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DIFFICULTIES IN OBTAINING FINITE TIME BLOWUP FOR FOURTH-ORDER SEMILINEAR SCHRÖDINGER EQUATIONS IN THE VARIATIONAL METHOD FRAME

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ABSTRACT. This article concerns the Cauchy problem for fourth-order semilinear Schrödinger equations. By constructing a variational problem and some invariant manifolds, we prove the existence of a global solution. Then we analyze the difficulties in proving the finite time blowup of the solution for the corresponding problem in the frame of the variational method. Understanding the finite time blowup of solutions, without radial initial data, still remains an open problem.

1. INTRODUCTION

In this article, we consider the Cauchy problem of the semilinear fourth-order Schrödinger equation

$$\begin{gathered} iu_t + \Delta u - \Delta^2 u = -|u|^p u,\ (x,t) \in \mathbb{R}^N \times [0,T), \\ u(0,x) = u_0(x), \end{gathered} \tag{1.1}$$

where $i = \sqrt{-1}$, $\Delta^2 = \Delta\Delta$ is the biharmonic operator, $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^N$; $u(x,t) : \mathbb{R}^N \times [0,T) \to \mathbb{C}$ denotes the complex valued function, T is the maximum existence time; N is the space dimension and p satisfies the embedding condition

$$0 < p < \begin{cases} +\infty, & 2 \le N \le 4, \\ \frac{8}{N-4}, & N > 4. \end{cases} \tag{1.2}$$

There has been a lot of interest in fourth-order semilinear Schrödinger equation, because of their strong physical background. Karpmam [12] investigated the fourth-order Schrödinger equation

$$iu_t + \frac{1}{2}\Delta u + \frac{1}{2}\gamma\Delta^2 u + |u|^{2p} u = 0, \tag{1.3}$$

where $\gamma \in \mathbb{R}$, $p \ge 1$, and the space dimension is no more than three. Equation (1.3) describes a stable soliton which is a wave pulse or wave beam, specially, there are solitons in magnetic materials for $p = 1$ in 3-D space. Karpmam and Shagalov

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[13] presented a numerical study on the axially symmetric fourth-order Schrödinger equation

$$i\frac{\partial u}{\partial \xi}+\frac{1}{2}S\Delta_{\perp}u+\lambda\Delta_{\perp}^{2}u+\mu|u|^{2}u=0, \tag{1.4}$$

where $S>0$, $\mu>0$, $\Delta_{\perp}=\partial^2/\partial\rho^2+(1/\rho)\partial/\partial\rho$, and ξ,ρ are properly normalized cylindrical variables. For $\lambda<0$, Equation (1.4) plays a crucial role in the self-focusing, here the fourth derivative term in (1.4) may give rise to an oscillatory approach to the asymptotically homogeneous wave beam. Fibich et al. [9] analyzed the self-focusing and singularity formation in the mixed-dispersion nonlinear Schrödinger equation

$$iu_t+\Delta u+\epsilon\Delta^2 u+|u|^{2p}u=0,\quad (x,t)\in\mathbb{R}^N\times[0,T), \tag{1.5}$$

where $\epsilon=\pm1$, $p\geq 1$, which occurs in propagation models for fiber arrays. The authors showed that the generic propagation dynamics for $\epsilon<0$ is focusing-defocusing oscillations. Davydova et al. [7] considered the Schrödinger equation in dimensionless variables

$$iu_t+D\Delta u+P\Delta^2 u+B|u|^2u+K|u|^4u=0, \tag{1.6}$$

where $D,P,B,K\in\mathbb{R}$ and $BK<0$. This equation was used for describing the dynamics of slowly varying wave packet envelope amplitude.

Given its mathematical interests, a lot of attention is paid to the existence and nonexistence of global solutions to the fourth-order semilinear Schrödinger equation. Pausader [14] studied the equation

$$iu_t+\Delta^2 u+\beta\Delta u+\lambda|u|^{p-1}u=0,\quad (x,t)\in\mathbb{R}^N\times[0,T), \tag{1.7}$$

where $\lambda,\beta\in\mathbb{R}$, $p\in(1,2^{\#}-1]$, and $2^{\#}=\frac{2N}{N-4}$ is the energy critical exponent for the embedding from H^2 into Lebesgue's spaces. Using the Strichartz-type estimates and Gagliardo-Nirenberg's inequalities, he proved the local well-posedness, and investigated global well-posedness and scattering with radially symmetrical initial data.

Fibich et al. [9] proved the existence of a global solution to the Cauchy problem

$$\begin{gathered} iu_t+\epsilon\Delta^2 u+|u|^{2p}u=0,\quad (x,t)\in\mathbb{R}^N\times[0,T),\\ u(0,x)=u_0(x), \end{gathered} \tag{1.8}$$

under each of the following three sets of conditions: (i) $\epsilon>0$, (ii) $\epsilon<0$ and $pN<4$, (iii) $\epsilon<0$, $pN=4$ and $||u_0||_2^2<\|R_B\|_2^2$, where R_B is the ground-state solution of $-\Delta^2R_B-R_B+R_B^{\frac{8}{N}+1}=0$. Furthermore, the authors gave the global well-posedness of problem (1.8) on a bounded domain $\Omega\subset\mathbb{R}^N$ with Dirichlet boundary condition when (i)–(iii) are satisfied. Based on the numerical simulations instead of rigorous mathematical proof, the blowup solution was showed. And they figured out sufficient conditions for existence of global solution for (1.8) and for (1.5).

Guo and Cui [10] studied the Cauchy problem of the equation

$$\begin{gathered} iu_t+\mu\Delta^2 u+\lambda\Delta u+f(|u|^2)u=0,\quad (x,t)\in\mathbb{R}^N\times[0,T),\\ u(0,x)=u_0(x), \end{gathered} \tag{1.9}$$

where $\lambda\in\mathbb{R}$, $\mu\neq0$, and f is a given real-valued nonlinear function. Let $N=1,2,3$, by the standard contraction mapping argument, a local solution for $u_0\in H^k$ and $k>\frac{N}{2}$ was obtained. Then the authors obtained a global solution of (1.9) with νu^{2p} instead of $f(|u|^2)$, for each of the following three sets of conditions: (i) $\mu\nu>0$;

(ii) $\mu\nu < 0$ and $0 < pN < 4$; (iii) $\mu\nu < 0$, $pN \geq 4$ and initial data $\|u_0\|_2^2 \leq c^*$, where $0 < c^* \leq 1$.

Later, Guo [11] consider the Cauchy problem (1.9) with νu^{2p} instead of $f(|u|^2)$ in the low regularity Sobolev space $H^s(\mathbb{R}^N)(s < 2)$. Assuming that $4 < (p-1)N/2 < p$ and

$$s > 1 + \frac{\frac{p-1}{2}N - 9 + \sqrt{(2p+5-\frac{p-1}{2}N)^2 + 16}}{2(p-1)},$$

by the I-method and $\|Iu_0\|_{L^{2(p+1)}}^{2(p+1)} \leq CN^{2(2-s)}\|u_0\|_{H^s}^{2(p+1)}$, where C is a constant, the global well-posedness of solution was established when $\mu > 0, \lambda < 0, \nu > 0$ or $\mu < 0, \lambda > 0, \nu < 0$.

There is literature devoted to blowup solutions. Consider the Cauchy problem of the fourth-order Schrödinger equation

$$\begin{gathered} iu_t - \Delta^2 u = -|u|^{2p}u,\ (x,t) \in \times\mathbb{R}^N \times [0,T), \\ u(x,0) = u_0(x). \end{gathered} \tag{1.10}$$

In the mass-critical case $Np = 4$, using an adaptive grid technique, Baruch et al. [1] proved the peak-type singular solution by numerical simulation when the space dimension is no more than three. Later, for the mass-supercritical case $Np > 4$, Baruch et al. [3] considered peak-type singular solution of (1.10) through asymptotic analysis and numerical simulations. Ring-type singular solution of (1.10) was studied numerically in [2]. Most recently, Dinh [8] proved the finite time blowup of solution with radial data $u_0 \in H^{\gamma_c}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ for negative initial energy, i.e., $E(u_0) < 0$ and $N \geq 5$, $4/N < p < \min\{2, 4/(N-4)\}$. Moreover, the additional condition $\sup_{t\in[0,T)} \|u(t)\|_H^{\gamma_c} < \infty$ is also needed, where the critical Sobolev exponent $\gamma_c := \frac{N}{2} - \frac{2}{p}$.

Boulenger and Lenzmann [4] studied the equation

$$iu_t - \Delta^2 u + \mu\Delta u = -|u|^{2p}u,\ (x,t) \in \mathbb{R}^N \times [0,T), \tag{1.11}$$

where $0 < p < \infty$ for $2 \leq N \leq 4$ and $0 < p < \frac{4}{N-4}$ for $N \geq 5$. When $\mu > 0$, in the mass-supercritical case $\frac{4}{N} < p < \frac{4}{N-4}$ $(N \geq 5)$ for negative initial energy $(E(u_0) < 0)$, they proved the blowup of solutions with radial initial data in $H^2(\mathbb{R}^N)$, which was also established for the mass-critical case $p = \frac{4}{N}$ $(N \geq 2)$ and the energy-critical case $p = \frac{4}{N-4}$ $(N \geq 5)$. Furthermore, for the mass-supercritical case and the energy-critical case, it was proved that: (i) when $\mu \neq 0$, finite time blowup of solution was obtained for negative initial energy $(E(u_0) < 0)$; (ii) when $\mu = 0$ (although this is not the case considered in the present paper), the finite time blowup of solution with radial initial data u_0 (not necessarily radial in the mass-supercritical case) can be obtained for the positive initial energy.

Cho et al. [6] studied the Cauchy problem

$$\begin{gathered} iu_t + \Delta u - \mu\Delta^2 u = -|x|^{-2}|u|^{\frac{4}{N}}u, \quad (x,t) \in \mathbb{R}^N \times [0,T), \\ u(x,0) = \psi(x), \end{gathered} \tag{1.12}$$

where $\mu > 0$ and $N \geq 3$, $-|x|^{-2}|u|^{\frac{4}{N}}$ works as an attractive self-reinforcing potential, $-|x|^{-2}|u|^{\frac{4}{N}}u$ is a Hartree type nonlinearity, $\psi(x)$ is a sufficiently smooth and decreasing function. They investigated existence and finite time blowup of local solution to (1.12) for negative initial energy $(E(u_0) < 0)$.

In summary for the Cauchy problem (1.10) of the fourth-order Schrödinger equation with term $-\Delta^2 u$, when u_0 is radially symmetric and $E(u_0) < 0$: for the mass-critical case, the solution of (1.10) either blows up in finite time, or blows up in infinite time [4]; for the mass-supercritical and energy-critical cases, the solution of (1.10) blows up in finite time [4]; because of the lack of conservation of mass, the solution of (1.10) blows up in finite time for $4/N < p < \min\{2, 4/(N-4)\}$ [8]. When u_0 is radially symmetric and $E(u_0) > 0$, the finite time blowup solution of (1.10) was proved in the energy-critical case and the mass-supercritical case (u_0 is not necessarily radial) in [4]. For the Cauchy problem (1.1) of the fourth-order Schrödinger equation with radial initial data, which contains both $-\Delta^2 u$ and Δu, $E(u_0) < 0$ is currently a sufficient condition for the finite time blowup of solution in mass-critical, mass-supercritical and energy-critical cases [4]. The above discussions indicate that there is no blowup result for problem (1.1) when the initial energy is non-negative, i.e., $E(u_0) \geq 0$. Hence we have no sharp condition for problem (1.1) in positive initial energy case, which even can be derived for the second-order nonlinear Schrödinger equation [22]. As the sharp condition is not only the sufficient condition of blowup, but also its necessary condition to some extent, and links the initial conditions of blowup and global existence, we desire to obtain it for problem (1.1), similar to the case of second-order nonlinear Schrödinger equation as follows: (i) If $u_0 \in \mathbb{B} := \{u \in H^1(\mathbb{R}^N) : I(u) < 0,\ E(u) < d\}$, where $d > 0$, then the solution $u(x,t)$ blows up in finite time; (ii) If $u_0 \in \mathbb{G} := \{u \in H^1(\mathbb{R}^N) : I(u) > 0,$ $E(u) < d\}$, then the solution $u(x,t)$ exists globally [22]. And the sharp conditions were also derived for the second-order Schrödinger equation with harmonic potential (cf. [16, 20, 21]). It is well known that all the sharp conditions on Schrödinger equations for positive initial energy were proved by the so-called potential well method or the mountain pass theory [16, 20, 21, 22]. It is natural to ask if we can applied the potential well method to problem (1.1) to derive the sharp condition. This paper deals with this problem by considering two points: existence of a global solution and finite time blowup of local solutions. First we construct the structure of the potential well method, and then prove the invariance of manifolds. By these tools, we shall prove the global existence first. But we find that it is too hard to prove the finite time blowup because of the failure of the standard route for the second-order Schrödinger equation. However, we do not like to just convince that we have given up to prove the blowup part of the sharp condition. Hence in this paper we like to show the main difficulties of proving the blowup part of the sharp condition by comparing to the case of second-order Schrödinger equation and the computation of the second-order derivative of $J(t) = \int |x|^2|u|^2 dx$. It should be mentioned that, we denote by $J''(t)$ the second order derivative of $J(t) = \int |x|^2|u|^2 dx$ for the fourth-order semilinear Schrödinger equation and by $\mathbf{J}''(t)$ the second-order derivative of $\mathbf{J}(t) = \int |x|^2|u|^2 dx$ for the second-order semilinear Schrödinger equation, where $J''(t)$ and $\mathbf{J}''(t)$ are essentially different for the different composition of equation. Generally speaking, the motivation of this paper is to prove the global existence part of the expected sharp condition and analyze the difficulties of proving the blowup part. Next we shall explain why the computation of $J''(t)$ is that important for proving the finite time blowup of the solution to problem (1.1).

Indeed in this article we have proved the existence of a global solution to (1.1) for $I(u_0) > 0$ and $0 < E(u_0) < d$. To derive the expected sharp condition, we need to prove the finite time blowup part of the sharp condition for $I(u_0) < 0$

and $0 < E(u_0) < d$, which seems too difficult to be obtained, and is very different from the case of second-order semilinear Schrödinger equation. In the progress of proving the finite time blowup as a part of the sharp condition, the inequality $\|u\|^2 \le C(N)\|\nabla u\| \cdot \|xu\|$ plays a very important role [17]. Because of the mass conservation (we shall prove this in Lemma 3.2), we can only expect that the finite time blowup happens to the term $\|\nabla u\|$, thus we need show that the term $\|xu\|$ must hit the zero line as time t approaches to a finite time $T < \infty$. To prove this, a convenient way is to verify that $J(t) = \int |x|^2|u|^2 dx$ is convex, i.e., $J''(t) < 0$ for $J(u_0) > 0$ and $0 < E(u_0) < d$ as [16, 20, 21, 22]. For the Cauchy problem (2.1) of the second-order semilinear Schrödinger equation for $0 < \mathbf{E}(u_0) < d$, we have

$$\mathbf{I}(u) = \int_{\mathbb{R}^N} \Big(|u|^2 + |\nabla u|^2 - \frac{Np}{2(p+2)}|u|^{p+2}\Big)dx$$

and

$$\mathbf{J}''(t) = 8\int_{\mathbb{R}^N} \Big(|\nabla u|^2 - \frac{Np}{2(p+2)}|u|^{p+2}\Big)dx.$$

Obviously, $\mathbf{J}''(t)$ has a very similar structure with the Nehari functional $\mathbf{I}(u)$, hence $\mathbf{I}(u) < 0$ can easily yield $\mathbf{J}''(t) < 0$ to prove the blowup of solution. But for the Cauchy problem (1.1) of the fourth-order semilinear Schrödinger equation, we do not have such luck. We shall derive in the main part of this paper that the corresponding $J''(t)$ for the fourth-order semilinear Schrödinger equation is

$$\begin{aligned} J''(t) =& 8\Big(4\int_{\mathbb{R}^N} |\nabla(\Delta u)|^2 dx + 4\int_{\mathbb{R}^N} |\Delta u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx\Big) \\ &+ 4\Big(-\frac{Np}{p+2}\int_{\mathbb{R}^N} |u|^{p+2}dx + (2N+4)\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u\Delta\bar{u}dx \\ &+ 4\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u x\cdot\nabla(\Delta\bar{u})dx\Big) \end{aligned}$$

by comparing it with

$$I(u) = \int_{\mathbb{R}^N} \Big(|u|^2 + |\nabla u|^2 + |\Delta u|^2 - \frac{Np}{2(p+2)}|u|^{p+2}\Big)dx.$$

As the structure of this $J''(t)$ is complex and very different from that of $I(u)$, up to now, we do not know how to derive $J''(t) < 0$ from $I(u) < 0$, so we have to leave the finite time blowup part of the possible sharp condition open. In this paper, we first prove the global existence part of the sharp condition for problem (1.1) with $0 < E(u_0) < d$. Then by a rather long but totally explicit computation for $J''(t)$, we analyze the difficulties of proving the blowup part of the sharp condition for problem (1.1) with $0 < E(u_0) < d$. As Cazenave [5] (Chapter 6.5) indicated, the calculation based on the variance of $\mathbf{J}(t) = \int |x|^2|u|^2 dx$ for the second-order semilinear Schrödinger equation is technically complicated, to calculate $J''(t)$ for the fourth-order semilinear Schrödinger equation is a much more difficult work. Unfortunately, there has been a work [15, Theorem 3.3] incorrectly using the $\mathbf{J}''(t)$ for the second-order semilinear Schrödinger equation as the $J''(t)$ for the fourth-order semilinear Schrödinger equation. Therefore, the calculation of $J''(t)$ will be a tough work.

The rest of this paper is organized as follows. In Section 2, the established results for the second-order semilinear Schrödinger equation is revisited to compare with the fourth order case. We also calculate $\mathbf{J}''(t)$ for the second-order semilinear

Schrödinger equation in detail to illustrate the relation between $\mathbf{I}(u)$ and $\mathbf{J}''(t)$ and show the detailed steps for the calculations of $J''(t)$ for the fourth order case. In Section 3, we first verify the energy and mass conservation laws for the fourth-order semilinear Schrödinger equation through accurate calculations. Meanwhile, the global existence of solution for the fourth-order semilinear Schrödinger equation is proved in the case of $0 < E(u_0) < d$ and $I(u_0) > 0$. Then the attentions quickly move to the analysis of the failure of proving the finite time blowup of the solution for the fourth order case in the frame of variational method by calculating $J''(t)$ and corresponding analysis.

2. Nonlinear second-order Schrödinger equation

In this section, we consider the Cauchy problem

$$\begin{gathered} iu_t + \Delta u = -|u|^p u, \quad (x,t) \in \mathbb{R}^N \times [0,T), \\ u(0,x) = u_0(x), \end{gathered} \tag{2.1}$$

where $i = \sqrt{-1}$, $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^N$, $u(x,t) : \mathbb{R}^N \times [0,T) \to \mathbb{C}$ denotes the complex valued function, T is the maximum existence time, N is the space dimension and p satisfies the embedding condition

$$\frac{4}{N} < p < \begin{cases} +\infty, & N = 1,2 \\ \frac{4}{N-2}, & N > 2. \end{cases}$$

Consider the Cauchy problem (2.1), we define the energy functional

$$\mathbf{E}(u) = \int_{\mathbb{R}^N} \Big(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+2}|u|^{p+2}\Big)dx$$

and the auxiliary functionals

$$\mathbf{P}(u) = \int_{\mathbb{R}^N} \Big(\frac{1}{2}|u|^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{p+2}|u|^{p+2}\Big)dx,$$

and

$$\mathbf{I}(u) = \int_{\mathbb{R}^N} \Big(|u|^2 + |\nabla u|^2 - \frac{Np}{2(p+2)}|u|^{p+2}\Big)dx.$$

For the above two functionals, $\mathbf{P}(u)$ is composed of both mass and energy, and $\mathbf{I}(u)$ can be considered as Nehari functional. Further we define the Hilbert space

$$\mathbf{H} = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^2|u|^2 dx < \infty\},$$

the Nehari manifold

$$\mathbf{M} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : \mathbf{I}(u) = 0\},$$

and the invariant manifolds

$$\mathbf{G} = \{u \in \mathbf{H} : \mathbf{P}(u) < \mathbf{d}, \mathbf{I}(u) > 0\} \cup \{0\}$$

and

$$\mathbf{B} = \{u \in \mathbf{H} : \mathbf{P}(u) < \mathbf{d}, \mathbf{I}(u) < 0\},$$

where

$$\mathbf{d} = \inf_{u \in \mathbf{M}} \mathbf{P}(u).$$

Remark 2.1. (i) For set $\mathbf{G}$, we can obtain $\mathbf{P}(u) > 0$ by $\mathbf{I}(u) > 0$. So the set $\mathbf{G}$ is equivalent to

$$\mathbf{G}' = \{u \in \mathbf{H} | 0 < \mathbf{P}(u) < \mathbf{d}, \mathbf{I}(u) > 0\} \cup \{0\}.$$

(ii). For set $\mathbf{B}$, if $\mathbf{P}(u) \leq 0$, we can get $\mathbf{E}(u) < 0$, which is a sufficient condition for finite time blowup, cf. [22]. Therefore, it is only necessary here to consider the case of $\mathbf{E}(u) > 0$, i.e., we only need

$$\mathbf{B}' = \{u \in \mathbf{H} | 0 < \mathbf{P}(u) < \mathbf{d}, \mathbf{I}(u) < 0\}.$$

The above remark is also applicable to sets G and B in Section 3.

For the Cauchy problem (2.1) of the second-order semilinear Schrödinger equation, we summarize some results established in [5, 16, 18, 19, 21, 22] as follows.

Theorem 2.2. *Assume that $u_0 \in \mathbf{H}$ and p satisfies the embedding condition*

$$\frac{4}{N} < p < \begin{cases} +\infty, & N = 1, 2 \\ \frac{4}{N-2}, & N > 2. \end{cases}$$

(i) *(Local existence [5, 19]) There exists $T > 0$ and a unique local solution $u(x,t)$ of problem (2.1) in $C([0, T_{\max}]; \mathbf{H})$. Moreover if*

$$T_{\max} = \sup\{T > 0 : u = u(x,t) \text{ exists on } [0,T]\} < \infty$$

then

$$\lim_{t \to T_{\max}} \|u\|_{\mathbf{H}} = \infty \quad \text{(blowup)},$$

otherwise $T_{\max} = \infty$ (global existence).

(ii) *(Conservation laws [5, 18]) For the solution in (i),*

$$\int_{\mathbb{R}^N} |u(t)|^2 = \int_{\mathbb{R}^N} |u_0|^2 dx \quad \text{(mass conservation)},$$
$$\mathbf{E}(u(t)) = \mathbf{E}(u_0) \quad \text{(energy conservation)},$$
$$\mathbf{P}(u(t)) \equiv \mathbf{P}(u_0).$$

(iii) $\mathbf{d} > 0$, *cf.* [16, 21].

(iv) *(Global existence [22]) If $u_0 \in \mathbf{G}$, then the solution of problem (2.1) is global.*

(v) *(Blowup [22]) If $u_0 \in \mathbf{B}$, then the solution of problem (2.1) blows up in finite time.*

In fact, although [22] proved the blowup solution by a variance of the argument in [18], there is no explicit computation of $\mathbf{J}''(t)$. Now we give the specific computation of $\mathbf{J}''(t)$ for the Cauchy problem (2.1).

Theorem 2.3. *Assume that $u_0 \in \mathbf{B}$, $u(x,t) \in ([0,T); \mathbf{H})$ is the solution of (2.1). Let $\mathbf{J}(t) = \int_{\mathbb{R}^N} |x|^2 |u|^2 dx$, then*

$$\mathbf{J}''(t) = 8 \int_{\mathbb{R}^N} \Big(|\nabla u|^2 - \frac{Np}{2(p+2)} |u|^{p+2} \Big) dx.$$

Proof. First

$$\mathbf{J}'(t) = \int_{\mathbb{R}^N} |x|^2 \left(u\bar{u}_t + \bar{u}u_t \right) dx = \int_{\mathbb{R}^N} |x|^2 \left(\overline{\bar{u}u_t} + \bar{u}u_t \right) dx = 2\,\mathrm{Re} \int_{\mathbb{R}^N} |x|^2 \bar{u}u_t \, dx. \tag{2.2}$$

Multiplying both sides of (2.1) by i, we have

$$u_t = i\left(\Delta u + |u|^p u\right).$$

Substituting the above equation into (2.2) we have

$$\begin{aligned}
\mathbf{J}'(t) &= 2\operatorname{Re}\int_{\mathbb{R}^N} i|x|^2\bar{u}\left(\Delta u + |u|^p u\right)dx\\
&= -2\operatorname{Im}\int_{\mathbb{R}^N} |x|^2\bar{u}\left(\Delta u + |u|^p u\right)dx\\
&= -2\operatorname{Im}\int_{\mathbb{R}^N} |x|^2\left(\bar{u}\Delta u + |u|^{p+2}\right)dx\\
&= -2\operatorname{Im}\int_{\mathbb{R}^N} |x|^2\bar{u}\Delta u dx\\
&= 2\operatorname{Im}\int_{\mathbb{R}^N} |x|^2 u\Delta\bar{u} dx,
\end{aligned}$$

further

$$\begin{aligned}
\mathbf{J}''(t) &= 2\operatorname{Im}\int_{\mathbb{R}^N} |x|^2\left(u_t\Delta\bar{u} + u\Delta\bar{u}_t\right)dx\\
&= 2\operatorname{Im}\int_{\mathbb{R}^N} \left(|x|^2 u_t\Delta\bar{u} + \Delta(|x|^2 u)\bar{u}_t\right)dx\\
&= 2\operatorname{Im}\int_{\mathbb{R}^N} \Big(|x|^2 u_t\Delta\bar{u} + \bar{u}_t\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}(|x|^2 u)\Big)dx\\
&= 2\operatorname{Im}\int_{\mathbb{R}^N} \Big(|x|^2 u_t\Delta\bar{u} + \bar{u}_t\sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(|x|^2\frac{\partial u}{\partial x_i} + 2x_i u\Big)\Big)dx\\
&= 2\operatorname{Im}\int_{\mathbb{R}^N} \Big(|x|^2 u_t\Delta\bar{u} + \bar{u}_t\Big(2Nu + 4\sum_{i=1}^N x_i\cdot\frac{\partial u}{\partial x_i} + |x|^2\sum_{i=1}^N\frac{\partial^2 u}{\partial x_i^2}\Big)\Big)dx\\
&= 2\operatorname{Im}\int_{\mathbb{R}^N} \left(|x|^2 u_t\Delta\bar{u} + \bar{u}_t\left(2Nu + 4x\cdot\nabla u + |x|^2\Delta u\right)\right)dx\\
&= 2\operatorname{Im}\int_{\mathbb{R}^N} \Big(|x|^2 u_t\Delta\bar{u} + \overline{|x|^2 u_t\Delta\bar{u}} + \bar{u}_t\left(2Nu + 4x\cdot\nabla u\right)\Big)dx\\
&= 4\operatorname{Im}\int_{\mathbb{R}^N} \left(Nu + 2x\cdot\nabla u\right)\bar{u}_t\,dx.
\end{aligned}\tag{2.3}$$

According to (2.1), we obtain

$$\bar{u}_t = -i(\Delta\bar{u} + |u|^p\bar{u}). \tag{2.4}$$

Substituting the above equation into (2.3), we can get

$$\begin{aligned}
\mathbf{J}''(t) &= -4\operatorname{Im}\int_{\mathbb{R}^N} i\,(Nu+2x\cdot\nabla u)\,(\Delta\bar{u}+|u|^p\bar{u})\,dx \\
&= -4\operatorname{Re}\int_{\mathbb{R}^N} (Nu+2x\cdot\nabla u)(\Delta\bar{u}+|u|^p\bar{u})\,dx \\
&= -4\Big(\operatorname{Re}\int_{\mathbb{R}^N} (Nu+2x\cdot\nabla u)\,\Delta\bar{u}dx+\operatorname{Re}\int_{\mathbb{R}^N} (Nu+2x\cdot\nabla u)\,|u|^p\bar{u}dx\Big) \\
&= -4(\mathbf{I_1}+\mathbf{I_2}),
\end{aligned} \tag{2.5}$$

where

$$\mathbf{I}_1 := \operatorname{Re}\int_{\mathbb{R}^N} (Nu+2x\cdot\nabla u)\,\Delta\bar{u}dx \quad\text{and}\quad \mathbf{I}_2 := \operatorname{Re}\int_{\mathbb{R}^N} (Nu+2x\cdot\nabla u)\,|u|^p\bar{u}dx.$$

For $\mathbf{I}_1$ and $\mathbf{I}_2$, we have

$$\begin{aligned}
\mathbf{I}_1 =& N\operatorname{Re}\int_{\mathbb{R}^N} u\Delta\bar{u}dx+2\operatorname{Re}\int_{\mathbb{R}^N}(x\cdot\nabla u)\Delta\bar{u}dx \\
=& -N\int_{\mathbb{R}^N}|\nabla u|^2dx-2\operatorname{Re}\int_{\mathbb{R}^N}\nabla(x\cdot\nabla u)\nabla\bar{u}dx \\
=& -N\int_{\mathbb{R}^N}|\nabla u|^2dx-2\operatorname{Re}\int_{\mathbb{R}^N}\sum_{i=1}^N\frac{\partial}{\partial x_i}\Big(\sum_{j=1}^N x_j\frac{\partial u}{\partial x_j}\Big)\frac{\partial\bar{u}}{\partial x_i}dx \\
=& -N\int_{\mathbb{R}^N}|\nabla u|^2dx-2\operatorname{Re}\int_{\mathbb{R}^N}\sum_{i=1}^N\sum_{j=1}^N\frac{\partial}{\partial x_i}\left(x_j\frac{\partial u}{\partial x_j}\right)\frac{\partial\bar{u}}{\partial x_i}dx \\
=& -N\int_{\mathbb{R}^N}|\nabla u|^2dx-2\operatorname{Re}\int_{\mathbb{R}^N}\sum_{i=1}^N\frac{\partial u}{\partial x_i}\frac{\partial\bar{u}}{\partial x_i}dx \\
&-2\operatorname{Re}\int_{\mathbb{R}^N}\sum_{i=1}^N\sum_{j=1}^N x_j\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial\bar{u}}{\partial x_i}dx \\
=& -N\int_{\mathbb{R}^N}|\nabla u|^2dx-2\int_{\mathbb{R}^N}|\nabla u|^2dx \\
&-\operatorname{Re}\int_{\mathbb{R}^N}\sum_{i=1}^N\sum_{j=1}^N x_j\Big(\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial\bar{u}}{\partial x_i}+\frac{\partial^2\bar{u}}{\partial x_i\partial x_j}\frac{\partial u}{\partial x_i}\Big)dx \\
=& -N\int_{\mathbb{R}^N}|\nabla u|^2dx-2\int_{\mathbb{R}^N}|\nabla u|^2dx-\operatorname{Re}\int_{\mathbb{R}^N}\sum_{i=1}^N\sum_{j=1}^N x_j\frac{\partial}{\partial x_j}\Big(\frac{\partial u}{\partial x_i}\frac{\partial\bar{u}}{\partial x_i}\Big)dx \\
=& -N\int_{\mathbb{R}^N}|\nabla u|^2dx-2\int_{\mathbb{R}^N}|\nabla u|^2dx-\operatorname{Re}\int_{\mathbb{R}^N}x\cdot\nabla|\nabla u|^2dx \\
=& -N\int_{\mathbb{R}^N}|\nabla u|^2dx-2\int_{\mathbb{R}^N}|\nabla u|^2dx+N\int_{\mathbb{R}^N}|\nabla u|^2dx \\
=& -2\int_{\mathbb{R}^N}|\nabla u|^2dx
\end{aligned} \tag{2.6}$$

and

$$
\begin{aligned}
\mathbf{I}_2 =& N\int_{\mathbb{R}^N}|u|^{p+2}dx+\operatorname{Re}\int_{\mathbb{R}^N}x\cdot\left(|u|^p\left(\bar{u}\nabla u+u\nabla\bar{u}\right)\right)dx\\
=&N\int_{\mathbb{R}^N}|u|^{p+2}dx+\operatorname{Re}\int_{\mathbb{R}^N}x\cdot\left(|u|^p\nabla\left(u\bar{u}\right)\right)dx\\
=&N\int_{\mathbb{R}^N}|u|^{p+2}dx+\operatorname{Re}\int_{\mathbb{R}^N}x\cdot\Big((|u|^2)^{p/2}\nabla|u|^2\Big)dx\\
=&N\int_{\mathbb{R}^N}|u|^{p+2}dx+\frac{2}{p+2}\operatorname{Re}\int_{\mathbb{R}^N}x\cdot\nabla(|u|^2)^{\frac{p+2}{2}}dx\\
=&N\int_{\mathbb{R}^N}|u|^{p+2}dx-\frac{2N}{p+2}\operatorname{Re}\int_{\mathbb{R}^N}|u|^{p+2}dx\\
=&\frac{Np}{p+2}\operatorname{Re}\int_{\mathbb{R}^N}|u|^{p+2}dx.
\end{aligned}
$$

Substituting the above equation and (2.6) into (2.5), finally we derive

$$
\begin{aligned}
\mathbf{J}''(t) =& -4\Big(-2\int_{\mathbb{R}^N}|\nabla u|^2dx+\frac{Np}{p+2}\operatorname{Re}\int_{\mathbb{R}^N}|u|^{p+2}dx\Big)\\
=&8\int_{\mathbb{R}^N}\Big(|\nabla u|^2-\frac{Np}{2(p+2)}|u|^{p+2}\Big)dx.
\end{aligned}
$$

Then the proof is complete. □

Remark 2.4. From the above structure of $\mathbf{J}''(t)$ and $\mathbf{I}(u)$, we easily judge that $\mathbf{J}''(t)<0$ in the case of $\mathbf{I}(u)<0$, further the blowup of solution for the second-order semilinear Schrödinger equation is derived.

3. Nonlinear fourth-order Schrödinger equation

In this section, we consider the nonlinear fourth-order Schrödinger equation for the Cauchy problem (1.1). First we define the space

$$
H^2=\big\{u\in H^2(\mathbb{R}^N):\int_{\mathbb{R}^N}|x|^2|u|^2dx<\infty\big\}, \tag{3.1}
$$

the energy functional

$$
E(u(t))=\int_{\mathbb{R}^N}\Big(\frac{1}{2}|\nabla u|^2+\frac{1}{2}|\Delta u|^2-\frac{1}{p+2}|u|^{p+2}\Big)dx \tag{3.2}
$$

and the auxiliary functionals

$$
P(u)=\int_{\mathbb{R}^N}\Big(\frac{1}{2}|u|^2+\frac{1}{2}|\nabla u|^2+\frac{1}{2}|\Delta u|^2-\frac{1}{p+2}|u|^{p+2}\Big)dx,
$$

and

$$
I(u)=\int_{\mathbb{R}^N}\Big(|u|^2+|\nabla u|^2+|\Delta u|^2-\frac{Np}{2(p+2)}|u|^{p+2}\Big)dx,
$$

where $P(u)$ is composed of both mass and energy, and $I(u)$ is considered as a Nehari functional. The Nehari manifold is

$$
M=\{u\in H^2\setminus\{0\}:I(u)=0\}.
$$

Then we introduce the stable set G and unstable set B:

$$
G=\{u\in H^2|P(u)<d,I(u)>0\}\cup\{0\}
$$

and

$$B = \{u \in H^2 | P(u) < d, I(u) < 0\},$$

where $d = \inf_{u\in M} P(u)$.

Now we state the local existence theory of solution for the Cauchy problem (1.1).

Lemma 3.1 (Local existence [10]). *Let $u_0 \in H^2$, there exists a value $T > 0$ and a unique local solution $u(x,t)$ of the problem (1.1) in $C([0,T]; H^2)$. Moreover if*

$$T_{\max} = \sup\{ T > 0 : u = u(x,t) \text{ exists on } [0, T]\} < \infty$$

then

$$\lim_{t\to T_{\max}} \|u\|_{H^2} = \infty \quad \text{(blowup)},$$

otherwise $T = \infty$ (global existence).

Next we consider the conservation laws for the fourth order case.

Lemma 3.2 (Conservation laws). *Let $u_0 \in H^2$ and $u \in ([0,T); H^2)$ be the unique solution of problem (1.1), then*

$$\int_{\mathbb{R}^N} |u(t)|^2 = \int_{\mathbb{R}^N} |u_0|^2 dx \quad \text{(mass conservation)}, \tag{3.3}$$

$$E(u(t)) = E(u_0) \quad \text{(energy conservation)}, \tag{3.4}$$

$$P(u(t)) \equiv P(u_0). \tag{3.5}$$

Proof. From the definition of the energy functional $E(u(t))$ and the auxiliary functional $P(u(t))$, we have

$$\begin{aligned} \frac{d}{dt}\Big(\int_{\mathbb{R}^N} |u|^2 dx\Big) &= \frac{d}{dt}\Big(\int_{\mathbb{R}^N} u\bar{u} dx\Big) \\ &= \int_{\mathbb{R}^N} (u\bar{u}_t + u_t\bar{u})\, dx \\ &= \int_{\mathbb{R}^N} (\overline{u_t\bar{u}} + u_t\bar{u})\, dx \\ &= 2\mathrm{Re}\int_{\mathbb{R}^N} \bar{u}u_t\, dx. \end{aligned} \tag{3.6}$$

Multiplying both sides of (1.1) by $\bar{u}$, we derive

$$\bar{u}u_t = i(\bar{u}\Delta u - \bar{u}\Delta^2 u + |u|^{p+2}). \tag{3.7}$$

Substituting (3.7) into (3.6) we have

$$\begin{aligned} \frac{d}{dt}\left(\int_{\mathbb{R}^N} |u|^2 dx\right) &= 2\,\mathrm{Re}\int_{\mathbb{R}^N} i\left(\bar{u}\Delta u - \bar{u}\Delta^2 u + |u|^{p+2}\right) dx \\ &= -\,2\,\mathrm{Im}\int_{\mathbb{R}^N} \left(\bar{u}\Delta u - \bar{u}\Delta^2 u + |u|^{p+2}\right) dx \\ &= 2\,\mathrm{Im}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\Delta u|^2 - |u|^{p+2}\right) dx = 0, \end{aligned}$$

so (3.3) holds.

Then we prove the energy conservation as follows

$$\begin{aligned}
\frac{d}{dt}(E(u(t))) =&\frac{d}{dt}\Big(\int_{\mathbb{R}^N}\Big(\frac{1}{2}|\nabla u|^2+\frac{1}{2}|\Delta u|^2-\frac{1}{p+2}|u|^{p+2}\Big)dx\Big)\\
=&\frac{d}{dt}\Big(\int_{\mathbb{R}^N}\Big(\frac{1}{2}\nabla u\cdot\nabla\bar u+\frac{1}{2}\Delta u\Delta\bar u-\frac{1}{p+2}(u\bar u)^{\frac{p+2}{2}}\Big)dx\Big)\\
=&\int_{\mathbb{R}^N}\Big(\frac{1}{2}\left(\nabla u_t\cdot\nabla\bar u+\nabla u\cdot\nabla\bar u_t\right)+\frac{1}{2}\left(\Delta u_t\Delta\bar u+\Delta u\Delta\bar u_t\right)\\
&-\frac{1}{2}(u\bar u)^{p/2}\left(u_t\bar u+u\bar u_t\right)\Big)dx\\
=&\int_{\mathbb{R}^N}\Big(\frac{1}{2}\left(\nabla u_t\cdot\nabla\bar u+\overline{\nabla\bar u\cdot\nabla u_t}\right)+\frac{1}{2}\left(\Delta u_t\Delta\bar u+\overline{\Delta\bar u\Delta u_t}\right)\\
&-\frac{1}{2}(u\bar u)^{p/2}\left(u_t\bar u+\overline{u\bar u_t}\right)\Big)dx\\
=&\operatorname{Re}\int_{\mathbb{R}^N}\Big((\nabla u_t\cdot\nabla\bar u)+(\Delta u_t\Delta\bar u)-(u\bar u)^{p/2}(u_t\bar u)\Big)dx\\
=&-\operatorname{Re}\int_{\mathbb{R}^N}\left(u_t\Delta\bar u-u_t\Delta^2\bar u+|u|^p\bar u u_t\right)dx\\
=&-\operatorname{Re}\int_{\mathbb{R}^N}u_t\left(\Delta\bar u-\Delta^2\bar u+|u|^p\bar u\right)dx.
\end{aligned}\tag{3.8}$$

Multiplying both sides of (1.1) by $\bar u_t$, we obtain

$$i|u_t|^2=-\bar u_t\left(\Delta u-\Delta^2u+|u|^pu\right).$$

Then substituting the above equation into (3.8) gives

$$\frac{d}{dt}(E(u(t)))=\operatorname{Re}\int_{\mathbb{R}^N}i|u_t|^2\,dx=-\operatorname{Im}\int_{\mathbb{R}^N}|u_t|^2\,dx=0,$$

thus (3.4) holds. Combining (3.3) and (3.4), we obtain (3.5). □

As shown in [4], the negative initial energy ($E(u_0)<0$) is currently the sufficient condition for blowup of the Cauchy problem (1.1), i.e., in this case, it is impossible to divide the initial condition to obtain the sharp condition of global existence and blowup in the frame of the variational method. Therefore, we only consider the case of $0<E(u_0)<d$ and try to build a similar result to the second-order semilinear Schrödinger equation. First we need to verify $d>0$.

Lemma 3.3. *The depth of the potential well is positive, i.e., $d>0$.*

Proof. When $u\in M$, according to Sobolev embedding inequalities, we have

$$\begin{aligned}
\int_{\mathbb{R}^N}\left(|\nabla u|^2+|u|^2\right)dx\le&\int_{\mathbb{R}^N}\left(|\nabla u|^2+|u|^2+|\Delta u|^2\right)dx\\
=&\frac{Np}{2(p+2)}\int_{\mathbb{R}^N}|u|^{p+2}dx\\
\le&\frac{Np}{2(p+2)}\Big(\int_{\mathbb{R}^N}c\left(|\nabla u|^2+|u|^2\right)dx\Big)^{\frac{p+2}{2}},
\end{aligned}$$

where c denotes positive Sobolev embedding constant. Let C be a positive constant that may vary from line to line. If $C = \big(\frac{2(p+2)}{Np}\big)^{2/p} c^{-\frac{p+2}{p}}$, then

$$0 < C \le \int_{\mathbb{R}^N} \big(|\nabla u|^2 + |u|^2\big)\,dx. \tag{3.9}$$

From the definition of $P(u)$ and (3.9), we have

$$\begin{aligned}
P(u) &= \int_{\mathbb{R}^N} \Big(\frac{1}{2}|u|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\Delta u|^2 - \frac{1}{p+2}|u|^{p+2}\Big)dx \\
&= \Big(\frac{1}{2} - \frac{1}{p+2}\cdot\frac{2(p+2)}{Np}\Big)\int_{\mathbb{R}^N} \big(|u|^2 + |\nabla u|^2 + |\Delta u|^2\big)\,dx \\
&\quad + \frac{1}{p+2}\cdot\frac{2(p+2)}{Np}\int_{\mathbb{R}^N} \Big(|u|^2 + |\nabla u|^2 + |\Delta u|^2 - \frac{Np}{2(p+2)}|u|^{p+2}\Big)dx \\
&= \frac{Np-2}{2Np}\int_{\mathbb{R}^N} \big(|u|^2 + |\nabla u|^2 + |\Delta u|^2\big)\,dx \\
&\ge \frac{Np-2}{2Np}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + |u|^2\big)\,dx \\
&\ge C > 0.
\end{aligned}$$

Then finally we obtain $d \ge C > 0$. □

Next we construct the invariant sets G and B under the flow generated by the Cauchy problem of the fourth-order semilinear Schrödinger equation. With these invariant sets, we desire to establish a criterion for global existence and blowup of solution to the Cauchy problem (1.1) similar to the case of second-order semilinear Schrödinger equation. Here the invariant set G is used to describe the initial data leading to the global solution, and the invariant set B is expected to be the manifold for the initial data yielding the blowup solution although in this paper we fail to prove it, and our aim is to explain the reason of this failure in detail. Hence the introduction of B and its invariance are important in this sense.

Theorem 3.4. *The sets G and B are invariant manifolds.*

Proof. We only prove that G is invariant; the proof for B is similar. Suppose $u_0 \in G$, we claim that $u(t) \in G$ for every $t \in (0, T)$.

(i) When $u_0 = 0$, according to the mass conservation law (3.3), we know that $u(x,t) = 0$ for any $t \in [0, T)$. Namely, $u(t) \equiv 0$ is the trivial solution of the problem (1.1), that is $u(t) \in G$ for $t \in (0, T)$.

(ii) When $u_0 \ne 0$, according to (3.5) we have

$$P(u(t)) \equiv P(u_0) < d \quad \text{for } t \in (0, T). \tag{3.10}$$

Arguing by contradiction, we assume that there exists a first time $t_1 \in (0, T)$, such that $I(u(t_1)) = 0$, and $I(u(t)) > 0$ for any $t \in (0, t_1)$. Obviously, $u(t_1) \ne 0$. In fact, if $u(t_1) = 0$, according to mass conservation law (3.3), we have $u_0 = 0$, which contradicts to $u_0 \ne 0$. Hence from the definition of d, we have

$$P(u(t_1)) \ge d,$$

which contradicts (3.10), and means $u(x,t) \in G$ for any $t \in (0, T)$. □

Theorem 3.5. *If $u_0 \in G$, the solution $u(x,t)$ of the initial value problem* (1.1) *is global, i.e., the maximum existence time $T = \infty$.*

Proof. When $u_0 \in G$, according to Theorem 3.4, for any $t \in [0,T)$, we have $u(x,t) \in G$, hence

$$\begin{aligned} d > P(u) =& \int_{\mathbb{R}^N} \Big(\frac{1}{2}|u|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\Delta u|^2 - \frac{1}{p+2}|u|^{p+2}\Big) dx \\ =&\Big(\frac{1}{2} - \frac{1}{p+2}\cdot\frac{2(p+2)}{Np}\Big) \int_{\mathbb{R}^N} \big(|u|^2 + |\nabla u|^2 + |\Delta u|^2\big)\, dx \\ &+ \frac{1}{p+2}\cdot\frac{2(p+2)}{Np}\int_{\mathbb{R}^N} \Big(|u|^2 + |\nabla u|^2 + |\Delta u|^2 - \frac{Np}{2(p+2)}|u|^{p+2}\Big)dx \\ \geq& \frac{Np-2}{2Np}\int_{\mathbb{R}^N} \big(|u|^2 + |\nabla u|^2 + |\Delta u|^2\big)\, dx, \end{aligned}$$

i.e.,

$$\int_{\mathbb{R}^N} \big(|\nabla u|^2 + |u|^2 + |\Delta u|^2\big)\, dx \leq \frac{2dNp}{Np-2}.$$

Then according to Lemma 3.1, the existence time of a local solution of the initial value problem (1.1) can be extended to infinity, thus the solution of the problem (1.1) is global. □

Indeed here we are supposed to give the finite time blowup theorem of problem (1.1) for $0 < E(u_0) < d$, but we have to concede that it is an impossible task in this paper. However, we do not want to stop in this way, instead, we like to analyze the reason. As the fourth-order semilinear Schrödinger equation has a different structure from that the second-order semilinear Schrödinger equation has, we can not directly employ the result of $\mathbf{J}''(t)$ for the second-order semilinear Schrödinger equation to the fourth order nonlinear Schrödinger equation. Next we shall show that the $J''(t)$ for the fourth-order nonlinear Schrödinger equation is very different from the $\mathbf{J}''(t)$ for the second-order semilinear Schrödinger equation, by giving the detailed computation of $J''(t)$. Remark 3.7 points out the main difficulties of proving the finite time blowup of fourth-order semilinear Schrödinger equation in frame of potential well theory, also the wrong proof in [15].

First we present the computation of $J''(t)$ related to fourth-order Schrödinger equation.

Theorem 3.6. *Assume that $u_0 \in B$ and $u \in C^2([0,T);H^2)$ is the solution of Problem (1.1). Let $J(t) = \int_{\mathbb{R}^N} |x|^2|u|^2dx$, then*

$$\begin{aligned} J''(t) =& 8\Big(4\int_{\mathbb{R}^N} |\nabla(\Delta u)|^2 dx + 4\int_{\mathbb{R}^N} |\Delta u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx\Big) \\ &+ 4\Big(-\frac{Np}{p+2}\int_{\mathbb{R}^N} |u|^{p+2}dx + (2N+4)\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u\Delta\bar{u}dx \\ &+ 4\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u x\cdot\nabla(\Delta\bar{u})dx\Big). \end{aligned}$$

Proof. First

$$\begin{aligned} J'(t) &= \int_{\mathbb{R}^N} |x|^2 \left(u\bar{u}_t + \bar{u}u_t\right) dx \\ &= \int_{\mathbb{R}^N} |x|^2 \left(\overline{\bar{u}u_t} + \bar{u}u_t\right) dx \\ &= 2\operatorname{Re}\int_{\mathbb{R}^N} |x|^2\bar{u}u_t\, dx. \end{aligned} \tag{3.11}$$

Multiplying both sides of (1.1) by i, we have

$$u_t = i\left(\Delta u - \Delta^2 u + |u|^p u\right). \tag{3.12}$$

Substituting the above equation into (3.11), we obtain

$$\begin{aligned}
J'(t) =& 2\operatorname{Re}\int_{\mathbb{R}^N} i|x|^2\bar{u}\left(\Delta u - \Delta^2 u + |u|^p u\right) dx\\
=& -2\operatorname{Im}\int_{\mathbb{R}^N} |x|^2\bar{u}\left(\Delta u - \Delta^2 u + |u|^p u\right) dx\\
=& -2\operatorname{Im}\int_{\mathbb{R}^N} |x|^2\left(\bar{u}\Delta u - \bar{u}\Delta^2 u + |u|^{p+2}\right) dx\\
=& -2\operatorname{Im}\int_{\mathbb{R}^N} |x|^2\left(\bar{u}\Delta u - \bar{u}\Delta^2 u\right) dx.
\end{aligned}$$

Differentiating the above equation with respect to t, we obtain

$$\begin{aligned}
J''(t) =& -2\operatorname{Im}\int_{\mathbb{R}^N} |x|^2\left(\bar{u}_t\Delta u + \bar{u}\Delta u_t - \bar{u}_t\Delta^2 u - \bar{u}\Delta^2 u_t\right) dx\\
=& -2\operatorname{Im}\int_{\mathbb{R}^N} |x|^2\left(\bar{u}_t\Delta u + \bar{u}\Delta u_t\right) dx\\
&+ 2\operatorname{Im}\int_{\mathbb{R}^N} |x|^2\left(\bar{u}_t\Delta^2 u + \bar{u}\Delta^2 u_t\right) dx\\
=& -2K_1 + 2K_2,
\end{aligned} \tag{3.13}$$

where

$$K_1 := \operatorname{Im}\int_{\mathbb{R}^N} |x|^2\left(\bar{u}_t\Delta u + \bar{u}\Delta u_t\right) dx, \quad K_2 := \operatorname{Im}\int_{\mathbb{R}^N} |x|^2\left(\bar{u}_t\Delta^2 u + \bar{u}\Delta^2 u_t\right) dx.$$

Further we have

$$\begin{aligned}
K_1 =& \operatorname{Im}\int_{\mathbb{R}^N} \left(|x|^2\bar{u}_t\Delta u + \Delta\left(|x|^2\bar{u}\right)u_t\right) dx\\
=& \operatorname{Im}\int_{\mathbb{R}^N} \Big(|x|^2\bar{u}_t\Delta u + u_t\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}\Big(|x|^2\bar{u}\Big)\Big)dx\\
=& \operatorname{Im}\int_{\mathbb{R}^N} \Big(|x|^2\bar{u}_t\Delta u + u_t\sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(|x|^2\frac{\partial\bar{u}}{\partial x_i} + 2x_i\bar{u}\Big)\Big)dx\\
=& \operatorname{Im}\int_{\mathbb{R}^N} \Big(|x|^2\bar{u}_t\Delta u + u_t\Big(2N\bar{u} + 4\sum_{i=1}^N x_i\cdot\frac{\partial\bar{u}}{\partial x_i} + |x|^2\sum_{i=1}^N\frac{\partial^2\bar{u}}{\partial x_i^2}\Big)\Big)dx\\
=& \operatorname{Im}\int_{\mathbb{R}^N} \left(|x|^2\bar{u}_t\Delta u + u_t\left(2N\bar{u} + 4x\cdot\nabla\bar{u} + |x|^2\Delta\bar{u}\right)\right) dx\\
=& \operatorname{Im}\int_{\mathbb{R}^N} \Big(|x|^2\bar{u}_t\Delta u + \overline{|x|^2\bar{u}_t\Delta u} + u_t(2N\bar{u} + 4x\cdot\nabla\bar{u})\Big) dx\\
=& 2\operatorname{Im}\int_{\mathbb{R}^N} u_t\left(N\bar{u} + 2x\cdot\nabla\bar{u}\right) dx
\end{aligned} \tag{3.14}$$

and

$$K_2 = \operatorname{Im}\int_{\mathbb{R}^N} \left(|x|^2\bar{u}_t\Delta^2 u + \Delta\left(|x|^2\bar{u}\right)\Delta u_t\right) dx$$

$$
\begin{aligned}
&= \operatorname{Im} \int_{\mathbb{R}^N} \Big(|x|^2 \bar{u}_t \Delta^2 u + \Delta u_t \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} \big(|x|^2 \bar{u} \big) \Big) dx \\
&= \operatorname{Im} \int_{\mathbb{R}^N} \Big(|x|^2 \bar{u}_t \Delta^2 u + \Delta u_t \sum_{i=1}^N \frac{\partial}{\partial x_i} \Big(2x_i \bar{u} + |x|^2 \frac{\partial \bar{u}}{\partial x_i} \Big) \Big) dx \\
&= \operatorname{Im} \int_{\mathbb{R}^N} \Big(|x|^2 \bar{u}_t \Delta^2 u + \Delta u_t \Big(2N\bar{u} + 4 \sum_{i=1}^N x_i \frac{\partial \bar{u}}{\partial x_i} + |x|^2 \sum_{i=1}^N \frac{\partial^2 \bar{u}}{\partial x_i^2} \Big) \Big) dx \\
&= \operatorname{Im} \int_{\mathbb{R}^N} \big(|x|^2 \bar{u}_t \Delta^2 u + \Delta u_t \big(2N\bar{u} + 4x \cdot \nabla \bar{u} + |x|^2 \Delta \bar{u} \big) \big) dx \\
&= \operatorname{Im} \int_{\mathbb{R}^N} \big(|x|^2 \bar{u}_t \Delta^2 u + u_t \big(2N\Delta\bar{u} + 4\Delta(x \cdot \nabla \bar{u}) + \Delta \big(|x|^2 \Delta \bar{u} \big) \big) \big) dx \\
&= \operatorname{Im} \int_{\mathbb{R}^N} \Big(|x|^2 \bar{u}_t \Delta^2 u + u_t \Big(2N\Delta\bar{u} + 4 \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} \Big(\sum_{j=1}^N \Big(x_j \frac{\partial \bar{u}}{\partial x_j} \Big) \Big) \\
&\quad + \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} (|x|^2 \Delta \bar{u}) \Big) \Big) dx \\
&= \operatorname{Im} \int_{\mathbb{R}^N} \big(|x|^2 \bar{u}_t \Delta^2 u + 2N u_t \Delta \bar{u} \big) dx \\
&\quad + 4 \operatorname{Im} \int_{\mathbb{R}^N} u_t \Big(\sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2}{\partial x_i^2} \Big(x_j \frac{\partial \bar{u}}{\partial x_j} \Big) + \sum_{i=1}^N \frac{\partial}{\partial x_i} \Big(2x_i \Delta \bar{u} + |x|^2 \frac{\partial \Delta \bar{u}}{\partial x_i} \Big) \Big) dx \\
&= \operatorname{Im} \int_{\mathbb{R}^N} \big(|x|^2 \bar{u}_t \Delta^2 u + 2N u_t \Delta \bar{u} \big) dx \\
&\quad + 4 \operatorname{Im} \int_{\mathbb{R}^N} u_t \sum_{i=1}^N \sum_{j=1}^N \frac{\partial}{\partial x_i} \Big(\frac{\partial \bar{u}}{\partial x_i} + x_j \frac{\partial^2 \bar{u}}{\partial x_i \partial x_j} \Big) dx \\
&\quad + \operatorname{Im} \int_{\mathbb{R}^N} u_t \Big(2N\Delta\bar{u} + 4 \sum_{i=1}^N x_i \frac{\partial \Delta \bar{u}}{\partial x_i} + |x|^2 \sum_{i=1}^N \frac{\partial^2 \Delta \bar{u}}{\partial x_i^2} \Big) dx \\
&= \operatorname{Im} \int_{\mathbb{R}^N} \Big(|x|^2 \bar{u}_t \Delta^2 u + 2N u_t \Delta \bar{u} \Big) dx \\
&\quad + 4 \operatorname{Im} \int_{\mathbb{R}^N} u_t \Big(2 \sum_{i=1}^N \frac{\partial^2 \bar{u}}{\partial x_i^2} + \sum_{i=1}^N \sum_{j=1}^N \Big(x_j \frac{\partial^3 \bar{u}}{\partial x_i^2 \partial x_j} \Big) \Big) dx \\
&\quad + \operatorname{Im} \int_{\mathbb{R}^N} u_t \Big(2N\Delta\bar{u} + 4x \cdot \nabla(\Delta \bar{u}) + |x|^2 \Delta^2 \bar{u} \Big) dx \\
&= \operatorname{Im} \int_{\mathbb{R}^N} \big(|x|^2 \bar{u}_t \Delta^2 u + 2N u_t \Delta \bar{u} \big) dx \\
&\quad + 4 \operatorname{Im} \int_{\mathbb{R}^N} u_t \Big(2\Delta \bar{u} + \sum_{i=1}^N \sum_{j=1}^N \Big(x_j \frac{\partial}{\partial x_j} \Big(\frac{\partial^2 \bar{u}}{\partial x_i^2} \Big) \Big) \Big) dx \\
&\quad + \operatorname{Im} \int_{\mathbb{R}^N} \Big(u_t \big(2N\Delta\bar{u} + 4x \cdot \nabla(\Delta \bar{u}) \big) + |x|^2 \overline{\bar{u}_t \Delta^2 u} \Big) dx
\end{aligned}
$$

$$
\begin{aligned}
&=4\operatorname{Im}\int_{\mathbb{R}^N} u_t\left(N\Delta\bar u + x\cdot\nabla(\Delta\bar u)\right)dx + 4\operatorname{Im}\int_{\mathbb{R}^N} u_t\left(2\Delta\bar u + x\cdot\nabla(\Delta\bar u)\right)dx\\
&=4\operatorname{Im}\int_{\mathbb{R}^N} u_t\left(N\Delta\bar u + 2x\cdot\nabla(\Delta\bar u) + 2\Delta\bar u\right)dx.
\end{aligned}
$$

Substituting (3.14) and the above expression into (3.13), we derive

$$
\begin{aligned}
J''(t) =& -4\operatorname{Im}\int_{\mathbb{R}^N} u_t\left(N\bar u + 2x\cdot\nabla\bar u\right)dx\\
&+ 8\operatorname{Im}\int_{\mathbb{R}^N} u_t\left(N\Delta\bar u + 2x\cdot\nabla(\Delta\bar u) + 2\Delta\bar u\right)dx\\
=&4\operatorname{Im}\int_{\mathbb{R}^N} u_t\left((2N+4)\Delta\bar u + 4x\cdot\nabla(\Delta\bar u) - N\bar u - 2x\cdot\nabla\bar u\right)dx.
\end{aligned}
\tag{3.15}
$$

Substituting (3.12) into the above equation, we have

$$
\begin{aligned}
J''(t) =&4\operatorname{Im}\int_{\mathbb{R}^N} i\left(\Delta u - \Delta^2 u + |u|^p u\right)\Big((2N+4)\Delta\bar u + 4x\cdot\nabla(\Delta\bar u)\\
&- N\bar u - 2x\cdot\nabla\bar u\Big)dx\\
=&4\operatorname{Re}\int_{\mathbb{R}^N}\left(\Delta u - \Delta^2 u + |u|^p u\right)\Big((2N+4)\Delta\bar u + 4x\cdot\nabla(\Delta\bar u)\\
&- N\bar u - 2x\cdot\nabla\bar u\Big)dx\\
=&4\left(I_1 - I_2 + I_3\right),
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
I_1 &:= \operatorname{Re}\int_{\mathbb{R}^N}\Delta u\left((2N+4)\Delta\bar u + 4x\cdot\nabla(\Delta\bar u) - N\bar u - 2x\cdot\nabla\bar u\right)dx,\\
I_2 &:= \operatorname{Re}\int_{\mathbb{R}^N}\Delta^2 u\left((2N+4)\Delta\bar u + 4x\cdot\nabla(\Delta\bar u) - N\bar u - 2x\cdot\nabla\bar u\right)dx,\\
I_3 &:= \operatorname{Re}\int_{\mathbb{R}^N}|u|^p u\left((2N+4)\Delta\bar u + 4x\cdot\nabla(\Delta\bar u) - N\bar u - 2x\cdot\nabla\bar u\right)dx.
\end{aligned}
$$

Further we derive

$$
\begin{aligned}
I_1 =&(2N+4)\int_{\mathbb{R}^N}|\Delta u|^2dx + \operatorname{Re}\int_{\mathbb{R}^N}\left(4x\cdot\nabla(\Delta\bar u)\Delta u - N\bar u\Delta u - 2x\cdot\nabla\bar u\Delta u\right)dx\\
=&(2N+4)\int_{\mathbb{R}^N}|\Delta u|^2dx + N\int_{\mathbb{R}^N}|\nabla u|^2dx\\
&+ \operatorname{Re}\int_{\mathbb{R}^N}\Big(4\sum_{i=1}^N x_i\Big(\frac{\partial\Delta\bar u}{\partial x_i}\Delta u\Big) + 2\nabla(x\cdot\nabla\bar u)\cdot\nabla u\Big)dx\\
=&(2N+4)\int_{\mathbb{R}^N}|\Delta u|^2dx + N\int_{\mathbb{R}^N}|\nabla u|^2dx\\
&+ \operatorname{Re}\int_{\mathbb{R}^N}\Big(2\sum_{i=1}^N x_i\Big(\frac{\partial\Delta\bar u}{\partial x_i}\Delta u + \frac{\partial\Delta u}{\partial x_i}\Delta\bar u\Big)\\
&+ 2\sum_{i=1}^N\frac{\partial}{\partial x_i}\Big(\sum_{j=1}^N\Big(x_j\frac{\partial\bar u}{\partial x_j}\Big)\Big)\frac{\partial u}{\partial x_i}\Big)dx
\end{aligned}
$$

$$
\begin{aligned}
=&(2N+4)\int_{\mathbb{R}^N}|\Delta u|^2dx+N\int_{\mathbb{R}^N}|\nabla u|^2dx\\
&+\operatorname{Re}\int_{\mathbb{R}^N}\Big(2\sum_{i=1}^N x_i\frac{\partial}{\partial x_i}(\Delta u\Delta\bar u)+2\sum_{i=1}^N\sum_{j=1}^N\frac{\partial}{\partial x_i}\Big(x_j\frac{\partial\bar u}{\partial x_j}\Big)\frac{\partial u}{\partial x_i}\Big)dx\\
=&(2N+4)\int_{\mathbb{R}^N}|\Delta u|^2dx+N\int_{\mathbb{R}^N}|\nabla u|^2dx\\
&+\operatorname{Re}\int_{\mathbb{R}^N}\Big(2x\cdot\nabla|\Delta u|^2+2\sum_{i=1}^N\frac{\partial\bar u}{\partial x_i}\frac{\partial u}{\partial x_i}+2\sum_{i=1}^N\sum_{j=1}^N x_j\frac{\partial^2\bar u}{\partial x_i\partial x_j}\frac{\partial u}{\partial x_i}\Big)dx\\
=&(2N+4)\int_{\mathbb{R}^N}|\Delta u|^2dx+N\int_{\mathbb{R}^N}|\nabla u|^2dx-2N\int_{\mathbb{R}^N}|\Delta u|^2dx\\
&+2\int_{\mathbb{R}^N}|\nabla u|^2dx+\operatorname{Re}\int_{\mathbb{R}^N}\Big(\sum_{i=1}^N\sum_{j=1}^N x_j\Big(\frac{\partial^2\bar u}{\partial x_i\partial x_j}\frac{\partial u}{\partial x_i}+\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial\bar u}{\partial x_i}\Big)\Big)dx\\
=&4\int_{\mathbb{R}^N}|\Delta u|^2dx+(N+2)\int_{\mathbb{R}^N}|\nabla u|^2dx\\
&+\operatorname{Re}\int_{\mathbb{R}^N}\Big(\sum_{i=1}^N\sum_{j=1}^N x_j\frac{\partial}{\partial x_j}\Big(\frac{\partial\bar u}{\partial x_i}\frac{\partial u}{\partial x_i}\Big)\Big)dx\\
=&4\int_{\mathbb{R}^N}|\Delta u|^2dx+(N+2)\int_{\mathbb{R}^N}|\nabla u|^2dx+\operatorname{Re}\int_{\mathbb{R}^N}x\cdot\nabla|\nabla u|^2dx\\
=&4\int_{\mathbb{R}^N}|\Delta u|^2dx+(N+2)\int_{\mathbb{R}^N}|\nabla u|^2dx-N\int_{\mathbb{R}^N}|\nabla u|^2dx\\
=&4\int_{\mathbb{R}^N}|\Delta u|^2dx+2\int_{\mathbb{R}^N}|\nabla u|^2dx,
\end{aligned}
$$

$$
\begin{aligned}
I_2=&-(2N+4)\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2dx-N\int_{\mathbb{R}^N}|\Delta u|^2dx\\
&+4\operatorname{Re}\int_{\mathbb{R}^N}\Delta^2ux\cdot\nabla(\Delta\bar u)dx-2\operatorname{Re}\int_{\mathbb{R}^N}\Delta^2ux\cdot\nabla\bar udx\\
=&-(2N+4)\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2dx-N\int_{\mathbb{R}^N}|\Delta u|^2dx\\
&-4\operatorname{Re}\int_{\mathbb{R}^N}\nabla(\Delta u)\cdot\nabla\left(x\cdot\nabla(\Delta\bar u)\right)dx-2\operatorname{Re}\int_{\mathbb{R}^N}\Delta u\Delta(x\cdot\nabla\bar u)dx\\
=&-(2N+4)\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2dx-N\int_{\mathbb{R}^N}|\Delta u|^2dx\\
&-4\operatorname{Re}\int_{\mathbb{R}^N}\sum_{i=1}^N\frac{\partial\Delta u}{\partial x_i}\frac{\partial}{\partial x_i}\Big(\sum_{j=1}^N x_j\frac{\partial\Delta\bar u}{\partial x_j}\Big)dx\\
&-2\operatorname{Re}\int_{\mathbb{R}^N}\Delta u\sum_{i=1}^N\frac{\partial^2}{\partial x_i^2}\Big(\sum_{j=1}^N x_j\frac{\partial\bar u}{\partial x_j}\Big)dx\\
=&-(2N+4)\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2dx-N\int_{\mathbb{R}^N}|\Delta u|^2dx
\end{aligned}
$$

$$
\begin{aligned}
&- 4\,\mathrm{Re}\int_{\mathbb{R}^N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial\Delta u}{\partial x_i}\frac{\partial}{\partial x_i}\left(x_j\frac{\partial\Delta\bar{u}}{\partial x_j}\right)dx\\
&- 2\,\mathrm{Re}\int_{\mathbb{R}^N}\Delta u\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_i^2}\left(x_j\frac{\partial\bar{u}}{\partial x_j}\right)dx\\
=&-(2N+4)\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2dx - N\int_{\mathbb{R}^N}|\Delta u|^2dx\\
&- 4\,\mathrm{Re}\int_{\mathbb{R}^N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial\Delta u}{\partial x_i}\left(\frac{\partial\Delta\bar{u}}{\partial x_i}+x_j\frac{\partial^2\Delta\bar{u}}{\partial x_i\partial x_j}\right)dx\\
&- 2\,\mathrm{Re}\int_{\mathbb{R}^N}\Delta u\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{\partial x_j}{\partial x_i}\frac{\partial\bar{u}}{\partial x_j}+x_j\frac{\partial^2\bar{u}}{\partial x_i\partial x_j}\right)dx\\
=&-(2N+4)\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2dx - N\int_{\mathbb{R}^N}|\Delta u|^2dx\\
&- 4\int_{\mathbb{R}^N}\sum_{i=1}^{N}\frac{\partial\Delta u}{\partial x_i}\frac{\partial\Delta\bar{u}}{\partial x_i}dx - 4\int_{\mathbb{R}^N}|\Delta u|^2dx\\
&- 2\,\mathrm{Re}\int_{\mathbb{R}^N}\sum_{i=1}^{N}\sum_{j=1}^{N}x_j\left(\frac{\partial\Delta u}{\partial x_i}\frac{\partial^2\Delta\bar{u}}{\partial x_j\partial x_i}+\frac{\partial\Delta\bar{u}}{\partial x_i}\frac{\partial^2\Delta u}{\partial x_j\partial x_i}\right)dx\\
&- 2\,\mathrm{Re}\int_{\mathbb{R}^N}\Delta u\sum_{i=1}^{N}\sum_{j=1}^{N}x_j\frac{\partial^3\bar{u}}{\partial^2 x_i\partial x_j}dx\\
=&-(2N+4)\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2dx - N\int_{\mathbb{R}^N}|\Delta u|^2dx\\
&- 4\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2dx - 4\int_{\mathbb{R}^N}|\Delta u|^2dx\\
&- 2\,\mathrm{Re}\int_{\mathbb{R}^N}\sum_{i=1}^{N}\sum_{j=1}^{N}x_j\frac{\partial}{\partial x_j}\left(\frac{\partial\Delta u}{\partial x_i}\frac{\partial\Delta\bar{u}}{\partial x_i}\right)dx - 2\,\mathrm{Re}\int_{\mathbb{R}^N}\Delta u\sum_{j=1}^{N}x_j\frac{\partial\Delta\bar{u}}{\partial x_j}dx\\
=&-(2N+8)\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2dx - (N+4)\int_{\mathbb{R}^N}|\Delta u|^2dx\\
&- 2\,\mathrm{Re}\int_{\mathbb{R}^N}x\cdot\nabla|\nabla(\Delta u)|^2dx - \mathrm{Re}\int_{\mathbb{R}^N}\sum_{j=1}^{N}x_j\left(\frac{\partial\Delta\bar{u}}{\partial x_j}\Delta u+\frac{\partial\Delta u}{\partial x_j}\Delta\bar{u}\right)dx\\
=&-(2N+8)\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2dx - (N+4)\int_{\mathbb{R}^N}|\Delta u|^2dx\\
&+ 2N\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2dx - \mathrm{Re}\int_{\mathbb{R}^N}\sum_{j=1}^{N}x_j\frac{\partial}{\partial x_j}(\Delta\bar{u}\Delta u)dx\\
=&-8\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2dx - (N+4)\int_{\mathbb{R}^N}|\Delta u|^2dx - \mathrm{Re}\int_{\mathbb{R}^N}x\cdot\nabla|\Delta u|^2dx\\
=&-8\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2dx - (N+4)\int_{\mathbb{R}^N}|\Delta u|^2dx + N\int_{\mathbb{R}^N}|\Delta u|^2dx
\end{aligned}
$$

$$= -8\int_{\mathbb{R}^N} |\nabla(\Delta u)|^2 dx - 4\int_{\mathbb{R}^N} |\Delta u|^2 dx$$

and

$$\begin{aligned}
I_3 =& -N\int_{\mathbb{R}^N} |u|^{p+2} dx + (2N+4)\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u\Delta\bar{u} dx\\
&+ 4\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u x\cdot\nabla(\Delta\bar{u}) dx - 2\operatorname{Re}\int_{\mathbb{R}^N} |u|^p x\cdot(u\nabla\bar{u}) dx\\
=& -N\int_{\mathbb{R}^N} |u|^{p+2} dx + (2N+4)\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u\Delta\bar{u} dx\\
&+ 4\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u x\cdot\nabla(\Delta\bar{u}) dx - \operatorname{Re}\int_{\mathbb{R}^N} |u|^p x\cdot(u\nabla\bar{u} + \bar{u}\nabla u)\, dx\\
=& -N\int_{\mathbb{R}^N} |u|^{p+2} dx + (2N+4)\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u\Delta\bar{u} dx\\
&+ 4\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u x\cdot\nabla(\Delta\bar{u}) dx - \operatorname{Re}\int_{\mathbb{R}^N} x\cdot\Big((u\bar{u})^{p/2}\nabla(u\bar{u})\Big)\, dx\\
=& -N\int_{\mathbb{R}^N} |u|^{p+2} dx + (2N+4)\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u\Delta\bar{u} dx\\
&+ 4\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u x\cdot\nabla(\Delta\bar{u}) dx - \frac{2}{p+2}\operatorname{Re}\int_{\mathbb{R}^N} x\cdot\nabla(u\bar{u})^{\frac{p+2}{2}} dx\\
=& -N\int_{\mathbb{R}^N} |u|^{p+2} dx + (2N+4)\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u\Delta\bar{u} dx\\
&+ 4\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u x\cdot\nabla(\Delta\bar{u}) dx + \frac{2N}{p+2}\operatorname{Re}\int_{\mathbb{R}^N} |u|^{p+2} dx\\
=& -\frac{Np}{p+2}\int_{\mathbb{R}^N} |u|^{p+2} dx + (2N+4)\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u\Delta\bar{u} dx\\
&+ 4\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u x\cdot\nabla(\Delta\bar{u}) dx.
\end{aligned}$$

Substituting the above equalities for I_1, I_2 and I_3 into (3.16), we have

$$\begin{aligned}
J''(t) =& 4\Big(4\int_{\mathbb{R}^N} |\Delta u|^2 dx + 2\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)\\
&+ 4\Big(8\int_{\mathbb{R}^N} |\nabla(\Delta u)|^2 dx + 4\int_{\mathbb{R}^N} |\Delta u|^2 dx\Big)\\
&+ 4\Big(-\frac{Np}{p+2}\int_{\mathbb{R}^N} |u|^{p+2} dx + (2N+4)\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u\Delta\bar{u} dx\\
&+ 4\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u x\cdot\nabla(\Delta\bar{u}) dx\Big)\\
=& 8\Big(4\int_{\mathbb{R}^N} |\nabla(\Delta u)|^2 dx + 4\int_{\mathbb{R}^N} |\Delta u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)\\
&+ 4\Big(-\frac{Np}{p+2}\int_{\mathbb{R}^N} |u|^{p+2} dx + (2N+4)\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u\Delta\bar{u} dx\\
&+ 4\operatorname{Re}\int_{\mathbb{R}^N} |u|^p u x\cdot\nabla(\Delta\bar{u}) dx\Big).
\end{aligned} \tag{3.17}$$

The proof is complete. □

Remark 3.7. Based on the explicit expression of $J''(t)$, obviously, we cannot determine the sign of $J''(t)$ from $I(u) < 0$. On the other hand, the form of $J''(t)$ is more complicated than $\mathbf{J}''(t)$ in Section 2, which leads to great difficulties in proving the finite time blowup of solution if we adapt the arguments in [18]. Meanwhile, we point out that the proof of [15, Theorem 3.3] does not hold because they used the $\mathbf{J}''(t)$ for the classical second-order Schrödinger equation, which is very different from $J''(t)$ for the fourth-order nonlinear Schrödinger equation. Hence the finite time blowup of solutions to the Cauchy problem of the fourth-order semilinear Schrödinger equation without radially initial data under $0 < E(u_0) < d(d > 0)$ is still an open problem.

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