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POTENTIAL LANDESMAN-LAZER TYPE CONDITIONS AND THE FUČÍK SPECTRUM

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Abstract. We prove the existence of solutions to the nonlinear problem

$$u''(x) + \lambda_+ u^+(x) - \lambda_- u^-(x) + g(x, u(x)) = f(x)\,, \quad x \in (0, \pi)\,,$$
$$u(0) = u(\pi) = 0$$

where the point $[\lambda_+, \lambda_-]$ is a point of the Fučík spectrum and the nonlinearity $g(x, u(x))$ satisfies a potential Landesman-Lazer type condition. We use a variational method based on the generalization of the Saddle Point Theorem.

1. Introduction

We investigate the existence of solutions for the nonlinear boundary-value problem

$$\begin{gathered} u''(x) + \lambda_+ u^+(x) - \lambda_- u^-(x) + g(x, u(x)) = f(x)\,, \quad x \in (0, \pi)\,, \\ u(0) = u(\pi) = 0\,. \end{gathered} \tag{1.1}$$

Here $u^\pm = \max\{\pm u, 0\}$, $\lambda_+, \lambda_- \in \mathbb{R}$, the nonlinearity $g : (0, \pi) \times \mathbb{R} \mapsto \mathbb{R}$ is a Caratheodory function and $f \in L^1(0, \pi)$. For $g \equiv 0$ and $f \equiv 0$ problem (1.1) becomes

$$\begin{gathered} u''(x) + \lambda_+ u^+(x) - \lambda_- u^-(x) = 0\,, \quad x \in (0, \pi)\,, \\ u(0) = u(\pi) = 0\,. \end{gathered} \tag{1.2}$$

We define $\Sigma = \{[\lambda_+, \lambda_-] \in \mathbb{R}^2 : (1.2) \text{ has a nontrivial solution}\}$. This set is called the Fučík spectrum (see [2]), and can be expressed as $\Sigma = \bigcup_{j=1}^{\infty} \Sigma_j$ where

$$\Sigma_1 = \big\{[\lambda_+, \lambda_-] \in \mathbb{R}^2 : \ (\lambda_+ - 1)(\lambda_- - 1) = 0\big\}\,,$$
$$\Sigma_{2i} = \Big\{[\lambda_+, \lambda_-] \in \mathbb{R}^2 : \ i\Big(\frac{1}{\sqrt{\lambda_+}} + \frac{1}{\sqrt{\lambda_-}}\Big) = 1\Big\}\,,$$
$$\Sigma_{2i+1} = \Sigma_{2i+1,1} \cup \Sigma_{2i+1,2} \quad \text{where}$$
$$\Sigma_{2i+1,1} = \Big\{[\lambda_+, \lambda_-] \in \mathbb{R}^2 : \ i\Big(\frac{1}{\sqrt{\lambda_+}} + \frac{1}{\sqrt{\lambda_-}}\Big) + \frac{1}{\sqrt{\lambda_+}} = 1\Big\}\,,$$
$$\Sigma_{2i+1,2} = \Big\{[\lambda_+, \lambda_-] \in \mathbb{R}^2 : \ i\Big(\frac{1}{\sqrt{\lambda_+}} + \frac{1}{\sqrt{\lambda_-}}\Big) + \frac{1}{\sqrt{\lambda_-}} = 1\Big\}\,.$$

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We suppose that

$$\begin{gathered}
[\lambda_+, \lambda_-] \in \Sigma_m\,, \text{ if } m \in \mathbb{N} \text{ is even} \\
[\lambda_+, \lambda_-] \in \Sigma_{m2}\,, \text{ if } m \in \mathbb{N} \text{ is odd} \\
\text{and } \lambda_- < \lambda_+ < (m+1)^2\,.
\end{gathered} \tag{1.3}$$

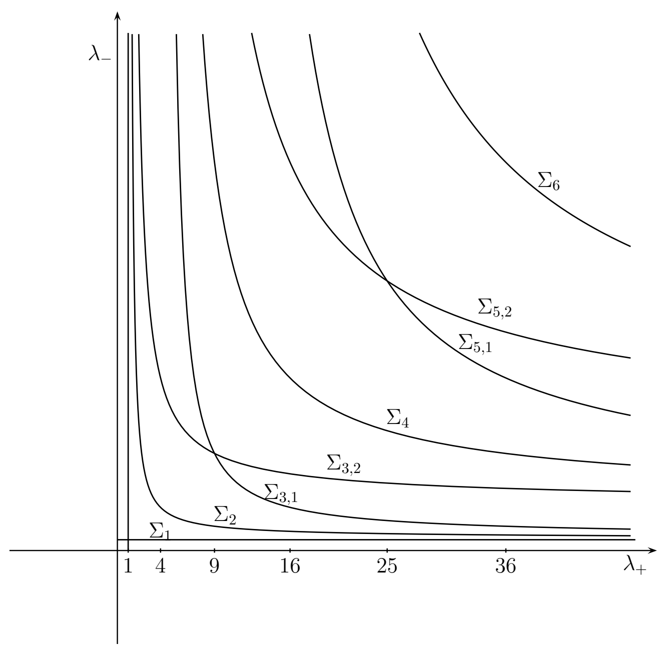

FIGURE 1. Fučík spectrum

Remark 1.1. Assuming that $(m+1)^2 > \lambda_+ > \lambda_-$, if $[\lambda_+, \lambda_-] \in \Sigma_m$, $m \in \mathbb{N}$, then $\lambda_- > (m-1)^2$.

We define the potential of the nonlinearity g as

$$G(x,s) = \int_0^s g(x,t)\,dt$$

and

$$G_+(x) = \liminf_{s\to+\infty} \frac{G(x,s)}{s}\,, \quad G_-(x) = \limsup_{s\to-\infty} \frac{G(x,s)}{s}\,.$$

We denote by φ_m a nontrivial solution of (1.2) corresponding to $[\lambda_+, \lambda_-]$ (see Remark 1.2). We assume that for any φ_m the following potential Landesman-Lazer type condition holds:

$$\int_0^\pi f(x)\varphi_m(x)\,dx < \int_0^\pi \big[G_+(x)(\varphi_m(x))^+ - G_-(x)(\varphi_m(x))^-\big]\,dx\,. \tag{1.4}$$

We suppose that the nonlinearity g is bounded, i.e. there exists $p(x) \in L^1(0,\pi)$ such that

$$|g(x,s)| \le p(x) \quad \text{for a.e. } x \in (0,\pi)\,,\ \forall\, s \in \mathbb{R} \tag{1.5}$$

and we prove the solvability of (1.1) in Theorem (3.1) below.

This article is inspired by a result in [3] where the author studies the case when $g(x,s)/s$ lies (in some sense) between Σ_1 and Σ_2 and by a result in [1] with the classical Landesman-Lazer type condition [1, Corollary 2].

Remark 1.2. First we note that if m is even then two different functions $\varphi_{m1}, \varphi_{m2}$ of norm 1 correspond to the point $[\lambda_+, \lambda_-] \in \Sigma_m$. For example for $m = 2$, $\lambda_+ > \lambda_-$ we have

$$\varphi_{21}(x) = \begin{cases} k_1\sqrt{\lambda_-}\sin(\sqrt{\lambda_+}x), & x \in \langle 0, \pi/\sqrt{\lambda_+}\rangle, \\ -k_1\sqrt{\lambda_+}\sin(\sqrt{\lambda_-}(x - \pi/\sqrt{\lambda_+})), & x \in \langle \pi/\sqrt{\lambda_+}, \pi\rangle, \end{cases}$$

where $k_1 > 0$, and

$$\varphi_{22}(x) = \begin{cases} -k_2\sqrt{\lambda_+}\sin(\sqrt{\lambda_-}x), & x \in \langle 0, \pi/\sqrt{\lambda_-}\rangle, \\ k_2\sqrt{\lambda_-}\sin(\sqrt{\lambda_+}(x - \pi/\sqrt{\lambda_-})), & x \in \langle \pi/\sqrt{\lambda_-}, \pi\rangle, \end{cases}$$

where $k_2 > 0$.

For $\lambda_+ = \lambda_- = 4$ we set $\varphi_{21}(x) = k_1 \sin 2x$ and $\varphi_{22}(x) = -k_2 \sin 2x$, where $k_1, k_2 > 0$.

FIGURE 2. Solutions corresponding to Σ_2

If m is odd, then $\Sigma_m = \Sigma_{m1} \cup \Sigma_{m2}$ and it corresponds only one function φ_{m1} od norm 1 to the point $[\lambda'_+, \lambda'_-] \in \Sigma_{m1}$, one function φ_{m2} of norm 1 to the point $[\lambda_+, \lambda_-] \in \Sigma_{m2}$, respectively.

For $m = 3$, $\lambda'_+ > \lambda'_-$, $\lambda_+ > \lambda_-$ we have

$$\varphi_{31}(x) = \begin{cases} k_1\sqrt{\lambda'_-}\sin(\sqrt{\lambda'_+}x), & x \in \langle 0, \pi/\sqrt{\lambda'_+}\rangle, \\ -k_1\sqrt{\lambda'_+}\sin(\sqrt{\lambda'_-}(x - \pi/\sqrt{\lambda'_+})), & x \in \langle \pi/\sqrt{\lambda'_+}, \pi/\sqrt{\lambda'_+} + \pi/\sqrt{\lambda'_-}\rangle, \\ k_1\sqrt{\lambda'_-}\sin(\sqrt{\lambda'_+}(x - \pi/\sqrt{\lambda'_+} - \pi/\sqrt{\lambda'_-})), & x \in \langle \pi/\sqrt{\lambda'_+} + \pi/\sqrt{\lambda'_-}, \pi\rangle, \end{cases}$$

where $k_1 > 0$.

$$\varphi_{32}(x) = \begin{cases} -k_2\sqrt{\lambda_+}\sin(\sqrt{\lambda_-}x), & x \in \langle 0, \pi/\sqrt{\lambda_-}\rangle, \\ k_2\sqrt{\lambda_-}\sin(\sqrt{\lambda_+}(x - \pi/\sqrt{\lambda_-})), & x \in \langle \pi/\sqrt{\lambda_-}, \pi/\sqrt{\lambda_-} + \pi/\sqrt{\lambda_+}\rangle, \\ -k_2\sqrt{\lambda_+}\sin(\sqrt{\lambda_-}(x - \pi/\sqrt{\lambda_-} - \pi/\sqrt{\lambda_+})), & x \in \langle \pi/\sqrt{\lambda_-} + \pi/\sqrt{\lambda_+}, \pi\rangle, \end{cases}$$

where $k_2 > 0$.

For $\lambda_+ = \lambda_- = m^2$ we set $\varphi_{m1}(x) = k_1 \sin mx$, and $\varphi_{m2}(x) = -k_2 \sin mx$, where $k_1, k_2 > 0$ and from the condition (1.4) we obtain

$$\int_0^\pi f(x)\sin mx\,dx < \int_0^\pi \big[G_+(x)(\sin mx)^+ - G_-(x)(\sin mx)^-\big]\,dx$$

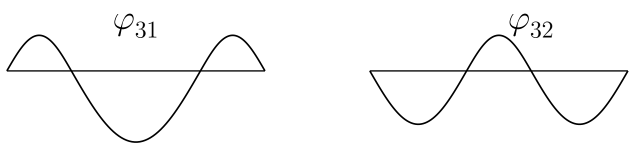

FIGURE 3. Solutions corresponding to Σ_3

and

$$\int_0^\pi f(x)(-\sin mx)\,dx < \int_0^\pi \big[G_+(x)(-\sin mx)^+ - G_-(x)(-\sin mx)^-\big]\,dx\,.$$

Hence it follows

$$\begin{aligned}&\int_0^\pi \big[G_-(x)(\sin mx)^+ - G_+(x)(\sin mx)^-\big]\,dx\\ &< \int_0^\pi f(x)\sin mx\,dx < \int_0^\pi \big[G_+(x)(\sin mx)^+ - G_-(x)(\sin mx)^-\big]\,dx\,.\end{aligned} \tag{1.6}$$

We obtained the potential Landesman-Lazer type condition (see [6]).

Remark 1.3. We have

$$\langle v, \sin mx\rangle = \int_0^\pi v'(x)(\sin mx)'\,dx = m^2\int_0^\pi v(x)\sin mx\,dx \quad \forall\, v \in H$$

(H is a Sobolev space defined below). Since and from the definition of the functions $\varphi_{m1}, \varphi_{m2}$ (see remark 1.2) it follows

$$\langle \varphi_{m1}, \sin mx\rangle > 0 \quad \text{and} \quad \langle \varphi_{m2}, \sin mx\rangle < 0\,. \tag{1.7}$$

2. Preliminaries

Notation. We shall use the classical spaces $C(0,\pi)$, $L^p(0,\pi)$ of continuous and measurable real-valued functions whose p-th power of the absolute value is Lebesgue integrable, respectively. H is the Sobolev space of absolutely continuous functions $u\colon (0,\pi)\to\mathbb{R}$ such that $u'\in L^2(0,\pi)$ and $u(0)=u(\pi)=0$. We denote by the symbols $\|\cdot\|$, and $\|\cdot\|_2$ the norm in H, and in $L^2(0,\pi)$, respectively. We denote $\langle\cdot,\cdot\rangle$ the pairing in the space H.

By a solution of (1.1) we mean a function $u \in C^1(0,\pi)$ such that u' is absolutely continuous, u satisfies the boundary conditions and the equations (1.1) holds a.e. in $(0,\pi)$.

Let $I: H \to \mathbb{R}$ be a functional such that $I \in C^1(H,\mathbb{R})$ (continuously differentiable). We say that u is a critical point of I, if

$$\langle I'(u), v\rangle = 0 \quad \text{for all } v \in H\,.$$

We say that γ is a critical value of I, if there is $u_0 \in H$ such that $I(u_0) = \gamma$ and $I'(u_0) = 0$.

We say that I satisfies Palais-Smale condition (PS) if every sequence (u_n) for which $I(u_n)$ is bounded in H and $I'(u_n) \to 0$ (as $n \to \infty$) possesses a convergent subsequence.

We study (1.1) by the use of a variational method. More precisely, we look for critical points of the functional $I : H \to \mathbb{R}$, which is defined by

$$I(u) = \frac{1}{2}\int_0^\pi \big[(u')^2 - \lambda_+(u^+)^2 - \lambda_-(u^-)^2\big]\,dx - \int_0^\pi \big[G(x,u) - fu\big]\,dx\,. \tag{2.1}$$

Every critical point $u \in H$ of the functional I satisfies

$$\int_0^\pi \big[u'v' - (\lambda_+u^+ - \lambda_-u^-)v\big]\,dx - \int_0^\pi \big[g(x,u)v - fv\big]\,dx = 0 \quad \text{for all } v \in H\,.$$

Then u is also a weak solution of (1.1) and vice versa.

The usual regularity argument for ODE yields immediately (see Fučík [2]) that any weak solution of (1.1) is also a solution in the sense mentioned above.

We will use the following variant of the Saddle Point Theorem (see [4]) which is proved in Struwe [5, Theorem 8.4].

Theorem 2.1. *Let S be a closed subset in H and Q a bounded subset in H with boundary ∂Q. Set $\Gamma = \{h : h \in \mathbf{C}(H,H),\, h(u) = u \text{ on } \partial Q\}$. Suppose $I \in C^1(H,\mathbb{R})$ and*

- *(i) $S \cap \partial Q = \emptyset$,*
- *(ii) $S \cap h(Q) \neq \emptyset$, for every $h \in \Gamma$,*
- *(iii) there are constants μ, ν such that $\mu = \inf_{u\in S} I(u) > \sup_{u\in\partial Q} I(u) = \nu$,*
- *(iv) I satisfies Palais-Smale condition.*

Then the number

$$\gamma = \inf_{h\in\Gamma}\sup_{u\in Q} I(h(u))$$

defines a critical value $\gamma > \nu$ of I.

We say that S and ∂Q *link* if they satisfy conditions (i), (ii) of the theorem above.

We denote the first integral in the functional I by

$$J(u) = \int_0^\pi \big[(u')^2 - \lambda_+(u^+)^2 - \lambda_-(u^-)^2\big]\,dx\,.$$

Now we present a few results needed later.

Lemma 2.2. *Let φ be a solution of (1.2) with $[\lambda_+,\lambda_-] \in \Sigma$, $\lambda_+ \geq \lambda_-$. We put $u = a\varphi + w$, $a \geq 0$, $w \in H$. Then the following relation holds*

$$\int_0^\pi \big[(w')^2 - \lambda_+w^2\big]\,dx \leq J(u) \leq \int_0^\pi \big[(w')^2 - \lambda_-w^2\big]\,dx\,. \tag{2.2}$$

Proof. We prove only the right inequality in (2.2), the proof of the left inequality is similar. Since φ is a solution of (1.2) we have

$$\int_0^\pi \varphi'w'\,dx = \int_0^\pi \big[\lambda_+\varphi^+w - \lambda_-\varphi^-w\big]\,dx \quad \text{for } w \in H \tag{2.3}$$

and

$$\int_0^\pi (\varphi')^2\,dx = \int_0^\pi \big[\lambda_+(\varphi^+)^2 + \lambda_-(\varphi^-)^2\big]\,dx\,. \tag{2.4}$$

By (2.3) and (2.4), we obtain

$$
\begin{aligned}
J(u) &= \int_0^\pi \Big[((a\varphi+w)')^2 - \lambda_+((a\varphi+w)^+)^2 - \lambda_-((a\varphi+w)^-)^2\Big]\,dx \\
&= \int_0^\pi \Big[(a\varphi')^2 + 2a\varphi' w' + (w')^2 - (\lambda_+ - \lambda_-)((a\varphi+w)^+)^2 \\
&\quad - \lambda_-(a\varphi+w)^2\Big]\,dx \\
&= \int_0^\pi \Big[(\lambda_+ - \lambda_-)(a\varphi^+)^2 + \lambda_-(a\varphi)^2 + 2a((\lambda_+ - \lambda_-)\varphi^+ + \lambda_-\varphi)w \\
&\quad + (w')^2 - (\lambda_+ - \lambda_-)((a\varphi+w)^+)^2 - \lambda_-((a\varphi)^2 + 2a\varphi w + w^2)\Big]\,dx \\
&= \int_0^\pi \Big\{(\lambda_+ - \lambda_-)\big[(a\varphi^+)^2 + 2a\varphi^+ w - ((a\varphi+w)^+)^2\big] \\
&\quad + (w')^2 - \lambda_- w^2\Big\}\,dx\,.
\end{aligned} \tag{2.5}
$$

For the function $(a\varphi^+)^2 + 2a\varphi^+ w - ((a\varphi+w)^+)^2$ in the last integral in (2.5) we have

$$
\begin{aligned}
&(a\varphi^+)^2 + 2a\varphi^+ w - ((a\varphi+w)^+)^2 \\
&= \begin{cases} -((a\varphi+w)^+)^2 \le 0 & \varphi < 0 \\ -w^2 \le 0 & \varphi \ge 0, a\varphi + w \ge 0 \\ a\varphi^+(a\varphi^+ + w + w) \le 0 & \varphi \ge 0, a\varphi + w < 0\,. \end{cases}
\end{aligned}
$$

By the assumption $\lambda_+ \ge \lambda_-$, we obtain the assertion of the Lemma 2.2. □

Remark 2.3. It follows from the previous proof that we obtain the equality

$$J(u) = \int_0^\pi \big[(w')^2 - \lambda_- w^2\big]\,dx$$

in (2.2) if $a\varphi + w \le 0$ when $\varphi < 0$, and $w = 0$ when $\varphi \ge 0$. Consequently, if the equality holds and if w in $\operatorname{span}\{\sin x, \dots, \sin kx\}$, $k \in \mathbb{N}$, then $w = 0$.

3. Main result

Theorem 3.1. *Under the assumptions (1.3), (1.4), and (1.5), Problem (1.1) has at least one solution in H.*

Proof. First we suppose that m is even. We shall prove that the functional I defined by (2.1) satisfies the assumptions in Theorem 2.1. Let $\varphi_{m1}, \varphi_{m2}$ be the normalized solutions of (1.2) described above (see Remark 1.2).

Let H^- be the subspace of H spanned by functions $\sin x, \dots, \sin(m-1)x$. We define $V \equiv V_1 \cup V_2$ where

$$
\begin{aligned}
V_1 &= \{u \in H : u = a_1\varphi_{m1} + w,\ 0 \le a_1,\ w \in H^-\}, \\
V_2 &= \{u \in H : u = a_2\varphi_{m2} + w,\ 0 \le a_2,\ w \in H^-\}.
\end{aligned}
$$

Let $K > 0$, $L > 0$ then we define $Q \equiv Q_1 \cup Q_2$ where

$$
\begin{aligned}
Q_1 &= \{u \in V_1 : 0 \le a_1 \le K,\ \|w\| \le L\}, \\
Q_2 &= \{u \in V_2 : 0 \le a_2 \le K,\ \|w\| \le L\}.
\end{aligned}
$$

Let S be the subspace of H spanned by functions $\sin(m+1)x, \sin(m+2)x, \dots$.

Next, we verify the assumptions of Theorem 2.1. We see that S is a closed subset in H and Q is a bounded subset in H.
(i) Firstly we note that for $z \in H^- \oplus S$ we have $\langle z, \sin mx\rangle = 0$. We suppose for contradiction that there is $u \in \partial Q \cap S$. Then

$$0 \overset{u\in S}{=} \langle u, \sin mx\rangle \overset{u\in\partial Q}{=} \langle a_i\varphi_{mi} + w, \sin mx\rangle \overset{w\in H^-}{=} a_i\langle\varphi_{mi}, \sin mx\rangle$$

$i = 1, 2$. From previous equalities and inequalities (1.7) it follows that $a_i = 0$, $i = 1, 2$ and $u = w$. For $u = w \in \partial Q$ we have $\|u\| = L > 0$ and we obtain a contradiction with $u \in H^- \cap S = \{o\}$.
(ii) We prove that $H = V \oplus S$. We can write a function $h \in H$ in the form

$$\begin{aligned} h &= \sum_{i=1}^{m-1} b_i \sin ix + b_m \sin mx + \sum_{i=m+1}^{\infty} b_i \sin ix \\ &= \overline{h} + b_m \sin mx + \widetilde{h},\ b_i \in \mathbb{R}, \end{aligned}$$

$i \in \mathbb{N}$. The inequalities (1.7) yield that there are constants $b_{m1}, b_{m2} > 0$ such that $\sin mx = b_{m1}(\varphi_{m1} - \overline{\varphi}_{m1} - \widetilde{\varphi}_{m1})$ and $-\sin mx = b_{m2}(\varphi_{m2} - \overline{\varphi}_{m2} - \widetilde{\varphi}_{m2})$. Hence we have for $b_m \geq 0$,

$$\begin{aligned} h &= \overline{h} + b_m b_{m1}(\varphi_{m1} - \overline{\varphi}_{m1} - \widetilde{\varphi}_{m1}) + \widetilde{h} \\ &= \underbrace{(\overbrace{\overline{h} - b_m b_{m1}\overline{\varphi}_{m1}}^{\in H^-} + \overbrace{b_m b_{m1}}^{\geq 0}\varphi_{m1})}_{\in V} + \underbrace{(\widetilde{h} - b_m b_{m1}\widetilde{\varphi}_{m1})}_{\in S}. \end{aligned}$$

Similarly for $b_m \leq 0$,

$$\begin{aligned} h &= \overline{h} + |b_m| b_{m2}(\varphi_{m2} - \overline{\varphi}_{m2} - \widetilde{\varphi}_{m2}) + \widetilde{h} \\ &= \underbrace{(\overbrace{\overline{h} - |b_m| b_{m2}\overline{\varphi}_{m2}}^{\in H^-} + \overbrace{|b_m| b_{m2}}^{\geq 0}\varphi_{m2})}_{\in V} + \underbrace{(\widetilde{h} - |b_m| b_{m2}\widetilde{\varphi}_{m2})}_{\in S}. \end{aligned}$$

We proved that H is spanned by V and S.

The proof of the assumption $S \cap h(Q) \neq \emptyset \quad \forall h \in \Gamma$ is similar to the proof in [5, example 8.2]. Let $\pi\colon H \to V$ be the continuous projection of H onto V. We have to show that $0 \in \pi(h(Q))$. For $t \in [0, 1]$, $u \in Q$ we define

$$h_t(u) = t\pi(h(u)) + (1 - t)u\,.$$

The function h_t defines a homotopy of $h_0 = id$ with $h_1 = \pi \circ h$. Moreover, $h_t|\partial Q =$ id for all $t \in [0, 1]$. Hence the topological degree $\deg(h_t, Q, 0)$ is well-defined and by homotopy invariance we have

$$\deg(\pi \circ h, Q, 0) = \deg(\mathrm{id}, Q, 0) = 1\,.$$

Hence $0 \in \pi(h(Q))$, as needed.
(iii) Firstly, we note that by assumption (1.5), one has

$$\lim_{\|u\|\to\infty} \int_0^\pi \frac{G(x, u) - fu}{\|u\|^2}\, dx = 0\,. \tag{3.1}$$

First we show that the infimum of functional I on the set S is a real number. We prove for this that

$$\lim_{\|u\|\to\infty} I(u) = \infty \quad \text{for all } u \in S \tag{3.2}$$

and I is bounded on bounded sets.

Because of the compact imbedding of H into $C(0,\pi)$ ($\|u\|_{C(0,\pi)} \le c_1\|u\|$), and of H into $\mathrm{L}^2(0,\pi)$ ($\|u\|_2 \le c_2\|u\|$), and the assumption (1.5) one has

$$\begin{aligned} I(u) &= \frac{1}{2}\int_0^\pi \big[(u')^2 - \lambda_+(u^+)^2 - \lambda_-(u^-)^2\big]\,dx - \int_0^\pi \big[G(x,u) - fu\big]\,dx \\ &\le \frac{1}{2}\left(\|u\|^2 + \lambda_+\|u^+\|_2^2 + \lambda_-\|u^-\|_2^2\right) + \int_0^\pi \big[\,(|p| + |f|)|u|\big]\,dx \\ &\le \frac{1}{2}\left(\|u\|^2 + \lambda_+ c_2\|u^+\|^2 + \lambda_- c_2\|u^-\|^2\right) + (\|p\|_1 + \|f\|_1)\,c_1\|u\|\,. \end{aligned}$$

Hence I is bounded on bounded subsets of S.

To prove (3.2), we argue by contradiction. We suppose that there is a sequence $(u_n) \subset S$ such that $\|u_n\| \to \infty$ and a constant c_3 satisfying

$$\liminf_{n\to\infty} I(u_n) \le c_3\,. \tag{3.3}$$

For $u \in S$ the following relation holds

$$\|u\|^2 = \int_0^\pi (u')^2\,dx \ge (m+1)^2\int_0^\pi u^2\,dx = (m+1)^2\|u\|_2^2\,. \tag{3.4}$$

The definition of I, (3.1), (3.3) and (3.4) yield

$$0 \ge \liminf_{n\to\infty}\frac{I(u_n)}{\|u_n\|^2} \ge \liminf_{n\to\infty}\frac{((m+1)^2-\lambda_+)\|u_n^+\|_2^2 + ((m+1)^2-\lambda_-)\|u_n^-\|_2^2}{2\|u_n\|^2}\,. \tag{3.5}$$

It follows from (3.5) and (1.3) that $\|u_n\|_2^2/\|u_n\|^2 \to 0$ and from the definition of I and (3.1) we have

$$\liminf_{\|u_n\|\to\infty}\frac{I(u_n)}{\|u_n\|^2} = \frac{1}{2}$$

a contradiction to (3.5). We proved that there is $\mu \in \mathbb{R}$ such that $\inf_{u\in S} I(u) = \mu$.

Second we estimate the value $I(u)$ for $u \in \partial Q$. We remark that $u \in \partial Q$ can be either of the form $K\varphi_m + w$, with $\|w\| \le L$ or of the form $a_i\varphi_{mi}$, with $0 \le a_i \le K$, $\|w\| = L$ $(i = 1,2)$. We prove that

$$\sup_{(K+L)\to\infty} I(K\varphi_m + w) = \sup_{\|u\|\to\infty} I(u) = -\infty \quad \text{for} \quad u \in \partial Q\,. \tag{3.6}$$

For (3.6), we argue by contradiction. Suppose that (3.6) is not true then there are a sequence $(u_n) \subset \partial Q$ such that $\|u_n\| \to \infty$ and a constant c_4 satisfying

$$\limsup_{n\to\infty} I(u_n) \ge c_4\,. \tag{3.7}$$

Hence, it follows

$$\limsup_{n\to\infty}\Big[\frac{1}{2}\int_0^\pi \frac{(u_n')^2 - \lambda_+(u_n^+)^2 - \lambda_-(u_n^-)^2}{\|u_n\|^2}\,dx - \int_0^\pi \frac{G(x,u_n) - fu_n}{\|u_n\|^2}\,dx\Big] \ge 0\,. \tag{3.8}$$

Set $v_n = u_n/\|u_n\|$. Since $\dim\partial Q < \infty$ there is $v_0 \in \partial Q$ such that $v_n \to v_0$ strongly in H (also strongly in $L^2(0,\pi)$). Then (3.8) and (3.1) yield

$$\frac{1}{2}\int_0^\pi \big[(v_0')^2 - \lambda_+(v_0^+)^2 - \lambda_-(v_0^-)^2\big]\,dx \ge 0\,. \tag{3.9}$$

Let $v_0 = a_0\varphi_m + w_0$, $a_0 \in \mathbb{R}_0^+, w_0 \in H^-$. It follows from Lemma 2.2 that

$$\int_0^\pi \big[(v_0')^2 - \lambda_+(v_0^+)^2 - \lambda_-(v_0^-)^2\big]\,dx \le \int_0^\pi \big[(w_0')^2 - \lambda_-(w_0)^2\big]\,dx\,. \tag{3.10}$$

For $w_0 \in H^-$ we have

$$\int_0^\pi \big[(w_0')^2 - \lambda_- w_0^2\big]\,dx \le \int_0^\pi \big[((m-1)^2 - \lambda_-)\,w_0^2\big]\,dx\,. \tag{3.11}$$

Since $(m-1)^2 < \lambda_-$ (see Remark 1.1) then (3.9), (3.10) and (3.11) yield

$$\int_0^\pi \big[(v_0')^2 - \lambda_+(v_0^+)^2 - \lambda_-(v_0^-)^2\big]\,dx = ((m-1)^2 - \lambda_-)\|w_0\|_2^2 = 0\,.$$

Hence we obtain $w_0 = 0$ and $v_0 = a_0\varphi_m$, $\|v_0\| = 1$. Now we divide (3.7) by $\|u_n\|$ then

$$\limsup_{n\to\infty}\Big[\frac{1}{2}\int_0^\pi \frac{(u_n')^2 - \lambda_+(u_n^+)^2 - \lambda_-(u_n^-)^2}{\|u_n\|}\,dx - \int_0^\pi \frac{G(x,u_n) - fu_n}{\|u_n\|}\,dx\Big] \ge 0\,. \tag{3.12}$$

By Lemma 2.2 the first integral in (3.12) is less then or equal to 0. Hence it follows

$$\limsup_{n\to\infty}\int_0^\pi \frac{-G(x,u_n) + fu_n}{\|u_n\|}\,dx = \limsup_{n\to\infty}\int_0^\pi \Big[\frac{-G(x,u_n)}{u_n}v_n + fv_n\Big]\,dx \ge 0\,. \tag{3.13}$$

Because of the compact imbedding $H^- \subset C(0,\pi)$, we have $v_n \to a_0\varphi_m$ in $C(0,\pi)$ and we get

$$\lim_{n\to\infty} u_n(x) = \begin{cases} +\infty & \text{for } x \in (0,\pi) \text{ such that } \varphi_m(x) > 0\,, \\ -\infty & \text{for } x \in (0,\pi) \text{ such that } \varphi_m(x) < 0\,. \end{cases}$$

We note that from (1.5) it follows that $-|p(x)| \le G_+(x)$, $G_-(x) \le |p(x)|$ for a.e. $x \in (0,\pi)$. We obtain from Fatou's lemma and (3.13)

$$\int_0^\pi f(x)\varphi_m(x)\,dx \ge \int_0^\pi \big[G_+(x)(\varphi_m(x))^+ - G_-(x)(\varphi_m(x))^-\big]\,dx\,,$$

a contradiction to (1.4). We proved that by choosing K, L sufficiently large there is $\nu \in \mathbb{R}$ such that $\sup_{u\in\partial Q} I(u) = \nu < \mu$. Then Assumption (iii) of Theorem 3.1 is verified.

(iv) Now we show that I satisfies the Palais-Smale condition. First, we suppose that the sequence (u_n) is unbounded and there exists a constant c_5 such that

$$\Big|\frac{1}{2}\int_0^\pi \big[(u_n')^2 - \lambda_+(u_n^+)^2 - \lambda_-(u_n^-)^2\big]\,dx - \int_0^\pi \big[G(x,u_n) - fu_n\big]\,dx\Big| \le c_5 \tag{3.14}$$

and

$$\lim_{n\to\infty}\|I'(u_n)\| = 0\,. \tag{3.15}$$

Let (w_k) be an arbitrary sequence bounded in H. It follows from (3.15) and the Schwarz inequality that

$$\begin{aligned}
&\Big|\lim_{\substack{n\to\infty\\k\to\infty}} \int_0^\pi \big[u_n' w_k' - (\lambda_+ u_n^+ - \lambda_- u_n^-)\, w_k\big]\, dx - \int_0^\pi \big[g(x,u_n)w_k - f w_k\big]\, dx\,\Big| \\
&= \big|\lim_{\substack{n\to\infty\\k\to\infty}} \langle I'(u_n), w_k\rangle\big| \\
&\le \lim_{\substack{n\to\infty\\k\to\infty}} \|I'(u_n)\|\cdot\|w_k\| = 0\,.
\end{aligned} \tag{3.16}$$

Put $v_n = u_n/\|u_n\|$. Due to compact imbedding $H \subset L^2(0,\pi)$ there is $v_0 \in H$ such that (up to subsequence) $v_n \rightharpoonup v_0$ weakly in H, $v_n \to v_0$ strongly in $L^2(0,\pi)$. We divide (3.16) by $\|u_n\|$ and we obtain

$$\lim_{n,k\to\infty} \int_0^\pi \big[v_n' w_k' - (\lambda_+ v_n^+ - \lambda_- v_n^-)\, w_k\big]\, dx = 0 \tag{3.17}$$

and

$$\lim_{i,k\to\infty} \int_0^\pi \big[v_i' w_k' - (\lambda_+ v_i^+ - \lambda_- v_i^-)\, w_k\big]\, dx = 0\,. \tag{3.18}$$

We subtract equalities (3.17) and (3.18) we have

$$\lim_{n,i,k\to\infty} \int_0^\pi \big[(v_n' - v_i')w_k' - (\lambda_+(v_n^+ - v_i^+) - \lambda_-(v_n^- - v_i^-)\,)\, w_k\big]\, dx = 0\,. \tag{3.19}$$

Because (w_k) is a arbitrary bounded sequence we can set $w_k = v_n - v_i$ in (3.19) and we get

$$\lim_{n,i\to\infty}\Big[\|v_n - v_i\|^2 - \int_0^\pi \big[\,[\lambda_+(v_n^+ - v_i^+) - \lambda_-(v_n^- - v_i^-)](v_n - v_i)\,\big]\ dx\Big] = 0\,. \tag{3.20}$$

Since $v_n \to v_0$ strongly in $L^2(0,\pi)$ the integral in (3.20) converges to 0 and then v_n is a Cauchy sequence in H and $v_n \to v_0$ strongly in H and $\|v_0\| = 1$.

It follows from (3.17) and the usual regularity argument for ordinary differential equations (see Fučík [2]) that v_0 is the solution of the equation

$$v_0'' + \lambda_+ v_0^+ - \lambda_- v_0^- = 0\,.$$

From the assumption $[\lambda_+, \lambda_-] \in \Sigma_m$ it follows that $v_0 = a_0\varphi_m, a_0 > 0$.

We set $u_n = a_n\varphi_m + \widehat{u}_n$, where $a_n \ge 0, \widehat{u}_n \in H^- \oplus S$. We remark that $u = u^+ - u^-$ and using (2.3) in the first integral in (3.16) we obtain

$$\begin{aligned}
I_1 &= \int_0^\pi \big[(a_n\varphi_m + \widehat{u}_n)' w_k' - (\lambda_+ u_n^+ - \lambda_- u_n^-) w_k\big]\ dx \\
&= \int_0^\pi \big[a_n \varphi_m' w_k' + (\widehat{u}_n)' w_k' - ((\lambda_+ - \lambda_-)u_n^+ + \lambda_- u_n)\, w_k\big]\ dx \\
&= \int_0^\pi \big[a_n(\lambda_+\varphi_m^+ - \lambda_-\varphi_m^-) w_k + (\widehat{u}_n)' w_k' - ((\lambda_+ - \lambda_-)u_n^+ + \lambda_- u_n)\, w_k\big]\ dx \\
&= \int_0^\pi \big\{a_n[(\lambda_+ - \lambda_-)\varphi_m^+ + \lambda_-\varphi_m] w_k + (\widehat{u}_n)' w_k' \\
&\quad - [(\lambda_+ - \lambda_-)(a_n\varphi_m + \widehat{u}_n)^+ + \lambda_-(a_n\varphi_m + \widehat{u}_n)]\, w_k\big\}\, dx \\
&= \int_0^\pi \big[(\lambda_+ - \lambda_-)(a_n\varphi_m^+ - (a_n\varphi_m + \widehat{u}_n)^+) w_k + (\widehat{u}_n)' w_k' - \lambda_- \widehat{u}_n w_k\big]\ dx\,.
\end{aligned} \tag{3.21}$$

Similarly we obtain

$$I_1 = \int_0^\pi \big[(\lambda_+ - \lambda_-)(a_n\varphi_m^- - (a_n\varphi_m + \widehat{u}_n)^-)w_k + (\widehat{u}_n)'w_k' - \lambda_+\widehat{u}_n w_k\big]\, dx\,. \tag{3.22}$$

Adding (3.21) and (3.22) and we have

$$2I_1 = \int_0^\pi [(\lambda_+ - \lambda_-)(|a_n\varphi_m| - |a_n\varphi_m + \widehat{u}_n|)w_k + 2(\widehat{u}_n)'w_k' - (\lambda_+ + \lambda_-)\widehat{u}_n w_k]\, dx\,. \tag{3.23}$$

We set $\widehat{u}_n = \overline{u}_n + \widetilde{u}_n$ where $\overline{u}_n \in H^-$, $\widetilde{u}_n \in S$ and we put in (3.23) $w_k = (\overline{u}_n - \widetilde{u}_n)/\|\widehat{u}_n\|$ then we have

$$\begin{aligned} 2I_1 = \frac{1}{\|\widehat{u}_n\|} \int_0^\pi \big[&(\lambda_+ - \lambda_-)(|a_n\varphi_m| - |a_n\varphi_m + \overline{u}_n + \widetilde{u}_n|)(\overline{u}_n - \widetilde{u}_n) \\ &+ 2\,(\overline{u}_n')^2 - 2(\widetilde{u}_n')^2 - (\lambda_+ + \lambda_-)(\overline{u}_n^2 - \widetilde{u}_n^2)\big]\, dx\,. \end{aligned} \tag{3.24}$$

Hence

$$\begin{aligned} 2I_1 &\le \frac{1}{\|\widehat{u}_n\|}\Big(\int_0^\pi [(\lambda_+ - \lambda_-)\,|\overline{u}_n + \widetilde{u}_n|\,|\overline{u}_n - \widetilde{u}_n|]\, dx \\ &\quad + 2\|\overline{u}_n\|^2 - 2\|\widetilde{u}_n\|^2 - (\lambda_+ + \lambda_-)(\|\overline{u}_n\|_2^2 - \|\widetilde{u}_n\|_2^2)\Big) \\ &= \frac{1}{\|\widehat{u}_n\|}\Big(\int_0^\pi [(\lambda_+ - \lambda_-)\,|\overline{u}_n^2 - \widetilde{u}_n^2|]\, dx \\ &\quad + 2\|\overline{u}_n\|^2 - (\lambda_+ + \lambda_-)\|\overline{u}_n\|_2^2 - 2\|\widetilde{u}_n\|^2 + (\lambda_+ + \lambda_-)\|\widetilde{u}_n\|_2^2\Big)\,. \end{aligned} \tag{3.25}$$

The inequality $|a^2 - b^2| \le \max\{a^2, b^2\}$, (3.25) and (1.3) yield

$$I_1 \le \max\big\{\|\overline{u}_n\|^2 - \lambda_-\|\overline{u}_n\|_2^2, -\|\widetilde{u}_n\|^2 + \lambda_+\|\widetilde{u}_n\|_2^2\big\}\,\frac{1}{\|\widehat{u}_n\|}\,. \tag{3.26}$$

We note that the following relations hold $\|\overline{u}_n\|^2 \le (m-1)^2\|\overline{u}_n\|_2^2$, $\|\widetilde{u}_n\|^2 \ge (m+1)^2\|\widetilde{u}_n\|_2^2$. Hence from assumption (1.3) and (3.26) it follows that there is $\varepsilon > 0$ such that

$$I_1 \le -\varepsilon \max\big\{\|\overline{u}_n\|^2, \|\widetilde{u}_n\|^2\big\}\,\frac{1}{\|\widehat{u}_n\|}\,. \tag{3.27}$$

From (3.16), (3.27) it follows

$$\lim_{n\to\infty} -\varepsilon\frac{\max\big\{\|\overline{u}_n\|^2, \|\widetilde{u}_n\|^2\big\}}{\|\widehat{u}_n\|} - \int_0^\pi \big[(g(x,u_n) - f)\frac{\overline{u}_n - \widetilde{u}_n}{\|\widehat{u}_n\|}\big]\, dx \ge 0. \tag{3.28}$$

Now we suppose that $\|\widehat{u}_n\| \to \infty$. We note that $\|\widehat{u}_n\|^2 = \|\overline{u}_n\|^2 + \|\widetilde{u}_n\|^2$, we divide (3.28) by $\|\widehat{u}_n\|$ and using (1.5) we have

$$-\frac{\varepsilon}{2} \ge \lim_{n\to\infty} -\varepsilon\frac{\max\big\{\|\overline{u}_n\|^2, \|\widetilde{u}_n\|^2\big\}}{\|\widehat{u}_n\|^2} - \int_0^\pi \frac{g(x,u_n) - f}{\|\widehat{u}_n\|}\,\frac{\overline{u}_n - \widetilde{u}_n}{\|\widehat{u}_n\|}\, dx \ge 0 \tag{3.29}$$

a contradiction to $\varepsilon > 0$. This implies that the sequence $(\widehat{u}_n)$ is bounded. We use (2.2) from Lemma 2.2 with $w = \widehat{u}_n$ and we obtain

$$\int_0^\pi \big[(\widehat{u}_n')^2 - \lambda_+\widehat{u}_n^2\big]\, dx \le J(u_n) \le \int_0^\pi \big[(\widehat{u}_n')^2 - \lambda_-\widehat{u}_n^2\big]\, dx\,.$$

Hence

$$\lim_{n\to\infty} \frac{J(u_n)}{\|u_n\|} = \lim_{n\to\infty} \frac{\int_0^\pi \big[(u_n')^2 - \lambda_+u_n^2 - \lambda_-u_n^2\big]\, dx}{\|u_n\|} = 0\,. \tag{3.30}$$

We divide (3.14) by $\|u_n\|$ and (3.30) yield

$$\lim_{n\to\infty}\int_0^\pi \Big[\frac{-G(x,u_n)+fu_n}{\|u_n\|}\Big]\,dx=0 \tag{3.31}$$

and using Fatou's lemma in (3.31) we obtain a contradiction to (1.4).

This implies that the sequence (u_n) is bounded. Then there exists $u_0\in H$ such that $u_n\rightharpoonup u_0$ in H, $u_n\to u_0$ in $L^2(0,\pi)$ (up to subsequence). It follows from the equality (3.16) that

$$\lim_{n,i,k\to\infty}\int_0^\pi \big[(u_n-u_i)'w_k'-[\lambda_+(u_n^+-u_i^+)-\lambda_-(u_n^--u_i^-)]w_k\big]\,dx=0\,. \tag{3.32}$$

We put $w_k=u_n-u_i$ in (3.32) and the strong convergence $u_n\to u_0$ in $L^2(0,\pi)$ and (3.32) imply the strong convergence $u_n\to u_0$ in H. This shows that the functional I satisfies Palais-Smale condition and the proof of Theorem 3.1 for m even is complete.

Now we suppose that m is odd. We have $[\lambda_+,\lambda_-]\in\Sigma_{m2}$ and the nontrivial solution φ_{m2} of (1.2) corresponding to $[\lambda_+,\lambda_-]$. Then there is $k>0$ such that $[\lambda_+-k,\lambda_--k]\in\Sigma_{m1}$ and solution φ_{m1} corresponding to $[\lambda_+-k,\lambda_--k]=[\lambda_+',\lambda_-']$ (see Remark 1.2).

We define the sets Q and S like for m even and the proof of the steps (i), (ii) of theorem 3.1 is the same. In the step (iii) we change inequality (3.10) if $v_0=a_0\varphi_{m1}$ as it follows

$$\begin{aligned}
&\int_0^\pi \big[(v_0')^2-\lambda_+(v_0^+)^2-\lambda_-(v_0^-)^2\big]\,dx\\
&=\int_0^\pi \big[(v_0')^2-(\lambda_+-k)(v_0^+)^2-(\lambda_--k)(v_0^-)^2\big]\,dx-k\int_0^\pi v_0^2\,dx \qquad (3.33)\\
&\le -k\int_0^\pi v_0^2\,dx+\int_0^\pi \big[(w_0')^2-\lambda_-(w_0)^2\big]\,dx\,.
\end{aligned}$$

Then by (3.9), (3.33) and (3.11) we obtain $k\int_0^\pi v_0^2\,dx=0$, a contradiction to $\|v_0\|=1$. The proof of the step (iv) is similar to the prove for m even. The proof of the theorem 3.1 is complete. □

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