

THE CRITICAL CASE FOR A SEMILINEAR WEAKLY HYPERBOLIC EQUATION

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ABSTRACT. We prove a global existence result for the Cauchy problem, in the three-dimensional space, associated with the equation

$$u_{tt} - a_\lambda(t)\Delta_x u = -u|u|^{p(\lambda)-1}$$

where $a_\lambda(t) \geq 0$ and behaves as $(t - t_0)^\lambda$ close to some $t_0 > 0$ with $a(t_0) = 0$, and $p(\lambda) = (3\lambda + 10)/(3\lambda + 2)$ with $3 \leq p(\lambda) \leq 5$. This means that we deal with the superconformal, critical nonlinear case. Moreover we assume a small initial energy.

1. INTRODUCTION

In this work we study the existence of global solutions to the Cauchy Problem

$$\begin{aligned} u_{tt}(x, t) - a(t)\Delta_x u(x, t) &= -u(x, t)|u(x, t)|^{p-1}, & x \in \mathbb{R}^3, \\ u(x, 0) &= u_0(x), & u_t(x, 0) = u_1(x), \end{aligned} \quad (1.1)$$

with $a(t) \geq 0$. We shall only consider real valued initial data and hence real solutions.

In the case $a(t) \equiv 1$ this equation is the standard wave equation for which a great deal of work has been developed starting from the pioneristic paper by Jörgens [6] (for a survey of these results see [11]). The interest in variable coefficients case corresponds to the change of the propagation speed.

In the case $a(t) > 0$ the global existence for (1.1) can be obtained with the same technique of the case $a(t) = 1$. The asymptotic behaviour of the solutions has been studied in [4] under the small initial data condition.

On the contrary, very few results are known concerning the global existence for nonlinear weakly hyperbolic equations. To our knowledge the weakly hyperbolic case with polynomial nonlinear term as been studied only in [2] for the space dimension $n = 3$ and in [3] for the case $n = 1, 2$. More precisely in [2], D'Ancona considers a real analytic function $a(t) \geq 0$ and a slight general forcing term $f(u)$ having right sign and polynomial growth at infinity:

$$uf(u) \geq 0 \text{ and } |f(u)| \leq C|u|^p, \text{ for } |u| \geq 1.$$

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In [2], a global existence result of a unique smooth solution is established provided the initial data are compactly supported, the greatest order of the zeroes of $a(t)$, $\lambda > 0$, is finite and

$$p < \frac{3\lambda + 10}{3\lambda + 2}. \quad (1.2)$$

Moreover, it is also possible to obtain D'Ancona result relaxing the analytic assumption on the time-dependent coefficient $a(t)$, assuming $a(t)$ is a positive continuous function, piecewise C^2 , with locally finite zeroes and each of them with finite order. For $\lambda = 0$, the restriction (1.2) gives $p < 5$, the subcritical range for the wave equations; for this reason it is possible to conjecture that

$$p_c(\lambda) := \frac{3\lambda + 10}{3\lambda + 2}$$

is the critical exponent for the semilinear weakly hyperbolic case.

For the critical semilinear wave equation (i.e. $\lambda = 0$, $p_c(\lambda) = 5$) the global existence result has been proved into three steps. Rauch in [8] established the global existence for this solution under a smallness assumption for the initial energy. Struwe in [12] removed this hypothesis requiring radial initial data. Finally, by means of Strichartz type estimates, the general case was covered by different authors; the interested reader can see for example [10]. A generalization of Strichartz estimates for some strictly hyperbolic operators with variable coefficients has been obtained in several works by Reissig, Yagdjian, Hirosawa and others (see the list of references in [5].) Due to the lack of such estimates for weakly hyperbolic equations we can not cover the general case and we come back to Rauch's approach. More precisely, we establish the following.

Theorem 1.1. *Consider (1.1) with initial data $u_0, u_1 \in C_0^\infty(\mathbb{R}^3)$. Let $\lambda \geq 0$ and $t_0 > 0$. Let $a(t)$ be a real continuous function, $a \in C^2([0, +\infty) \setminus \{t_0\})$ such that*

$$\begin{aligned} a(t) &= (t_0 - t)^\lambda b(t) \quad \text{on } [0, t_0], b \in C^2, b > 0. \\ a(t) &> 0 \text{ on } (t_0, +\infty), \end{aligned} \quad (1.3)$$

Let the nonlinear exponent satisfy

$$3 \leq p_c(\lambda) := \frac{3\lambda + 10}{3\lambda + 2} \leq 5.$$

Then, there exists $0 < \varepsilon \leq 1$, such that if

$$a(0) \int_{\mathbb{R}^3} \frac{|\nabla u_0(x)|^2}{2} dx + \int_{\mathbb{R}^3} \frac{|u_1(x)|^2}{2} dx + \int_{\mathbb{R}^3} \frac{|u_0(x)|^{p_c(\lambda)+1}}{p_c(\lambda) + 1} dx \leq \varepsilon, \quad (1.4)$$

then the Cauchy Problem (1.1) has a unique real solution $u(x, t) \in C^2(\mathbb{R}^3 \times [0, +\infty))$.

We notice that $3 \leq p_c(\lambda) \leq 5$ is equivalent to $0 \leq \lambda \leq 2/3$. Since λ belongs to this finite interval, the smallness rate ε is taken uniform with respect λ . The restriction $3 \leq p_c(\lambda) \leq 5$ comes from the employed technique: we obtain a-priori estimates combining (1.4) with a suitable variant of Hardy's inequality. However, $3 \leq p_c(\lambda)$ means that $p_c(\lambda)$ in the superconformal range, and this is the interesting case, since the nonlinear exponent is high. Conversely $p_c(\lambda) < 5$, is the subcritical assumption also needed in the strictly hyperbolic case. We mention again that our result for $\lambda = 0$ coincides with the existence result given in [8].

In a final remark we shall see how to extend this theorem to the case in which $a(t)$ vanishes in more than one point. It is also possible to generalize our result to a more general nonlinear term, that is to consider the equation

$$u_{tt} - a(t)\Delta u = -f(u),$$

where

$$|f(u)| + |uf'(u)| + |u^2 f''(u)| \leq C(1 + |u|)^p,$$

$$(1 + |u|)^{p+1} \leq C \int_0^u f(s) ds.$$

In the strictly hyperbolic critical case, one also assumes

$$uf(u) - 4 \int_0^u f(s) ds \geq 0.$$

In particular this condition implies $p \geq 3$ when $f(u) = u|u|^{p-1}$.

With suitable initial data condition, we can deal with the Grushin type operator

$$\partial_{tt} - |t|^\lambda \Delta_x.$$

For the corresponding equation $\partial_{tt}u - |t|^\lambda \Delta_x u = |u|^p$, a non-existence result is known when $p \leq (3\lambda + 8)/(3\lambda + 4)$ (see [1]). Under this restriction on p , such a result assures that the solution of $\partial_{tt}u - |t|^\lambda \Delta_x u = -|u|^{p-1}u$, given by D'Ancona's result, changes sign.

Notation.

- Given $t > 0$, we set $B_t = \{y \in \mathbb{R}^3 : |y| \leq t\}$.
- The surface measure on a sphere is denoted by $d\omega$. The surface measure on a truncated cone is denoted by $d\Sigma$.
- We omit to write \mathbb{R}^3 if it is a domain of a function space, denoting by $\|\cdot\|_p$ the $L^p(\mathbb{R}^3)$ -norm. The homogeneous Sobolev space \dot{H}^k is endowed with the seminorm

$$\|f\|_{\dot{H}^k} := \sum_{|\alpha|=k} \|D^\alpha f\|_2.$$

2. PRELIMINARY LEMMA

Weighted Hardy's inequality on the backward cone. In order to pass from the subcritical result by Jørgens to the critical one by Rauch, Hardy's inequality comes into play. In our case we need a weighted localized variant of Hardy's inequality on the backward cone. The unweighted variant of the next lemma on the ball can be found in [12].

Lemma 2.1. *Let $\varphi \in C^1(\mathbb{R}^3 \times \mathbb{R})$ and $t > 0$. If $\alpha > -1$, then*

$$\begin{aligned} & \int_0^t \int_{|x-y|=t-s} (t-s)^{\alpha-2} \varphi^2(y, s) d\omega_y ds \\ & < C_H(\alpha) \int_0^t \int_{|x-y|=t-s} (t-s)^\alpha \left| \nabla_y \varphi(y, s) - \frac{y-x}{|y-x|} \partial_s \varphi(y, s) \right|^2 d\omega_y ds \quad (2.1) \\ & + C_H(\alpha) \int_0^{t/2} \int_{|x-y|=t-s} (t-s)^{\alpha-2} \varphi^2(y, s) d\omega_y ds, \end{aligned}$$

where $C_H(\alpha)$ is given by (2.3).

Proof. First we observe that the transformation

$$\int_0^t \int_{|x-y|=t-s} v(y, s) \, d\omega_y \, ds = \sqrt{2} \int_{B_t} v(x+y, t-|y|) \, dy \quad (2.2)$$

gives

$$\int_0^t \int_{|x-y|=t-s} (t-s)^{\alpha-2} \varphi^2(y, s) \, d\omega_y \, ds = \sqrt{2} \int_{B_t} |y|^{\alpha-2} \psi_{x,t}^2(y) \, dy,$$

where $\psi_{x,t}(y) := \varphi(x+y, t-|y|)$. Since $\alpha > -1$, we can use the following weighted Hardy's inequality (see for example [7]):

$$\int_{\mathbb{R}^3} |y|^{\alpha-2} \psi^2(y) \, dy \leq \frac{4}{(\alpha+1)^2} \int_{\mathbb{R}^3} |y|^\alpha |\nabla \psi(y)|^2 \, dy, \quad \psi \in C_0^1(\mathbb{R}^3).$$

The constant in this inequality is sharp.

To study the noncompactly supported ψ , we use a localizing function. We notice that for any $\delta > 0$ there exists a C_0^1 function $\bar{\eta}_\delta : [0, +\infty) \rightarrow \mathbb{R}$ such that $\bar{\eta}_\delta \equiv 1$ on $[0, 1]$, $\bar{\eta}_\delta \equiv 0$ on $[2, +\infty)$, $0 \leq \bar{\eta}_\delta \leq 1$ and $\|\bar{\eta}'_\delta\|_\infty \leq 1 + \delta$. In fact, the assumptions $\bar{\eta}_\delta \equiv 1$ on $[0, 1]$, $\bar{\eta}_\delta \equiv 0$ on $[2, +\infty)$ imply $\|\bar{\eta}'_\delta\|_\infty \geq 1$. Hence is not possible to take $\delta = 0$ unless $\bar{\eta}_\delta(s) = 2 - s$ in the interval $[1, 2]$, losing the smoothness requirement. Let us put $\eta_\delta = \bar{\eta}_\delta(\frac{2}{\rho}|x|)$, then

$$\begin{aligned} \int_{B_\rho} |y|^{\alpha-2} \psi(y)^2 \, dy &\leq \int_{\mathbb{R}^3} |y|^{\alpha-2} (\psi \eta_\delta)^2(y) \, dy + \int_{B_\rho \setminus B_{\rho/2}} |y|^{\alpha-2} \psi^2(y) \, dy \\ &\leq \frac{4}{(\alpha+1)^2} \int_{\mathbb{R}^3} |y|^\alpha |\nabla(\psi \eta_\delta)|^2 \, dy + \int_{B_\rho \setminus B_{\rho/2}} |y|^{\alpha-2} \psi^2(y) \, dy. \end{aligned}$$

At this point we use the inequality $(a+b)^2 \leq C_1 a^2 + C_2 b^2$ which holds for $C_1 > 1$ and $(C_1 - 1)C_2 - C_1 \geq 0$. We get

$$\begin{aligned} &\int_{B_\rho} |y|^{\alpha-2} \psi^2(y) \, dy \\ &\leq \frac{4C_1}{(\alpha+1)^2} \int_{B_\rho} |y|^\alpha |\nabla \psi|^2 \, dy + \left(\frac{16C_2(1+\delta)^2}{(\alpha+1)^2} + 1 \right) \int_{B_\rho \setminus B_{\rho/2}} |y|^{\alpha-2} \psi^2(y) \, dy. \end{aligned}$$

For $\delta \rightarrow 0$ and $4C_1 = 16C_2(1+\delta)^2 + (\alpha+1)^2$, we can take

$$C_H(\alpha) = \frac{\sqrt{(\alpha+1)^4 + 104(\alpha+1)^2 + 400}}{2(\alpha+1)^2} + \frac{10}{(\alpha+1)^2} + \frac{1}{2} \quad (2.3)$$

arriving to

$$\int_{B_\rho} |y|^{\alpha-2} \psi^2(y) \, dy < C_H(\alpha) \left(\int_{B_\rho} |y|^\alpha |\nabla \psi(y)|^2 \, dy + \int_{B_\rho \setminus B_{\rho/2}} |y|^{\alpha-2} \psi^2(y) \, dy \right).$$

This constant is sharp for this kind of localization of Hardy's inequality on the ball. Coming back to $\psi_{x,t}$, we see that

$$\nabla \psi_{x,t}(y, s) = (\nabla \varphi)(x+y, t-|y|) - \frac{y}{|y|} \partial_s \varphi(x+y, t-|y|).$$

Applying the inverse transformation of (2.2) we get the estimate (2.1). \square

Reduction to a nonlinear wave equation. A relevant difference between the wave operator $\partial_{tt} - \Delta$ and $\partial_{tt} - a(t)\Delta$ is the lack of a representation formula for the fundamental solution of

$$u_{tt}(x, t) - a(t)\Delta_x u(x, t) = -f(u(x, t)).$$

In order to overcome this difficulty, it is possible to employ the Liouville transformation, obtaining a suitable wave equation with a mass term.

Given $a(t) : [0, t_0] \rightarrow \mathbb{R}$, such that $a(t) > 0$ for $t \neq t_0$, $a(t_0) = 0$ and $a \in \mathcal{C}^1([0, t_0])$, we associate a function ϕ which satisfies

$$\begin{aligned} \phi'(s) &= a(\phi(s))^{-1/2} \quad s \in [0, r_0], \\ \phi(0) &= 0, \end{aligned} \tag{2.4}$$

with

$$r_0 = \int_0^{t_0} a(s)^{1/2} ds.$$

In particular $\phi \in \mathcal{C}^2([0, r_0]) \cap \mathcal{C}([0, r_0])$ is a strictly increasing function on $[0, r_0]$, and $\phi(r_0) = t_0$. For $t \in [0, r_0)$, we define $v(x, t) = u(x, \phi(t))$. We can check that v is solution of

$$v_{tt} - \Delta v = -(\phi')^2 f(u(x, \phi(t))) + v_t \frac{\phi''}{\phi'}. \tag{2.5}$$

To avoid the term containing v_t , we write

$$w(x, t) = \psi(t)v(x, t) = \psi(t)u(x, \phi(t)).$$

Then w solves the equation

$$w_{tt} = \psi'' v_t + \psi \Delta v - (\phi')^2 \psi f(u(x, \phi(t))) + \psi v_t \frac{\phi''}{\phi'} + 2\psi' v_t.$$

If ψ never vanishes in $[0, r_0)$, then

$$w_{tt} - \Delta w = -(\phi')^2 \psi f(u(x, \phi(t))) + \left(\frac{\psi''}{\psi} - \frac{\psi'}{\psi} \frac{\phi''}{\phi'} - 2 \left(\frac{\psi'}{\psi} \right)^2 \right) w + \left(\frac{\phi''}{\phi'} + 2 \frac{\psi'}{\psi} \right) w_t.$$

In order to erase the term containing w_t we choose

$$\psi(t) = (\phi'(t))^{-1/2} = a(\phi(t))^{1/4}.$$

In this way, if $t \in [0, t_0)$, we can write

$$u(x, t) = a(t)^{-1/4} w(x, \phi^{-1}(t)). \tag{2.6}$$

Finally, for $t \in [0, r_0)$, we obtain

$$w_{tt} - \Delta w = \psi'' \psi^{-1} w - \psi^{-3} f(u(x, \phi(t)))$$

with initial data

$$\begin{aligned} w(x, 0) &= a(0)^{1/4} u_0, \\ w_t(x, 0) &= \frac{1}{4} a(0)^{-1/4} a'(0) u_0 + a(0)^{-1/4} u_1. \end{aligned} \tag{2.7}$$

We apply Kirchhoff's formula to get

$$\begin{aligned} w(x, t) &= w_0(x, t) + \frac{1}{4\pi} \int_0^t \frac{\psi''(s)}{\psi(s)(t-s)} \int_{|x-y|=t-s} w(y, s) \, d\omega_y \, ds \\ &\quad - \frac{1}{4\pi} \int_0^t \frac{1}{\psi^3(s)(t-s)} \int_{|x-y|=t-s} f(u(y, \phi(s))) \, d\omega_y \, ds \\ &= w_0(x, t) + I + II, \end{aligned} \quad (2.8)$$

here $w_0(x, t)$ solves the homogeneous equation $w_{tt} - \Delta w = 0$ with initial data (2.7). \square

3. PROOF OF MAIN THEOREM

Reduction to a decay estimate. The local existence result follows from the strictly hyperbolic case. For other details see pp. 249-250 in [2].

Similarly, still using the linear theory of weakly hyperbolic equations, one can establish the uniqueness of the smooth solution of (1.1) and the finite speed of propagation property. We focus our attention on the existence result.

If $a'(t) > 0$ or $a(t) > 0$, then the existence result follows from the strictly hyperbolic case provided $1 \leq p \leq 5$. Being $p_c(\lambda) \leq 5$, it remains to analyze the case $a(t)$ decreasing and vanishing at the end point of the interval. Hence it suffices to prove that the solution $u(x, t) : \mathbb{R}^3 \times [0, t_0[\rightarrow \mathbb{R}$ admits a finite limit for $t \rightarrow t_0^-$.

Using Sobolev embedding theorem, we see that it is sufficient to show that there exists a continuous function $C(t) > 0$ such that, in the close interval $[0, t_0]$,

$$\|u(t)\|_{H^2(\mathbb{R}_x^3)} \leq C(t). \quad (3.1)$$

First we prove that if there exists $\alpha < (p_c(\lambda) - 1)^{-1}$ such that

$$|u(x, t)| \leq C(t_0 - t)^{-\alpha}, \quad \text{for any } 0 \leq t < t_0, x \in \mathbb{R}^3 \quad (3.2)$$

then (3.1) holds.

Let us introduce the high-order energy

$$E_3(t) = \frac{1}{2} \|u_t\|_{\dot{H}^2(\mathbb{R}_x^3)}^2 + \frac{1}{2} a(t) \|\nabla u\|_{\dot{H}^2(\mathbb{R}_x^3)}^2 + \frac{1}{2} \|u\|_{\dot{H}^2(\mathbb{R}_x^3)}^2.$$

After integration by parts, we find

$$E_3'(t) = \frac{1}{2} a'(t) \|\nabla u\|_{\dot{H}^2(\mathbb{R}_x^3)}^2 + \sum_{|\alpha|=2} \int D_x^\alpha u_t \left(D_x^\alpha u - D_x^\alpha (u|u|^{p_c(\lambda)-1}) \right) \, dx.$$

Recalling $a'(t) \leq 0$, we get

$$\left(\sqrt{E_3(t)} \right)' \leq \frac{\sqrt{2}}{2} \left(\|u\|_{\dot{H}^2(\mathbb{R}_x^3)} + \|u|u|^{p_c(\lambda)-1}\|_{\dot{H}^2(\mathbb{R}_x^3)} \right).$$

Due to finite speed of propagation $\|u(t)\|_{\dot{H}^2(\mathbb{R}_x^3)}$ is equivalent to $\|u(t)\|_{H^2(\mathbb{R}_x^3)}$. We can apply Moser-type inequality (see [9, Sec. 5.2.5]), obtaining

$$\left(\sqrt{E_3(t)} \right)' \leq C \|u\|_{\dot{H}^2(\mathbb{R}_x^3)} \left(1 + \|u\|_{L^\infty(\mathbb{R}_x^3)}^{p_c(\lambda)-1} \right)$$

when $p_c(\lambda) > 3/2$. Suppose we have proved (3.2), we have

$$\left(\sqrt{E_3(t)} \right)' \leq C \sqrt{E_3(t)} \left(1 + (t_0 - t)^{-\alpha(p_c(\lambda)-1)} \right).$$

Gronwall's lemma gives (3.1) when $\alpha < (p_c(\lambda) - 1)^{-1}$.

Energy estimates. In order to establish (3.2), we start deriving the basic a-priori energy estimate for the C^2 solution of the Cauchy Problem (1.1). In what follows we always denote by u such solution. Let us define the energy density

$$e[u](x, t) := \frac{1}{2}|u_t(x, t)|^2 + a(t) \frac{|\nabla_x u(x, t)|^2}{2} + \frac{|u(x, t)|^{p_c(\lambda)+1}}{p_c(\lambda) + 1};$$

the corresponding energy is

$$E(t) := \int_{\mathbb{R}^3} e[u](x, t) \, dx.$$

We put $E_0 := E(0)$. We have

$$\partial_t e[u] = a(t) \operatorname{div}(u_t \nabla_x u) + a'(t) \frac{|\nabla_x u|^2}{2}, \quad (3.3)$$

hence

$$E'(t) = a'(t) \int_{\mathbb{R}^3} \frac{|\nabla_x u(x, t)|^2}{2} \, dx. \quad (3.4)$$

The lack of conservation law for the energy is an important difference between the considered equation and the wave equation. Since $a(t) : [0, t_0] \rightarrow \mathbb{R}$ is a decreasing function, then $E(t)$ is a decreasing function in the same interval, that is

$$E(t) \leq E_0.$$

The next point is to associate to $a(t)$ a curved cone. Let $\phi : [0, r_0] \rightarrow [0, t_0]$ be defined by means of (2.4); being $a(t)$ decreasing, ϕ is convex.

Let us fix $\bar{x} \in \mathbb{R}^3$ and $S, T, \bar{t} \in [0, r_0]$. For $\bar{z} = (\bar{x}, \phi(\bar{t}))$, $S \leq T \leq \bar{t}$, we introduce the truncated curved cone

$$K_S^T(\bar{z}) = \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : \exists \sigma \in [S, T] \text{ s.t. } \phi(\sigma) = t, |x - \bar{x}| \leq \bar{t} - \sigma\}.$$

We will use the notation $K_S^T(\bar{z}) = K_S^{\bar{t}}(\bar{z})$. The mantle of $K_S^T(\bar{z})$ is

$$M_S^T(\bar{z}) = \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : \exists \sigma \in [S, T] \text{ s.t. } \phi(\sigma) = t, |x - \bar{x}| = \bar{t} - \sigma\}.$$

The outward normal to the mantle of the cone is

$$\vec{n} = \frac{1}{\sqrt{1 + \phi'(\sigma)^2}} \left(\phi'(\sigma) \frac{x - \bar{x}}{|x - \bar{x}|}, 1 \right).$$

The standard measure on $M_S^T(\bar{z})$ is given by $d\Sigma = \sqrt{1 + \phi'(\sigma)^2} \, d\omega_x \, d\sigma$. The section of $K_0(\bar{z})$ at the time $t \in [0, \phi(\bar{t})]$ is denoted by $D(t, \bar{z})$, that is

$$D(t, \bar{z}) := \{x \in \mathbb{R}^3 \mid (x, t) \in K_0(\bar{z})\}.$$

For the local energy at the time t , we set

$$E(u, t) := \int_{D(t, \bar{z})} e[u](x, t) \, dx.$$

We shall see that since $a(t)$ is decreasing, then $E(u, \phi(t))$ is decreasing too. To this aim we introduce the energy flux of u : for any $0 \leq t < t_0$, $\bar{x} \in \mathbb{R}^3$, we set

$$d_{\bar{x}}[u](x, t) = \frac{1}{\sqrt{1 + (a(t))^{-1}}} \left(\frac{1}{2} |u_t - \sqrt{a(t)} \frac{x - \bar{x}}{|x - \bar{x}|} \cdot \nabla u|^2 + \frac{|u|^{p_c(\lambda)+1}}{p_c(\lambda) + 1} \right).$$

Lemma 3.1. *Let $0 \leq S < T < r_0$. Then the following relation holds*

$$E(u, \phi(S)) = E(u, \phi(T)) + \int_{M_S^T(\bar{z})} d_{\bar{x}}[u] \, d\Sigma - \frac{1}{2} \int_{K_S^T(\bar{z})} a'(t) |\nabla_x u|^2 \, dx \, dt. \quad (3.5)$$

Proof. Explicitly, we have

$$\begin{aligned} & \int_{K_S^T(\bar{z})} \partial_t e[u](x, t) \, dx \, dt \\ &= \int_{M_S^T(\bar{z})} e[u] n_t \, d\Sigma + \int_{D(\phi(T), \bar{z})} e[u](x, \phi(T)) \, dx - \int_{D(\phi(S), \bar{z})} e[u](x, \phi(S)) \, dx. \end{aligned}$$

On the other hand, the relation (3.3) implies

$$\begin{aligned} \int_{K_S^T(\bar{z})} \partial_t e[u](x, t) \, dx \, dt &= \int_{K_S^T(\bar{z})} a(t) \operatorname{div}(u_t \nabla_x u) + a'(t) \frac{|\nabla_x u|^2}{2} \, dx \, dt \\ &= \int_{M_S^T(\bar{z})} a u_t \nabla_x u \cdot n_x \, d\Sigma + \int_{K_S^T(\bar{z})} a'(t) \frac{|\nabla_x u|^2}{2} \, dx \, dt. \end{aligned}$$

Then

$$\begin{aligned} & E(u, \phi(T)) - E(u, \phi(S)) \\ &= - \int_{M_S^T(\bar{z})} (e[u] n_t - a u_t \nabla_x u \cdot n_x) \, d\Sigma + \int_{K_S^T(\bar{z})} a'(t) \frac{|\nabla_x u|^2}{2} \, dx \, dt. \end{aligned}$$

Being $e[u] n_t - a u_t \nabla_x u \cdot n_x = d_{\bar{x}}[u]$ on $M_S^T(\bar{z})$, the proof is complete. \square

Corollary 3.2. *Let $S, T, \bar{t} \in [0, r_0)$. If $S < T < \bar{t}$, then $E(u, \phi(T)) \leq E(u, \phi(S))$.*

In particular, for all $t \in [0, t_0)$, it holds $E(u, t) \leq E_0$. In order to control the nonlinear term, we need to estimate the energy flux of u close to the vertex of the cone.

Corollary 3.3. *Under the same assumptions of the previous corollary, we have*

$$\lim_{S \rightarrow \bar{t}} \int_{M_S^T(\bar{z})} d_{\bar{x}}[u] \, d\Sigma = 0.$$

Proof. Denoting by χ_A the characteristic function of a measurable set A , we can write

$$E(u, \phi(S)) = \int_{\mathbb{R}^3} \chi_{D(\phi(S), \bar{z})}(x) e[u](x, \phi(S)) \, dx = \int_{\mathbb{R}^3} g(x, S) \, dx.$$

It is evident that $\lim_{S \rightarrow \bar{t}} g(x, S) = 0$. From $E(u, \phi(S)) \leq E_0$, it follows that $g(\cdot, S) \in L^1(\mathbb{R}^3)$. The Lebesgue convergence theorem gives

$$E(u, \phi(S)) \rightarrow 0, \quad \text{for } S \rightarrow \bar{t}. \quad (3.6)$$

On the other hand, we prove that

$$\int_{K_S^T} -a'(t) \frac{|\nabla u|^2}{2} \, dx \, dt \rightarrow 0, \quad (3.7)$$

when $S \rightarrow \bar{t}$. In fact, employing (3.4), we see that

$$\int_{K_S^T} -a'(t) \frac{|\nabla u|^2}{2} \, dx \, dt \leq \int_{\phi(S)}^{\phi(T)} -E'(t) \, dt = E(\phi(S)) - E(\phi(T)).$$

The convergence (3.7) will be a consequence of the continuity of $E(t)$. Finally, combining (3.6), (3.7) with (3.5) we get our thesis. \square

Corollary 3.4. *Assume that the hypotheses of Corollary 3.2 hold. Then*

$$\int_{M_S^T(\bar{z})} \frac{|u|^{p_c(\lambda)+1}}{\sqrt{1 + \phi'(\sigma)^2}} d\Sigma \leq (p_c(\lambda) + 1)E_0. \tag{3.8}$$

Proof. Being $a'(t) \leq 0$ and $E(u, \phi(T)) \geq 0$, from (3.5), we have

$$\frac{1}{p_c(\lambda) + 1} \int_{M_S^T(\bar{z})} \frac{|u|^{p_c(\lambda)+1}}{\sqrt{1 + \phi'(\sigma)^2}} d\Sigma \leq \int_{M_S^T(\bar{z})} d_{\bar{x}}[u] d\Sigma \leq E(u, \phi(S)) \leq E_0. \tag{3.9}$$

This is our claim. □

The inequality (3.8) gives the conical energy estimate

$$\int_S^T \int_{|\bar{x}-y|=\bar{t}-s} |u(y, \phi(s))|^{p_c(\lambda)+1} d\omega_y ds \leq (p_c(\lambda) + 1)E_0. \tag{3.10}$$

We shall often use the explicit version of (3.9):

$$\int_S^T \int_{|\bar{x}-y|=\bar{t}-s} \sqrt{1 + \phi'(s)^2} d_{\bar{x}}[u](y, \phi(s)) d\omega_y ds \leq E_0. \tag{3.11}$$

Combining Corollary 3.4 and the weighted Hardy’s inequality we can prove our main a-priori estimate for the solution of (1.1). Starting from (2.4), we see that there exists a C^1 strictly positive function $K_\lambda : [0, t_0] \rightarrow \mathbb{R}$ such that

$$a(\phi(t)) = K_\lambda(\phi(t))(r_0 - t)^{\frac{2\lambda}{\lambda+2}}. \tag{3.12}$$

In order to show this we observe that $(\phi^{-1}(s))' = a(s)^{1/2}$. Hence

$$r_0 - \phi^{-1}(s) = (t_0 - s)^{\frac{\lambda}{2}+1} \int_0^1 \tau^{\lambda/2} b^{1/2}(t_0 - (t_0 - s)\tau) d\tau. \tag{3.13}$$

The above identity implies (3.12) with

$$K_\lambda(s) = \left[\int_0^1 \tau^{\lambda/2} b^{1/2}(t_0 - (t_0 - s)\tau) d\tau \right]^{-2\lambda/(\lambda+2)} b(s).$$

Finally, we have

$$m(\lambda)(r_0 - s)^{\frac{2\lambda}{\lambda+2}} \leq a(\phi(s)) \leq M(\lambda)(r_0 - s)^{\frac{2\lambda}{\lambda+2}}.$$

where

$$0 < m(\lambda) = \min_{[0, r_0]} K_\lambda(\phi(t)), \quad M(\lambda) = \max_{[0, r_0]} K_\lambda(\phi(t)). \tag{3.14}$$

Lemma 3.5. *Let $h_0 \in \mathbb{R}$, $h_1 \geq 0$, $\lambda \geq 0$. Consider $u(x, t)$ a solution of (1.1). For any $0 \leq t \leq r_0$, we set*

$$\mathcal{I} := \int_0^t \int_{|x-y|=t-s} (t - s)^{-h_0} a(\phi(s))^{-h_1} u^2(y, \phi(s)) d\omega_y ds.$$

If

$$-\frac{2\lambda}{\lambda + 2}(h_1 + 1) + 2 - h_0 = 0, \tag{3.15}$$

then there exists a constant $C(\lambda, h_1) > 0$ such that

$$\mathcal{I} \leq C(\lambda, h_1) \left(E_0 + E_0^{2/(p_c(\lambda)+1)} \right). \tag{3.16}$$

Proof. Using (3.14), we have

$$\mathcal{I} \leq m(\lambda)^{-h_1} \int_0^t \int_{|x-y|=t-s} (t-s)^{-h_0 - \frac{2\lambda h_1}{\lambda+2}} \varphi^2(y, s) \, d\omega_y \, ds,$$

where $\varphi(y, s) = u(y, \phi(s))$. Assuming

$$2 - h_0 - \frac{2\lambda h_1}{\lambda + 2} > -1, \quad (3.17)$$

we can apply the Lemma 2.1 obtaining

$$\begin{aligned} \mathcal{I} &\leq C_1(\lambda, h_0, h_1) \int_0^t \int_{|x-y|=t-s} (t-s)^{2-h_0 - \frac{2\lambda h_1}{\lambda+2}} \left| \nabla \varphi - \frac{y-x}{|y-x|} \partial_s \varphi \right|^2 \, d\omega_y \, ds \\ &\quad + C_1(\lambda, h_0, h_1) \int_0^{t/2} \int_{|x-y|=t-s} (t-s)^{-h_0 - \frac{2\lambda h_1}{\lambda+2}} \varphi^2(y, s) \, d\omega_y \, ds \\ &=: C_1(\lambda, h_0, h_1) [I_1 + I_2], \end{aligned}$$

where $C_1(\lambda, h_0, h_1) = m(\lambda)^{-h_1} C_H(2 - h_0 - (2\lambda h_1)/(\lambda + 2))$.

In order to estimate I_1 , we compute

$$\left| \nabla \varphi - \frac{y-x}{|y-x|} \partial_s \varphi \right|^2 = \left| \nabla_y u(y, \phi(s)) - \frac{y-x}{|y-x|} \phi'(s) u_t(y, \phi(s)) \right|^2.$$

Hence

$$\left| \nabla \varphi - \frac{y-x}{|y-x|} \partial_s \varphi \right|^2 \leq 2[a(\phi(s))]^{-1} \sqrt{1 + [\phi'(s)]^2} d_x[u](y, \phi(s)).$$

Recalling that $t \leq r_0$, we have

$$\begin{aligned} I_1 &\leq 2m(\lambda)^{-1} \int_0^t \int_{|x-y|=t-s} (t-s)^{2-h_0 - \frac{2\lambda(h_1+1)}{\lambda+2}} \sqrt{1 + [\phi'(s)]^2} d_x[u] \, d\omega_y \, ds \\ &\leq 2m(\lambda)^{-1} t^{2-h_0 - \frac{2\lambda(h_1+1)}{\lambda+2}} E_0. \end{aligned}$$

Here we used (3.11) and the assumption

$$2 - h_0 - \frac{2\lambda h_1}{\lambda + 2} \geq \frac{2\lambda}{\lambda + 2}. \quad (3.18)$$

To our aim it is sufficient to take the equality in (3.18). In particular (3.17) is satisfied. We turn to the estimate of I_2 observing that $-h_0 - 2h_1\lambda/(\lambda + 2) = -4/(\lambda + 2)$.

$$\begin{aligned} I_2 &\leq \int_0^{t/2} \int_{|x-y|=t-s} (t-s)^{\frac{-4}{\lambda+2}} u^2(y, \phi(s)) \, d\omega_y \, ds \\ &\leq \left(\int_0^{t/2} \int_{|x-y|=t-s} (t-s)^{\frac{-4}{\lambda+2} \frac{p_c(\lambda)+1}{p_c(\lambda)-1}} \, d\omega_y \, ds \right)^{\frac{p_c(\lambda)-1}{p_c(\lambda)+1}} \\ &\quad \times \left(\int_0^t \int_{|x-y|=t-s} |u(y, \phi(s))|^{p_c(\lambda)+1} \, d\omega_y \, ds \right)^{\frac{2}{p_c(\lambda)+1}}. \end{aligned}$$

An explicit computation gives $\frac{-4}{\lambda+2} \frac{p_c(\lambda)+1}{p_c(\lambda)-1} + 2 = -1$; hence by the aid of (3.10), we arrive at

$$I_2 \leq (4\pi \lg 2)^{\frac{p_c(\lambda)-1}{p_c(\lambda)+1}} [(p_c(\lambda) + 1) E_0]^{2/(p_c(\lambda)+1)}.$$

In particular we have (3.16) with

$$C(\lambda, h_1) = m(\lambda)^{-h_1} C_H \left(\frac{2\lambda}{\lambda + 2} \right) \max \left\{ \frac{2}{m(\lambda)}, (4\pi \lg 2)^{\frac{p_c(\lambda)-1}{p_c(\lambda)+1}} (p_c(\lambda) + 1)^{\frac{2}{p_c(\lambda)+1}} \right\}.$$

This concludes the proof. □

Proof of the decay estimate. Combining (1.3) and (2.6), we see that the decay estimate (3.2) is equivalent to

$$|w(x, t)| \leq Cb(\phi(t))^{1/4} (t_0 - \phi(t))^{-\alpha+\lambda/4},$$

for any $0 \leq t < r_0$, $\alpha < (p_c(\lambda) - 1)^{-1}$. Recalling that $b > 0$ and $b(t)$ is bounded on the close interval $[0, t_0]$, by using (3.13), we see that the previous inequality will follow from

$$\mu(t) \leq C(r_0 - t)^{-\delta}, \text{ for any } 0 \leq t < r_0, 0 < \delta < 1/4 \tag{3.19}$$

where

$$\mu(t) := \sup_{\mathbb{R}^3 \times [0, t]} |w(x, s)|.$$

From (2.8) we deduce

$$\begin{aligned} \mu(t) &\leq |w_0(x, t)| + \int_0^t (t - s) |\psi''(s)| \psi^{-1}(s) \mu(s) \, ds \\ &\quad + \frac{\mu(t)}{4\pi} \int_0^t (t - s)^{-1} a(\phi(s))^{-1} \int_{|x-y|=t-s} |u(y, \phi(s))|^{p_c(\lambda)-1} \, d\omega_y \, ds. \end{aligned} \tag{3.20}$$

Lemma 3.6. *Let $u(x, t)$ be a solution of (1.1). Assume $3 \leq p_c(\lambda) := \frac{3\lambda+10}{3\lambda+2} \leq 5$. For any $A > 0$ there exists $0 < \varepsilon(A) \leq 1$, such that if $E_0 \leq \varepsilon(A)$, then*

$$\mathcal{I} := \int_0^t (t - s)^{-1} a(\phi(s))^{-1} \int_{|x-y|=t-s} |u(y, \phi(s))|^{p_c(\lambda)-1} \, d\omega_y \, ds \leq A.$$

Proof. First, we consider the case $p_c(\lambda) = 3$, that is $\lambda = 2/3$. We can directly use Lemma 3.5 with $h_0 = h_1 = 1$ and get

$$\int_0^t (t - s)^{-1} a(\phi(s))^{-1} \int_{|x-y|=t-s} |u(y, \phi(s))|^2 \, d\omega_y \, ds \leq C(2/3, 1)(E_0 + E_0^{1/2}). \tag{3.21}$$

If $p_c(\lambda) > 3$, we apply Hölder inequality and find

$$\begin{aligned} \mathcal{I} &\leq \left(\int_0^t \int_{|x-y|=t-s} |u(y, \phi(s))|^{p_c(\lambda)+1} \, d\omega_y \, ds \right)^{1/p} \\ &\quad \times \left(\int_0^t \int_{|x-y|=t-s} a(\phi(s))^{-\frac{p}{p-1}} (t - s)^{-\frac{p}{p-1}} |u(y, \phi(s))|^2 \, d\omega_y \, ds \right)^{(p-1)/p} \end{aligned}$$

provided $p = \frac{p_c(\lambda)-1}{p_c(\lambda)-3}$.

The first term can be directly estimated by means of (3.10); for the second term, we employ Lemma 3.5 with $h_0 = h_1 = p/(p - 1)$. The conditions (3.15) and $0 < 1/p < 1$ are satisfied if and only if $\lambda < 2/3$. Hence we conclude

$$\mathcal{I} \leq C(\lambda, 4/(3\lambda + 2))^{\frac{3\lambda+2}{4}} (p_c(\lambda) + 1)^{\frac{p_c(\lambda)-3}{p_c(\lambda)-1}} E_0^{\frac{p_c(\lambda)-3}{p_c(\lambda)-1}} (E_0 + E_0^{\frac{2}{p_c(\lambda)+1}})^{\frac{2}{p_c(\lambda)-1}}. \tag{3.22}$$

This formula coincides with (3.21) when $\lambda = 2/3$.

It is not difficult to see that $C(\lambda, 4/(3\lambda + 2))^{3\lambda+2} (p_c(\lambda) + 1)^{\frac{p_c(\lambda)-3}{p_c(\lambda)-1}}$ is a continuous function of λ , then we can put

$$\mathcal{M} := \max_{\lambda \in [0, 2/3]} C(\lambda, 4/(3\lambda + 2))^{3\lambda+2} (p_c(\lambda) + 1)^{\frac{p_c(\lambda)-3}{p_c(\lambda)-1}}.$$

Moreover $2/(p_c(\lambda) - 1) \leq 1$, hence (3.22) gives

$$\mathcal{I} \leq \mathcal{M}(E_0 + E_0^{\frac{p_c(\lambda)-1}{p_c(\lambda)+1}}).$$

Choosing $\sqrt{\varepsilon} \leq \min\{1, \frac{A}{2\mathcal{M}}\}$, we find

$$\mathcal{I} \leq \mathcal{M}(\varepsilon + \varepsilon^{\frac{4}{3\lambda+6}}) \leq 2\mathcal{M} \min\{1, \frac{A}{2\mathcal{M}}\} \leq A.$$

This concludes our proof. □

Using this estimate in (3.20), we arrive to

$$\left(1 - \frac{A}{4\pi}\right) \mu(t) \leq |w_0(x, t)| + \int_0^t (t - s) |\psi''(s)| \psi^{-1}(s) \mu(s) ds.$$

At this point the proof proceeds as in [3] page 20. For sake of completeness we sketch it. Differentiating twice (3.12), we find

$$|\psi''(s)| \psi^{-1}(s) \leq \frac{\lambda(\lambda + 4)}{4(\lambda + 2)^2} (r_0 - s)^{-2} (1 + \delta(s)).$$

with $\delta(t) \rightarrow 0$ as $t \rightarrow r_0^-$. We can find $\bar{t} \in [0, r_0)$ such that $\delta(s) \leq 1/9$ in $[\bar{t}, r_0)$; moreover we choose $A/4\pi = 1/9$; hence

$$\mu(t) \leq C + \frac{5\lambda(\lambda + 4)}{16(\lambda + 2)^2} \int_{\bar{t}}^t (t - s)(r_0 - s)^{-2} \mu(s) ds \quad \bar{t} < t < r_0.$$

By comparison with the solution of the Euler type integral-equation

$$m(t) = C + \frac{5\lambda(\lambda + 4)}{16(\lambda + 2)^2} \int_{\bar{t}}^t (t - s)(r_0 - s)^{-2} m(s) ds$$

one finds $\mu(t) \lesssim C(\lambda, r_0)(r_0 - t)^{-\gamma}$ with

$$\gamma = \frac{1}{2} \sqrt{1 + \frac{5\lambda(\lambda + 4)}{4(\lambda + 2)^2}} - \frac{1}{2}.$$

To conclude our proof, it suffices to observe that $0 < \gamma < 1/4$.

Final remark. It is possible to extend our theorem dealing with a positive function $a(t) \in \mathcal{C}^2([0, +\infty) \setminus A)$ with A a discrete set of zeros for $a(t)$:

$$A = \{t_1 < t_2 < t_3 < \dots < t_n < \dots\}.$$

We assume that there exists a uniform $\delta > 0$ such that for any $t_j \in A$ it holds

$$a(t) = (t_j - t)^{\lambda_j} b_j(t) \text{ on } [0, t_j], b_j \in \mathcal{C}^2, b_j > 0 \text{ and } 3 \leq p_c(\lambda_j) \leq 5.$$

If $t_1 > 0$ the local existence result follows from the strictly hyperbolic case. The same is true when $t_1 = 0$ since $a'(t) > 0$ in a suitable neighborhood of zero. So we can consider $t_1 > 0$. Our theorem is then available when the assumption $E_0 < \varepsilon$ is replaced by $E(t_1 - \delta) < \varepsilon$. This gives a solution on $[0, T_1]$ with $T_1 < t_2 - \delta$. Again we can apply our theorem assuming $E(t_2 - \delta) < \varepsilon$. Iterating this argument we find a global smooth solution of (1.1) provided in each step $E(t_j - \delta) < \varepsilon$.

REFERENCES

- [1] L. D'AMBROSIO, S. LUCENTE, *Nonlinear Liouville theorems for Grushin and Tricomi operators*, J. Differential Equations, **123** (2003) 511-541.
- [2] P. D'ANCONA, *A Note on a Theorem of Jörgens*, Math. Z., **218** (1995), 239-252.
- [3] P. D'ANCONA, A. DI GIUSEPPE, *Global Existence with Large Data for a Nonlinear Weakly Hyperbolic Equation*, Math. Nachr., **231** (2001), 5-23.
- [4] A. DI GIUSEPPE, *Global existence with small data for a nonlinear wave equation with time dependent coefficients*, personal communication.
- [5] F. HIROSAWA, M. REISSIG, *From Wave to Klein-Gordon Type Decay Rates*, In "Nonlinear Hyperbolic Equations, Spectral Theory, and Wavelet transformations", Oper. Theory Adv. Appl. **145**, Birkhäuser, Basel, 2003, 95-155.
- [6] K. JÖRGENS, *Das Anfangswertproblem in Groem fr eine Klasse nichtlinearen Wellengleichungen*, Math. Z., **77** (1961), 295-308.
- [7] R. T. LEWIS, *Singular Elliptic Operators of Second Order with Purely Discrete Spectra*, Trans. Amer. Math. Soc., **271** (1982), 653-666.
- [8] J. RAUCH, *The u^5 -Klein-Gordon Equation*, In "Nonlinear partial differential equations and their applications", Coll. de France Semin., Vol. I, Res. Notes Math. Pitman Research Notes in Math. **53**, Pitman 1981, 335-364.
- [9] T. RUNST, W. SICKEL, *Sobolev Spaces of Fractional Order, Nemitskij operators and Nonlinear Partial Differential Equations*, Walther de Gryter, Berlin 1996.
- [10] J. SHATAH, M. STRUWE, *Regularity Result for Nonlinear Wave Equation*, Ann. Math. **138** (1993), 503-518.
- [11] M. STRUWE, *Semilinear Wave Equations*, Bull. Am. Math. Soc., **26** (1992), 53-84.
- [12] M. STRUWE, *Globally regular solutions to the u^5 -Klein-Gordon Equation*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV Ser. 15 (1988), 495-513.

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