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WEIGHTED FUNCTION SPACES OF FRACTIONAL DERIVATIVES FOR VECTOR FIELDS

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Abstract. We introduce and study weighted function spaces for vector fields from the point of view of the regularity theory for quasilinear subelliptic PDEs.

section

1. Results

We consider a bounded domain $\Omega \subset \mathbb{R}^n$ and a system of smooth vector fields $X = (X_1, \dots, X_m)$, $m \leq n$, defined on Ω. Denote by $Xf = (X_1 f, \dots, X_m f)$ the X-gradient of a function f and use the notation $|Xf|^2 = \sum_{i=1}^m (X_i f)^2$.

In terms of the vector fields $X_1, \dots, X_m$, in the theory of second order PDE, usually we have one of the following two cases:

(1) $X_i = \frac{\partial}{\partial x_i}$, $1 \leq i \leq n$ and we refer to it as the (classical) elliptic case.
(2) There are points in Ω where the linear subspace of the tangent space spanned by the vector fields $X_1, \dots, X_m$ has dimension strictly less then n, but at the same time Hörmander's condition is satisfied, which means that there exists a positive integer $\nu \geq 2$ such that the vector fields X_i and their commutators
$$[X_{i_1}, [X_{i_2}, \dots, X_{i_k}] \dots], \quad 2 \leq k \leq \nu$$
of length at most $\nu \in \mathbb{N}$ span the tangent space at every point of Ω. We refer to this case as the subelliptic case and the vector fields X_i are called horizontal vector fields.

Let $2 \leq p < \infty$ and $K \subset \Omega$ be a compact subset of Ω. Consider the Sobolev space
$$XW^{1,p}(\Omega) = \Big\{ f \in L^p(\Omega) : X_i f \in L^p(\Omega) \text{ for all } \ i \in \{1, \dots, m\} \Big\}.$$
In the elliptic case we use the usual $W^{1,p}(\Omega)$ notation.

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If Z is a smooth vector field then we define its flow as the mapping $F(x,s) = e^{sZ}x$ which solves the initial value problem

$$\begin{gathered} \frac{\partial F}{\partial s}(x,s) = ZF(x,s) \\ F(x,0) = x\,. \end{gathered} \tag{1.1}$$

For $f \in XW^{1,p}(\Omega)$, we define the weight

$$w(Xf,s,x) = \Big(1 + |Xf(x)|^2 + |Xf(e^{sZ}x)|^2\Big)^{1/2}$$

and the following first and second order differences:

$$\begin{gathered} \Delta_{Z,s} f(x) = f(e^{sZ}x) - f(x)\,, \\ \Delta_{Z,-s} f(x) = f(x) - f(e^{-sZ}x)\,, \\ \Delta^2_{Z,s} f(x) = f(e^{sZ}x) + f(e^{-sZ}x) - 2f(x)\,. \end{gathered}$$

Notice that

$$\Delta^2_{Z,s} f(x) = \Delta_{Z,-s}\Delta_{Z,s} f(x) = \Delta_{Z,s}\Delta_{Z,-s} f(x)\,.$$

Let $0 < \theta < 2$, $0 \le \alpha \le p-2$ and $2 \le q \le p-\alpha$. Consider $s_K > 0$ sufficiently small such that

$$e^{sZ}x \in \Omega\,, \quad \text{for all } 0 < |s| < s_K \text{ and } x \in K\,,$$

and the Jacobian of the transformation $x \mapsto e^{sZ}x$ to be bounded in the following way:

$$0 < a^q \le \big|J\big(e^{sZ}x\big)\big| \le b^q\,, \quad \text{for all } 0 < |s| < s_K \text{ and } x \in K\,,$$

where $0 < a \le 1 \le b$.

Consider the following two pseudo-norms:

$$\begin{gathered} \|f\|^{\theta,1}_{Z,\alpha,p,q} = \|f\|_{L^p(\Omega)} + \sup_{0<|s|<s_K} \Big(\int_\Omega w^\alpha(Xf,s,x)\frac{|\Delta_{Z,s}f(x)|^q}{|s|^{\theta q}}\,dx\Big)^{1/q}, \\ \|f\|^{\theta,2}_{Z,\alpha,p,q} = \|f\|_{L^p(\Omega)} + \sup_{0<|s|<s_K} \Big(\int_\Omega w^\alpha(Xf,s,x)\frac{|\Delta^2_{Z,s}f(x)|^q}{|s|^{\theta q}}\,dx\Big)^{1/q}. \end{gathered}$$

Define the following function spaces which help us to handle the fractional derivatives in the Z direction:

$$B^{\theta,1}_{Z,\alpha,p,q}(K,\Omega) = \big\{f \in XW^{1,p}(\Omega) : \operatorname{supp} f \subset K \text{ and } \|f\|^{\theta,1}_{Z,\alpha,p,q} < \infty\big\}\,,$$

and

$$B^{\theta,2}_{Z,\alpha,p,q}(K,\Omega) = \big\{f \in XW^{1,p}(\Omega) : \operatorname{supp} f \subset K \text{ and } \|f\|^{\theta,2}_{Z,\alpha,p,q} < \infty\big\}\,.$$

If $\alpha = 0$ then these are linear normed spaces. Also, in the elliptic case, for $\alpha = 0$, $q = p$ we get similar spaces to the fractional order Besov spaces [5, 6]

$$B^\theta_{p,\infty}(\Omega) = \Big\{f \in L^p(\Omega) : \|f\|_{L^p(\Omega)} + \sup_{0\neq\|z\|\le\delta,\ z\in\mathbb{R}^n} \frac{\|\triangle^2_z f\|_{L^p(\Omega_z)}}{|z|^\theta} < \infty\Big\}\,,$$

where $\triangle^2_z f(x) = f(x+z) + f(x-z) - 2f(x)$, and $\Omega_z = \big\{x \in \Omega : x+z \in \Omega\big\}$. In the elliptic case the vector fields $\frac{\partial}{\partial x_i}$ generate a commuting family of strongly continuous semigroup of operators and by their isotropic nature, we can have a uniform treatment of the difference quotients in every direction. In the subelliptic case, using the Carnot-Carathéodory metric, a generalization of the elliptic setting is possible [2]. However, this approach does not allow us to study fractional derivatives in the direction of one vector field at a time.

Let us list a few evident properties of our function spaces:

(i) By [4, Theorem 4.3], if Z is a commutator of length k of the horizontal vector fields X_i, then
$$XW^{1,p}(\Omega) \subset B^{\frac{1}{k},1}_{Z,0,p,p}(K,\Omega)\,.$$
(ii) By [1, Lemma 2.3], if $f \in B^{1,1}_{Z,0,p,p}(K,\Omega)$ then $Zf \in L^p(K)$.

(iii) Using the fact that $\Delta^2_{Z,s}f(x) = \Delta_{Z,s}f(x) - \Delta_{Z,-s}f(x)$ we easily get that
$$B^{\theta,1}_{Z,\alpha,p,q}(K,\Omega) \subset B^{\theta,2}_{Z,\alpha,p,q}(K,\Omega)\,.$$
The reversed inclusion is not elementary, and for the proof we use a method of Zygmund [7] which already proved to be useful in the Heisenberg group [3].

Theorem 1.

(a) *For* $0 < \theta < 1$ *we have* $B^{\theta,2}_{Z,\alpha,p,q}(K,\Omega) \subset B^{\theta,1}_{Z,\alpha,p,q}(K,\Omega)$.

(b) *For every* $0 < \gamma < 1$ *we have* $B^{1,2}_{Z,\alpha,p,q}(K,\Omega) \subset B^{\gamma,1}_{Z,\alpha,p,q}(K,\Omega)$.

(c) *For* $1 < \theta < 2$ *we have* $B^{\theta,2}_{Z,\alpha,p,q}(K,\Omega) \subset B^{1,1}_{Z,\alpha,p,q}(K,\Omega)$.

Proof. (a) Let $f \in B^{\theta,2}_{Z,\alpha,p,q}(K,\Omega)$. Then
$$\int_\Omega \big(1 + |Xf(x)|^2 + |Xf(e^{sZ}x)|^2\big)^{\alpha/2} |f(e^{sZ}x) + f(e^{-sZ}x) - 2f(x)|^q dx \le M^q|s|^{\theta q}$$
for all $0 < |s| < s_K$. Therefore,
$$\int_\Omega \big(1 + |Xf(e^{sZ}x)|^2\big)^{\alpha/2} |f(e^{sZ}x) + f(e^{-sZ}x) - 2f(x)|^q dx \le M^q|s|^{\theta q}$$
and then changing s to $-s/2$ we get
$$\int_\Omega \big(1 + |Xf(e^{-\frac{s}{2}Z}x)|^2\big)^{\alpha/2} |f(e^{\frac{s}{2}Z}x) + f(e^{-\frac{s}{2}Z}x) - 2f(x)|^q dx \le \frac{M^q}{2^{\theta q}}|s|^{\theta q}\,.$$
We use now the change of variables $x \mapsto e^{\frac{s}{2}Z}x$ to get
$$\int_\Omega \big(1 + |Xf(x)|^2\big)^{\alpha/2}\ \big|f(e^{sZ}x) + f(x) - 2f(e^{\frac{s}{2}Z}x)\big|^q\, dx \le \frac{M^q}{a^q 2^{\theta q}}|s|^{\theta q}\,.$$
In this way we have obtained the inequality
$$\int_\Omega \big(1 + |Xf(x)|^2\big)^{\alpha/2} \big|\triangle_{Z,s}(f)(x) - 2\triangle_{Z,\frac{s}{2}}(f)(x)\big|^q\, dx \le \frac{M^q}{a^q 2^{\theta q}}|s|^{\theta q}$$
and repeating n-times the process of changing s to $s/2$ and multiplying the inequality by 2^q we get
$$\begin{aligned}&\int_\Omega \big(1 + |Xf(x)|^2\big)^{\alpha/2} \big|2^{n-1}\triangle_{Z,\frac{s}{2^{n-1}}}f(x) - 2^n\triangle_{Z,\frac{s}{2^n}}f(x)\big|^q dx\\ &\le \frac{M^q}{a^q 2^{\theta q}}|s|^{\theta q}\,2^{(1-\theta)q(n-1)}\,.\end{aligned}$$

These inequalities give

$$\Big(\int_\Omega \big(1+|Xf(x)|^2\big)^{\alpha/2}\big|\triangle_{Z,s}f(x)-2^n\triangle_{Z,\frac{s}{2^n}}f(x)\big|^q\,dx\Big)^{1/q}\le \frac{M}{a2^\theta}|s|^\theta\sum_{k=0}^{n-1}2^{(1-\theta)k} \tag{1.2}$$

and hence by our assumptions on q, p and α it follows that, for a constant $C>0$ depending on the $XW^{1,p}$ norm of f, we have

$$\begin{aligned}&\Big(\int_\Omega \big(1+|Xf(x)|^2\big)^{\alpha/2}\big|\triangle_{Z,\frac{s}{2^n}}f(x)\big|^q\,dx\Big)^{1/q}\\&\le \frac{1}{2^n}\Big(\int_\Omega \big(1+|Xf(x)|^2\big)^{\alpha/2}|\triangle_{Z,s}f(x)|^q\,dx\Big)^{1/q}+c\frac{M}{a2^\theta}|s|^\theta 2^{-\theta n}\\&\le C\Big(\frac{1}{2^n}+|s|^\theta 2^{-\theta n}\Big)\,.\end{aligned} \tag{1.3}$$

For all h with $0<|h|<s_K/2$ there exist $n\in\mathbb{N}$ and $s\in\mathbb{R}$ such that $|s|\in[s_K/2,s_K]$ and $h=s/2^n$. In this way we get

$$\frac{1}{|h|^\theta}\Big(\int_\Omega \big(1+|Xf(x)|^2\big)^{\alpha/2}\ |\triangle_{Z,h}f(x)|^q\,dx\Big)^{1/q}\le C\Big(\frac{|h|^{1-\theta}}{s_K}+1\Big)\,.$$

Also, for $s_K/2\le|h|\le s_K$ we have

$$\frac{1}{|h|^\theta}\Big(\int_\Omega \big(1+|Xf(x)|^2\big)^{\alpha/2}|\triangle_{Z,h}f(x)|^q\,dx\Big)^{1/q}\le C\,,$$

and therefore,

$$\sup_{0<|h|<s_K}\Big(\int_\Omega \big(1+|Xf(x)|^2\big)^{\alpha/2}\frac{|\triangle_{Z,h}f(x)|^q}{|h|^{\theta q}}dx\Big)^{1/q}\le C\,.$$

The change of variables $x\mapsto e^{-hZ}x$ shows that, for a possible different C and sufficiently small h, we have

$$\Big(\int_\Omega \big(1+|Xf(e^{-hZ}x)|^2\big)^{\alpha/2}\ \frac{|\triangle_{Z,-h}f(x)|^q}{|h|^{q\theta}}dx\Big)^{1/q}\le C\,.$$

Changing h to $-h$ gives

$$\Big(\int_\Omega \big(1+|Xf(e^{hZ}x)|^2\big)^{\alpha/2}\ \frac{|\triangle_{Z,h}f(x)|^q}{|h|^{q\theta}}dx\Big)^{1/q}\le C\,. \tag{1.4}$$

and therefore,

$$\sup_{0<|h|<s_K}\Big(\int_\Omega \big(1+|Xf(x)|^2+|Xf(e^{hZ}x)|^2\big)^{\alpha/2}\frac{|\triangle_{Z,h}f(x)|^q}{|h|^{q\theta}}dx\Big)^{1/q}\le C\,.$$

(b) Let $f\in B^{1,2}_{Z,\alpha,p,q}(K,\Omega)$ and start in a similar way to the proof of the part (a). Inequality (1.2) for $\theta=1$ gives

$$\Big(\int_\Omega \big(1+|Xf(x)|^2\big)^{\alpha/2}\ \big|\triangle_{Z,s}f(x)-2^n\triangle_{Z,\frac{s}{2^n}}f(x)\big|^q\,dx\Big)^{1/q}\le\frac{M}{a2^\theta}|s|n. \tag{1.5}$$

Again, for $0<|h|<s_K/2$ consider $n\in\mathbb{N}$ and $s\in\mathbb{R}$ such that $|s|\in[s_K/2,s_K]$ and $h=s/2^n$ and get

$$\frac{1}{|h|^\gamma}\Big(\int_\Omega \big(1+|Xf(x)|^2\big)^{\alpha/2}|\triangle_{Z,h}f(x)|^q\,dx\Big)^{1/q}\le C\Big(\frac{|h|^{1-\gamma}}{s_K}+|h|^{1-\gamma}|\ln h|\Big)\,.$$

This leads to $f \in B^{\gamma,1}_{Z,\alpha,p,q}(K,\Omega)$.

(c) Let $f \in B^{\theta,2}_{Z,\alpha,p,q}(K,\Omega)$. Taking into consideration that we suppose now $1<\theta<2$, inequality (1.2) has the form

$$\Big(\int_\Omega \big(1+|Xf(x)|^2\big)^{\alpha/2}\ \big|\triangle_{Z,s}f(x)-2^n\triangle_{Z,\frac{s}{2^n}}f(x)\big|^q\,dx\Big)^{1/q} \le \frac{M}{a2^\theta}|s|. \tag{1.6}$$

and this leads to

$$\frac{1}{|h|}\Big(\int_\Omega \big(1+|Xf(x)|^2\big)^{\alpha/2}\ |\triangle_{Z,h}f(x)|^q\,dx\Big)^{1/q} \le C\frac{1}{s_K}\left(1+s_K^{\theta-1}\right).$$

It easily follows now that $f \in B^{1,1}_{Z,\alpha,p,q}(K,\Omega)$. $\square$

Remark 2. As will be shown in the Examples 3 and 5 below, slight variations of these weighted function spaces might also appear. To define them consider the pseudo-norms:

$$\|f\|^{\theta,1}_{XZ,\alpha,p,q} = \|f\|_{L^P(\Omega)} + \sup_{0<|s|<s_K}\Big(\int_\Omega w^\alpha(Xf,s,x)\frac{|\Delta_{Z,s}Xf(x)|^q}{|s|^{\theta q}}\,dx\Big)^{1/q},$$

$$\|f\|^{\theta,2}_{XZ,\alpha,p,q} = \|f\|_{L^P(\Omega)} + \sup_{0<|s|<s_K}\Big(\int_\Omega w^\alpha(Xf,s,x)\frac{|\Delta^2_{Z,s}Xf(x)|^q}{|s|^{\theta q}}\,dx\Big)^{1/q},$$

and the function spaces

$$XB^{\theta,1}_{Z,\alpha,p,q}(K,\Omega) = \left\{f\in XW^{1,p}(\Omega) : \operatorname{supp} f\subset K \text{ and } \|f\|^{\theta,1}_{XZ,\alpha,p,q}<\infty\right\},$$

and

$$XB^{\theta,2}_{Z,\alpha,p,q}(K,\Omega) = \left\{f\in XW^{1,p}(\Omega) : \operatorname{supp} f\subset K \text{ and } \|f\|^{\theta,2}_{XZ,\alpha,p,q}<\infty\right\}.$$

If we follow the proof of Theorem 1, we realize that it remains valid in the case of $XB^{\theta,1}_{Z,\alpha,p,q}(K,\Omega)$ and $XB^{\theta,2}_{Z,\alpha,p,q}(K,\Omega)$, too. Another inclusion which will be used in Examples 3 and 5 is that if $f \in XB^{\theta,1}_{Z,p-2,p,2}(K,\Omega)$ then $f \in XB^{\frac{2\theta}{p},1}_{Z,0,p,p}(K,\Omega)$ (see also the proof of [3, Lemma 3.1]) .

In the following two examples we show that our function spaces naturally appear when we study the regularity of the minimizers to the problem

$$\min_{u\in XW^{1,p}(\Omega)} \int_\Omega \big(1+|Xu(x)|^2\big)^{p/2}\,dx \tag{1.7}$$

subject to a boundary condition of type $u-v\in XW^{1,p}_0(\Omega)$, where $v\in XW^{1,p}(\Omega)$ is fixed. A minimizing function u is a weak solutions of the following nondegenerate p-Laplacian equation

$$\sum_{i=1}^m X_i\Big(\big(1+|Xu|^2\big)^{\frac{p-2}{2}}X_iu\Big)=0\,, \quad \text{in } \Omega \tag{1.8}$$

which means that

$$\int_\Omega \big(1+|Xu|^2\big)^{\frac{p-2}{2}}X_1u\,X_1\varphi + \big(1+|Xu|^2\big)^{\frac{p-2}{2}}X_2u\,X_2\varphi\,dx=0\,, \tag{1.9}$$

for all $\varphi\in XW^{1,p}(\Omega)$ with support compactly included in Ω.

Example 3. In this example we refer to the proof of [3, Lemma 3.1]. Consider the the Heisenberg group $\mathbb{H}$ as $\mathbb{R}^3$ endowed with the group multiplication

$$(x_1, x_2, t) \cdot (y_1, y_2, s) = \Big(x_1 + y_1, x_2 + y_2, t + s - \frac{1}{2}(x_2 y_1 - x_1 y_2)\Big)\,.$$

The horizontal vector fields are

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial t}\,.$$

Denote

$$T = \frac{\partial}{\partial t}$$

and observe that $[X_1, X_2] = T$. To study the regularity of weak solutions first we have to prove the differentiability in the direction of T. The vector fields X_1, X_2 and T span the tangent space at every point and according to [4, Theorem 4.3] we have

$$\eta^2 u \in B^{\frac{1}{2},1}_{T,0,p,p}(\Omega)$$

for every $\eta \in C_0^\infty(\Omega)$. Use now a test function

$$\varphi = \frac{\triangle_{T,-s}}{s^{1/2}}\Big(\frac{\triangle_{T,s}(\eta^2 u)}{s^{1/2}}\Big)$$

to get

$$\eta^2 u \in XB^{\frac{1}{2},1}_{T,p-2,p,2}(\operatorname{supp}\eta, \Omega).$$

This implies that

$$\eta^2 u \in XB^{\frac{1}{p},1}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)$$

and by the fact that T commutes with the horizontal vector fields X_1 and X_2 we can use again [4, Theorem 4.3] to get

$$\eta^2 u \in B^{\frac{1}{2}+\frac{1}{p},2}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)\,.$$

For $p = 2$ we have $\eta^2 u \in B^{1,2}_{T,p-2,p,2}(\operatorname{supp}\eta, \Omega)$ which implies

$$\eta^2 u \in B^{\gamma,1}_{T,p-2,p,2}(\operatorname{supp}\eta, \Omega)$$

for any $\frac{1}{2} < \gamma < 1$. Restarting our proof on the bases of the previous line we get

$$\eta^2 u \in B^{\frac{1}{2}+\gamma,2}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)\,,$$

and this leads to $Tu \in L^p_{\rm loc}(\Omega)$.
For $p > 2$, by Theorem 1, the inequality $\frac{1}{2} + \frac{1}{p} < 1$ implies that

$$\eta^2 u \in B^{\frac{1}{2}+\frac{1}{p},1}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)\,,$$

and hence we can restart the whole process again with $\frac{1}{2} + \frac{1}{p}$ instead of $\frac{1}{2}$ and a new cut-off function η with a conveniently chosen support to get

$$\eta^2 u \in B^{\frac{1}{2}+\frac{1}{p}+\frac{2}{p^2},1}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)\,.$$

In general, after k iterations we get $\eta^2 u \in B^{\gamma_k,2}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)$, with

$$\gamma_k = \frac{1}{2} + \frac{1}{p}\Big(1 + \frac{2}{p} + \dots + \frac{2^{k-1}}{p^{k-1}}\Big)\,.$$

If $2 \le p < 4$ then for a sufficiently large k we have $\gamma_k > 1$ and then

$$\eta^2 u \in B^{1,1}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)$$

which implies that $Tu \in L^p_{\rm loc}(\Omega)$. Of course, there is the question of what is happening if, for a $k \in \mathbb{N}$, we get $\gamma_k = 1$. In this case, we can choose a $\gamma_{k+1} < 1$ sufficiently close to 1 such that after repeating the iteration to get $\gamma_{k+2} > 1$.

Remark 4. We study the case $p \ge 2$ in order to be able to give a uniform approach to our function spaces in various cases of horizontal vector fields. In [3] it is also proved that $Tu \in L^p_{\rm loc}(\Omega)$ for $1 < p < 2$. The proof of this result is connected to Heisenberg group and does not work for other Carnot groups of step 3 or higher. However, let us give the sequence of spaces in which we include $\eta^2 u$. So, we start with $B^{\frac12,1}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)$ and continue with

$$\begin{gathered} XB^{\frac14,1}_{T,p-2,p,2}(\operatorname{supp}\eta, \Omega)\,, \quad XB^{\frac14,1}_{T,0,p,p}(\operatorname{supp}\eta, \Omega), \\ B^{\frac34,2}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)\,, \quad B^{\frac34,1}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)\,, \dots\,, \\ B^{\frac{2^{k+1}-1}{2^{k+1}},1}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)\,, \quad B^{\frac12+\gamma_k,2}_{T,0,p,p}(\operatorname{supp}\eta, \Omega), \end{gathered}$$

where $\gamma_k = \frac{2^k-1}{2^{k+2}}(p-1) + \frac{2^{k+1}-1}{2^{k+2}} > 1/2$ for k sufficiently large.

Example 5. We consider now an example involving commutators of length higher than 2. Our preference goes with Grushin type vector fields, but we could use T from the center of any nilpotent Lie Algebra generated by a system of horizontal vector fields. Consider $\Omega \subset \mathbb{R}^2$ intersecting the line $x_1 = 0$ and the vector fields $X_1 = \frac{\partial}{\partial x_1}$ and $X_2 = x_1^3 \frac{\partial}{\partial x_2}$. At the points $(0, x_2) \in \Omega$ the vector fields X_1 and X_2 span a 1 dimensional subspace, so we need their commutator of length 4

$$T = [X_1, [X_1, [X_1, X_2]]] = 6\frac{\partial}{\partial x_2}$$

to span the whole tangent space.

According to [4] we have

$$\eta^2 u \in B^{\frac14,1}_{T,0,p,p}(\Omega)$$

for every $\eta \in C_0^\infty(\Omega)$ and we can start the iteration process with the test function

$$\varphi = \frac{\triangle_{T,-s}}{s^{1/4}}\Big(\frac{\triangle_{T,s}(\eta^2 u)}{s^{1/4}}\Big)\,.$$

In a similar to way to Example 3 we get the series of inclusions

$$\begin{gathered} \eta^2 u \in XB^{\frac14,1}_{T,p-2,p,2}(\operatorname{supp}\eta, \Omega), \\ \eta^2 u \in XB^{\frac{1}{2p},1}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)\,, \\ \eta^2 u \in B^{\frac14+\frac{1}{2p},2}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)\,. \end{gathered}$$

By Theorem 1, the inequality $\frac14 + \frac{1}{2p} < 1$ implies that

$$\eta^2 u \in B^{\frac14+\frac{1}{2p},1}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)\,,$$

and hence we can restart the whole process again with $\frac14 + \frac{1}{2p}$ instead of $\frac14$ and get

$$\eta^2 u \in B^{\frac14+\frac{1}{2p}+\frac{1}{p^2},1}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)\,.$$

Therefore, after k iterations we get

$$\eta^2 u \in B^{\gamma_k,2}_{T,0,p,p}(\operatorname{supp}\eta, \Omega),$$

with

$$\gamma_k = \frac{1}{4} + \frac{1}{2p}\Big(1 + \frac{2}{p} + \dots + \frac{2^{k-1}}{p^{k-1}}\Big)\,.$$

If $2 \le p < 8/3$ then for a sufficiently large k we have $\gamma_k > 1$ and then

$$\eta^2 u \in B^{1,1}_{T,0,p,p}(\operatorname{supp}\eta, \Omega)$$

which implies that $Tu \in L^p_{\rm loc}(\Omega)$.

References

[1] L. Capogna, *Regularity of quasilinear equations in the Heisenberg group*, Comm. Pure Appl. Math. **50** (1997) 867-889.

[2] D. Danielli, N. Garofalo and D.-M. Nhieu; *Sub-elliptic Besov spaces and the characterization of traces on lower dimensional manifolds*, Contemporary Mathematics 277(2001), 19-37.

[3] A. Domokos, *Differentiability of solutions for the non-degenerate p-Laplacian in the Heisenberg group*, J. Differential Equations, 204 (2004), 439-470.

[4] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. 119(1967), 147-171.

[5] J. Peetre, *New Thoughts on Besov Spaces*, Duke Univ. Math. Series, Durham Univ. 1976.

[6] H. Triebel, *Interpolation Theory*, Function Spaces, Differential Operators, North-Holland Publishing Company, Amsterdam-New-York-Oxford, 1978.

[7] A. Zygmund, *Trigonometric series*, Volume 1, Cambridge University Press, 1977.

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