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NONLINEAR PERTURBATIONS OF THE KIRCHHOFF EQUATION

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Abstract. In this article we study the existence and uniqueness of local solutions for the initial-boundary value problem for the Kirchhoff equation

$$
\begin{gathered}
u'' - M(t, \|u(t)\|^2)\Delta u + |u|^\rho = f \quad \text{in } \Omega \times (0, T_0), \\
u = 0 \quad \text{on } \Gamma_0 \times]0, T_0[, \\
\frac{\partial u}{\partial \nu} + \delta h(u') = 0 \quad \text{on } \Gamma_1 \times]0, T_0[,
\end{gathered}
$$

where Ω is a bounded domain of $\mathbb{R}^n$ with its boundary constiting of two disjoint parts Γ_0 and Γ_1; $\rho > 1$ is a real number; $\nu(x)$ is the exterior unit normal vector at $x \in \Gamma_1$ and $\delta(x), h(s)$ are real functions defined in Γ_1 and $\mathbb{R}$, respectively. Our result is obtained using the Galerkin method with a special basis, the Tartar argument, the compactness approach, and a Fixed-Point method.

1. Introduction

Frist we do some preliminary considerations to justify the mixed problem we want to study. Milla Miranda and Medeiros [20] analyzed the existence of solutions for problem

$$
\begin{gathered}
u'' - \mu(t)\Delta u = 0 \quad \text{in } \Omega \times (0, \infty), \\
u = 0 \quad \text{on } \Gamma_0 \times (0, \infty), \\
\mu(t)\frac{\partial u}{\partial \nu} + \delta(x)u' = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \\
u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x), \quad x \in \Omega.
\end{gathered}
\tag{1.1}
$$

When μ is a positive constant, existence and uniqueness of global solutions for (1.1) has been proved by Komornik and Zuazua [5], Lasiecka and Triggiane [9] and Quinn and Russell [22], Goldstein [4] applying semigroup theory. This method does not work for (1.1) because the boundary condition $(1.1)_3$ brings serious difficulties. For this reason, the authors of [20] defined a special basis of the space where lie the approximations of the initial data and apply the Galerkin method. This approach works well for problem (1.1).

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Motivated by (1.1), Milla Miranda and Jutuca [21] analized the initial-boundary value problem for the Kirchhoff equation

$$
\begin{gathered}
u'' - M\Big(t, \int_\Omega |\nabla u|^2 dx\Big)\Delta u = f \quad \text{in } \Omega \times (0,\infty),\\
u = 0 \quad \text{on } \Gamma_0 \times (0,\infty),\\
\mu(t)\frac{\partial u}{\partial \nu} + \delta(x) u' = 0 \quad \text{on } \Gamma_1 \times (0,\infty),\\
u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad x \in \Omega.
\end{gathered} \tag{1.2}
$$

Following the ideas in [20] but having much more difficulty, the authors of [21], succeeded in the construction of a special basis and the Galerkin method works well for (1.2). They proved existence and uniqueness of solutions for (1.2). See also [3, 7].

An extensive list of references about the Kirchhoff equation can be found in Medeiros, Limaco and Menezes [17]. In Medeiros et al. [16] was investigated the existence and uniqueness of global solutions for the problem

$$
\begin{gathered}
u'' - \Delta u + |u|^\rho = f \quad \text{in } \Omega \times (0,\infty)\\
u = 0 \quad \text{on } \Gamma \times (0,\infty)\\
u(x,0) = u^0(x), \quad u'(x,0) = u^1(x), \quad x \in \Omega
\end{gathered} \tag{1.3}
$$

There, Galerkin method and Tartar argument [23] were applied.

Motivated by the studies of (1.1)-(1.3), we investigate the existence and uniqueness of local solutions of the initial value problem for the nonlinear mixed problem of Kirchhoff type:

$$
\begin{gathered}
u'' - M\Big(t, \int_\Omega |\nabla u|^2 dx\Big)\Delta u + |u|^\rho = f \quad \text{in } \Omega \times (0,T_0),\\
u = 0 \quad \text{on } \Gamma_0 \times (0,T_0),\\
\frac{\partial u}{\partial \nu} + \delta(x) h(u') = 0 \quad \text{on } \Gamma_1 \times (0,T_0),\\
u(x,0) = u^0(x), \quad u'(x,0) = u^1(x), \quad x \in \Omega.
\end{gathered} \tag{1.4}
$$

By applying the Galerkin method with a special basis, a modification of the Tartar approach, compactness method and fixed-point theorem, we obtain our result.

Note that the existence of global solutions for (1.4) without the term $|u|^\rho = 0$, null Dirichlet boundary condition on Γ and $u^0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u^1 \in H_0^1(\Omega)$ is a open question.

2. Notation and statement of main results

Let Ω be bounded open set of $\mathbb{R}^n$ with boundary Γ of class C^2. It is assumed that Γ is constituted by two disjoint parts Γ_0 and Γ_1, Γ_0 and Γ_1 with positive measures, such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. By $\nu(x)$ represents the unit normal vector at $x \in \Gamma_1$.

We denote by $H^m(\Omega)$ the Sobolev space of order m and by (u, v) and $|u|$, the scalar product and norm, respectively, in $L^2(\Omega)$. We define the Hilbert space

$$
V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\},
$$

equipped with the scalar product

$$((u,v)) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_i}(x)\, dx$$

and norm $\|u\|^2 = ((u,u))$. All scalar functions considered in this article will be real-valued. To state our main result, we introduce the following hypotheses:

(H1) The function $M(t,\lambda)$ satisfies $M \in W^{1,\infty}_{\rm loc}([0,\infty[^2)$, $M(t,\lambda) \geq m_0 > 0$ for all $\{t,\lambda\} \in ([0,\infty[)^2$ with m_0 constant.

(H2) The function h is a Lipschitz continuous, $h(0) = 0$, and h is strongly monotonous, that is, for a positive constant d_0,

$$(h(r) - h(s))(r-s) \geq d_0(r-s)^2, \quad \forall r,s \in \mathbb{R}.$$

(H3) $\delta \in W^{1,\infty}(\Gamma_1)$ and $\delta(x) \geq \delta_0$ for all $x \in \Gamma_1$ and a positive constant δ_0.

(H4) The real number ρ satisfies the following restrictions

$$\rho > 1 \text{ if } n = 1,2; \quad \frac{n+1}{n} \leq \rho \leq \frac{n}{n-2} \text{ if } n \geq 3. \tag{2.1}$$

Let $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with $h(0) = 0$. In Marcus and Mizel [14] (see also [2]) it is shown that $h(v) \in H^{1/2}(\Gamma_1)$ for $v \in H^{1/2}(\Gamma_1)$ and $h : H^{1/2}(\Gamma_1) \to H^{1/2}(\Gamma_1)$, $v \mapsto h(v)$, is continuous.

Remark 2.1. Consider the trace of order zero $\gamma_0 : V \to H^{1/2}(\Gamma_1)$. Then the map

$$\widetilde{h} = h \circ \gamma_0, \quad \widetilde{h} : V \to H^{1/2}(\Gamma_1)$$

is continuous.

Throughout the article, to facilitate the notation, the mapping $\widetilde{h}(v)$, $v \in V$, will be denoted by $h(v)$.

Remark 2.2. Let $\delta : \Gamma_1 \to \mathbb{R}$ be a function such that $\delta \in W^{1,\infty}(\Gamma_1)$. Then $\delta v \in H^{1/2}(\Gamma_1)$ for $v \in H^{1/2}(\Gamma_1)$, and the linear operator

$$\delta : H^{1/2}(\Gamma_1) \to H^{1/2}(\Gamma_1), \quad v \mapsto \delta v$$

is continuous.

Also, the linear operators

$$\delta : H^1(\Gamma_1) \to H^1(\Gamma_1), \quad v \mapsto \delta v,$$
$$\delta : L^2(\Gamma_1) \to L^2(\Gamma_1),\ v \mapsto \delta v$$

are continuous. The statements in this remark follow from the theory of interpolation of Hilbert spaces, see Lions-Magenes [12].

Next, we state our main result.

Theorem 2.3. *Assume that hypotheses* (H1)–(H4) *are satisfied. Consider* $\{u^0, u^1\}$ *in* $V \cap H^2(\Omega) \times V$ *satisfying the compatibility condition*

$$\frac{\partial u^0}{\partial \nu} + \delta h(u^1) = 0, \tag{2.2}$$

and the norm condition

$$\|u^0\| < \lambda^* := \Big(\frac{m_0}{3k_0^{\rho+1}}\Big)^{\frac{1}{\rho-1}}, \tag{2.3}$$

where k_0 *is the immersion constant of* V *in* $L^{\rho+1}(\Omega)$*, and*

$$f \in L^1(0,T;L^2(\Omega)), \quad f' \in L^1(0,T;L^2(\Omega))\,. \tag{2.4}$$

Then there exist a real number $0 < T_0 \leq T$*, and a unique function* u *with*

$$\begin{gathered} u \in L^\infty(0,T_0;V \cap H^2(\Omega)), \\ u' \in L^\infty(0,T_0;V), \\ u'' \in L^\infty(0,T_0;L^2(\Omega)) \cap L^2(0,T_0;L^2(\Gamma_1)), \end{gathered} \tag{2.5}$$

such that u *satisfies*

$$u'' - M(\cdot, \|u\|^2)\Delta u + |u|^\rho = f \quad \text{in } L^\infty(0,T_0;L^2(\Omega)), \tag{2.6}$$

$$\begin{gathered} \frac{\partial u}{\partial \nu} + \delta h(u') = 0 \quad \text{in } L^2(0,T_0;H^{1/2}(\Gamma_1)), \\ \frac{\partial u'}{\partial \nu} + \delta h'(u')u'' = 0 \quad \text{in } L^2(0,T_0;L^2(\Gamma_1)), \end{gathered} \tag{2.7}$$

and

$$u(0) = u^0, \quad u'(0) = u^1, \tag{2.8}$$

Remark 2.4. By Remarks 2.1 and 2.2, the function $\delta h(u^1)$ belongs to $H^{1/2}(\Gamma_1)$. Then condition (2.2) makes sense.

3. Existence of Solutions

To apply Banach Fixed-Point Theorem in the proof of our result, we introduce an auxiliary problem related to (1.4).

3.1. **Auxiliary Problem.** Consider the problem

$$\begin{gathered} u'' - \mu\Delta u + |u|^\rho = f \quad \text{in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \\ \frac{\partial u}{\partial \nu} + \delta h(u') = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \\ u(0) = u^0, \quad u'(0) = u^1 \quad \text{in } \Omega. \end{gathered} \tag{3.1}$$

Where $\mu(t), h(s)$ and δ are real functions defined in $[0,\infty)$, $\mathbb{R}$ and Γ_1, respectively.

The existence of solutions of (3.1) is derived by applying the Galerkin method with a special basis of $V \cap H^2(\Omega)$ and a modification of the Tartar method. To obtain this basis we introduce some results.

Lemma 3.1. *Let* m *and* n *be functions in* $L^1(0,T)$ *with* $m(t) \geq 0$ *and* $n(t) \geq 0$ *a.e.* t *in* $(0,T)$ *and let* $a \geq 0$ *be a constant. Consider* $\varphi : [0,T] \to \mathbb{R}$ *continuous,* $\varphi(t) \geq 0$*, for all* $t \in [0,T]$*, and satisfying*

$$\frac{1}{2}\varphi^2(t) \leq \frac{1}{2}a^2 + \int_0^t m(\tau)\varphi(\tau)d\tau + \int_0^t n(\tau)\varphi^2(\tau)d\tau, \quad \forall t \in [0,T].$$

Then

$$\varphi(t) \leq \Big(a + \int_0^T m(\tau)d\tau\Big) \exp\Big(\int_0^t n(\tau)d\tau\Big), \quad \forall t \in [0,T].$$

The above result is a consequence of a lemma provided in Brezis [1, p. 157]. Milla Miranda and Medeiros [20] showed the following three results:

Proposition 3.2. *Let us consider $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma_1)$. Then, the solution u of the problem*

$$\begin{gathered} -\Delta u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma_1, \end{gathered} \tag{3.2}$$

belongs to $V \cap H^2(\Omega)$ and satisfies

$$\|u\|^2_{H^2(\Omega)} \le c\big[|f|^2 + \|g\|^2_{H^{1/2}(\Gamma_1)}\big],$$

where the constant $c > 0$ is independent of u, f and g.

Proposition 3.3. *In $V \cap H^2(\Omega)$ the norms $H^2(\Omega)$ and*

$$\Big[|\Delta u|^2 + \|\frac{\partial u}{\partial \nu}\|^2_{H^{1/2}(\Gamma_1)}\Big]^{1/2},$$

are equivalent.

We equipp $V \cap H^2(\Omega)$ with the preceding norm.

Remark 3.4. The space $V \cap H^2(\Omega)$ is dense in V. In fact, we consider the operator $A = -\Delta$ defined by the triplet $\{V, L^2(\Omega), ((u,v))\}$. Then its domain $D(-\Delta)$ is

$$D(-\Delta) = \big\{v \in V \cap H^2(\Omega); \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_1\big\},$$

is dense in V (see [11]). As $D(-\Delta)$ is contained in $V \cap H^2(\Omega)$, the conclusion follows.

Lemma 3.5. Consider a function δ satisfying hypothesis (H3), and a Lipschitz continuous function $h(s)$, $s \in \mathbb{R}$, with $h(0) = 0$. Take $u^0 \in V \cap H^2(\Omega)$ and $u^1 \in V$ satisfying the condition

$$\frac{\partial u^0}{\partial \nu} + \delta h(u^1) = 0 \quad \text{on } \Gamma_1. \tag{3.3}$$

Then, for each $\varepsilon > 0$, there exist w and z in $V \cap H^2(\Omega)$ such that

$$\begin{gathered} \|w - u^0\|_{V\cap H^2(\Omega)} < \varepsilon, \quad \|z - u^1\| < \varepsilon, \\ \frac{\partial w}{\partial \nu} + \delta h(z) = 0 \quad \text{on } \Gamma_1. \end{gathered}$$

With respect to the function μ we make the following assumptions:

$$\mu \in W^{1,1}_{\rm loc}(0,\infty), \quad 0 < \mu_0 \le \mu(t) \le \mu_1, \quad \forall t \ge 0, \quad \mu' \in L^1(0,\infty) \tag{3.4}$$

for some constants μ_0, μ_1.

Consider the real number ρ satisfying the restrictions (H4). Then

$$V \hookrightarrow L^{p^*}(\Omega) \hookrightarrow L^{2\rho}(\Omega) \hookrightarrow L^{\rho+1}(\Omega) \hookrightarrow L^{\rho}(\Omega) \tag{3.5}$$

where $p^* = \frac{2n}{n-2}$, $n \ge 3$. In what follows $X \hookrightarrow Y$ denotes that injection of the space X into the space Y is continuous. Note that when $p > 1$ and $n = 1$ or $n = 2$, the continuous injections (3.5) without $L^{p^*}(\Omega)$ is true.

With respect to the above injections, we introduce the following notation:

$$\begin{gathered} \|v\|_{L^{\rho+1}(\Omega)} \le k_0\|v\|, \quad \|v\|_{L^{\rho}(\Omega)} \le k_1\|v\|, \\ \|v\|_{L^{2\rho}(\Omega)} \le k_2\|v\|, \quad \|v\|_{L^{(\rho-1)n}(\Omega)} \le k_3\|v\|, \\ \|v\|_{L^{p^*}(\Omega)} \le k_4\|v\| \end{gathered} \tag{3.6}$$

for all $v \in V$.

Consider

$$\|u^0\| < \lambda_1^* := \Big(\frac{\mu_0}{3k_0^{\rho+1}}\Big)^{\frac{1}{\rho-1}}, \tag{3.7}$$

$$G(s) = \frac{1}{\rho+1}|s|^\rho s. \tag{3.8}$$

Recall that $G(s) = \int_0^s |\tau|^\rho d\tau$. With the above assumptions, we have the following result.

Theorem 3.6. *Assume hypotheses* (H1), (H3), (H4) *and* (3.4). *Consider*

$$u^0 \in V \cap H^2(\Omega), \quad u^1 \in V, f \in L^1(0,\infty; L^2(\Omega)), \quad f' \in L^1_{\rm loc}(0,\infty; L^2(\Omega)) \tag{3.9}$$

satisfying (2.2) *and*

$$\begin{gathered} \|u^0\| < \lambda_1^*, \\ \big(\frac{2}{\mu_0}\big)^{1/2}\Big[(2N)^{1/2} + \int_0^\infty |f(t)|dt\Big] \exp\Big(\frac{2}{\mu_0}\int_0^\infty |\mu'(t)|dt\Big) < \lambda_1^*, \end{gathered} \tag{3.10}$$

where

$$N = \frac{1}{2}|u^1|^2 + \frac{1}{2}\mu(0)\|u^0\|^2 + \frac{k_0^{\rho+1}}{\rho+1}\|u^0\|^{\rho+1}. \tag{3.11}$$

and the real number λ_1^* *defined in* (3.7). *Then there exists a function* u *with*

$$\begin{gathered} u \in L^\infty(0,\infty; V), \quad u' \in L^\infty(0,\infty; L^2(\Omega)) \cap L^\infty_{\rm loc}(0,\infty; V) \\ u'' \in L^\infty_{\rm loc}(0,\infty; L^2(\Omega)), \quad u' \in L^\infty(0,\infty; L^2(\Gamma_1)); \\ u'' \in L^\infty_{\rm loc}(0,\infty; L^2(\Gamma_1)) \end{gathered} \tag{3.12}$$

satisfying

$$u'' - \mu\Delta u + |u|^\rho = f \quad \text{in } L^2_{\rm loc}(0,\infty; L^2(\Omega)), \tag{3.13}$$

$$\frac{\partial u}{\partial \nu} + \delta h(u') = 0 \quad \text{in } L^2_{\rm loc}(0,\infty; H^{1/2}(\Gamma_1)), \tag{3.14}$$

$$\frac{\partial u'}{\partial \nu} + \delta h'(u')u'' = 0 \quad \text{in } L^2_{\rm loc}(0,\infty; L^2(\Gamma_1)), \tag{3.15}$$

$$u(0) = u^0, \quad u'(0) = u^1. \tag{3.16}$$

Proof of Theorem 3.6. By Lemma 3.5, we obtain sequences $(u_l^0), (u_l^1)$ of vectors of $V \cap H^2(\Omega)$ satisfying

$$\begin{gathered} \lim_{l\to\infty} u_l^0 = u^0 \quad \text{in } V \cap H^2(\Omega) \\ \lim_{l\to\infty} u_l^1 = u^1 \quad \text{in } V \\ \frac{\partial u_l^0}{\partial \nu} + \delta h(u_l^1) = 0 \quad \text{on } \Gamma_1,\ \forall l \in \mathbb{N}. \end{gathered} \tag{3.17}$$

We construct a special basis of $V \cap H^2(\Omega)$ as follows: Fix $l \in \mathbb{N}$. Consider the basis

$$\{w_1^l, w_2^l, \dots, w_j^l, \dots\},$$

of $V \cap H^2(\Omega)$ satisfying $u^0, u^1 \in [w_1^l, w_2^l]$, where $[w_1^l, w_2^l]$ denotes the subspace generated by w_1^l, w_2^l. With this basis determine approximate solutions $u_{lm}(t)$ of

Problem (3.1), that is,

$$
\begin{gathered}
u_{lm}(t)=\sum_{j=1}^{m} g_{jlm}(t)w_j^l,\\
(u''_{lm}(t),v)+\mu(t)((u_{lm}(t),v))+(|u_{lm}(t)|^\rho,v)\\
+\mu(t)\int_{\Gamma_1}\delta h(u'_{lm}(t))v d\Gamma=(f(t),v),\quad \forall v\in V_m^l,\\
u_{lm}(0)=u_l^0,\quad u'_{lm}(0)=u_l^1,
\end{gathered}
\tag{3.18}
$$

where V_m^l is the subspace generated by $w_1^l,w_2^l,\dots,w_m^l$.

The above finite-dimensional system has a solution u_{lm} defined in $[0,t_{lm})$. The following estimates allow us to extend this solution to the interval $[0,\infty)$

First Estimate. Set $v=u'_{lm}$ in $(3.18)_1$. We have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u'_{lm}(t)|^2+\frac{1}{2}\frac{d}{dt}\big[\mu(t)\|u'_{lm}(t)\|^2\big]+\frac{d}{dt}\int_\Omega G(u_{lm}(t))dx\\
&+\mu(t)\int_{\Gamma_1}\delta h(u'_{lm}(t))u'_{lm}(t)d\Gamma\\
&=(f(t),u'_{lm}(t))+\frac{1}{2}\mu'(t)\|u'_{lm}(t)\|^2.
\end{aligned}
$$

Integrating on $[0,t]$, $0<t<t_{lm}$, we obtain

$$
\begin{aligned}
&\frac{1}{2}|u'_{lm}(t)|^2+\frac{\mu(t)}{2}\|u'_{lm}(t)\|^2+\int_\Omega G(u_{lm}(t))dt\\
&+\int_0^t\int_{\Gamma_1}\mu(t)h(u'_{lm}(\tau))u'_{lm}(\tau)d\Gamma d\tau\\
&=\int_0^t(f(\tau),u'_{lm}(\tau))d\tau+\frac{1}{2}\int_0^t\mu'(\tau)\|u'_{lm}(\tau)\|^2d\tau\\
&\quad+\frac{1}{2}|u_l^1|^2+\frac{\mu(0)}{2}\|u_l^0\|^2+\int_\Omega G(u_l^0)dx.
\end{aligned}
\tag{3.19}
$$

Using (3.8), it follows that

$$
\begin{gathered}
\Big|\int_\Omega G(u_{lm}(t))dx\Big|\le\frac{1}{\rho+1}k_0^{\rho+1}\|u_{lm}(t)\|^{\rho+1},\\
\Big|\int_\Omega G(u_l^0)dx\Big|\le\frac{1}{\rho+1}k_0^{\rho+1}\|u_l^0\|^{\rho+1}.
\end{gathered}
$$

Taking into account the last two inequalities in (3.19), and using hypotheses $(3.4)_2$ and the fact $h_l(s)s\ge d_0$, we find

$$
\begin{aligned}
&\frac{1}{2}|u'_{lm}(t)|^2+\frac{\mu_0}{2}\|u_{lm}(t)\|^2-\frac{1}{\rho+1}k_0^{\rho+1}\|u_{lm}(t)\|^{\rho+1}\\
&\le\frac{1}{2}|u'_{lm}(t)|^2+\frac{\mu(t)}{2}\|u_{lm}(t)\|^2+\int_\Omega G(u_{lm}(t))dx\\
&\quad+\mu_0d_0\int_0^t\int_{\Gamma_1}[u'_{lm}(\tau)]^2d\Gamma d\tau\\
&\le\int_0^t|f(\tau)||u'_{lm}(\tau)|d\tau+\frac{1}{2}\int_0^t|\mu'(\tau)|\|u'_{lm}(\tau)\|^2d\tau+N_{1l}
\end{aligned}
\tag{3.20}
$$

where

$$N_l = \frac{1}{2}|u_l^1|^2 + \frac{\mu(0)}{2}\|u^0\|^2 + \frac{1}{\rho+1}k_0^{\rho+1}\|u^0\|^{\rho+1}. \tag{3.21}$$

Motivated by the expression

$$\frac{\mu_0}{2}\|u_{lm}(t)\|^2 - \frac{1}{\rho+1}k_0^{\rho+1}\|u_{lm}(t)\|^{\rho+1}$$

we introduce the function

$$J(\lambda) = \frac{1}{4}\mu_0\lambda^2 - \frac{3}{2}\frac{k_0^{\rho+1}}{\rho+1}\lambda^{\rho+1}, \quad \lambda \geq 0. \tag{3.22}$$

That is,

$$J'(\lambda) = \frac{1}{2}\mu_0\lambda - \frac{3}{2}k_0^{\rho+1}\lambda^{\rho}.$$

We are interested in $\lambda \geq 0$ such that $J'(\lambda) \geq 0$, that is,

$$\frac{3}{2}k_0^{\rho+1}\lambda^{\rho-1} \leq \frac{1}{2}\mu_0 \tag{3.23}$$

or

$$0 \leq \lambda^{\rho-1} \leq \frac{\mu_0}{3k_0^{\rho+1}}. \tag{3.24}$$

This inequality is equivalent to $0 \leq \lambda \leq \lambda_1^*$, where λ_1^* was defined in (2.3). Thus

$$J(\lambda) \geq 0 \quad \text{for } \lambda \in [0, \lambda_1^*]. \tag{3.25}$$

As consequence of (3.25) and hypothesis $(2.3)_1$, we obtain

$$\frac{\mu_0}{4}\|u_{lm}(t)\|^2 - \frac{3}{2}\frac{k_0^{\rho+1}}{\rho+1}\|u_{lm}(t)\|^{\rho+1} \geq 0, \tag{3.26}$$

for $\|u_{lm}(t)\| < \lambda_1^*,\ t \in [0, t_{lm})$. Inequality (3.26) implies

$$\frac{1}{4}\mu_0\|u_{lm}(t)\|^2 + \frac{1}{2}\frac{k_0^{\rho+1}}{\rho+1}\|u_{lm}(t)\|^{\rho+1} \leq \frac{1}{2}\mu_0\|u_{lm}(t)\|^2 - \frac{k_0^{\rho+1}}{\rho+1}\|u_{lm}(t)\|^{\rho+1}.$$

Taking into account this inequality and (3.26), we have

$$\begin{aligned}
&\frac{1}{2}|u'_{lm}(t)|^2 + \frac{1}{4}\mu_0\|u_{lm}(t)\|^2 + \frac{1}{2}\frac{k_0^{\rho+1}}{\rho+1}\|u_{lm}(t)\|^{\rho+1} \\
&\leq \frac{1}{2}|u'_{lm}(t)|^2 + \frac{\mu(t)}{2}\|u_{lm}(t)\|^2 + \int_\Omega G(u_{lm}(t))dx \\
&\quad + \mu_0 d_0 \int_0^t \int_{\Gamma_1} [u'_{lm}(\tau)]^2 d\Gamma d\tau \\
&\leq \int_0^t |f(\tau)||u'_{lm}(\tau)|d\tau + \frac{1}{2}\int_0^t |\mu'(\tau)|\|u_{lm}(\tau)\|^2 d\tau + N_l.
\end{aligned} \tag{3.27}$$

Note that

$$N_l < N \quad \text{for all } l \geq l_0 \tag{3.28}$$

where N was introduced in (3.11).

We set

$$\varphi(t) = |u'_{lm}(t)|^2 + \frac{1}{2}\mu_0\|u_{lm}(t)\|^2 + \frac{k_0^{\rho+1}}{\rho+1}\|u_{lm}(t)\|^{\rho+1}.$$

Then taking into account (3.28) in (3.27) and noting that $\frac{1}{\mu_1} \leq \frac{1}{\mu_0}$, we obtain

$$\varphi^2(t) \leq \frac{[(2N)^{1/2}]^2}{2} + \int_0^t |f(\tau)||\varphi(\tau)|d\tau + \int_0^t 2\frac{|\mu'(\tau)|}{\mu_0}\varphi^2(\tau)d\tau.$$

Then by Lemma 3.1, we obtain

$$\varphi(t) \leq \Big[(2N)^{1/2} + \int_0^\infty |f(t)|dt\Big] \exp\Big(\frac{2}{\mu_0}\int_0^\infty |\mu'(t)|dt\Big) = P. \tag{3.29}$$

So

$$|u'_{lm}(t)| \leq P \quad \text{and} \quad \|u_{lm}(t)\| \leq \big(\frac{2}{\mu_0}\big)^{1/2} P \tag{3.30}$$

for each $t \in [0, t_{lm})$ and $\|u_{lm}(t)\| < \lambda_1^*$. The following result ensures that inequalities (3.30) hold for all $t \in [0, \infty)$.

Lemma 3.7. *Let $[0, t_{lm})$ be an interval of existence of the solution $u_{lm}(t)$ of (3.18). Then*

$$\|u_{lm}(t)\| < \lambda_1^*, \quad \forall t \in [0, \infty),\ \forall l \geq l_0,\ \forall m.$$

Proof. First, we note that by hypothesis (2.3), we have

$$\|u_{lm}(0)\| = \|u_l^0\| < \lambda_1^*, \quad \forall l \geq l_0,\ \forall m.$$

Reasoning by contradiction, we assume that there exists $t_1 \in (0, t_{lm})$ such that $\|u_{lm}(t_1)\| = \lambda_1^*$. Let

$$t^* = \inf\{t_1 \in (0, t_{lm}) : \|u_{lm}(t_1)\| = \lambda_1^*\}.$$

By the continuity of $\|u_{lm}(t)\|$, we obtain $\|u_{lm}(t^*)\| = \lambda_1^*$. Note that $0 < t^* < t_{lm}$. Consider $t \in [0, t^*)$. Then $\|u_{lm}(t)\| < \lambda_1^*$. So inequality (3.30) provides

$$\|u_{lm}(t)\| \leq \Big(\frac{2}{\mu_0}\Big)^{1/2} P,\ \forall t \in [0, t^*)$$

that implies

$$\lambda_1^* = \|u_{lm}(t^*)\| \leq \Big(\frac{2}{\mu_0}\Big)^{1/2} P$$

But this is a contradiction because by hypothesis $(2.3)_2$, $\big(\frac{2}{\mu_0}\big)^{1/2} P < \lambda_1^*$. This concludes the proof. ☐

Lemma 3.7 provides the estimates

$$|u'_{lm}(t)| \leq P, \quad \|u_{lm}(t)\| \leq \Big(\frac{2}{\mu_0}\Big)^{1/2} P, \quad \forall t \in [0, \infty),\ \forall l \geq l_0,\ \forall m. \tag{3.31}$$

Also inequalities (3.29), (3.31) and (3.20) gives us

$$\int_0^\infty \|u'_{lm}(t)\|_{L^2(\Gamma_1)} dt \leq K, \quad \forall t \in [0, \infty),\ \forall l \geq l_0,\ \forall m. \tag{3.32}$$

Second Estimate. In this part, to facilitate the notation we do not write the variable t and the subscripts l and m. Differentiating with respect to t equation $(3.18)_1$ and then setting $w = u''$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u''|^2 + \frac{1}{2}\frac{d}{dt}[\mu\|u'\|^2] + \mu'((u,u'')) + (\rho|u|^{\rho-2}uu', u'') \\
&+ \mu\int_{\Gamma_1}\delta h'(u')[u'']^2 d\Gamma + \mu'\int_{\Gamma_1} h(u')u'' d\Gamma \\
&= (f', u'') + \frac{1}{2}\mu'\|u'\|^2.
\end{aligned}$$

Considering $w = \frac{\mu'}{\mu}u''$ in approximate equation $(3.18)_1$, we find

$$\mu'((u,u'')) + \mu'\int_{\Gamma_1} h(u')u'' d\Gamma = \big(f', \frac{\mu'}{\mu}u''\big) - \big(u'', \frac{\mu'}{\mu}u''\big) - \big(|u|^\rho, \frac{\mu'}{\mu}u''\big).$$

Combining the last two equalities, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u''|^2 + \frac{1}{2}\frac{d}{dt}\left[\mu\|u'\|^2\right] + \mu\int_{\Gamma_1}\delta h'(u)[u'']^2 d\Gamma \\
&= (f', u'') + \frac{1}{2}\mu'\|u\|^2 - \big(f, \frac{\mu'}{\mu}u''\big) + \big(u'', \frac{\mu'}{\mu}u''\big) \\
&\quad + \big(|u|^\rho, \frac{\mu'}{\mu}u''\big) - (\rho|u|^{\rho-2}uu', u'').
\end{aligned} \tag{3.33}$$

Fix a real number $T > 0$. We bound the last terms of the second member of (3.33). By $C = C(T) > 0$ is denoted a generic constant which is independent of l and m. By (3.8), $(3.6)_1$ and estimate (3.33), we obtain

$$\big|\big(|u|^\rho, \frac{\mu'}{\mu}u''\big)\big| \le k_2^\rho\|u\|^\rho\frac{|\mu'|}{\mu_0}|u''| \le C\frac{|\mu'|}{\mu_0}|u''|.$$

By $(3.6)_2$, $(3.6)_3$, estimates (3.31) and noting that $\frac{1}{n} + \frac{1}{p^*} + \frac{1}{2} = 1$ (p^* introduced in (3.5)), we find

$$|(\rho|u|^{\rho-2}uu', u'')| \le \rho k_3^{\rho-1}k_4\|u'\||u''| \le C\|u'\||u''| \le \frac{C}{2}\|u'\|^2 + \frac{C}{2}|u''|^2.$$

Taking into account the last two inequalities (3.33) and integrating on $[0,t]$, we obtain

$$\begin{aligned}
&\frac{1}{2}|u''_{lm}(t)|^2 + \frac{1}{2}\mu(t)\|u'_{lm}(t)\|^2 + \mu_0 d_0\int_0^t\int_{\Gamma_1}[u''_{lm}(\tau)]^2 d\Gamma d\tau \\
&\le \int_0^t\left[|f'(\tau)| + \frac{|\mu'(\tau)|}{\mu_0}|f(\tau)| + \frac{C|\mu'(\tau)|}{\mu_0}\right]|u''_{lm}(\tau)|d\tau \\
&\quad + \int_0^t\frac{C}{2}|u''_{lm}(\tau)|^2 d\tau + \int_0^t\frac{C}{2}\|u'_{lm}(\tau)\|^2 d\tau \\
&\quad + \frac{1}{2}\int_0^t\frac{|\mu'(\tau)|}{\mu_0}\mu(\tau)\|u(\tau)\|^2 d\tau + \frac{1}{2}|u''_{lm}(0)|^2 + \frac{\mu(0)}{2}\|u_l^1\|^2.
\end{aligned} \tag{3.34}$$

For this inequality provides an estimate, we need to bound $|u''_{lm}(0)|$. This is possible thanks to the choice of the special basis of $V \cap H^2(\Omega)$ and $(3.17)_3$.

We bound $|u''_{lm}(0)|$. Set $t = 0$ in approximate equation $(3.18)_1$ and then take $v = u''_{lm}(0)$. The Gauss theorem and $(3.17)_3$ gives us

$$|u''_{lm}(0)|^2 + \mu(0)(-\Delta u_l^0, u''_{lm}(0)) + (|u_l^0|^\rho, u''_{lm}(0)) = (f(0), u''_{lm}(0)).$$

This equality and (3.17) gives us

$$|u''_{lm}(0)|^2 \le K_1.$$

Taking into account this inequality in (3.34) and using Lemma 3.7, follows that

$$\begin{gathered}
\|u'_{lm}(t)\| \le C, \quad \forall t \in [0,T],\ \forall l \ge l_0,\ \forall m \\
|u''_{lm}(t)| \le C, \quad \forall t \in [0,T],\ \forall l \ge l_0,\ \forall m \\
\int_0^t \|u''_{lm}(t)\|_{L^2(\Gamma_1)} \le C, \quad \forall t \in [0,T],\ \forall l \ge l_0,\ \forall m
\end{gathered} \tag{3.35}$$

Passage to the Limit in m. Estimates (3.31), (3.32), (3.35) and diagonal process allows to find a function u_k and a subsequence of (u_{lm}), still denoted by (u_{lm}), such that

$$\begin{gathered}
u_{lm} \to u_l \quad \text{weak star in } L^\infty(0,\infty,V); \\
u'_{lm} \to u'_l \quad \text{weak star in } L^\infty(0,\infty,L^2(\Omega)) \cap L^\infty_{\rm loc}(0,\infty,V); \\
u''_{lm} \to u''_l \quad \text{weak star in } L^\infty_{\rm loc}(0,\infty,L^2(\Omega)); \\
u'_{lm} \to u'_l \quad \text{weak star in } L^\infty(0,\infty,L^2(\Gamma_1)); \\
u''_{lm} \to u''_l \quad \text{weak star in } L^\infty_{\rm loc}(0,\infty,L^2(\Gamma_1)).
\end{gathered} \tag{3.36}$$

Estimates $(3.36)_1$, $(3.36)_2$ and Aubin-Lions Theorem provides us

$$u_{lm}(x,t) \to u_l(x,t) \quad \text{a.e. in } Q = \Omega \times (0,T).$$

Then

$$|u_{lm}(x,t)|^\rho \to |u_l(x,t)|^\rho \quad \text{a.e. in } Q = \Omega \times (0,T). \tag{3.37}$$

By (3.8), $(3.6)_2$ and (3.31), we find

$$\int_\Omega |u_{lm}|^{2\rho} dx \le k_2^{2\rho} \|u_{lm}\|^{2\rho} \le C. \tag{3.38}$$

Expressions (3.37), (3.38), Lions Lema [10] and diagonal process provide

$$|u_{lm}|^\rho \ \to |u_l|^\rho \quad \text{weak star in } L^\infty_{\rm loc}(0,\infty;L^2(\Omega)). \tag{3.39}$$

Estimate $(3.36)_3$ yields

$$u'_{lm} \to u'_l \quad \text{weak star in } L^\infty(0,\infty;H^{1/2}(\Gamma_1)).$$

This, convergence $(3.36)_5$ and Aubin-Lions Theorem and fact h Lipchitizian function gives us

$$h(u'_{lm}(x,t)) \to h(u'_l(x,t)) \quad \text{a.e. in } Q$$

and by trace theorem and (3.36), we obtain

$$(h(u'_{lm})) \text{ bounded in } L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)).$$

Therefore, by Lions Lemma, we conclude that

$$h(u'_{lm}) \to h(u'_l) \text{ weak star in } L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)). \tag{3.40}$$

Convergences (3.36), (3.39)-(3.40) allows us to pass to the limit in approximate equation $(3.18)_1$. Then by density of $V \cap H^2(\Omega)$ in V, we obtain

$$\begin{aligned} &\int_0^\infty (u_l''(t), v)\theta(t)dt + \mu \int_0^\infty ((u_l(t), v))\theta(t)dt + \int_0^\infty (|u_l(t)|^\rho, v)\theta(t)dt \\ &+ \int_0^\infty \int_{\Gamma_1} \mu(t)\delta h(u_l'(t))v\theta(t)d\Gamma dt \\ &= \int_0^\infty (f(t), v)\theta(t)dt, \quad v \in V, \ \forall \theta \in C_0^\infty(\Omega). \end{aligned} \tag{3.41}$$

Taking $v \in \mathcal{D}(\Omega)$ in (3.41), and observing the regularities of u_l'', $|u_l|^\rho$ and f, follows that

$$u_l'' - \mu\Delta u_l + |u_l|^\rho = f \quad \text{in } L^2_{\rm loc}(0,\infty; L^2(\Omega)). \tag{3.42}$$

This equation provides $\Delta u_l \in L^\infty(0,\infty; L^2(\Omega))$ and $(3.36)_1$, $u_l \in L^\infty(0,\infty; V)$. Then

$$\frac{\partial u_l}{\partial \nu} \in L^\infty_{\rm loc}(0,\infty; H^{1/2}(\Gamma_1)). \tag{3.43}$$

Multiply both sides of (3.42) by $v\theta, v \in V$ and $\theta \in C_0^\infty(0,\infty)$, and integrate on $\Omega \times (0,\infty)$. Using regularity (3.43) of $\frac{\partial u}{\partial \nu}$, we conclude

$$\begin{aligned} &\int_0^\infty (u_l''(t), v)\theta(t)dt + \mu \int_0^\infty ((u_l(t), v))\theta(t)dt - \int_0^\infty \mu(t)\langle \frac{\partial u_l}{\partial \nu}, v\rangle\theta(t)dt \\ &+ \int_0^\infty (|u_l(t)|^\rho, v)\theta(t)dt \\ &= \int_0^\infty (f(t), v)\theta(t)dt, \quad v \in V, \ \forall \theta \in C_0^\infty(\Omega). \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality paring between $H^{-\frac{1}{2}}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$. Comparing this equality with (3.41) and observing the regularity of $h(u_l')$, we find (see [19])

$$\frac{\partial u_l}{\partial \nu} + \delta h(u_l') = 0 \quad \text{in } L^2_{\rm loc}(0,\infty; H^{1/2}(\Gamma_1)). \tag{3.44}$$

Passage to the Limit in l. Estimates (3.31), (3.32), (3.35) and convergence (3.36) provide

$$\begin{gathered} |u_l'(t)| \le P, \|u_l(t)\| \le \Big(\frac{2}{\mu_0}\Big)^{1/2} \quad \forall t \in [0,\infty), \ \forall l \ge l_0, \\ \int_0^\infty \|u_l''(t)\|^2_{L^2(\Gamma_1)} dt \le C, \quad \forall t \in [0,\infty), \ \forall l \ge l_0; \\ \|u_l'(t)\| \le C, \quad |u_l''(t)| \le C \quad \forall t \in [0,T],]; \forall l \ge l_0, \\ \int_0^t \|u_l''(\tau)\|^2_{L^2(\Gamma_1)} d\tau \le C, \quad \forall t \in [0,T], \ \forall l \ge l_0. \end{gathered} \tag{3.45}$$

These estimates allows to obtain similar convergence to those obtained in (3.36). So there exists a function u and subsequence of (u_l), still denoted by (u_l), such that

$$\begin{gathered} u_l \to u \quad \text{weak star in } L^\infty(0,\infty,V); \\ u_l' \to u' \quad \text{weak star in } L^\infty(0,\infty,L^2(\Omega)) \cap L^\infty_{\rm loc}(0,\infty;V); \\ u_l'' \to u'' \quad \text{weak star in } L^\infty_{\rm loc}(0,\infty,L^2(\Omega)); \\ u_l' \to u' \quad \text{weak in } L^2(0,\infty,L^2(\Gamma_1)); \\ u_l'' \to u'' \quad \text{weak in } L^2_{\rm loc}(0,\infty,L^2(\Gamma_1)). \end{gathered} \tag{3.46}$$

By arguments similar to those used for (3.39), we find

$$|u_l|^\rho \to |u|^\rho \quad \text{weak in } L^2_{\rm loc}(0,\infty;L^2(\Omega)). \tag{3.47}$$

This convergence, $(3.46)_3$ and (3.42) provide

$$\Delta u_l \to \Delta u \quad \text{weak in } L^2_{\rm loc}(0,\infty;L^2(\Omega)) \tag{3.48}$$

and therefore

$$u'' - \mu\Delta u + |u|^\rho = f \quad \text{in } L^2_{\rm loc}(0,\infty;L^2(\Omega)). \tag{3.49}$$

Also convergences $(3.46)_1$ and (3.48) provide us with

$$\frac{\partial u_l}{\partial \nu} \to \frac{\partial u}{\partial \nu} \quad \text{weak in } L^2_{\rm loc}(0,\infty;H^{-\frac12}(\Gamma_1)). \tag{3.50}$$

As done in (3.40), we find

$$\delta h(u_l') \to \delta h(u') \quad \text{weak star in } L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)). \tag{3.51}$$

So these two convergences and (3.44), we met

$$\frac{\partial u}{\partial \nu} + \delta h(u') = 0 \quad \text{in } L^2_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1)). \tag{3.52}$$

From the regularity

$$u \in L^\infty_{\rm loc}(0,\infty;V), \quad \Delta u \in L^\infty_{\rm loc}(0,\infty;L^2(\Omega)), \quad \frac{\partial u}{\partial \nu} \in L^\infty_{\rm loc}(0,\infty;H^{1/2}(\Gamma_1))$$

and by Proposition 3.2, we obtain

$$u \in L^\infty_{\rm loc}(0,\infty;V \cap H^2(\Omega)). \tag{3.53}$$

Also, by estimate $(3.46)_4$ and noting that h is a Lipschitz continuous function we find

$$\frac{\partial u'}{\partial \nu} + \delta h'(u')u'' = 0 \quad \text{in } L^2_{\rm loc}(0,\infty;L^2(\Gamma_1)). \tag{3.54}$$

The verification of initial conditions follows in the usual way.

In what follows, we prove the uniqueness of solutions. Let u and v two functions in class (3.12) which satisfy equations (3.13), (3.14) and initial conditions (3.16). Consider $w = u - v$. Then

$$\begin{gathered} w'' - \mu\Delta w + |u|^\rho - |v|^\rho = 0 \quad \text{in } L^\infty(0,T;L^2(\Omega)), \\ \frac{\partial w}{\partial \nu} + \delta[h(u') - h(v')] = 0 \quad \text{in } L^\infty(0,T;H^{1/2}(\Gamma_1)), \\ w(0) = 0, \quad w'(0) = 0 \end{gathered} \tag{3.55}$$

Multiplying both sides of $(3.55)_1$ by w' integrating on Ω and using Gauss Theorem, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|w'(t)|^2 + \frac{1}{2}\|w(t)\|^2 + \int_{\Gamma_1} \delta[h(u'(t)) - h(v'(t))]d\Gamma \\ &= -(|u(t)|^\rho - |v(t)|^\rho, w'(t)). \end{aligned} \tag{3.56}$$

We have

$$|u(x,t)|^\rho - |v(x,t)|^\rho = \rho|\xi|^{\rho-2}\xi w(x,t)$$

where ξ is between $u(x,t)$ and $v(x,t)$. Then

$$||u(x,t)|^\rho - |v(x,t)|^\rho| = \rho|\xi|^{\rho-1}|w(x,t)|$$

that provides

$$\begin{aligned} ||u(t)|^\rho - |v(t)|^\rho| &\le \rho[|u(x,t)| + |v(x,t)|]^{\rho-1}|w(x,t)| \\ &\le C(\rho)[|u(x,t)|^{\rho-1}|w(x,t)| + |v(x,t)|^{\rho-1}|w(x,t)|]. \end{aligned} \tag{3.57}$$

We obtain

$$\begin{aligned} \int_\Omega |u(x,t)|^{\rho-1}|w(x,t)||w'(x,t)|dx &\le \|u(t)\|^{\rho-1}_{L^{(\rho-1)n}(\Omega)}\|w(t)\|_{L^{p^*}(\Omega)}|w'(t)| \\ &\le k_3k_4\|u(t)\|^{\rho-1}\|w(t)\||w'(t)|. \end{aligned}$$

Thus

$$|(|u(t)|^\rho - |v(t)|^\rho, w'(t))| \le C\|w(t)\||w'(t)| \le \frac{C}{2}\|w(t)\|^2 + \frac{C}{2}|w'(t)|^2.$$

This inequality, (3.56) and property of monotony of h, imply

$$\frac{1}{2}\frac{d}{dt}|w'(t)|^2 + \frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + \delta_0 d_0 \int_{\Gamma_1} w'(t)^2 d\Gamma \le \frac{C}{2}\|w(t)\|^2 + \frac{C}{2}|w'(t)|.$$

Then the Gronwall inequality provides $w'(t) = 0$ and $w(t) = 0$. This concludes the proof of Theorem 3.6.

3.2. **Proof of Theorem 2.3.** We introduce some notation to apply the Banach Fixed-Point Theorem. Consider a real number $R > 0$ such that

$$R > M_0 \tag{3.58}$$

where $M_0 = \max\{M_1, M_2\}$ is defined in (3.71), M_1, M_2 are defined by (3.65) and (3.69) respectively. Let

$$R_1^2 = N_1^2 = |u^1|^2 + M(0, \|u^0\|^2)\|u^0\|^2 + \frac{1}{\rho+1}k_0\|u_0\|^{\rho+1}, \tag{3.59}$$

$$R_2^2 = M(0, \|u^0\|^2)\|u^1\|^2 + M(0, \|u^0\|^2)|\Delta u^0| + |u^0|^\rho + |f(0)|. \tag{3.60}$$

We define B_{R,T_0} as the set of vectors

$$\begin{aligned} B_{R,T_0} = \Big\{ &u : u \in L^\infty(0,T_0;V),\ u' \in L^\infty(0,T_0;V) \cap C^0([0,T_0];L^2(\Omega)), \\ &\|u\|_{L^\infty(0,T_0;V)} + \|u'\|_{L^\infty(0,T_0;V)} \le R, \\ &u(0) = u^0,\ u'(0) = u^1. \Big\} \end{aligned}$$

The real number T_0 with $0 < T_0 \le 1$ will be determined later. We equipped B_{R,T_0} with the metric

$$d(u,v) = \|u - v\|_{L^\infty(0,T_0;V)} + \|u' - v'\|_{C^0([0,T_0];L^2(\Omega))}$$

where u and v belong to B_{R,T_0}. In [21] is proved that $(B_{R,T_0}, d(u,v))$ is a complete metric space.

Consider the map $S : B_{R,T_0} \to \mathcal{H}, z \mapsto S(z) = \varphi$, where $\mathcal{H}$ denotes the set of solutions φ, of the problem

$$
\begin{gathered}
\varphi'' - M(\cdot, \|z\|^2)\Delta\varphi + |\varphi|^\rho = f \quad \text{in } \Omega \times (0,T_0) \\
\varphi = 0 \quad \text{on } \Gamma_0 \times (0,T_0) \\
\frac{\partial\varphi}{\partial\nu} + \delta h(\varphi') = 0 \quad \text{on } \Gamma_1 \times (0,T_0) \\
\varphi(0) = u^0, \quad \varphi'(0) = u^1 \quad \text{in } \Omega
\end{gathered}
\tag{3.61}
$$

We prove that the map S is well defined. Set

$$
K = \max\Big\{\big|\frac{\partial M}{\partial t}(t,\lambda)\big|, \big|\frac{\partial M}{\partial \lambda}(t,\lambda)\big|; t \in [0,1],\ \lambda \in [0,R^2]\Big\}. \tag{3.62}
$$

Consider

$$
\mu(t) = M(t, \|z(t)\|^2), \quad t \in [0,T_0]. \tag{3.63}
$$

We have that $\mu \in W^{1,\infty}(0,T_0)$. In fact,

$$
\mu'(t) = \frac{\partial M}{\partial t}(t, \|z(t)\|^2) + \frac{\partial M}{\partial \lambda}(t, \|z(t)\|^2)\frac{d}{dt}\|z(t)\|^2.
$$

As $z \in B_{R,T_0}$, we find that

$$
|\mu'(t)| \le K(1+4R^2), \quad \text{a.e. } t \in]0,T_0[. \tag{3.64}
$$

Thus, $\mu \in W^{1,\infty}(0,T_0)$ with $\mu_0 = m_0$. Theorem 3.6 says that there exists a unique solution φ of system (3.61) and this solution has the regularity of the vectors of B_{R,T_0}.

Our objective now is to show that $S(B_{R,T_0})$ is contained B_{R,T_0} and that S is a strict contraction.

Let φ be a solution of the problem (3.61) given by the Theorem 3.6 with $\mu(t)$ defined in (3.63). Let φ_{lm} be the approximate solution given in the proof of Theorem 3.6. Then by first a priori estimate given the proof of Theorem 3.6, we obtain

$$
\|\varphi_{lm}(t)\|^2 \le M_1 \exp\Big(\frac{2}{m_0}\int_0^t |\mu'(\tau)|d\tau\Big), \quad 0 \le t \le T_0,
$$

where

$$
M_1 = (2R_1)^{1/2} + \int_0^{T_0} |f(t)|dt. \tag{3.65}
$$

This and (3.64) gives

$$
\|\varphi_{lm}(t)\| \le M_1 \exp(\mathcal{K}_1 T_0), \quad 0 \le t \le T_0, \text{ for } m \ge 2 \text{ and } l \ge l_0(1). \tag{3.66}
$$

where

$$
\mathcal{K}_1 = \frac{2K(1+R^2)}{m_0}. \tag{3.67}
$$

The second priori estimates Theorem 2.3 gives us

$$
\|\varphi'_{lm}(t)\| \le M_2 \exp(\mathcal{K}_2 T_0), \quad 0 \le t \le T_0, \text{ for } m \ge 2 \text{ and } l \ge l_0(1). \tag{3.68}
$$

where

$$\begin{aligned} M_2 &= 2R_2^{1/2} + \int_0^{T_0} \Big[|f'(t)| + \frac{|\mu'(t)|}{m_0}|f(t)| + \frac{C}{m_0}|\mu'(t)|\Big]\,dt \\ &\le 2R_2^{1/2} + \int_0^{T_0} \big[|f'(t)| + \frac{K(1+4R^2)}{m_0}|f(t)| + \frac{C}{m_0}K(1+4R^2)\big]dt \end{aligned} \tag{3.69}$$

and

$$\mathcal{K}_2 = \frac{(2+m_0)K(1+4R^2)}{2m_0} + \frac{3C}{2}. \tag{3.70}$$

Consider

$$M_0 = \max\{M_1, M_2\}, \quad \mathcal{K} = \max\{\mathcal{K}_1, \mathcal{K}_2\}. \tag{3.71}$$

From (3.66), (3.68) and (3.71) and taking the maximum on $[0, T_0]$ of both of members the (3.66) and (3.68) and then the limit inferior, first with respect to m and later with respect to l, we obtain

$$\|\varphi\|_{L^\infty(0,T_0;V)} + \|\varphi'\|_{L^\infty(0,T_0;V)} \le M_0 exp(\mathcal{K}T_0). \tag{3.72}$$

We will choose $T_0 > 0$ so that the second member of the preceding inequality be less than or equal to R. In fact, set

$$q(t) = M_0 e^{\mathcal{K}t}, \quad t \ge 0.$$

Then q is continuous, increasing, $q(t) \to \infty$ when $t \to \infty$ and $q(0) = M_0 < R$ (see (3.58)). Then by the Intermediate Value Theorem there exists $T_1^* > 0$ such that $q(T_1^*) = R$, that is,

$$T_1^* = \frac{1}{\mathcal{K}} \ln\Big(\frac{R}{M_0}\Big). \tag{3.73}$$

We choose

$$0 < T_0 \le \min\{1, T_1^*\}. \tag{3.74}$$

Then expression (3.72) with T_0 given by (3.74) satisfies

$$\|\varphi\|_{L^\infty(0,T_0;V)} + \|\varphi'\|_{L^\infty(0,T_0;V)} \le R.$$

Therefore φ belongs to B_{R,T_0}. Thus $S(B_{R,T_0})$ is contained in B_{R,T_0}.

In the sequel we prove that S is a strict contraction. Set $r_1, y_1 \in B_{R,T_0}$ and $S(r_1) = r$, $S(y_1) = y$. Introduce the notation

$$\varphi = r - y. \tag{3.75}$$

We have

$$\begin{gathered} \varphi'' - M(\cdot, \|r_1\|^2)\Delta r + M(\cdot, \|y_1\|^2)\Delta y + |r|^\rho - |y|^\rho = 0 \quad \text{in } \Omega\times]0,T_0[, \\ \varphi = 0, \quad \psi = 0 \quad \text{on } \Gamma_0\times]0,T_0[, \\ \frac{\partial\varphi}{\partial\nu} + \delta[h(r') - h(y')] = 0 \quad \text{on } \Gamma_1\times]0,T_0[, \\ \varphi(0) = 0, \quad \varphi'(0) = 0 \quad \text{in } \Omega. \end{gathered} \tag{3.76}$$

Taking the scalar product in $L^2(\Omega)$ of $(3.76)_1$ with $\varphi'(t)$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{d}|\varphi'(t)|^2 - M(t, \|r_1(t)\|^2)(\Delta r(t), \varphi'(t)) \\ &+ M(t, \|y_1(t)\|^2)(\Delta y(t), \varphi'(t)) + (|r|^\rho - |y|^\rho, \varphi'(t)) = 0. \end{aligned} \tag{3.77}$$

We modify (3.77), to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{d}|\varphi'(t)|^2 - M(t,\|r_1(t)\|^2)(\Delta\varphi(t),\varphi'(t)) \\ &= [M(t,\|r_1(t)\|^2) - M(t,\|y_1(t)\|^2)](\Delta y(t),\varphi'(t)) - (|y|^\rho - |r|^\rho,\varphi'(t)). \end{aligned}$$

We abbreviate the notation and write this expression in the form

$$\frac{1}{2}\frac{d}{dt}|\varphi'(t)|^2 + A(t) = B(t). \tag{3.78}$$

• Analysis of $A(t)$. Using the Green's Theorem and the boundary condition in $(3.76)_3$, we find that

$$\begin{aligned} A(t) &= M(t,\|r_1(t)\|^2)\frac{1}{2}\frac{d}{dt}\|\varphi(t)\|^2 \\ &\quad + M(t,\|r_1(t)\|^2)\int_{\Gamma_1}\delta[h(r'(t)) - h(y'(t))]\varphi'(t)d\Gamma. \end{aligned}$$

Note that, $\delta(x) \geq \delta_0 > 0$ and $\varphi'(t) = r'(t) - y'(t)$ then by the strong monotonicity of h, follows that

$$\int_{\Gamma_1}\delta[h(r'(t)) - h(y'(t))]\varphi'(t)d\Gamma \geq 0.$$

Combining the last two expressions we conclude that

$$A(t) \geq M(t,\|r_1(t)\|^2)\frac{1}{2}\frac{d}{dt}\|\varphi(t)\|^2 \text{ a.e. } t \in]0,T_0[. \tag{3.79}$$

• Analysis of $B(t)$. To facilitate the notation in this part we do not write the variable t. We have

$$B = \left[M(\cdot,\|r_1\|^2) - (M(\cdot,\|y_1\|^2)\right](\Delta y(t),\varphi'(t)) - (|y|^\rho - |r|^\rho,\varphi'(t)). \tag{3.80}$$

- As $M \in C^1$ we have
$$\left|M(\cdot,\|r_1\|^2) - M(\cdot,\|y_1\|^2)\right| \leq 2KM_0\|r_1 - y_1\|,$$
where K and M_0 were defined in (3.62) and (3.71), respectively.
- Analysis of $(|y(t)|^\rho - |r(t)|^\rho,\varphi'(t))$. We have
$$|y(x,t)|^\rho - |r(x,t)|^\rho = \rho|\xi|^{\rho-2}\xi\varphi(x,t)$$
where ξ is between $y(x,t)$ and $r(x,t)$. Then
$$||y(x,t)|^\rho - |r(x,t)|^\rho| \leq \rho|\xi|^{\rho-1}|\varphi(x,t)|$$
which implies
$$||y(x,t)|^\rho - |r(x,t)|^\rho| \leq C[|y(x,t)|^{\rho-1} + |r(x,t)|^{\rho-1}]|\varphi(x,t)|.$$
Thus
$$|(|y(t)|^\rho - |r(t)|^\rho,\varphi'(t))| \leq C\|y(t)\|^{\rho-1}_{L^{(\rho-1)n}(\Omega)}\|\varphi(t)\|_{L^{p^*}(\Omega)}|\varphi'(t)|.$$
By (3.6), we find that
$$\begin{gathered} \|y(t)\|^{\rho-1}_{L^{(\rho-1)n}(\Omega)} \leq k_3^{\rho-1}\|y(t)\|^{\rho-1} \leq C, \quad \forall t \in [0,T_0], \\ \|r(t)\|^{\rho-1}_{L^{(\rho-1)n}(\Omega)} \leq C,\ \forall t \in [0,T_0]. \end{gathered}$$

Combining the last tree inequalities, we obtain

$$|(|y(t)|^\rho - |r(t)|^\rho, \varphi'(t))| \le C\|\varphi(t)\||\varphi'(t)| \le \frac{C}{2}\|\varphi(t)\|^2 + \frac{C}{2}|\varphi'(t)|^2.$$

Taking into account the last two inequalities in (3.80), we obtain

$$|B(t)| \le C|\Delta y(t)||\varphi'(t)|d(r_1, y_1) + \frac{C}{2}\|\varphi(t)\|^2 + \frac{C}{2}|\varphi'(t)|^2. \tag{3.81}$$

Next we find a bound for $|\Delta y(t)|$. We have

$$\varphi'' - M(\cdot, \|z\|^2)\Delta\varphi + |\varphi|^\rho = f \quad \text{in } L^\infty(0, T_0; L^2(\Omega)).$$

By estimates (3.66), (3.68) and following the same reasoning used for (3.68), we obtain

$$|y''(t)| \le M_0 \exp(\mathcal{K}T_0) \quad \text{a.e. } t \in]0, T_0[. \tag{3.82}$$

Hence,

$$\begin{aligned} |M(t, \|z(t))\|||\Delta\varphi(t)| &\le |f(t)| + |u(t)|^\rho + |\varphi'(t)| \\ &\le \Big(\frac{C_1 + C_2}{m_0}\Big) + \frac{M_0}{m_0}\exp(\mathcal{K}T_0). \end{aligned} \tag{3.83}$$

These last two expressions give

$$|\Delta y(t)| \le M_3 + M_3 exp(\mathcal{K}T_0) \quad \text{a.e. } t \in]0, T_0[, \tag{3.84}$$

where

$$M_3 = \max\big\{\frac{C_1 + C_2}{m_0}, \frac{M_0}{m_0}\big\}.$$

Note that $e^{\mathcal{K}T_0} > 1$, therefore $M_3 \le M_3 e^{\mathcal{K}T_0}$. Hence Combining $(3.81) and (3.84)$ we derive

$$|B(t)| \le P_0[exp(\mathcal{K}T_0)]|\varphi'(t)|d(r_1, y_1) \quad \text{a.e. } t \in]0, T_0[\tag{3.85}$$

where

$$P_0 = 4KM_0M_3. \tag{3.86}$$

Combining (3.79) and (3.85) with (3.78), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\varphi'(t)|^2 + M(t, \|r_1(t)\|^2)\frac{1}{2}\frac{d}{dt}\|\varphi(t)\|^2 \\ &\le P_0[exp(\mathcal{K}T_0)]^2 d^2(r_1, y_1) + |\varphi'(t)|^2 \quad \text{a.e } t \in]0, T_0[. \end{aligned} \tag{3.87}$$

We have

$$\begin{aligned} M(\cdot, \|r_1\|^2)\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 &= \frac{1}{2}\frac{d}{dt}[M(\cdot, \|r_1\|^2)\|\varphi(t)\|^2] \\ &- \frac{1}{2}\Big[\frac{\partial M}{\partial t}(\cdot, \|r_1\|^2) + \frac{\partial M}{\partial \lambda}(\cdot, \|r_1\|^2)\frac{d}{dt}\|r_1\|^2\Big]\|\varphi\|^2. \end{aligned}$$

Substituting this equality in (3.87), and using boundedness (3.62) and (3.60), we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{d}\left[|\varphi'(t)|^2 + M(t, \|r_1(t)\|^2)\|\varphi(t)\|^2\right] \\ &\le \frac{K(1 + 2R^2)}{2}\|\varphi(t)\|^2 + P_0^2[\exp(\mathcal{K}T_0)]^2 d^2(r_1, y_1) + |\varphi'(t)|^2 \quad \text{a.e. } t \in]0, T_0[. \end{aligned}$$

Integrating on $[0,t]$, $0 < t \le T_0$, and noting that $M(t,\lambda) \ge m_0$ and $\varphi(0) = \varphi'(0) = 0$, we obtain

$$
\begin{aligned}
&\frac{1}{2}[|\varphi'(t)|^2 + m_0\|\varphi(t)\|^2] \\
&\le P_1 \int_0^t \|\varphi(s)\|^2 ds + T_0 P_0^2 [exp(\mathcal{K}T_0)]^2 d^2(r_1, y_1) + \int_0^t |\varphi'(s)|^2 ds,
\end{aligned} \tag{3.88}
$$

where

$$
P_1 = \frac{K(1+2R^2)}{2}. \tag{3.89}
$$

Considering

$$
b_1^2 = \frac{P_0[exp(\mathcal{K}T_0)]^2}{\min\{\frac{1}{2}, \frac{m_0}{2}\}}, \quad b_2 = \frac{\max\{P_1, 1\}}{\min\{\frac{1}{2}, \frac{m_0}{2}\}}, \tag{3.90}
$$

where P_0 was defined in (3.86), we have

$$
\|\varphi(t)\|^2 + |\varphi'(t)|^2 \le b_1^2 T_0 d^2(r_1, y_1) + b_2 \int_0^t [\|\varphi(s)\|^2 + |\varphi'(s)|^2] ds.
$$

Then Gronwall's lemma gives

$$
\|\varphi(t)\|^2 + |\varphi'(t)|^2 \le 4 b_1^2 T_0 d^2(r_1, y_1) \exp(b_2 T_0),
$$

which implies

$$
\|\varphi(t)\| + |\varphi'(t)| \le 2 b_1 T_0^{1/2} d(r_1, y_1) \exp(b_2 T_0),
$$

Recalling that $S(r_1) = r$, $S(y_1) = y$ and $\varphi = r - y$, from the above inequality it follows that

$$
d(S(r_1), S(y_1)) \le [2 b_1 T_0^{1/2} \exp(b_2 T_0)] d((r_1, y_1). \tag{3.91}
$$

Note that K given in (3.62) is independent of T_0, therefore $\mathcal{K}$, P_0 and P_1 defined in (3.71), (3.86) and (3.89) respectively, are independent of T_0. Thus the constants b_1 and b_2 given in (3.90) are also independent of T_0.

Consider $\psi(t) = 2b_1 t \exp(b_2 t)$, $t \ge 0$. Then ψ is continuous, increasing and $\psi(0) = 0$. So there exists $T_2^* > 0$ such that $\psi(T_2^*) < 1$. Take

$$
T_0 = \min\{1, T_1^*, T_2^*\} > 0,
$$

where T_1^* was defined in (3.73). Then T_0 satisfies (3.74) and

$$
2 b_1 T_0 \exp(b_2 T_0) = \alpha_0 < 1.
$$

Substituting this constant in (3.91), we conclude that

$$
d(S(r_1), S(y_1)) \le \alpha_0 d(r_1, y_1), \quad \forall r_1, y_1 \in B_{R,T_0}.
$$

Thus d is a strict contraction. By the Banach Fixed-Point Theorem there exists a unique point $u \in B_{R,T_0}$ such that $S(u) = u$. This fixed point satisfies all conditions required in the theorem.

The uniqueness of solutions follows as in [21].

The existence of global solutions to problem (2.6) and their asymptotic behavior with small data will be published in a future article.

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