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A non-local problem with integral conditions for hyperbolic equations *

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Abstract

A linear second-order hyperbolic equation with forcing and integral constraints on the solution is converted to a non-local hyperbolic problem. Using the Riesz representation theorem and the Schauder fixed point theorem, we prove the existence and uniqueness of a generalized solution.

1 Introduction

Certain problems arising in: plasma physics [1], heat conduction [2, 3], dynamics of ground waters [4, 5], thermo-elasticity [6], can be reduced to the non-local problems with integral conditions. The above-mentioned papers consider problems with parabolic equations. However, some problems concerning the dynamics of ground waters are described in terms of hyperbolic equations [4]. Motivated by this, we study the equation

$$Lu \equiv u_{xy} + A(x,y)u_x + B(x,y)u_y + C(x,y)u = f(x,y) \tag{1}$$

with smooth coefficients in the rectangular domain

$$D = \{(x,y) : 0 < x < a, 0 < y < b\},$$

bounded by the characteristics of equation (1), with the conditions

$$\int_0^\alpha u(x,y)\,dx = \psi(y), \quad \int_0^\beta u(x,y)\,dy = \phi(x). \tag{2}$$

where $\phi(x)$, $\psi(y)$ are given functions and $0 < \alpha < a, 0 < \beta < b$. The special case $\alpha = a$, $\beta = b$ is considered by author in [7]. The consistency condition assumes the form

$$\int_0^\alpha \phi(x)\,dx = \int_0^\beta \psi(y)\,dy.$$

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2 A problem for a loaded equation

Since the integral conditions (2) are not homogeneous, we construct a function $K(x,y)=\frac{1}{\alpha}\psi(y)+\frac{1}{\beta}\phi(x)-\frac{1}{\alpha\beta}\int_0^\alpha \phi(x)\,dx$, satisfying the conditions (2), and introduce a new unknown function $\bar{u}(x,y)=u(x,y)-K(x,y)$. Then (1) is converted into a similar equation $L\bar{u}=\bar{f}$, where $\bar{f}=f-LK$, while the corresponding integral data are now homogeneous. Now we construct another function

$$M(x,y)=\frac{1}{a}\int_\alpha^a \bar{u}(x,y)\,dx+\frac{1}{b}\int_\beta^b \bar{u}(x,y)\,dy-\frac{1}{ab}\int_\beta^b\int_\alpha^a \bar{u}(x,y)\,dx\,dy\,,$$

which satisfies the conditions

$$\int_0^a M(x,y)\,dx=\int_\alpha^a \bar{u}(x,y)\,dx,\qquad \int_0^b M(x,y)\,dy=\int_\beta^b \bar{u}(x,y)\,dy\,.$$

Let $\bar{u}(x,y)=w(x,y)+M(x,y)$, where $w(x,y)$ satisfies a differential equation to be determined. To find the form of this equation, we consider the previous equality as an integral equation with respect to $\bar{u}$

$$\bar{u}(x,y)-\frac{1}{a}\int_\alpha^a \bar{u}(x,y)\,dx-\frac{1}{b}\int_\beta^b \bar{u}(x,y)\,dy+\frac{1}{ab}\int_\beta^b\int_\alpha^a \bar{u}(x,y)\,dx\,dy=w(x,y)\,. \tag{3}$$

It is not difficult to show that

$$\bar{u}(x,y)=w(x,y)+\frac{1}{\alpha}\int_\alpha^a w(x,y)\,dx+\frac{1}{\beta}\int_\beta^b w(x,y)\,dy+\frac{1}{\alpha\beta}\int_\beta^b\int_\alpha^a w(x,y)\,dx\,dy\,. \tag{4}$$

If we substitute (4) into the left-hand side of the equation $L\bar{u}=\bar{f}$, then we obtain the so called loaded equation with respect to $w(x,y)$,

$$\begin{aligned}\bar{L}w\equiv w_{xy}+A(w+\frac{1}{\beta}\int_\beta^b w(x,y)\,dy)_x+B(w+\frac{1}{\alpha}\int_\alpha^a w(x,y)\,dx)_y&\\ +C(w+\frac{1}{\alpha}\int_\alpha^a w(x,y)\,dx+\frac{1}{\beta}\int_\beta^b w(x,y)\,dy&\\ +\frac{1}{\alpha\beta}\int_\beta^b\int_\alpha^a w(x,y)\,dx\,dy)&=\bar{f}(x,y)\end{aligned} \tag{5}$$

and integral conditions

$$\int_0^a w(x,y)\,dx=0,\int_0^b w(x,y)\,dy=0. \tag{6}$$

3 Generalized solution

Define the function S by

$$Sw = A(w+\frac{1}{\beta}\int_\beta^b w\,dy)_x+B(w+\frac{1}{\alpha}\int_\alpha^a w\,dx)_y$$

$$+C(w+\frac{1}{\alpha}\int_\alpha^a w\,dx+\frac{1}{\beta}\int_\beta^b w\,dy+\frac{1}{\alpha\beta}\int_\beta^b\int_\alpha^a w\,dx\,dy)$$

and $F(x,y,Sw)=\bar{f}(x,y)-Sw$. Then (5) can be assumed to have the form

$$w_{xy}=F(x,y,Sw).$$

We introduce the function space

$$V=\{w: w\in C^1(\bar{D}), \exists w_{xy}\in C(\bar{D}), \int_0^a w\,dx=\int_0^b w\,dy=0\}\,.$$

The completion of this space, with respect to the norm

$$\|w\|_1^2=\int_0^b\int_0^a(w^2+w_x^2+w_y^2)\,dx\,dy$$

is denoted by $\tilde{H}^1(D)$. Notice that $\tilde{H}^1(D)$ is Hilbert space with

$$(w,v)_1=\int_0^b\int_0^a(wv+w_xv_x+w_yv_y)\,dx\,dy\,.$$

For $v\in\tilde{H}^1$ define the operator l by

$$lv\equiv\int_0^y v_x(x,\tau)d\tau+\int_0^x v_y(t,y)dt-\int_0^y\int_0^x v(t,\tau)\,dt\,d\tau\,.$$

Consider the scalar product $(w_{xy},lv)_{L_2}$. Employing integration by parts and taking account of $w\in V, v\in\tilde{H}^1$, we can see that $(w_{xy},v)_{L_2}=(w,v)_1$.

Definition. A function $w\in\tilde{H}^1(D)$ is called a generalized solution of the problem (5)-(6), if $(w,v)_1=(F(x,y,Sw),lv)_{L_2}$ for every $v\in\tilde{H}^1(D)$.

4 Subsidiary problem

Consider the problem with integral conditions (6) for the equation

$$w_{xy}=F(x,y).$$

Theorem 1 *Let $F(x,y)\in L_2(D)$. Then there exists one and only one generalized solution w_0 of the problem*

$$w_{xy}=F(x,y)$$
$$\int_0^a w\,dx=0,\qquad \int_0^b w\,dy=0,$$

where for some positive constant c_1,

$$c_1\|w_0\|_1\le\|F\|_{L_2}\,. \tag{7}$$

Proof. For $F(x,y) \in L_2(D)$, $\Psi(v) = (F, lv)_{L_2}$ is a bounded linear functional on $\tilde{H}^1(D)$. Indeed,

$$|(F, lv)| \leq \|F\|_{L_2} \|lv\|_{L_2} \leq 3\max\{a^2, b^2, a^2b^2\} \|F\|_{L_2} \|v\|_1 .$$

Thus by the Riesz-representation theorem there exists a unique $w_0 \in \tilde{H}^1(D)$ such that $\Psi(v) = (F, lv)_{L_2} = (w_0, v)_1$. Hence $(w, v)_1 = (w_0, v)_1$ for every $v \in \tilde{H}^1(D)$, i.e., w_0 is generalized solution. Letting $\frac{1}{c_1} = 3\max\{a^2, b^2, a^2b^2\}$, we obtain inequality (7). $\diamondsuit$

Lemma 1 *Operator $S : \tilde{H}^1 \to L_2$ is bounded, that is, there exists a positive constant c_2 such that $\|Sw\|_{L_2} \leq c_2 \|w\|_1$.*

Proof. Let $|A(x,y)| \leq A_0$, $|B(x,y)| \leq B_0$, and $|C(x,y)| \leq C_0$. Then $Sw = A\bar{u}_x + B\bar{u}_y + C\bar{u}$, and

$$\begin{aligned} \|Sw\|^2_{L_2} &= \int_0^b \int_0^a (A\bar{u}_x + B\bar{u}_y + C\bar{u})^2 \, dx \, dy \\ &\leq 3(A_0^2 \|\bar{u}_x\|^2_{L_2} + B_0^2 \|\bar{u}_y\|^2_{L_2} + C_0^2 \|\bar{u}\|^2_{L_2}) . \end{aligned}$$

Now by straightforward calculation, using the inequality $2ab \leq a^2 + b^2$, and Hölder's inequality, we find that

$$\begin{gathered} \|\bar{u}\|^2_{L_2} \leq c_3 \|w\|^2_{L_2}, \\ \text{with } c_3 = 4\left(1 + \frac{(a-\alpha)a}{\alpha^2} + \frac{(b-\beta)b}{\beta^2} + \frac{(b-\beta)(a-\alpha)ab}{\alpha^2\beta^2}\right); \\ \|\bar{u}_x\|^2_{L_2} \leq c_4 \|w_x\|^2_{L_2}, \text{ with } c_4 = 2\left(1 + \frac{(b-\beta)b}{\beta^2}\right); \\ \|\bar{u}_y\|^2_{L_2} \leq c_5 \|w_y\|^2_{L_2}, \text{ with } c_5 = 2\left(1 + \frac{(a-\alpha)a}{\alpha^2}\right). \end{gathered}$$

Hence $\|Sw\|^2_{L_2} \leq c_2 \|w\|^2_1$, where $c_2 = 3\max\{A_0^2 c_4,\ B_0^2 c_5,\ C_0^2 c_3\}$. Indeed,

$$\begin{aligned} \|Sw\|^2_{L_2} &\leq 3(A_0^2 c_4 \|w_x\|^2_{L_2} + B_0^2 c_5 \|w_y\|^2_{L_2} + C_0^2 c_3 \|w\|^2_{L_2}) \\ &\leq c_2(\|w_x\|^2_{L_2} + \|w_y\|^2_{L_2} + \|w\|^2_{L_2}) \\ &= c_2 \|w\|^2_1 . \end{aligned}$$

$\diamondsuit$

As S is linear $S(\sqrt{2}\lambda w) = \sqrt{2}\lambda S(w)$ for arbitrary λ. Let $\lambda > \frac{1}{c_1}$, and let

$$S_\lambda(w) = S(\sqrt{2}\lambda w) .$$

Theorem 2 *If $\bar{f}(x,y) \in L_2(D)$ and $|\bar{f}(x,y)| \leq \frac{P}{\sqrt{2}}$, then there exists at least one generalized solution $w_0 \in \tilde{H}^1(D)$ to problem (5)-(6), where $\|w_0\|^2_1 \leq \frac{P^2}{\eta^2}$, with $\eta^2 = c_1^2 - \frac{1}{\lambda^2}$. Furthermore, the solution is uniquely determined, if $c_2 < c_1$.*

Proof. Consider the closed ball

$$W=\{S_\lambda\omega: S_\lambda\omega\in L_2(D),\ \|S_\lambda\omega\|^2_{L_2}\le\frac{P^2ab}{\eta^2}\}\,.$$

Then

$$|F(x,y,S\omega)|\le|\bar f(x,y)|+\sqrt{\frac{c_1^2-\eta^2}{2}}|S_\lambda\omega|\,,$$

and for all $S_\lambda\omega\in W$ we have

$$\|F(x,y,S\omega)\|^2\le\frac{c_1^2P^2ab}{\eta^2}.$$

From Theorem 1 there exists a unique generalized solution of the problem

$$w_{xy}=F(x,y,S\omega),\ \int_0^a w(x,y)\,dx=0,\ \int_0^b w(x,y)\,dy=0$$

so that $(w,v)_1=(F,lv)_{L_2}$ and $\|w\|_1^2\le\frac{1}{c_1^2}\|F\|^2\le\frac{P^2ab}{\eta^2}$. Define an operator $T: S\omega\in W\to w=TS\omega\in\tilde H^1(D)$, $T(W)\subset W$. Notice that T is a continuous operator. To see this, let $(S\omega)_n,\ (S\omega)_0\in W$ and $\|(S\omega)_n-(S\omega)_0\|\to0$ as $n\to\infty$. Then for $w_n=T(S\omega)_n,\ w_0=T(S\omega)_0$ we have

$$(w_n-w_0,v)=(F(x,y,(S\omega)_n)-F(x,y,(S\omega)_0),\ lv)_{L_2}=((S\omega)_n-(S\omega)_0,lv)_{L_2}\,.$$

Now from Theorem 1

$$\|w_n-w_0\|_1\le\frac{1}{c_1}\|(S\omega)_n-(S\omega)_0\|_{L_2}\to0,\quad n\to\infty\,.$$

Furthermore, T is a compact operator. In order to show this, we take a sequence $\{(S\omega)_n\}\subset W$, that is $\|(S\omega)_n\|^2_{L_2}\le\frac{P^2ab}{\eta^2}$. For $w_n=T(S\omega)_n$ we have $\|w_n\|^2\le\frac{P^2ab}{\eta^2}$, so a sequence $\{w_n\}$ is bounded in $\tilde H^1(D)$, therefore there exists a subsequence weakly convergent in $\tilde H^1(D)$. Since any bounded set in $\tilde H^1$ is compact in L_2, then there exists a subsequence, which we again denote by $\{w_n\}$, strongly convergent in $L_2(D)$ to w_0, as $n\to\infty$. Now w_0 satisfies the inequality $\|w_0\|^2_{L_2}\le P^2ab/\eta^2$. As S is a bounded operator, T is completely continuous and so TS is completely continuous. Thus from Schauder's fixed-point theorem there exists at least one $w_0\in W$ such that $w_0=TSw_0$ and

$$(w_0,v)_1=(F(x,y,Sw_0),lv)_{L_2}$$

for all$v\in\tilde H^1(D)$.

Assume that $w_1,\ w_2$ are distinct generalized solutions, then

$$(w_1-w_2,v)_1=(F(x,y,Sw_1)-F(x,y,Sw_2),lv)_{L_2}.$$

¿From (7) and Lemma 1 we have

$$\|w_1-w_2\|_1\le\frac{1}{c_1}\|Sw_1-Sw_2\|_{L_2}\le\frac{c_2}{c_1}\|w_1-w_2\|_1.$$

Thus, if $c_2<c_1$ then it gives a contradiction; therefore, $w_1=w_2$.

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