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Existence and boundary stabilization of a nonlinear hyperbolic equation with time-dependent coefficients *

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Abstract

In this article, we study the hyperbolic problem

$$\begin{gathered} K(x,t)u_{tt} - \sum_{j=1}^{n} \left(a(x,t)u_{x_j}\right) + F(x,t,u,\nabla u) = 0 \\ u = 0 \quad \text{on } \Gamma_1, \quad \tfrac{\partial u}{\partial \nu} + \beta(x)u_t = 0 \quad \text{on } \Gamma_0 \\ u(0) = u^0, \quad u_t(0) = u^1 \quad \text{in } \Omega\,, \end{gathered}$$

where Ω is a bounded region in $\mathbb{R}^n$ whose boundary is partitioned into two disjoint sets Γ_0, Γ_1. We prove existence, uniqueness, and uniform stability of strong and weak solutions when the coefficients and the boundary conditions provide a damping effect.

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ with C^2 boundary Γ. Assume that Γ has a partition Γ_0, Γ_1, such that each set has positive measure, and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ is empty. See the definition of these two sets in (2.1) below, and note that this condition excludes domains with connected boundary. Our objective is to study the problem

$$\begin{gathered} K(x,t)\tfrac{\partial^2 u}{\partial t^2} + A(t)u + F(x,t,u,\nabla u) = 0 \quad \text{in} \quad Q = \Omega\times]0,\infty[\\ u = 0 \quad \text{on} \quad \Sigma_1 = \Gamma_1\times]0,\infty[\\ \tfrac{\partial u}{\partial \nu_A} + \beta(x)\tfrac{\partial u}{\partial t} = 0 \quad \text{on} \quad \Sigma_0 = \Gamma_0\times]0,\infty[\\ u(0) = u^0, \quad \tfrac{\partial u}{\partial t}(0) = u^1 \quad \text{in} \quad \Omega\,, \end{gathered} \tag{1.1}$$

where $A(t) = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(a(x,t)\frac{\partial}{\partial x_j}\right)$.

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Stability of solutions for this problem with $K(x,t) = 1$, $A(t) = -\Delta$ and $F = 0$ has been studied by many authors; see for example J. P. Quinn & D. L. Russell [10], G. Chen [2,3,4], J. Lagnese [6,7], and V. Komornik & E. Zuazua [5] who also studied the nonlinear problem with $F = F(x,t,u)$. To the best of our knowledge, this is the first publication on boundary stabilization with time-dependent coefficients and the nonlinear term $F = F(x,t,u,\nabla u)$.

Stability of problems with the nonlinear term $F(x,t,u,\nabla u)$ require a careful treatment, because we do not have any information about the influence of integral $\int_\Omega F(x,t,u,\nabla u)u'\,dx$ on the energy

$$e(t) = \frac{1}{2}\int_\Omega (K(x,t)|u'(x,t)|^2 + a(x,t)|\nabla u(x,t)|^2)\,dx\,, \tag{1.2}$$

or about the sign of the derivative $e'(t)$.

When the coefficients depend on time, there are some technical difficulties that we need to overcome. First, semigroup arguments are not suitable for finding solutions to (1.1); therefore, we make use of a Galerkin approximation. For strong solutions, this approximation requires a change of variables to transform (1.1) into an equivalent problem with initial value equals zero. Secondly, the presence of ∇u in the nonlinear part brings up serious difficulties when passing to the limit.

The goal of this work is to investigate conditions on the coefficients that lead to exponential decay of an energy determined by the solution. To this end, we use the perturbed-energy method developed by V. Komornik & E. Zuazua in [5]. By establishing adequate hypotheses on $K(x,t)$, $a(x,t)$ and $F(x,t,u,\nabla u)$, the above method allow us to solve (1.1) when $\beta(x) = (x - x^0)\cdot\nu(x)$ with x^0 a point in $\mathbb{R}^n$ and $\nu(x)$ the exterior unit normal.

Our paper is divided in 4 sections. In §2, we establish notation and state our results. In §3, we prove solvability of (1.1) using the Galerkin method. In §4, we prove exponential decay of solutions.

2 Notation and statement of results

For the rest of this article, let x^0 be a fixed point in $\mathbb{R}^n$. Then put

$$m = m(x) = x - x^0\,,$$

and partition the boundary Γ into two sets:

$$\Gamma_0 = \{x \in \Gamma \,:\, m(x)\cdot\nu(x) \geq 0\}\,,\quad \Gamma_1 = \{x \in \Gamma \,:\, m(x)\cdot\nu(x) < 0\}\,. \tag{2.1}$$

Consider the Hilbert space

$$V = \{v \in H^1(\Omega) \,:\, v = 0 \quad \text{on} \quad \Gamma_1\}\,,$$

and define the following:

$$(u,v) = \textstyle\int_\Omega u(x)v(x)\,dx, \quad |u|^2 = \int_\Omega |u(x)|^2\,dx,$$
$$(u,v)_{\Gamma_0} = \textstyle\int_{\Gamma_0} u(x)v(x)\,d\Gamma, \quad |u|^2_{\Gamma_0} = \int_{\Gamma_0} |u(x)|^2\,dx,$$
$$\|u\|_\infty = \text{ess sup}_{t\geq 0}\,\|u(t)\|_{L^\infty(\Omega)}, \quad u' = u_t = \tfrac{\partial u}{\partial t}, \quad u_{x_i} = \tfrac{\partial u}{\partial x_i}$$

and

$$R(x^0) = \max_{x\in\overline{\Omega}} \|x - x^0\| \tag{2.2}$$

Now, we state the general hypotheses.

(A.1) Assumptions on $F(x,t,u,\nabla u)$. Suppose $F : \overline{\Omega}\times[0,\infty[\times\mathbb{R}^{n+1} \to \mathbb{R}$ is an element of the space $C^1(\overline{\Omega}\times[0,\infty[\times\mathbb{R}^{n+1})$ and satisfies

$$|F(x,t,\xi,\zeta)| \le C_0(1 + |\xi|^{\gamma+1} + |\zeta|) \tag{2.3}$$

where C_0 is a positive constant, and $\zeta = (\zeta_1, ..., \zeta_n)$.

Let γ be a constant such that $\gamma > 0$ for $n = 1, 2$, and $0 < \gamma \le 2/(n-2)$ for $n \ge 3$. Assume that there is a non-negative function $C(t)$ in the space $L^\infty(0,\infty)\cap L^1(0,\infty)$, such that

$$F(x,t,\xi,\zeta)\eta \ge |\xi|^\gamma\xi\eta - C(t)(1 + |\eta||\zeta|), \quad \forall \eta \in \mathbb{R}\,, \tag{2.4}$$

$$F(x,t,\xi,\zeta)\,(m\cdot\zeta) \ge |\xi|^\gamma\xi\;(m\cdot\zeta) - C(t)(1 + |\zeta||m\cdot\zeta|)\,. \tag{2.5}$$

Assume that there exist positive constants $C_0,\dots,C_n$, such that

$$|F_t(x,t,\xi,\zeta)| \le C_0\left(1 + |\xi|^{\gamma+1} + |\zeta|\right), \tag{2.6}$$

$$|F_\xi(x,t,\xi,\zeta)| \le C_0(1 + |\xi|^\gamma), \tag{2.7}$$

$$|F_{\zeta_i}(x,t,\xi,\zeta)| \le C_i \quad \text{for } i = 1,2,\dots,n\,. \tag{2.8}$$

We also assume that there exist positive constants D_1, D_2, such that for all η, $\hat\eta$ in $\mathbb{R}$ and for all ζ, $\hat\zeta$ in $\mathbb{R}^n$,

$$(F(x,t,\xi,\zeta)-F(x,t,\hat\xi,\hat\zeta))(\eta-\hat\eta) \ge -D_1(|\xi|^\gamma+|\hat\xi|^\gamma)|\xi-\hat\xi||\eta-\hat\eta|-D_2|\eta-\hat\eta||\zeta-\hat\zeta|\,. \tag{2.9}$$

The following is an example of a function F that satisfies the above conditions.

$$F(x,t,u,\nabla u) = |u|^\gamma u + \varphi(t)\sum_{i=1}^n \sin\left(\frac{\partial u}{\partial x_i}\right),$$

where φ is a function sufficiently regular.

(A.2) Assumptions on the initial data.

$$u^0, u^1 \in V\cap H^2(\Omega) \quad \text{and} \quad \frac{\partial u^0}{\partial\nu_A} + \beta(x)u^1 = 0 \text{ on } \Gamma_0\,.$$

(A.3) Assumptions on the coefficients.

$$K \in W^{1,\infty}(0,\infty; C^1(\overline{\Omega})), \quad a \in W^{1,\infty}(0,\infty; C^1(\overline{\Omega}))\cap W^{2,\infty}(0,\infty; L^\infty(\Omega))$$

$$a_t, K_t \in L^1(0,\infty; L^\infty(\Omega)), \quad \beta \in W^{1,\infty}(\Gamma_0)\,.$$

Also assume that there exist positive constants a_0, k_0, such that

$$K \geq k_0, \quad a \geq a_0, \quad \text{in } Q, \quad \text{and} \quad \beta(x) \geq 0 \quad \text{a.e. on } \Gamma_0. \tag{2.10}$$

For short notation, define

$$\begin{aligned} a(t,u,v) &= \textstyle\sum_{j=1}^n \int_\Omega a(x,t) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j}\, dx, \\ a'(t,u,v) &= \textstyle\sum_{j=1}^n \int_\Omega a_t(x,t) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j}\, dx, \\ a''(t,u,v) &= \textstyle\sum_{j=1}^n \int_\Omega a_{tt}(x,t) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j}\, dx\,. \end{aligned}$$

We observe that from the above assumptions on a, there exist positive constants a_1, a_2, and a_3 such that,

$$a_0|\nabla u|^2 \leq a(t,u,u) \leq a_1 |\nabla u|^2 \quad \forall u \in V \quad \text{and} \quad t \geq 0\,, \tag{2.11}$$

$$|a'(t,u,v)| \leq a_2 |\nabla u| |\nabla v| \quad \forall u \in V \quad \text{and} \quad t \geq 0\,, \tag{2.12}$$

$$|a''(t,u,v)| \leq a_3 |\nabla u| |\nabla v| \quad \forall u \in V \quad \text{and} \quad t \geq 0\,. \tag{2.13}$$

Now, we are in a position to state our results.

Theorem 2.1 *Under Assumptions (A1, A2, A3), Problem (1.1) possesses a unique strong solution,* $u :]0,\infty[\times\Omega \to \mathbb{R}$*, such that*

$$u \in L^\infty(0,\infty; V \cap H^2(\Omega)),\ u' \in L^\infty(0,\infty;V),\ \textit{and}\ u'' \in L^\infty(0,\infty;L^2(\Omega)).$$

Now, we present a result on stability of strong solutions, which will be extended to weak solutions. Let

$$\begin{aligned} H(t) \quad = \quad & \|\nabla a(t)\|_{L^\infty(\Omega)} + \|\nabla K(t)\|_{L^\infty(\Omega)} \\ & + \|a_t(t)\|_{L^\infty(\Omega)} + \|K_t(t)\|_{L^\infty(\Omega)} + C(t)\,. \end{aligned}$$

Theorem 2.2 *Assume that there are positive constants* $\alpha, r, \epsilon, \theta_0$*, such that for all* t *sufficiently large,*

$$\int_0^t \exp(\epsilon\theta_0 s) H(s)\, ds \leq \alpha t^r\,. \tag{2.14}$$

Then the energy (1.2) determined by the strong solution u *decays exponentially. This is, for some positive constants* $\delta, \epsilon, \theta_1$*,*

$$E(t) = e(t) + \frac{1}{\gamma+2} \int_\Omega |u(x,t)|^2\, dx \leq \delta \exp(-\epsilon\theta_1 t)\,. \tag{2.15}$$

Notice that (2.14) requires the integral to have polynomial growth. Therefore, each term in $H(t)$ behaves as a function of the form $Q(t)\exp(-\beta t)$ with $Q(t)$ a polynomial and $\beta > \epsilon\theta_0$.

An example of a function that satisfies (2.14) is $H(t) = t\exp(-\beta t)$. In fact,

$$\begin{aligned}
&\int_0^t \exp(\epsilon\theta_0 s)s\exp(-\beta s)\,ds \\
&= -\frac{t}{\beta-\theta_0\epsilon}\exp(-(\beta-\theta_0\epsilon)t) - \frac{1}{(\beta-\theta_0\epsilon)^2}\exp(-(\beta-\theta_0\epsilon)t) + \frac{1}{(\beta-\theta_0\epsilon)^2} \\
&\le \alpha t + \delta\,,
\end{aligned}$$

for some positive constants α and δ.

Theorem 2.3 *Suppose that $\{u^0, u^1\}$ is in $V \times L^2(\Omega)$, and that assumptions (A1), (A3) hold. Then (1.1) has a unique weak solution, $u : \Omega\times]0,\infty[\to \mathbb{R}$, in the space*

$$C([0,\infty);V) \cap C^1([0,\infty);L^2(\Omega))\,.$$

Furthermore, Theorem 2.2 holds for the weak solution u.

Remark Notice that as t increases, (1.1) converges to an equation of constant coefficients, and $F = |u|^\gamma u$. Hence, (1.1) can be seen as a disturbance of a much better known problem, which was studied in [5]. Also note that both equations have solutions with the same exponential decay, (2.15).

3 Existence of strong and weak solutions

In this section, we prove the existence and uniqueness of strong and weak solutions to (1.1). First we consider strong solutions, and then using a density argument we extend the same result to weak solutions.

A variational formulation of Problem (1.1) leads to the equation

$$\int_\Omega Ku''w\,dx + \int_\Omega a(x,t)\nabla u\nabla w\,dx + \int_\Omega F(x,t,u,\nabla u)w\,dx + \int_{\Gamma_0}\beta u'w\,d\Gamma = 0\,,$$

for all w in the space V.

Strong solutions to (1.1) with the boundary condition $\int_{\Gamma_0}\beta u'w\,d\Gamma$ can not be obtained by the method of "special basis"; therefore, bases formed with eigenfunctions can not be used for (1.1). Differentiating the above expression with respect to t does not help, because of the technical difficulties when estimating $u''(0)$. To avoid these difficulties, we transform (1.1) into an equivalent problem with initial value equal to zero. In fact, the change of variables

$$v(x,t) = u(x,t) - \phi(x,t) \tag{3.1}$$

$$\phi(x,t) = u^0(x) + tu^1(x), \quad t\in[0,T] \tag{3.2}$$

leads to

$$\begin{aligned}
K(x,t)v'' + A(t)v + F(x,t,\phi+v,\nabla\phi+\nabla v) = f \quad &\text{in } Q=\Omega\times(0,T), &(3.3)\\
v = 0 \quad &\text{on } \Sigma_1=\Gamma_1\times(0,T),\\
\tfrac{\partial v}{\partial\nu_A} + \beta(x)v' = g \quad &\text{on } \Sigma_0=\Gamma_0\times(0,T),\\
v(x,0) = v'(x,0) = 0\,, & &(3.4)
\end{aligned}$$

where $f(x,t) = -A(t)u^0(x) - tA(t)u^1(x)$, $(x,t) \in \Omega \times [0,T]$, and $g(x,t) = -t\frac{\partial u^1}{\partial \nu_A}$.

Note that if v is a solution of (3.3) on [0,T], then $u = v + \phi$ is a solution of (1.1) in the same interval. From estimates obtained below, we are able to prove that

$$|A(t)v(t)|^2 + |\nabla v'(t)|^2 \le C, \quad \forall t \in [0,T]\,. \tag{3.5}$$

Thus, from (3.1) and (3.2) we obtain the same inequality (3.5) for the solution u. Then using standard methods, we extend u to the interval $(0,\infty)$. Hence, it is sufficient to prove that (3.3) has a local solution, which shall be done by using the Galerkin method.

Let $(\omega_\nu)_{\nu\in\mathbf{N}}$ be a set of functions in $V \cap H^2(\Omega)$, that form and orthonormal basis for $L^2(\Omega)$. Let V_m be the space generated by $\omega_1, \omega_2, \dots, \omega_m$ and let

$$v_m(t) = \sum_{i=1}^{m} g_{jm}(t)\omega_j \tag{3.6}$$

be the solution to the Cauchy problem

$$\begin{aligned} &(K(t)v''_m(t), w) + a(t, v_m(t), w) + (\beta v'_m(t), w)_{\Gamma_0} \\ &+ \int_\Omega F(x,t,v_m+\phi, \nabla v_m + \nabla\phi)w\,dx \\ &\quad = \quad (f(t),w) + (g(t),w)_{\Gamma_0}, \quad \forall w \in V_m\,, \\ &\qquad\quad v_m(0) = v'_m(0) = 0\,. \end{aligned} \tag{3.7}$$

Observe that all the terms in the above expression are well defined. In particular, $\int_\Omega F(x,t,v_m+\phi,\nabla v_m+\nabla\phi)w\,dx$ exists because of (2.3).

By standard methods in differential equations, we can prove the existence of a solution to (3.7) on some interval $[0,t_m)$. Then this solution can be extended to the close interval by the use of the first estimate below.

A priori estimates

First Estimate: Taking $w = 2v'_m(t)$ in (3.7), we have

$$\begin{aligned} &\frac{d}{dt}\{|\sqrt{K(t)}v'_m(t)|^2 + a(t,v_m(t),v_m(t))\} + 2(\beta, v'^2_m(t))_{\Gamma_0} \\ &+2\int_\Omega F(x,t,v_m+\phi,\nabla v_m+\nabla\phi)v'_m\,dx \\ &= \quad (K_t(t), v'^2_m(t)) + a'(t,v_m(t),v_m(t)) + 2(f(t),v'_m(t)) \\ &\qquad +2\frac{d}{dt}(g(t),v_m(t))_{\Gamma_0} - 2(g'(t),v_m(t))_{\Gamma_0}\,. \end{aligned}$$

Integrating the above expression over [0,t], we obtain

$$|\sqrt{K(t)}v'_m(t)|^2 + a(t,v_m(t),v_m(t)) + 2\int_0^t (\beta, {v'_m}^2(s))_{\Gamma_0}\,ds \tag{3.8}$$

$$
\begin{aligned}
&+2\int_0^t\int_\Omega F(x,s,v_m+\phi,\nabla v_m+\nabla\phi)v_m'\,dx\,ds\\
&= \int_0^t (K_s(s),v_m'^2(s))\,ds+\int_0^t a'(s,v_m(s),v_m(s))\,ds\\
&\quad +2\int_0^t (f(s),v_m'(s))\,ds+2(g(t),v_m(t))_{\Gamma_0}-2\int_0^t (g'(s),v_m(s))_{\Gamma_0}\,ds\,.
\end{aligned}
$$

Estimate for $I_1 := 2\int_0^t\int_\Omega F(x,s,v_m+\phi,\nabla v_m+\nabla\phi)v_m'\,dx\,ds$**.** We have

$$
\begin{aligned}
I_1 &= 2\int_0^t\int_\Omega F(x,s,v_m+\phi,\nabla v_m+\nabla\phi)(v_m'+\phi')\,dx\,ds\\
&\quad -2\int_0^t\int_\Omega F(x,s,v_m+\phi,\nabla v_m+\nabla\phi)\phi'\,dx\,ds\,.
\end{aligned}
$$

From (2.3) and (2.4) it follows that

$$
\begin{aligned}
I_1 &\geq \frac{2}{\gamma+2}\|v_m(t)+\phi(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)}-\frac{2}{\gamma+2}\|\phi(0)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)}\\
&\quad -2C\int_0^t\int_\Omega (1+|v_m'+\phi'||\nabla v_m+\nabla\phi|)\,dx\,ds\\
&\quad -2C\int_0^t\int_\Omega (1+|v_m+\phi|^{\gamma+1}+|\nabla v_m+\nabla\phi|)|\phi'|\,dx\,ds\,.
\end{aligned}
\tag{3.9}
$$

Substituting (3.9) in (3.8), observing that (2.10), (2.11), (2.12) hold, and noting that $v_m(0)=v_m'(0)=0$, it follows that

$$
\begin{aligned}
&k_0|v_m'(t)|^2+a_0|\nabla v_m(t)|^2+\frac{2}{\gamma+2}\|v_m(t)+\phi(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)}+2\int_0^t(\beta,v_m'(s)^2)_{\Gamma_0}\,ds\\
&\leq \frac{2}{\gamma+2}\|\phi(0)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)}+\int_0^t (K_s(s),v_m'^2(s))\,ds+a_2\int_0^t|\nabla v_m(s)|^2\,ds\\
&\quad +2\int_0^t (f(s),v_m'(s))\,ds+2(g(t),v_m(t))_{\Gamma_0}-2\int_0^t (g'(s),v_m(s))_{\Gamma_0}\,ds\\
&\quad +2C\int_0^t\int_\Omega (1+|v_m'+\phi'||\nabla v_m+\nabla\phi|)\,dx\,ds\\
&\quad +2C\int_0^t\int_\Omega (1+|v_m+\phi|^{\gamma+1}+|\nabla v_m+\nabla\phi|)|\phi'|\,dx\,ds\,.
\end{aligned}
$$

Using Young, Hölder and the Schwarz inequalities we obtain

$$
\begin{aligned}
&k_0|v_m'(t)|^2+\frac{a_0}{2}|\nabla v_m(t)|^2+\frac{2}{\gamma+2}\|v_m(t)+\phi(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)}+2\int_0^t(\beta,v_m'^2(s))_{\Gamma_0}\,ds\\
&\leq L_0+L_1\int_0^t\left(|v_m'(s)|^2+|\nabla v_m(s)|^2+\|v_m(s)+\phi(s)\|^{\gamma+2}_{L^{\gamma+2}}\right)\,ds\,.
\end{aligned}
$$

From this inequality and the Gronwall's inequality, we obtain the first estimate,

$$|v_m'(t)|^2 + |\nabla v_m(t)|^2 + \|v_m(t)+\phi(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)} + \int_0^t (\beta, v_m'(s)^2)_{\Gamma_0}\, ds \le L\,, \quad (3.10)$$

where L is a positive constant independent of m and $t \in [0,T]$.

Second Estimate: First, we prove that $v_m''(0)$ is bounded in the $L^2(\Omega)$ norm. Indeed, considering $t=0$ in (3.7) we obtain

$$\begin{gathered}(K(0)v_m''(0), w) + a(0, v_m(0), w) + (\beta v_m'(0), w)_{\Gamma_0} + \textstyle\int_\Omega F(x,0,u^0,\nabla u^0) w\, dx \\ = (-A(0)u^0, w) \quad \forall w \in V_m\,.\end{gathered}$$

From this inequality and the fact that $v_m(0) = v_m'(0) = 0$, we get

$$(K(0)v_m''(0), w) = -\int_\Omega F(x,0,u^0,\nabla u^0) w\, dx - (A(0)u^0, w), \quad \forall w \in V_m\,.$$

With $w = v_m''(0)$ in the above equation, we obtain

$$(K(0), v_m''^2(0)) = -\int_\Omega F(x,0,u^0,\nabla u^0) v_m''(0)\, dx - (A(0)u^0, v_m''(0))$$

From this equation, (2.3), and (2.10), we conclude that

$$\begin{aligned} k_0 |v_m''(0)|^2 &\le C\int_\Omega (1 + |u^0|^{\gamma+1} + |\nabla u^0|)|v_m''(0)|\, dx + |A(0)u^0\| v_m''(0)| \\ &\le C(\Omega)[1 + |\nabla u^0|^{\gamma+1} + |\nabla u^0| + |A(0)u^0|]|v_m''(0|\,. \end{aligned}$$

That is

$$|v_m''(0)| \le C(\Omega, k_0)[1 + |\nabla u^0|^{\gamma+1} + |\nabla u^0| + |A(0)u^0|], \quad \forall m \in \mathbb{N}\,.$$

Therefore,

$$v_m''(0) \text{ is bounded in } L^2(\Omega). \quad (3.11)$$

Taking the derivative of (3.7) with respect to t, it follows that

$$\begin{aligned} &(K_t(t)v_m''(t), w) + (K(t)v_m'''(t), w) + a'(t, v_m(t), w) + a(t, v_m'(t), w) \\ &(\beta v_m''(t), w)_{\Gamma_0} + \int_\Omega F_t(x,t,v_m+\phi, \nabla v_m + \nabla\phi) w\, dx \\ &+ \int_\Omega F_{v_m+\phi}(x,t,v_m+\phi,\nabla v_m + \nabla\phi)(v_m' + \phi') w\, dx \\ &+ \sum_{i=1}^n \int_\Omega F_{v_{m x_i} + \phi x_i}(x,t,v_m+\phi,\nabla v_m+\nabla\phi)(v'_{m x_i} + \phi'_{x_i}) w\, dx \\ &\quad = (f'(t), w) + (g'(t), w)_{\Gamma_0}\,. \end{aligned}$$

Substituting w by $2v''_m(t)$ in the above expression it results that

$$\begin{aligned}
&\frac{d}{dt}\{|\sqrt{K(t)}v''_m(t)|^2 + a(t, v'_m(t), v'_m(t)) + 2a'(t, v_m(t), v'_m(t))\} + 2(\beta, v''^2_m(t))_{\Gamma_0} \\
= \quad & -(K_t(t), v''^2_m(t)) + 2a'(t, v'_m(t), v'_m(t)) + 2a''(t, v_m(t), v'_m(t)) \\
& + a'(t, v'_m(t), v'_m(t)) - 2\int_\Omega F_t(x, t, v_m + \phi, \nabla v_m + \nabla\phi)v''_m\, dx \\
& -2\int_\Omega F_{v_m+\phi}(x, t, v_m + \phi, \nabla v_m + \nabla\phi)(v'_m + \phi')v''_m\, dx \\
& -2\sum_{i=1}^n \int_\Omega F_{v_{m}x_i+\phi x_i}(x, t, v_m + \phi, \nabla v_m + \nabla\phi)(v'_{mx_i} + \phi'_{x_i})v''_m\, dx \\
& +2(f'(t), v''_m(t)) + 2\frac{d}{dt}(g'(t), v'_m(t))_{\Gamma_0}\,.
\end{aligned}$$

Integrating both sides of this equation over [0,t] and observing that $v'_m(0) = 0$, we obtain

$$\begin{aligned}
&|\sqrt{K(t)}v''_m(t)|^2 + a(t, v'_m(t), v'_m(t)) + 2\int_0^t (\beta, v''^2_m(s))_{\Gamma_0}\, ds \qquad (3.12)\\
= \quad & |\sqrt{K(0)}v''_m(0)|^2 - 2a'(t, v_m(t), v'_m(t)) - \int_0^t (K_s(s), v''^2_m(s))\, ds \\
& +3\int_0^t a'(s, v'_m(s), v'_m(s))\, ds + 2\int_0^t a''(s, v_m(s), v'_m(s))\, ds \\
& -2\int_0^t\int_\Omega F_s(x, s, v_m + \phi, \nabla v_m + \nabla\phi)v''_m\, dx\, ds \\
& -2\int_0^t\int_\Omega F_{v_m+\phi}(x, s, v_m + \phi, \nabla v_m + \nabla\phi)(v'_m + \phi')v''_m\, dx\, ds \\
& -2\sum_{i=1}^n \int_0^t\int_\Omega F_{v_{m}x_i+\phi x_i}(x, s, v_m + \phi, \nabla v_m + \nabla\phi)(v'_{mx_i} + \phi'_{x_i})v''_m\, dx\, ds \\
& +2\int_0^t (f'(s), v''_m(s))\, ds + 2(g'(t), v'_m(t))_{\Gamma_0}\,.
\end{aligned}$$

From (2.6), (2.7), (2.8), (2.10), (2.11), (2.12), (2.13), (3.10), (3.11), (3.12) and using Young, Hölder and Schwarz inequalities, and the Sobolev injection, we have

$$\begin{aligned}
&k_0|v''_m(t)|^2 + \frac{a_0}{2}|\nabla v'_m(t)|^2 + 2\int_0^t (\beta, v''^2_m(s))\, ds \\
\leq \quad & L_2 + L_3\int_0^t (|v''_m(s)|^2 + |\nabla v'_m(s)|^2)\, ds\,.
\end{aligned}$$

Then using the Gronwall's inequality, we obtain the second estimate,

$$|v''_m(t)|^2 + |\nabla v'_m(t)|^2 + \int_0^t (\beta, v''^2_m(s))\, ds \leq L\,,$$

where L is a positive constant independent of $m \in \mathbb{N}$ and $t \in [0,T]$.

The above estimates, allows us passing to the limit in the linear terms. Next we analyze the nonlinear term.

Analysis of the nonlinear term F

From (2.3) there is positive constant M such that

$$\begin{aligned}&\int_\Omega |F(x,t,v_m+\phi,\nabla v+\nabla\phi)|^2\,dx\\ &\quad\le\quad M\left(1+\|v_m(t)+\phi(t)\|^{2(\gamma+1)}_{L^{2(\gamma+1)}}+|\nabla v_m(t)+\nabla\phi(t)|^2\right).\end{aligned}$$

Therefore, from the first estimate it follows that

$$\{F(x,t,v_m+\phi,\nabla v_m+\nabla\phi)\}_{m\in\mathbf{N}} \text{ is bounded in } L^2(0,T;L^2(\Omega))\,. \tag{3.13}$$

Consequently, there exists a subsequence of $\{v_m\}_{m\in\mathbf{N}}$ (which we still denote by the same symbol) and a function χ in $L^2(0,T;L^2(\Omega))$ such that

$$F(x,t,v_m+\phi,\nabla v_m+\nabla\phi) \rightharpoonup \chi \quad \text{weak in } L^2(0,T;L^2(\Omega))\,. \tag{3.14}$$

From the above estimates after passing to the limit, we conclude that

$$Kv''+A(t)v+\chi=f \quad \text{ in } L^2(0,T;L^2(\Omega)). \tag{3.15}$$

We observe that

$$v\in L^\infty(0,T;V),\quad v'\in L^\infty(0,T;V),\quad v''\in L^\infty(0,T;L^2(\Omega))\,.$$

Moreover,

$$\frac{\partial v}{\partial\nu_A}+\beta v'=g \quad \text{in } L^\infty(0,TH^{1/2}(\Gamma_0))\,.$$

On the other hand, integrating the approximate problem (3.7) over $[0,T]$ and considering that $w=v_m$, we obtain

$$\begin{aligned}&\int_0^T (K(t)v_m''(t),v_m(t))\;dt+\int_0^T a(t,v_m(t),v_m(t))\,dt \\ &+\int_0^T (\beta v_m'(t),v_m(s))_{\Gamma_0}\;ds+\int_0^T\int_\Omega F(x,t,v_m+\phi,\nabla v_m+\nabla\phi)v_m\,dx,\,dt\\ &=\quad\int_0^T (f(t),v_m(t))\,dt+\int_0^T (g(t),v_m(t))_{\Gamma_0}\,dt\,.\end{aligned} \tag{3.16}$$

To simplify notation, subsequences will be denoted by the same symbol as the corresponding original sequences.

Notice that from the first and second estimates, and the Aubin-Lions Theorem there exists a subsequence of $\{v_m\}_{m\in\mathbb{N}}$, such that

$$v_m\to v \quad \text{strong in } L^2(0,T;L^2(\Omega))\,, \tag{3.17}$$

$$v_m'\to v' \quad \text{strong in } L^2(0,T;L^2(\Omega))\,. \tag{3.18}$$

Now, the first estimate yields

$$\left|\sqrt{\beta}v_m'(s)\right|^2_{H^{1/2}(\Gamma_0)} \leq C_0 \left|\nabla v_m'(s)\right|^2 \leq L; \quad s \in [0,T]\,, \tag{3.19}$$

and from the second estimate we get

$$\left|\sqrt{\beta}v_m''(s)\right|^2_{\Gamma_0} \leq L; \quad s \in [0,T]\,. \tag{3.20}$$

Combining (3.19) and (3.20), noting that the injection $H^{1/2}(\Gamma_0) \hookrightarrow L^2(\Gamma_0)$ is compact, and considering Aubin-Lions Theorem it follows that

$$\sqrt{\beta}v_m' \to \sqrt{\beta}v' \quad \text{in} \quad L^2(0,T;L^2(\Gamma_0)). \tag{3.21}$$

In a similar way

$$\sqrt{\beta}v_m \to \sqrt{\beta}v \quad \text{in} \quad L^2(0,T;L^2(\Gamma_0)).$$

Moreover, because of the second estimate

$$v_m'' \rightharpoonup v'' \quad \text{weak} \quad \text{in} \quad L^2(0,T;L^2(\Omega)).$$

Then, considering the strong convergences given in (3.17), (3.18) and (3.21) and the corresponding weak converges, we are able to pass to the limit in (3.16).

$$\begin{aligned} &\lim_{m\to\infty}\int_0^T a(t,v_m(t),v_m(t))\,dt \\ &= \; -\int_0^T (K(t)v''(t),v(t))\;dt - \int_0^T (\beta v'(t),v(t))_{\Gamma_0}\;dt \\ &\quad -\int_0^T\int_\Omega \chi(t)v(t)\,dx\,dt + \int_0^T (f(t),v(t))\,dt + \int_0^T (g(t),v)_{\Gamma_0}\,dt\,. \end{aligned} \tag{3.22}$$

Substituting (3.15) in (3.22), applying Green formula and noting that

$$\frac{\partial v}{\partial \nu_A} = -\beta v' + g \quad \text{a.e. on } \Gamma_0$$

we deduce that

$$\lim_{m\to\infty}\int_0^T a(t,v_m(t),v_m(s))\,dt = \int_0^T a(t,v(t),v(t))\;dt$$

and that

$$\lim_{m\to\infty}\int_0^T (\nabla v_m(t),\nabla v_m(t))\;dt = \int_0^T (\nabla v(t),\nabla v(t))\;dt\,. \tag{3.23}$$

Finally, taking into account that

$$\begin{aligned}&\int_0^T |\nabla v_m(t) - \nabla v(t)|^2\, dt \\ &= \int_0^T |\nabla v_m(t)|^2\, dt - 2\int_0^T (\nabla v_m(t), \nabla v(t))\, dt + \int_0^T |\nabla v(t)|^2\, dt\,,\end{aligned}$$

from (3.23) and the first estimate we deduce that

$$\lim_{m\to\infty} \int_0^T |\nabla v_m(t) - \nabla v(t)|^2\, dt = 0\,.$$

Therefore,

$$\nabla v_m \to \nabla v \quad \text{in} \quad L^2(0,T;L^2(\Omega))\,,$$

and consequently

$$\nabla v_m \to \nabla v \quad \text{a.e.} \quad \text{in} \quad Q_T = \Omega \times (0,T)\,.$$

From (3.17) and the above convergence, we obtain

$$F(x,t,v_m+\phi,\nabla v_m+\nabla\phi) \to F(x,t,v+\phi,\nabla v+\nabla\phi) \quad \text{a. e. in } Q_T\,.$$

Applying Lemma 1.3 in [8, Chant. 1], it follows from the above convergence, (3.13) and (3.14) that

$$F(x,t,v_m+\phi,\nabla v_m+\nabla\phi) \rightharpoonup F(x,t,v+\phi,\nabla v+\nabla\phi) \quad \text{weak in } L^2(0,T;L^2(\Omega))\,.$$

Note that the function $v:\Omega\to\mathbb{R}$ is a weak solution to the Dirichlet-Neumann problem

$$\begin{gathered} A(t)v = f^* \quad \text{in } \Omega\,, \\ v = 0 \quad \text{in } \Gamma_1\,, \qquad \tfrac{\partial v}{\partial \nu_A} = g^* \quad \text{in } \Gamma_0\,, \end{gathered}$$

where $f^* = f - Kv'' - F(x,t,v+\phi,\nabla v+\nabla\phi)$, $f^*\in L^2(\Omega)$, $g^* = -\beta v' + g$, $g^* \in H^{1/2}(\Gamma_0)$, and t is a fixed value in [0,T].

The theory of elliptic problems states that the solution v belongs to the space $L^\infty(0,T;H^2(\Omega))$; therefore, $v\in L^\infty(0,T;V\cap H^2(\Omega))$.

Uniqueness of the solution

Let u and $\hat u$ be two solutions of (1.1), and put $z = u-\hat u$. From (2.9), (2.10), (2.11) and (2.12), it follows that

$$\begin{aligned}&k_0\,|z'(t)|^2 + a_0\,|\nabla z(t)|^2 + 2\int_0^t \left(\beta, z'^2(s)\right)_{\Gamma_0}\, ds \\ &\le\; 2D_1\int_0^t\int_\Omega (|u|^\gamma + |\hat u|^\gamma)\,|z|\,|z'|\,dx\,dt + 2D_2\int_0^t\int_\Omega |z'|\,|\nabla z|\,dx\,ds \\ &\quad + \|K_1\|_\infty \int_0^t |z'(s)|^2\, ds + a_2\int_0^t |\nabla z(s)|^2\, ds\,.\end{aligned}$$

Since $0 < \gamma \leq 2/(n-2)$, for $n \geq 3$, we have the Sobolev immersion $H^1(\Omega) \hookrightarrow L^{2(\gamma+1)}(\Omega)$. This immersion is also true for all $\gamma > 0$ when $n = 1, 2$. Therefore, with

$$\frac{\gamma}{2(\gamma+1)} + \frac{1}{2(\gamma+1)} + \frac{1}{2} = 1\,,$$

and using the generalized Hölder and the Poincaré inequalities, we conclude that

$$|z'(t)|^2 + a_0\,|\nabla z(t)|^2 + 2\int_0^t \left(\beta, z'^2(s)\right)_{\Gamma_0} ds \leq C\int_0^t \left\{|z'(s)|^2 + |\nabla z(s)|^2\right\} ds\,.$$

Applying Gronwall's lemma in the last inequality we obtain $z = 0$ and therefore, $u = \hat{u}$. This concludes the proof of Theorem (2.1).

Existence of weak solutions. We have just proved the existence of strong solutions to (1.1) when u^0 and u^1 are smooth. Now by a density argument and a procedure analogous to the one in the third estimate, we prove the existence of a weak solution. The main step in this approach is obtaining a sequence that satisfy the hypothesis of compatibility (A.2). For this purpose, we define the following sequence. Given $\{u^0, u^1\}$ in $V \times L^2(\Omega)$, consider

$$u^1_\mu \in H^1_0(\Omega) \cap H^2(\Omega) \quad \text{such that} \quad u^1_\mu \to u^1 \quad \text{in } L^2(\Omega)\,,$$

and

$$u^0_\mu \in D(-\Delta) = \{u \in V \cap H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_0\}\,, \quad \text{such that } u^0_\mu \to u^0 \quad \text{in } V.$$

Uniqueness of a weak solution is guaranteed by the Visik-Ladyshenskaya method. See for example Lions and Magenes [9, section 8].

4 Asymptotic behaviour

In this section we prove exponential decay for strong solutions of (1.1), and by a density argument we obtain the same results for weak solutions.

Let us consider the modified energy

$$E(t) = e(t) + \frac{1}{\gamma+2}\int_\Omega |u(x,t)|^{\gamma+2}\,dx\,,$$

which by (2.4) satisfies

$$\begin{aligned} E'(t) \quad \leq \quad & \frac{1}{2}a'(t,u,u) + \frac{1}{2}\int_\Omega K_t(x,t)|u'|^2\,dx \\ & - \int_{\Gamma_0}(m\cdot\nu)|u'|^2\,d\Gamma + C(t)\int_\Omega (1 + |u'||\nabla u|)\,dx\,. \end{aligned} \tag{4.1}$$

Let μ and λ be positive constants such that

$$\int_{\Gamma_0}(m\cdot\nu)v^2\,d\Gamma \le \mu\int_\Omega |\nabla v|^2\,dx \quad \forall v\in V \tag{4.2}$$

$$|v|^2 \le \lambda|\nabla v|^2 \quad \forall v\in V\,. \tag{4.3}$$

For an arbitrary $\epsilon>0$ define the perturbed energy

$$E_\epsilon(t) = E(t)+\epsilon\psi(t)\,, \tag{4.4}$$

where

$$\psi(t) = 2\int_\Omega K(x,t)u'(m\cdot\nabla u)\,dx + \theta\int_\Omega K(x,t)u'u\,dx\,, \tag{4.5}$$

$\theta\in]n-2,n[$, and $\theta>\frac{2n}{\gamma+2}$. For short notation, put

$$k_1 = \min\left\{2(\theta-n+2), 2(n-\theta), (\gamma+2)(\theta-\frac{2n}{\gamma+2})\right\} > 0\,. \tag{4.6}$$

Proposition 4.1 *There exists $\delta_0>0$ such that*

$$|E_\epsilon(t)-E(t)| \le \epsilon\delta_0 E(t),\ \forall t\ge 0\ \forall\epsilon>0\,.$$

Proof: From (2.2), (2.11), (4.3), and (4.5) we obtain

$$\begin{aligned}|\psi(t)| &\le 2a_0^{-1/2}\|K\|_\infty^{1/2}R(x^0)|\sqrt{K}u'(t)|a^{1/2}(t,u,u)\\ &\quad + a_0^{-1/2}\lambda^{1/2}\theta\|K\|_\infty^{1/2}|\sqrt{K}u'(t)|a^{1/2}(t,u,u)\\ &\le a_0^{-1/2}\|K\|_\infty^{1/2}(2R(x^0)+\lambda^{1/2}\theta)E(t)\,.\end{aligned}$$

Putting $\delta_0 = a_0^{-1/2}\|K\|_\infty^{1/2}(2R(x^0)+\lambda^{1/2}\theta)$, we deduce

$$|E_\epsilon(t)-E(t)| = \epsilon|\psi(t)| \le \epsilon\delta_0 E(t)\,.$$

Which proves this proposition.

For a positive constant M, let

$$\begin{aligned}H(t) &= M\big(\|\nabla a(t)\|_{L^\infty(\Omega)} + \|\nabla K(t)\|_{L^\infty(\Omega)}\\ &\quad + \|a_t(t)\|_{L^\infty(\Omega)} + \|K_t(t)\|_{L^\infty(\Omega)} + C(t)\big)\,.\end{aligned}$$

Proposition 4.2 *There exist positive constants $\delta_1,\delta_2,\epsilon_1$ such that*

$$E'_\epsilon(t) \le -\epsilon\delta_1 E(t) + H(t)E(t) + \delta_2 C(t)\,,$$

for all $t\ge 0$ and for all $\epsilon\in(0,\epsilon_1]$.

Proof: Differentiating each term in (4.5) with respect to t and substituting $Ku'' = -A(t)u - F(x,t,u,\nabla u)$ in the expression obtained,

$$\begin{aligned}
&\psi'(t)\\
&= 2\int_\Omega K_t u'(m\cdot\nabla u)\,dx - 2\int_\Omega A(t)u(m\cdot\nabla u)\,dx\\
&\quad -2\int_\Omega F(x,t,u,\nabla u)(m\cdot\nabla u)\,dx + 2\int_\Omega Ku'(m\cdot\nabla u')\,dx + \theta\int_\Omega K_t u'u\,dx\\
&\quad -\theta\int_\Omega A(t)uu\,dx - \theta\int_\Omega F(x,t,u,\nabla u)u\,dx + \theta\int_\Omega K|u'|^2\,dx\,.
\end{aligned}$$

From (2.5) and the above identity we have

$$\begin{aligned}
\psi'(t) \le\; & 2\int_\Omega K_t u'(m\cdot\nabla u)\,dx - 2\int_\Omega A(t)u(m\cdot\nabla u)\,dx\\
& -2\int_\Omega |u|^\gamma u(m\cdot\nabla u)\,dx + 2C(t)\int_\Omega(1+|\nabla u||m\cdot\nabla u|)\,dx\\
& +2\int_\Omega Ku'(m\cdot\nabla u')\,dx + \theta\int_\Omega K_t u'u\,dx - \theta\int_\Omega A(t)uu\,dx \quad (4.7)\\
& -\theta\int_\Omega F(x,t,u,\nabla u)u\,dx + \theta\int_\Omega K|u'|^2\,dx\,.
\end{aligned}$$

Now, we estimate one by one the terms on the right-hand side of the above inequality.

Estimate for $I_1 := -2\int_\Omega A(t)u(m\cdot\nabla u)\,dx$. Using Green and Gauss formula, we obtain

$$\begin{aligned}
I_1 &= (n-2)\int_\Omega a(x,t)|\nabla u|^2\,dx + \int_\Omega(\nabla a\cdot m)|\nabla u|^2\,dx\\
&\quad - \int_\Gamma a(x,t)(m\cdot\nu)|\nabla u|^2\,d\Gamma + 2\int_\Gamma \frac{\partial u}{\partial\nu_A}(m\cdot\nabla u)\,d\Gamma\,. \qquad (4.8)
\end{aligned}$$

Estimate for $I_2 := -2\int_\Omega |u|^\gamma u(m\cdot\nabla u)\,dx$. By the Gauss formula,

$$\begin{aligned}
I_2 &= -\frac{2}{\gamma+2}\int_\Omega \nabla(|u|^{\gamma+2})\cdot m\,dx \qquad (4.9)\\
&= \frac{2n}{\gamma+2}\int_\Omega |u|^{\gamma+2}\,dx - \frac{2}{\gamma+2}\int_\Gamma(m\cdot\nu)|u|^{\gamma+2}\,d\Gamma\,.
\end{aligned}$$

From (2.1) and noting that $u|_{\Gamma_1}=0$, we have

$$-\frac{2}{\gamma+2}\int_\Gamma(m\cdot\nu)|u|^{\gamma+2}\,d\Gamma \le 0\,. \qquad (4.10)$$

Estimate for $I_3 := 2\int_\Omega Ku'(m\cdot\nabla u')\,dx$. By Gauss Theorem we get

$$\begin{aligned} I_3 &= \int_\Omega K(x,t)\; m\cdot\nabla(|u'|^2)\,dx \qquad (4.11)\\ &= -\int_\Omega(\nabla K\cdot m)|u'|^2\,dx - n\int_\Omega K(x,t)|u'|^2dx + \int_{\Gamma_0}(m\cdot\nu)K(x,t)|u'|^2d\Gamma. \end{aligned}$$

Estimate for $I_4 := -\theta\int_\Omega A(t)uu\,dx$. By Green's formula and observing that $\frac{\partial u}{\partial\nu_A} = -(m\cdot\nu)u'$ on Γ_0, it follows that

$$I_4 = -\theta\int_\Omega a(x,t)|\nabla u|^2\,dx - \theta\int_{\Gamma_0}(m\cdot\nu)u'u\,d\Gamma\,. \qquad (4.12)$$

Estimate for $I_5 := -\theta\int_\Omega F(x,t,u,\nabla u)u\,dx$. From (2.4) we deduce that

$$I_5 \le -\theta\int_\Omega |u|^{\gamma+2}\,dx + \theta C(t)\int_\Omega(1+|u||\nabla u|)\,dx\,. \qquad (4.13)$$

Thus, substituting (4.8)–(4.13) in (4.7) we conclude that

$$\begin{aligned} \psi'(t) \;\le\; & (\theta-n)\int_\Omega K(x,t)|u'|^2\,dx + (n-2-\theta)\int_\Omega a(x,t)|\nabla u|^2\,dx \qquad (4.14)\\ &+(\frac{2n}{\gamma+2}-\theta)\int_\Omega|u|^{\gamma+2}\,dx + \int_\Omega(\nabla a\cdot m)|\nabla u|^2\,dx\\ &-\int_\Omega(\nabla K\cdot m)|u'|^2\,dx + 2\int_\Omega K_tu'(m\cdot\nabla u)\,dx + \theta\int_\Omega K_tu'u\,dx\\ &+2C(t)\int_\Omega(1+|\nabla u||m\cdot\nabla u|)\,dx + \theta C(t)\int_\Omega(1+|u||\nabla u|)\,dx\\ &-\int_\Gamma(m\cdot\nu)a(x,t)|\nabla u|^2\,d\Gamma + 2\int_\Gamma\frac{\partial u}{\partial\nu_A}(m\cdot\nabla u)\,d\Gamma\\ &+\int_{\Gamma_0}(m\cdot\nu)K(x,t)|u'|^2\,d\Gamma - \theta\int_{\Gamma_0}(m\cdot\nu)u'u\,d\Gamma\,. \end{aligned}$$

On the other hand, $\frac{\partial u}{\partial x_k} = \frac{\partial u}{\partial\nu}\nu_k$ on Γ_1 implies

$$m\cdot\nabla u = (m\cdot\nu)\frac{\partial u}{\partial\nu} \quad\text{and}\quad |\nabla u|^2 = (\frac{\partial u}{\partial\nu})^2 \quad\text{on}\quad \Gamma_1\,.$$

Consequently,

$$\begin{aligned} &-\int_\Gamma(m\cdot\nu)a(x,t)|\nabla u|^2\,d\Gamma \qquad (4.15)\\ &= -\int_{\Gamma_0}(m\cdot\nu)a(x,t)|\nabla u|^2\,d\Gamma - \int_{\Gamma_1}(m\cdot\nu)a(x,t)(\frac{\partial u}{\partial\nu})^2\,d\Gamma \end{aligned}$$

and

$$2\int_\Gamma \frac{\partial u}{\partial \nu_A}(m\cdot\nabla u)\,d\Gamma = -2\int_{\Gamma_0}(m\cdot\nu)u'(m\cdot\nabla u)\,d\Gamma + 2\int_{\Gamma_1} a(x,t)(m\cdot\nu)(\frac{\partial u}{\partial \nu})^2\,d\Gamma\,. \tag{4.16}$$

In the above equality, we used that $\frac{\partial u}{\partial \nu_A} = -(m\cdot\nu)u'$ on Γ_0. Replacing (4.15) and (4.16) in (4.14), and using that $\int_{\Gamma_1} a(x,t)(m\cdot\nu)(\frac{\partial u}{\partial\nu})^2\,d\Gamma \le 0$, we obtain

$$\begin{aligned}
\psi'(t) \quad \le \quad & (\theta-n)\int_\Omega K(x,t)|u'|^2\,dx + (n-2-\theta)\int_\Omega a(x,t)|\nabla u|^2\,dx \qquad (4.17)\\
&+(\frac{2n}{\gamma+2}-\theta)\int_\Omega |u|^{\gamma+2}\,dx + \int_\Omega(\nabla a\cdot m)|\nabla u|^2\,dx\\
&-\int_\Omega(\nabla K\cdot m)|u'|^2\,dx + 2\int_\Omega K_t u'(m\cdot\nabla u)\,dx + \theta\int_\Omega K_t u' u\,dx\\
&+2C(t)\int_\Omega(1+|\nabla u||m\cdot\nabla u|)\,dx + \theta\int_\Omega C(t)(1+|u||\nabla u|)\,dx\\
&-\int_{\Gamma_0}(m\cdot\nu)a(x,t)|\nabla u|^2\,d\Gamma - 2\int_{\Gamma_0}(m\cdot\nu)u'(m\cdot\nabla u)\,d\Gamma\\
&+\int_{\Gamma_0}(m\cdot\nu)K(x,t)|u'|^2\,d\Gamma - \theta\int_{\Gamma_0}(m\cdot\nu)u'u\,d\Gamma\,.
\end{aligned}$$

However, since

$$\begin{aligned}
&-2\int_{\Gamma_0}(m\cdot\nu)u'(m\cdot\nabla u)\,d\Gamma\\
&\quad\le\quad a_0^{-1}\mathbb{R}^2(x^0)\int_{\Gamma_0}(m\cdot\nu)|u'|^2\,d\Gamma + \int_{\Gamma_0}(m\cdot\nu)a(x,t)|\nabla u|^2\,d\Gamma\,,
\end{aligned}$$

from (4.17) it results that

$$\begin{aligned}
&\psi'(t)\\
&\le\quad (\theta-n)\int_\Omega K(x,t)|u'|^2\,dx + (n-2-\theta)\int_\Omega |\nabla u|^2\,dx \qquad (4.18)\\
&\quad +(\frac{2n}{\gamma+2}-\theta)\int_\Omega |u|^{\gamma+2}\,dx + \int_\Omega(\nabla a\cdot m)|\nabla u|^2\,dx - \int_\Omega(\nabla K\cdot m)|u'|^2\,dx\\
&\quad +2\int_\Omega K_t u'(m\cdot\nabla u)\,dx + \theta\int_\Omega K_t u'u\,dx + 2C(t)\int_\Omega(1+|\nabla u||m\cdot\nabla u|)\,dx\\
&\quad +\theta\int_\Omega C(t)(1+|u||\nabla u|)\,dx + a_0^{-1}\mathbb{R}^2(x^0)\int_{\Gamma_0}(m\cdot\nu)|u'|^2\,d\Gamma\\
&\quad +\int_{\Gamma_0}(m\cdot\nu)K(x,t)|u'|^2\,d\Gamma - \theta\int_{\Gamma_0}(m\cdot\nu)u'u\,d\Gamma\,.
\end{aligned}$$

Let k_2 be a positive real number such that $0<k_2<k_1$. Then from (4.2),

$$-\theta\int_{\Gamma_0}(m\cdot\nu)u'u\,d\Gamma \le \frac{\mu\theta^2}{2a_0k_2}\int_{\Gamma_0}(m\cdot\nu)|u'|^2\,d\Gamma + k_2E(t)\,. \tag{4.19}$$

Therefore, from (4.6), (4.18), and (4.19) it follows that

$$\begin{aligned}\psi'(t) & \\ \leq\ & -(k_1-k_2)E(t)+\int_\Omega(\nabla a\cdot m)|\nabla u|^2dx-\int_\Omega(\nabla K\cdot m)|u'|^2\,dx \qquad (4.20)\\ & +2\int_\Omega K_t u'(m\cdot\nabla u)\,dx+\theta\int_\Omega K_t u' u\,dx+2C(t)\int_\Omega(1+|\nabla u\|m\cdot\nabla u|)\,dx\\ & +\theta\int_\Omega C(t)(1+|u\|\nabla u|)\,dx+a_0^{-1}\mathbb{R}^2(x^0)\int_{\Gamma_0}(m\cdot\nu)|u'|^2\,d\Gamma\\ & +\int_{\Gamma_0}(m\cdot\nu)K(x,t)|u'|^2\,d\Gamma+\frac{\mu\theta^2}{2a_0k_2}\int_{\Gamma_0}(m\cdot\nu)|u'|^2 d\Gamma\,.\end{aligned}$$

From (4.20), we obtain

$$\begin{aligned}\psi'(t)\ \leq\ & -(k_1-k_2)E(t)+(M_1\|\nabla a(t)\|_{L^\infty(\Omega)}+M_2\|\nabla K(t)\|_{L^\infty(\Omega)}\\ & +M_3\|K_t(t)\|_{L^\infty(\Omega)}+M_4C(t))E(t)+(\theta+2)\,\mathrm{meas}(\Omega)C(t)\\ & +(a_0^{-1}R^2(x^0)+\frac{\mu\theta^2}{2a_0k_2}+\|K\|_\infty)\int_{\Gamma_0}m\cdot\nu|u'|^2\,d\Gamma\,, \qquad (4.21)\end{aligned}$$

where

$$\begin{gathered}M_1=2a_0^{-1}R(x^0),\quad M_2=2k_0^{-1}R(x^0),\\ M_3=2k_0^{-1/2}a_0^{-1/2}R(x^0)+\theta\lambda^{1/2}k_0^{-1/2}a_0^{-1/2},\\ M_4=4a_0^{-1}R(x^0)+2\theta\lambda^{1/2}a_0^{-1}.\end{gathered}$$

Define

$$G(t)=M_1\|\nabla a(t)\|_{L^\infty(\Omega)}+M_2\|\nabla K(t)||_{L^\infty(\Omega)}+M_3\|K_t(t)\|_{L^\infty(\Omega)}+M_4C(t)\,. \tag{4.22}$$

Then from (4.1), (4.4), (4.21), and (4.22), we obtain

$$\begin{aligned}E'_\epsilon(t)\ =\ & E'(t)+\epsilon\psi'(t) \qquad (4.23)\\ \leq\ & \frac{1}{2}a'(t,u,u)+\frac{1}{2}\int_\Omega K_t|u'|^2\,dx+C(t)\int_\Omega(1+|u'\|\nabla u|)\,dx\\ & -\epsilon(k_1-k_2)E(t)+\epsilon G(t)E(t)+\epsilon(\theta+2)\,\mathrm{meas}(\Omega)C(t)\\ & -\int_{\Gamma_0}(m\cdot\nu)\left[1-(a_0^{-1}\mathbb{R}^2(x^0)+\frac{\mu\theta^2}{2a_0k_2}+\|K\|_\infty)\epsilon\right]|u'|^2\,d\Gamma\,.\end{aligned}$$

By setting $\delta_1=k_1-k_2$, from (4.23) we conclude that

$$\begin{aligned}E'_\epsilon(t)\ \leq\ & -\epsilon\delta_1E(t)+\epsilon G(t)E(t)+\epsilon(\theta+2)\,\mathrm{meas}(\Omega)C(t)+J(t)E(t)\\ & +\mathrm{meas}(\Omega)C(t)+k_0^{-1/2}a_0^{-1/2}C(t)E(t) \qquad (4.24)\\ & -\int_{\Gamma_0}(m\cdot\nu)\left[1-(a_0^{-1}\mathbb{R}^2(x^0)+\frac{\mu\theta^2}{2a_0k_2}+\|K\|_\infty)\epsilon\right]|u'|^2\,d\Gamma\,,\end{aligned}$$

where

$$J(t) = a_0^{-1}\|a_t(t)\|_{L^\infty(\Omega)} + k_0^{-1}\|K_t(t)||_{L^\infty(\Omega)}\,.$$

Let $\epsilon_1 = \min\{(a_0^{-1}\mathbb{R}^2(x^0) + \frac{\mu\theta^2}{2a_0k_2} + \|K\|_\infty)^{-1}, 1\}$. Then from (4.24), we obtain that for all $\epsilon \in (0, \epsilon_1]$,

$$\begin{aligned} E_\epsilon'(t) &\le -\epsilon\delta_1 E(t) + (G(t) + J(t) + k_0^{-1/2}a_0^{-1/2}C(t))E(t) \\ &\quad +(\theta+3)\,\mathrm{meas}(\Omega)C(t) \\ &\le -\epsilon\delta_1 E(t) + H(t)E(t) + \delta_2 C(t)\,, \end{aligned}$$

where $M = \max\{M_1, M_2, M_3 + k_0^{-1}, a_0^{-1}, M_4 + k_0^{-1/2}a_0^{-1/2}\}$ and $\delta_2 = (\theta + 3)\,\mathrm{meas}(\Omega)$. Which completes the proof of Proposition 4.2.

Proposition 4.3 *There exists a positive constant δ_3 such that*

$$E(t) \le \delta_3 \quad \forall t \ge 0\,.$$

Proof: We shall show that the constant is given by

$$\delta_3 = (E(0) + \mathrm{meas}(\Omega)\|C\|_{L^1(0,\infty)})\exp(\int_0^\infty \mathcal{F}(t)\,dt)\,,$$

where $\mathcal{F}(t) = a_0^{-1}\|a_t(t)\|_{L^\infty(\Omega)} + k_0^{-1}||K_t(t)\|_{L^\infty(\Omega)} + k_0^{-1/2}a_0^{-1/2}C(t)$.

From (4.1) we have

$$E'(t) \le \mathcal{F}(t)E(t) + \mathrm{meas}(\Omega)C(t).$$

Hence

$$\frac{d}{dt}(E(t)exp\left(-\int_0^t \mathcal{F}(s)\,ds\right)) \le \mathrm{meas}\,(\Omega)C(t)\exp\left(-\int_0^t \mathcal{F}(s)\,ds\right)\,.$$

Therefore,

$$E(t) \le E(0)\exp\left(\int_0^\infty \mathcal{F}(s)\,ds\right) + \mathrm{meas}(\Omega)\exp\left(\int_0^\infty \mathcal{F}(s)\,ds\right)\int_0^t C(s)\,ds\,.$$

Which completes the proof of this porposition.

Now, we prove exponential decay. In what follows, let

$$\epsilon_0 = \min\{\epsilon_1, \frac{1}{2\delta_0}\}$$

where δ_0 is the constant obtained in Proposition 4.1.

For all $\epsilon \in (0, \epsilon_0]$, we have

$$\frac{1}{2}E(t) \le E_\epsilon(t) \le \frac{3}{2}E(t) \le 2E(t), \quad \forall t \ge 0\,. \tag{4.25}$$

Consequently, from (4.25) and Proposition 4.2 we obtain

$$E'_\epsilon(t) \le -\frac{\epsilon}{2}\delta_1 E_\epsilon(t) + H(t)E(t) + \delta_2 C(t)\,. \tag{4.26}$$

From Proposition 4.3 and (4.26), we get

$$E'_\epsilon(t) \le -\frac{\epsilon}{2}\delta_1 E_\epsilon(t) + \delta_3 H(t) + \delta_2 C(t)\,.$$

Therefore,

$$\frac{d}{dt}(E_\epsilon(t)\exp(\frac{\epsilon}{2}\delta_1 t)) \le \exp(\frac{\epsilon}{2}\delta_1 t)(\delta_3 H(t) + \delta_2 C(t))\,. \tag{4.27}$$

Integrating (4.27) over [0,t] and using (4.25), we conclude

$$\begin{aligned}
\frac{1}{2}E(t) \quad &\le \quad \frac{3}{2}E(0)\exp(-\frac{\epsilon}{2}\delta_1 t) \\
&\quad + \left[\delta_3\int_0^t \exp(\frac{\epsilon}{2}\delta_1 s)H(s)\,ds + \delta_2\int_0^t \exp(\frac{\epsilon}{2}\delta_1 s)C(s)\,ds\right]\exp(-\frac{\epsilon}{2}\delta_1 t) \\
&\le \quad \frac{3}{2}E(0)\exp(-\frac{\epsilon}{2}\delta_1 t) \\
&\quad + \max\{\delta_2,\delta_3\}\left[\int_0^t \; exp(\frac{\epsilon}{2}\delta_1 s)(H(s)+C(s))\,ds\right]\exp(-\frac{\epsilon}{2}\delta_1 t)\,.
\end{aligned}$$

From the above inequality and (2.14), we obtain exponential decay, which completes the proof of Theorem 2.2.

Remark 1. Exponential decay for weak solutions can be proved using a density argument.

Remark 2. Theorem 2.3 remains valid for $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ not empty when $A(t) = -\Delta$ and $n \le 3$. In which case, the Rellich identity given in (4.8) can be replaced by the Grisvard inequality,

$$I_1 \le (n-2)\int_\Omega |\nabla u|^2\,dx - \int_\Gamma (m\cdot\nu)|\nabla u|^2\,d\Gamma + 2\int_\Gamma \frac{\partial u}{\partial \nu}(m\cdot\nabla u)\,d\Gamma\,.$$

The proof of this inequality can be found in Komornik and Zuazua [5].

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