^* s)\,dy\,ds \\ &\le \int_0^\infty \int_{\xi - \xi^* - c^* s}^{+\infty} J(y,s) I_+(\xi - y - c^* s)\,dy\,ds \\ &= -\rho \xi e^{\lambda^* \xi} m - \rho e^{\lambda^* \xi} n. \end{aligned}$$

Thus, we only need to verify that

$$c^* S_-'(\xi) \le d_1 S_-''(\xi) + \beta \rho \xi e^{\lambda^* \xi} m + \beta \rho e^{\lambda^* \xi} n.$$

Based on direct calculations, (2.3) holds once

$$-c^* \lambda_3 p e^{\lambda_3 \xi} \le -d_1 \lambda_3^2 p e^{\lambda_3 \xi} + \beta \rho \xi e^{\lambda^* \xi} m + \beta \rho e^{\lambda^* \xi} n, \xi < \xi_1,$$

which is true by the definition of p.

Now we verify (2.3) with $\xi > \xi_1$; it suffices to confirm that

$$c^* S_-'(\xi) \le d_1 S_-''(\xi) - \beta S_-(\xi),$$

which is equivalent to

$$-c^* \lambda_4 \epsilon e^{-\lambda_4 \xi} \le d_1 \lambda_4^2 \epsilon e^{-\lambda_4 \xi} - \beta \epsilon e^{-\lambda_4 \xi},$$

this is also evident by the definition of λ_4.

(4) Finally, we verify (2.4). For $\rho > 0$ and $\xi_2 < 0$ defined in (2), let $L \ge M_1 \ge \rho\sqrt{-\xi_2}$ such that

$$S_-(\xi) \ge S_0/2, \quad \xi < \xi_3 := -\big(\frac{L}{\rho}\big)^2 \le \xi_2,$$

then ξ_3 is well defined, see Figure 1.4.

Now we verify that $I_-(\xi)$ satisfies (2.4). Clearly, the definition of ξ_3 implies that $I_-(\xi) \le I_+(\xi)$ for all $\xi \in \mathbb{R}$. If $\xi < \xi_3$, then

$$I_-(\xi) = -\rho \xi e^{\lambda^* \xi} - L(-\xi)^{1/2} e^{\lambda^* \xi} \le -\rho \xi e^{\lambda^* \xi} = I_+(\xi)$$

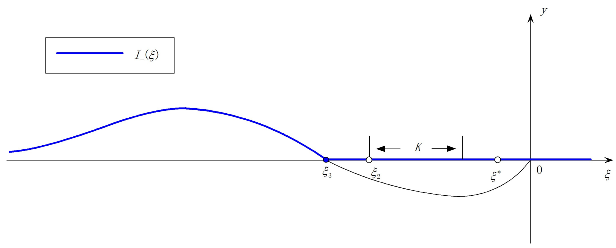

FIGURE 3. $I_-(\xi)$.

such that

$$\begin{aligned}(J*I_-)(\xi) &= \int_0^\infty \int_{\mathbb{R}} J(y,s) I_-(\xi - y - c^*s)\,dy\,ds\\ &\le \int_0^\infty \int_{\mathbb{R}} J(y,s) I_+(\xi - y - c^*s)\,dy\,ds\\ &= -\rho\xi e^{\lambda^*\xi} m - \rho e^{\lambda^*\xi} n.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{\beta S_-(\xi)(J*I_-)(\xi)}{S_-(\xi)+(J*I_-)(\xi)} - \beta(J*I_-)(\xi)\\ &\ge \frac{\beta\frac{S_0}{2}(J*I_-)(\xi)}{\frac{S_0}{2}+(J*I_-)(\xi)} - \beta(J*I_-)(\xi)\\ &\ge -\frac{2\beta}{S_0}\left[(J*I_-)(\xi)\right]^2\\ &\ge -\frac{2\beta\rho^2}{S_0} e^{2\lambda^*\xi}(\xi m + n)^2.\end{aligned}$$

Hence, (2.4) is true provided that

$$c^* I_-'(\xi) \le d_2 I_-''(\xi) + \beta(J*I_-)(\xi) - \gamma I_-(\xi) - \frac{2\beta\rho^2}{S_0} e^{2\lambda^*\xi}(\xi m + n)^2.$$

For any $\xi < \xi_3$, a direct calculation yields

$$\begin{aligned}I_-'(\xi) &= I_+'(\xi) + Le^{\lambda^*\xi}\big[\frac{1}{2}(-\xi)^{-1/2} - \lambda^*(-\xi)^{1/2}\big],\\ I_-''(\xi) &= I_+''(\xi) + Le^{\lambda^*\xi}\big[\frac{1}{4}(-\xi)^{-3/2} + \lambda^*(-\xi)^{-1/2} - (\lambda^*)^2(-\xi)^{1/2}\big].\end{aligned}$$

Since $-\xi + y + c^*s \ge 0$ for any $\xi < \xi_3, |y| < K$ and $s \ge 0$, applying the Taylor's Theorem, we have

$$\begin{aligned}&[-\xi + (y + c^*s)]^{1/2}\\ &= (-\xi)^{1/2} + \frac{1}{2}(-\xi)^{-1/2}(y + c^*s) - \frac{1}{8}[-\xi + \theta(y + c^*s)]^{-3/2}(y + c^*s)^2\\ &\le (-\xi)^{1/2} + \frac{1}{2}(-\xi)^{-1/2}(y + c^*s)\end{aligned}$$

with some $\theta \in (0,1)$. This implies

$$\begin{aligned}
(J * I_-)(\xi) &= \int_0^\infty \int_{\mathbb{R}} J(y,s) I_-(\xi - y - c^* s)\, dy\, ds \\
&= \int_0^\infty \int_{-K}^{K} J(y,s) I_-(\xi - y - c^* s)\, dy\, ds \\
&\geq \int_0^\infty \int_{-K}^{K} J(y,s) \Big[-\rho(\xi - y - c^* s) e^{\lambda^*(\xi - y - c^* s)} \\
&\quad - L(-(\xi - y - c^* s))^{1/2} e^{\lambda^*(\xi - y - c^* s)} \Big]\, dy\, ds \\
&\geq -\rho \int_0^\infty \int_{-K}^{K} J(y,s)(\xi - y - c^* s) e^{\lambda^*(\xi - y - c^* s)}\, dy\, ds \\
&\quad - L \int_0^\infty \int_{-K}^{K} J(y,s) \big[(-\xi)^{1/2} + \frac{1}{2}(-\xi)^{-1/2}(y + c^* s)\big] e^{\lambda^*(\xi - y - c^* s)}\, dy\, ds \\
&= -\rho\xi e^{\lambda^*\xi} m - \rho e^{\lambda^*\xi} n - L(-\xi)^{1/2} e^{\lambda^*\xi} m + \frac{1}{2} L(-\xi)^{-1/2} e^{\lambda^*\xi} n.
\end{aligned}$$

Therefore, (2.4) holds if

$$\begin{aligned}
& c^* I_+'(\xi) + L c^* e^{\lambda^*\xi} \big[\frac{1}{2}(-\xi)^{-1/2} - \lambda^*(-\xi)^{1/2}\big] \\
& \leq d_2 I_+''(\xi) + d_2 L e^{\lambda^*\xi} \big[\frac{1}{4}(-\xi)^{-3/2} + \lambda^*(-\xi)^{-1/2} - (\lambda^*)^2(-\xi)^{1/2}\big] \\
& \quad - \beta\rho\xi e^{\lambda^*\xi} m - \beta\rho e^{\lambda^*\xi} n - \beta L(-\xi)^{1/2} e^{\lambda^*\xi} m + \frac{\beta}{2} L(-\xi)^{-1/2} e^{\lambda^*\xi} n \\
& \quad - \gamma I_+(\xi) + \gamma L(-\xi)^{1/2} e^{\lambda^*\xi} - \frac{2\beta\rho^2}{S_0} e^{2\lambda^*\xi}(\xi m + n)^2,
\end{aligned}$$

which is true provided that

$$d_2 L e^{\lambda^*\xi} \frac{1}{4}(-\xi)^{-3/2} - \frac{2\beta\rho^2}{S_0} e^{2\lambda^*\xi}(\xi m + n)^2 \geq 0.$$

Taking

$$M_2 := \sup_{\xi < 0} \frac{8\beta\rho^2(\xi m + n)^2(-\xi)^{3/2} e^{\lambda^*\xi}}{d_2 S_0} + 1,$$

for any $\xi < \xi_3$, (2.4) is satisfied with $L := M_1 + M_2$. When $\xi > \xi_3$, it is straightforward to show (2.4). The proof is complete. $\square$

Remark 3.2. We now show the logical sequence on the parameters in Lemma 3.1. Choose $\rho > 0$ such that there are two negative constants $\xi_2 = \xi_2(\rho)$ and $\xi^* = \xi^*(\rho)$. Then we can select $p = p(\xi^*, \rho) > S_0$ and $\epsilon = \epsilon(p) > 0$ such that $\xi_1 = \xi_1(p, \epsilon)$ exists. For any $\rho > 0, \xi_2 < 0$ given above, let $L = L(\rho, \xi_2) > 0$ be a positive constant large enough and $\xi_3 = -(L/\rho)^2$, then $S_+(\xi), S_-(\xi), I_+(\xi), I_-(\xi)$ are well defined.

Theorem 3.3. *Assume that $\beta > \gamma$ and* (A1)–(A5) *hold. Then* (1.3) *with $c = c^*$ admits a nontrivial positive solution satisfying* (1.4).

Proof. From Lemmas 2.2 and 3.1, (1.3) with $c = c^*$ has a nonnegative solution (S, I) such that

$$0 \leq S(\xi) \leq S_0, \quad 0 \leq I(\xi) \leq \frac{\beta - \gamma}{\gamma} S_0, \quad \xi \in \mathbb{R}.$$

Thanks to Li et al. [11, Theorem 2.5], in light of the strongly positivity of the solution operator, the nonnegative solution (S, I) satisfies

$$0 < S(\xi) < S_0, \quad 0 < I(\xi) < \frac{\beta - \gamma}{\gamma} S_0, \quad \xi \in \mathbb{R}.$$

Moreover, by following exactly the same arguments as that in Li et al. [11, Theorem 3.6], the asymptotic behavior (1.4) is obtained and we omit it here. The proof is complete. □

Before ending this paper, we make the following remark by the invariant form of traveling wave solutions.

Remark 3.4. In Li et al. [11], they proved that if $c > c^*$, then the system admits a positive solution such that $I(\xi) \sim Ae^{\lambda_1(c)\xi}$, $\xi \to -\infty$ for any given constant $A > 0$. Our results imply that (1.3) with $c = c^*$ has a solution satisfying

$$I(\xi) \sim -C\xi e^{\lambda^* \xi}, \quad \xi \to -\infty,$$

where $C > 0$ is any given constant.

The model here admits the similar monotonicity of predator-prey system. Recently, Pan [17] estimated the spreading speed of a predator-prey system [13], from which it is possible to study the asymptotic spreading of this model. But the limit behavior of this model is different from that in [13], so some new techniques are needed, and we shall further study this question.

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