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PARADIGM FOR THE CREATION OF SCALES AND PHASES IN NONLINEAR EVOLUTION EQUATIONS

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Abstract. The transition from regular to apparently chaotic motions is often observed in nonlinear flows. The purpose of this article is to describe a deterministic mechanism by which several smaller scales (or higher frequencies) and new phases can arise suddenly under the impact of a forcing term. This phenomenon is derived from a multiscale and multiphase analysis of nonlinear differential equations involving stiff oscillating source terms. Under integrability conditions, we show that the blow-up procedure (a type of normal form method) and the Wentzel-Kramers-Brillouin approximation (of supercritical type) introduced in [7, 8] still apply. This allows to obtain the existence of solutions during long times, as well as asymptotic descriptions and reduced models. Then, by exploiting transparency conditions (coming from the integrability conditions), by implementing the Hadamard's global inverse function theorem and by involving some specific WKB analysis, we can justify in the context of Hamilton-Jacobi equations the onset of smaller scales and new phases.

1. Introduction

The aim is to exhibit fundamental mechanisms explaining how the complexity of a nonlinear flow can suddenly increase. The phenomenon is illustrated by Theorem 1.5 and Figure 1. We explain how we could have transfer and creation of oscillations. In Subsection 1.1, we present a class of Hamilton-Jacobi equations and related assumptions. In Subsection 1.2, we state our main result. Then, in Subsection 1.3, we explain the link between this class of Hamilton-Jacobi equations and a special class of nonlinear differential equations. The main outcomes concerning the latter class of nonlinear differential equations are detailed in Section 2.

1.1. A class of Hamilton-Jacobi equations.

Given some $\varepsilon_0 \in]0,1]$, the effects will be assessed quantitatively by a small positive real parameter $\varepsilon \in]0,\varepsilon_0]$ which is intended to go to 0. We consider functions u depending on the time variable $\tau \in \mathbb{R}_+$ and on the space variable $x=(x_1,\dots,x_d)\in\mathbb{R}^d$ with $d\in\mathbb{N}^*$. Denote by $\mathbb{T}$ the torus $\mathbb{R}/(2\pi\mathbb{Z})$. Select two smooth scalar functions

$$\mathcal{U}_0 : [0,\varepsilon_0]\times\mathbb{R}^d \to \mathbb{R}, \tag{1.1}$$

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$$\mathsf{H} : [0, \varepsilon_0] \times \mathbb{T} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{T} \to \mathbb{R}. \tag{1.2}$$

We can expand $\mathcal{U}_0$ and H in powers of ε near $\varepsilon = 0$ up to the order $N \in \mathbb{N}^*$ according to

$$\mathcal{U}_0(\varepsilon, x) = \sum_{j=0}^{N} \varepsilon^j \mathcal{U}_{0j}(x) + \mathcal{O}(\varepsilon^{N+1}), \tag{1.3}$$

$$\mathsf{H}(\varepsilon, \theta_\tau, x, u, p, \theta_u) = \sum_{j=0}^{N} \varepsilon^j \mathsf{H}_j(\theta_\tau, x, u, p, \theta_u) + \mathcal{O}(\varepsilon^{N+1}). \tag{1.4}$$

Then, we consider the family of local solutions $u \equiv u_\varepsilon$ (indexed by ε) to the Cauchy problems

$$\partial_\tau u + \frac{1}{\varepsilon} \mathsf{H}\Big(\varepsilon, \frac{\tau}{\varepsilon}, x, \varepsilon u, \nabla_x u, \frac{u}{\varepsilon}\Big) = 0, \quad u(0, x) = u_0(x) := \mathcal{U}_0(\varepsilon, x). \tag{1.5}$$

The above equation involves fast variations in τ (due to the substitution of θ_τ for τ/ε) and strong nonlinear effects linked to the substitution of εu for $u \in \mathbb{R}$, $\nabla_x u$ for $p \in \mathbb{R}^d$ and of u/ε for $\theta_u \in \mathbb{T}$. The Hamilton-Jacobi equation (1.5) is highly oscillating, and therefore the same should apply to its solution u_ε. It follows that the asymptotic description of u_ε should require the use of several *scales* and *phases*:

(i) A *scale* is a power ε^ℓ with $\ell \in \mathbb{R}_+$. The *scale* ε^ℓ appears in the solution (resp. in the Cauchy problem) when it is needed (even in composite form) for the multiscale description of u_ε (resp. of u_0 or H). For instance, the three scales ε^0, ε^1 and ε^2 are needed when dealing with the function $g_\varepsilon : (\tau, x) \mapsto \sin\big[\big(\cos(x + \tau/\varepsilon)\big)/\varepsilon^2\big]$.

(ii) A *phase* φ is a smooth real valued scalar function occurring in the solution (or in the Cauchy problem) when, after multiplication by some negative power of ε, it is substituted for a periodic variable (like θ_τ or θ_u) in a profile. The phase $\varphi \equiv \varphi_\varepsilon$ may depend on ε and contain oscillations (like in the case of *chirped pulses* [12]) but it must be (locally) uniformly bounded with bounded first order derivatives. For example, the expression $\varepsilon \cos(x + \tau/\varepsilon)$ acts as a phase (associated with the frequency ε^{-3}) in the above function g_ε. The weight ε is here important to recover a time bounded derivative.

To tackle (1.5), restrictions on the leading term H_0 issued from (1.4) are needed. To this end, we first consider the following simplification of the nonlinear interaction in H_0.

Assumption 1.1 (Independence on the periodic variable θ_u)**.** The leading term H_0 does not depend on the last periodic variable θ_u. In other words

$$\partial_{\theta_u} \mathsf{H}_0 \equiv 0. \tag{1.6}$$

We also impose the following positivity condition on H_0.

Assumption 1.2 (Positivity condition)**.** We have for all $(\theta_\tau, x, u, p) \in \mathbb{T} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$

$$\big[(p \cdot \nabla_p)\mathsf{H}_0 - \mathsf{H}_0\big](\theta_\tau, x, u, p) > 0. \tag{1.7}$$

Note that Assumption 1.2 implies that $\mathsf{H}_0 \not\equiv 0$. In addition, we implement an integrability condition which will serve a lot, for instance to prove the uniform (in ε) local existence of the solutions u_ε.

Assumption 1.3 (Complete integrability)**.** For all position $\mathfrak{z} = {}^t(\mathfrak{z}_x, \mathfrak{z}_u, \mathfrak{z}_p) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, the solution ${}^t(\Xi_{0x}, \Xi_{0p})(\mathfrak{z}; \cdot)$ to the Cauchy problem

$$\partial_s \begin{pmatrix} \Xi_{0x} \\ \Xi_{0p} \end{pmatrix} = \begin{pmatrix} +\nabla_p \mathsf{H}_0 \\ -\nabla_x \mathsf{H}_0 \end{pmatrix} (s, \Xi_{0x}, \mathfrak{z}_u, \Xi_{0p}), \quad {}^t(\Xi_{0x}, \Xi_{0p})(\mathfrak{z}; 0) = {}^t(\mathfrak{z}_x, \mathfrak{z}_p), \tag{1.8}$$

is globally defined in s, and it is periodic in s of period 2π.

For reasons that will become apparent later in Subsection 6.4, we need to impose the following smallness restriction on some second order derivatives of the scalar function H_0.

Assumption 1.4 (Smallness condition on H_0)**.** For a well-adjusted small parameter $\delta \in \mathbb{R}_+^*$, we have

$$|\!|\!| \mathrm{D}_{x,u,p} \nabla_p \mathsf{H}_0 |\!|\!| := \sup \big\{ \| \mathrm{D}_{x,u,p} \nabla_p \mathsf{H}_0(\theta_\tau, z) \| ; \, \theta_\tau \in \mathbb{T} \, , z \in K \big\} \leq \delta, \tag{1.9}$$

where $\| \cdot \|$ is a matrix norm and K is a compact set which is identified in (6.19).

The initial data in the right hand side of (1.5) is of amplitude 1. It is smooth with uniformly bounded derivatives. But the source term H is large, of size $1/\varepsilon$ due to (1.7), and it implies very rapid oscillations (involving both τ and u). By this way, strong nonlinear processes are implemented when solving (1.5), see Remark 6.1. Such aspects are often studied in an isolated or partial manner. They appear for instance in the references [5, 6, 17, 18, 19, 23].

1.2. **Onset of smaller scales and new phases.**

The next result proves that the above nonlinear interactions can suddenly generate additional scales and phases. Denote by $B(0, R]$ the closed ball in $\mathbb{R}^d$ of center 0 and radius $R \in \mathbb{R}_+^*$.

Theorem 1.5 (WKB description of the flow)**.** *Under Assumptions 1.1, 1.2, 1.3, and 1.4, for all $\varepsilon \in]0, \varepsilon_0]$ with ε_0 small enough, there exists a local smooth solution $u \equiv u_\varepsilon$ to* (1.5) *on the product of a time interval and a spatial domain having the form $[0, \mathcal{T}] \times B(0, R]$ with $0 < \mathcal{T}$ and $R > 0$. Moreover, on this uniform region, for all $N \in \mathbb{N}$ with $N \geq 3$, the expression u_ε can be described in the sup norm by the following multiscale and multiphase expansion*

$$\begin{aligned} u_\varepsilon(\tau, x) = & \frac{1}{\varepsilon} \, \overline{\mathscr{U}}_{-1}\big(x; \tau, \frac{\tau}{\varepsilon}\big) + \overline{\mathscr{U}}_0\big(x; \tau, \frac{\tau}{\varepsilon}\big) \\ & + \sum_{j=1}^{N} \varepsilon^j \, \mathscr{U}_j \Big(x; \tau, \frac{\tau}{\varepsilon}, \frac{\psi_\varepsilon(x;\tau)}{\varepsilon^3} \Big) + \mathcal{O}(\varepsilon^{N+1}), \end{aligned} \tag{1.10}$$

where ψ_ε is a phase in the sense of (ii), looking like

$$\psi_\varepsilon(x; \tau) := \varepsilon \, \overline{\mathscr{U}}_{-1}\big(x; \tau, \frac{\tau}{\varepsilon}\big) + \varepsilon^2 \, \overline{\mathscr{U}}_0\big(x; \tau, \frac{\tau}{\varepsilon}\big). \tag{1.11}$$

The two profiles $\overline{\mathscr{U}}_{-1}(x, \tau, \theta_\tau)$ and $\overline{\mathscr{U}}_0(x, \tau, \theta_\tau)$ which appear both in (1.10) *and* (1.11) *are smooth functions on the domain $B(0, R] \times [0, \mathcal{T}] \times \mathbb{T}$. For $j \geq 1$, the profiles $\mathscr{U}_j(x; \tau, \theta_\tau, \hat{\theta}_{\mathrm{r}}^\circ)$ are smooth with respect to the variables $(x, \tau) \in B(0, R] \times [0, \mathcal{T}]$ and smooth (periodic) with respect to the two last variables $\theta_\tau \in \mathbb{T}$ and $\hat{\theta}_{\mathrm{r}}^\circ \in \mathbb{T}$.*

When solving (1.5), a number of new patterns are generated (creation of oscillations). When looking at smaller details that are at smaller amplitudes, the flow is growing in complexity. This cascade towards more and more scales and phases is put in concrete form at the level of (1.10) both quantitatively in terms of frequencies (larger derivatives) and qualitatively in terms of phases (extra directions of fast variations). Theorem 1.5 shows clearly that the description of the solution u_ε with an incrementally precision (expressed in terms of powers of ε) is associated with an increasing agitation. This principle may also be illustrated through the Table 1.

Table 1. Growing complexity of the flow as time passes and when looking at smaller details

Aspect of the flow modulo a precision of size ε^j with	Involved profile	Number of scales	Number of phases
$\tau = 0$ and $j \in \mathbb{N}$	$\mathcal{U}_{0j}$	1	0
$\tau > 0$ and $j = -1, 0$	$\overline{\mathscr{U}}_{-1}, \overline{\mathscr{U}}_0$	2	1
$\tau > 0$ and $j \geq 1$	$\mathscr{U}_j$	3	2

While it is commonly believed that nonlinear evolution equations can instantaneously develop microstructures (like in turbulent flows), concrete mechanisms for this and rigorous proofs are rarely exhibited (or in very specific contexts due to subsequent instabilities). Theorem 1.5 is a step forward in this direction. It is proved in Section 6, and appears as a difficult corollary of a WKB analysis which is developed in Sections 3, 4, and 5.

1.3. **From Hamilton-Jacobi equations to nonlinear differential equations.**

For smooth solutions, Hamilton-Jacobi equations can be solved by the method of characteristics. In the context of (1.5), this yields the following system of nonlinear differential equations

$$\partial_\tau \begin{pmatrix} z \\ \nu \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} \mathsf{A} \\ \mathsf{V} \end{pmatrix} \big(\varepsilon, z_0; z; \frac{\tau}{\varepsilon}, \frac{\nu}{\varepsilon}\big), \quad \begin{pmatrix} z \\ \nu \end{pmatrix}(0) = \begin{pmatrix} z_0 \\ \nu_0 \end{pmatrix}, \tag{1.12}$$

where, given $n \in \mathbb{N}^*$, the dependent variables $z \in \mathbb{R}^n$ and $\nu \in \mathbb{R}$ depend on the time $\tau \in \mathbb{R}$. Here, $\varepsilon \in]0, \varepsilon_0]$ is a small positive parameter which is intended to go to zero, whereas $z_0 \in \mathbb{R}^n$ and $\nu_0 \in \mathbb{R}$ stand for initial data (which may depend smoothly on $\varepsilon \in [0, \varepsilon_0]$). The precise content of (1.12) is described in next Section 2.

The strategy to study (1.12) is inspired from [7, 8]. But the works [7, 8] are devoted to the characteristics of the Vlasov equation, and they rely crucially on the conservation of the kinetic energy (of charged particles). They cannot be applied directly to the system (1.12) or in the presence of an electric field (see [9] to this end). With this in mind, the framework of [7, 8] needs to be extended; and the tools of [7, 8] must be revisited. In fact, in comparison to [7, 8], we replace some invariant quantities by (more general) integrability conditions (which will operate at all levels to move forward). As a consequence, we have to adapt (in Sections 3, 4 and 5) our preceding arguments to the new difficulties thus generated.

The proof of Theorem 1.5 depends on studying the above system of nonlinear differential equations (1.12). It is derived from an existence result (Theorem 2.5), asymptotic descriptions (Theorem 2.7) and reduced models (Theorem 2.10) that apply to the above class of differential equations (1.12). The contents of the latter theorems are identified in the next section.

2. A class of nonlinear differential equations

In this section, we present the underlying framework. In Subsection 2.1, we state the assumptions and the main results. In Subsection 2.2, we make some general comments about the content of our theorems. In Subsection 2.3, we outline the plan, we highlight some innovative ideas, and we come back to the possible applications.

The expressions A and V in the source term of (1.12) depend on ε. They are defined up to $\varepsilon = 0$, and they are smooth near $\varepsilon = 0$. Thus, for all $N \in \mathbb{N}$, they can be expressed as Taylor series expansions of order N in terms of ε, near $\varepsilon = 0$:

$$
\begin{aligned}
&C^\infty([0,\varepsilon_0] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}; \mathbb{R}^n) \\
&\ni \mathsf{A}(\varepsilon, z_0; z; \theta_\tau, \theta_{\mathrm{r}}) = \sum_{j=0}^{N} \varepsilon^j \mathsf{A}_j(z_0; z; \theta_\tau, \theta_{\mathrm{r}}) + \mathcal{O}(\varepsilon^{N+1}), \\
&C^\infty([0,\varepsilon_0] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}; \mathbb{R}_+^*) \\
&\ni \mathsf{V}(\varepsilon, z_0; z; \theta_\tau, \theta_{\mathrm{r}}) = \sum_{j=0}^{N} \varepsilon^j \mathsf{V}_j(z_0; z; \theta_\tau, \theta_{\mathrm{r}}) + \mathcal{O}(\varepsilon^{N+1}).
\end{aligned}
$$

As indicated, the functions A and V may imply $z_0 \in \mathbb{R}^n$. By this way, they can take into account the influence of the component z_0 in the initial condition, as it may be required in the applications. They may imply the unknown z. But they do not involve ν (the scalar variable ν appears in the right hand side only after substitution of θ_{r} by ν/ε). Most importantly, they are periodic with respect to the two last variables θ_τ and θ_{r}, of period respectively 2π and $T_{\mathrm{r}}(z_0)$. More precisely

$$\theta_\tau \in \mathbb{T} := \mathbb{R}/\big(2\pi\mathbb{Z}\big), \tag{2.1}$$

$$\theta_{\mathrm{r}} \in \mathbb{T}_{\mathrm{r},z_0} := \mathbb{R}/\big(T_{\mathrm{r}}(z_0)\mathbb{Z}\big), \quad T_{\mathrm{r}} \in C^\infty(\mathbb{R}^n; \mathbb{R}_+^*). \tag{2.2}$$

Remark 2.1. With no loss of generality, we work with (2.1). We can also start with source terms A and V that are periodic in θ_τ of period $T_\tau(z_0)$ with $T_\tau \in C^\infty(\mathbb{R}^n; \mathbb{R}_+^*)$. This means to deal with $\theta_\tau \in \mathbb{T}_{\tau,z_0} := \mathbb{R}/(T_\tau(z_0)\mathbb{Z})$. Suppose this is the case, then we can always replace θ_τ by $2\pi\theta_\tau/T_\tau(z_0)$. This modification has the effect to substitute A and V for

$$\frac{T_\tau(z_0)}{2\pi} \begin{pmatrix} \mathsf{A} \\ \mathsf{V} \end{pmatrix} \Big(\varepsilon, z_0; z; \frac{T_\tau(z_0)}{2\pi}\theta_\tau, \theta_{\mathrm{r}}\Big).$$

This gives rise to (2.1) without affecting the general form that has been introduced at the level of (1.12). But, when the original functions A and V do not imply z_0, this does produce a (smooth) dependence on z_0. This is why, care is taken to incorporate z_0 when defining A and V.

The source term inside (1.12) is:

- stiff since the large weight ε^{-1} is put in factor of the (locally bounded) functions A and V which may satisfy $\mathsf{A}_0 \not\equiv 0$ and $\mathsf{V}_0 \not\equiv 0$. In fact, we will assume that V_0 is positive.
- strongly oscillating due to the large amplitude oscillations which are generated by A and V after substitution of θ_τ and θ_{r} for respectively τ/ε and ν/ε;
- nonlinear because A and V depend on z and θ_{r} (and therefore ν/ε) in a non-trivial way.

Let us examine more precisely what happens at the level of v/ε. To this end, fix $\varepsilon \in]0, \varepsilon_0]$ small enough, and look at the scalar component $\mathsf{v}(\tau)$. Since the function V_0 will be assumed to be positive (see Assumption 2.6), by the mean value theorem, we can find some $c = c(\tau) \in]0, \tau[$ such that

$$\begin{gathered} \frac{\mathsf{v}(\tau) - \mathsf{v}_0}{\varepsilon} = \mathsf{v}_{\varepsilon,\tau} \frac{\tau}{\varepsilon^2}, \\ 0 < \mathsf{v}_{\varepsilon,\tau} := \varepsilon \partial_\tau \mathsf{v}(c) = \mathsf{V}\Big(\varepsilon, z_0; z(c); \frac{c}{\varepsilon}, \frac{\mathsf{v}(c)}{\varepsilon}\Big) = \mathcal{O}(1). \end{gathered} \tag{2.3}$$

This indicates that the description of the solutions ${}^t(z, \mathsf{v})(\varepsilon, z_0, \mathsf{v}_0; \tau)$ to the system (1.12) should involve (at least) three time scales:

- τ for the current time variable and normal variations (in the case of electrons in tokamaks, the value $\tau \sim 1$ represents a few seconds);
- $s := \varepsilon^{-1}\tau$ for the quick time variable and for quick variations (in comparison to changes in τ). The substitution of the periodic variable θ_τ by $\varepsilon^{-1}\tau$ furnishes oscillations at high frequencies of size ε^{-1};
- $t := \varepsilon^{-2}\tau$ for the rapid time variable and for rapid variations (in comparison to changes in τ and s). In view of line (2.3), the substitution of the periodic variable θ_{r} by $\varepsilon^{-1}\mathsf{v}(\tau)$ should produce oscillations at very high frequencies of size ε^{-2}. The subscript r in θ_{r} is introduced to refer to these rapid variations.

To state our results, some basic operations on periodic functions $Z(\theta_\tau, \theta_{\mathrm{r}}) \in L^1(\mathbb{T} \times \mathbb{T}_{\mathrm{r},z_0})$, like above $\mathsf{A}(\varepsilon, z_0; z; \cdot)$ or $\mathsf{V}(\varepsilon, z_0; z; \cdot)$, must be introduced. As a preliminary point, to sort out the different oscillating features, we need to define:

- The *rapid mean value* of Z is the periodic function $\overline{Z}$ given by
$$\overline{Z}(\theta_\tau) := \frac{1}{T_{\mathrm{r}}(z_0)} \int_0^{T_{\mathrm{r}}(z_0)} Z(\theta_\tau, \theta_{\mathrm{r}})\, d\theta_{\mathrm{r}}. \tag{2.4}$$
- The *rapid oscillating part* of Z is the periodic function Z^* defined by
$$Z^*(\theta_\tau, \theta_{\mathrm{r}}) := Z(\theta_\tau, \theta_{\mathrm{r}}) - \overline{Z}(\theta_\tau). \tag{2.5}$$
- The *double mean value* of Z (in both variables θ_τ and θ_{r}) or the *quick mean value* of $\overline{Z}$ is the constant
$$\langle \overline{Z} \rangle := \frac{1}{2\pi T_{\mathrm{r}}(z_0)} \int_0^{2\pi} \int_0^{T_{\mathrm{r}}(z_0)} Z(\theta_\tau, \theta_{\mathrm{r}})\, d\theta_\tau d\theta_{\mathrm{r}} = \frac{1}{2\pi} \int_0^{2\pi} \overline{Z}(\theta_\tau)\, d\theta_\tau. \tag{2.6}$$
- The *quick oscillating part* of $\overline{Z}$ is the periodic function $\overline{Z}^\star$ given by
$$\overline{Z}^\star(\theta_\tau) := \overline{Z}(\theta_\tau) - \langle \overline{Z} \rangle. \tag{2.7}$$

Recall that this induces a decomposition of Z according to

$$Z(\theta_\tau, \theta_{\mathrm{r}}) = \langle \overline{Z} \rangle + \overline{Z}^\star(\theta_\tau) + Z^*(\theta_\tau, \theta_{\mathrm{r}}). \tag{2.8}$$

Our purpose in Sections 3, 4 and 5 is to find a setting under which the system (1.12) can be solved on a time interval that is uniform in $\varepsilon \in]0, \varepsilon_0]$. It is also to exhibit conditions leading to a *three-scale* asymptotic description (when ε goes to zero) of the flow, showing the three frequencies of size 1, ε^{-1} and ε^{-2} that are associated to the underlying presence of the time variables τ, s and t.

2.1. **Assumptions and results.** When solving (1.12), the oscillations at frequencies 1, ε^{-1}, and ε^{-2} are closely interlinked. They are difficult to disentangle. To guess what happens, as a first step, it is interesting to interpret (1.12) in terms of the quick time variable s. By this way, with

$$\begin{pmatrix} \mathring{z} \\ \mathring{\nu} \end{pmatrix} (\varepsilon, z_0, \nu_0; s) := \begin{pmatrix} z \\ \nu \end{pmatrix} (\varepsilon, z_0, \nu_0; \varepsilon s),$$

we obtain

$$\partial_s \begin{pmatrix} \mathring{z} \\ \mathring{\nu} \end{pmatrix} = \begin{pmatrix} \mathsf{A} \\ \mathsf{V} \end{pmatrix} (\varepsilon, z_0; \mathring{z}; s, \frac{\mathring{\nu}}{\varepsilon}), \quad \begin{pmatrix} \mathring{z} \\ \mathring{\nu} \end{pmatrix} (0) = \begin{pmatrix} z_0 \\ \nu_0 \end{pmatrix}. \tag{2.9}$$

By Picard-Lindelöf theorem, a local solution to the Cauchy problem (2.9) exists on a maximal time interval $[0, \mathrm{S}(\varepsilon, z_0, \nu_0)[$ with $\mathrm{S}(\varepsilon, z_0, \nu_0) \in \mathbb{R}_+^* \cup \{+\infty\}$. Fix z_0 and ν_0. The right hand side of (2.9) is no more stiff and it is periodic with respect to θ_{r}. Thus, it is (locally in $\mathring{z}$) uniformly bounded with respect to $\varepsilon \in]0, \varepsilon_0]$. Taking into account the explosion behavior that a maximal solution must have at the boundary of its domain of definition, it follows that

$$\exists\, \mathrm{S}(z_0, \nu_0) \in \mathbb{R}_+^*; \quad \forall \varepsilon \in]0, \varepsilon_0], \quad 0 < \mathrm{S}(z_0, \nu_0) \leq \mathrm{S}(\varepsilon, z_0, \nu_0). \tag{2.10}$$

Now, the source term of the system (2.9) is still oscillating. In view of (2.3) with τ replaced by εs, it contains oscillations at high frequencies of size

$$\varepsilon^{-1} \mathring{\nu}(s) = \varepsilon^{-1} \nu(\varepsilon s) = \varepsilon^{-1} (\nu_0 + \mathsf{v}_{\varepsilon, \varepsilon s}\, s) = \mathcal{O}(\varepsilon^{-1}).$$

Since V_0 (and therefore V for small values of ε) is assumed to be a positive valued function, the function $\mathring{\nu}$ is strictly increasing, and the oscillations with respect to θ_{r} are certainly effective. After integration in time (in s), these oscillations compensate each other to deliver some average evolution. During quick times s, at leading order, the behavior of $\mathring{z}$ inside (2.9) may be approximated by the mean flow introduced below.

Definition 2.2 (Mean flow associated with the quick time evolution of z)**.** We define

$$\begin{aligned} \mathsf{A}_{\mathrm{mf}}(z_0; z; \theta_\tau) := \frac{1}{T_{\mathrm{r}}(z_0)} \Big(\frac{1}{T_{\mathrm{r}}(z_0)} \int_0^{T_{\mathrm{r}}(z_0)} \frac{1}{\mathsf{V}_0(z_0; z; \theta_\tau, \theta_{\mathrm{r}})} \, d\theta_{\mathrm{r}} \Big)^{-1} \\ \times \int_0^{T_{\mathrm{r}}(z_0)} \frac{\mathsf{A}_0(z_0; z; \theta_\tau, \theta_{\mathrm{r}})}{\mathsf{V}_0(z_0; z; \theta_\tau, \theta_{\mathrm{r}})} \, d\theta_{\mathrm{r}}. \end{aligned}$$

The *mean flow* associated with (2.9) is the mapping $\Xi_{\mathrm{mf}}(z_0; \cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ obtained by solving

$$\partial_s \Xi_{\mathrm{mf}}(z_0; \mathfrak{z}; s) = \mathsf{A}_{\mathrm{mf}}\big(z_0; \Xi_{\mathrm{mf}}(z_0; \mathfrak{z}; s); s\big), \quad \Xi_{\mathrm{mf}}(z_0; \mathfrak{z}; 0) = \mathfrak{z} \in \mathbb{R}^n \tag{2.11}$$

on its maximal interval of existence $[0, \mathrm{S}_{\mathrm{mf}}(z_0, \mathfrak{z})[$ where $\mathrm{S}_{\mathrm{mf}}(z_0, \mathfrak{z}) \in \mathbb{R}_+^* \cup \{+\infty\}$.

Equation (2.11) is derived in Subsection 3.4 from some homological equation (Lemma 3.10). Observe that when V_0 does not depend on θ_{r}, the function A_{mf} is extracted from $\mathsf{A}(0, z_0; z; \cdot)$ as it is indicated in (2.4), that is

$$\mathsf{A}_{\mathrm{mf}}(z_0; z; \theta_\tau) = \overline{\mathsf{A}}_0(z_0; z; \theta_\tau) = \frac{1}{T_{\mathrm{r}}(z_0)} \int_0^{T_{\mathrm{r}}(z_0)} \mathsf{A}_0(z_0; z; \theta_\tau, \theta_{\mathrm{r}}) \, d\theta_{\mathrm{r}}\,. \tag{2.12}$$

In general, the lifespan $\mathrm{S}_{\mathrm{mf}}(z_0, \mathfrak{z})$ is finite, and the same holds concerning $\mathrm{S}(\varepsilon, z_0, \nu_0)$.

Example 2.3. For instance, just take $n = 1$ and $\mathsf{A} \equiv z^2$ together with $\partial_{\theta_{\mathrm{r}}}\mathsf{V}_0 \equiv 0$ so that

$$\mathsf{A} \equiv \mathsf{A}_0 \equiv \mathsf{A}_{\mathrm{mf}} \equiv z^2, \quad \mathring{z}(\varepsilon, z_0, \nu_0; s) = \Xi_{\mathrm{mf}}(z_0; z_0; s). \tag{2.13}$$

In this simple case, we can see that

$$\forall (z_0, \nu_0) \in \mathbb{R}_+^* \times \mathbb{R}, \quad \mathrm{S}(\varepsilon, z_0, \nu_0) = \mathrm{S}_{\mathrm{mf}}(z_0, z_0) = z_0^{-1} < +\infty.$$

It follows that the lifespan $\mathcal{T}(\varepsilon, z_0, \nu_0) = \varepsilon \mathrm{S}(\varepsilon, z_0, \nu_0)$ associated with (1.12) is finite, and shrinks to 0 like εz_0^{-1} when ε goes to zero. Then, there is no way to guarantee that

$$\exists\, \mathcal{T}(z_0, \nu_0) \in \mathbb{R}_+^*; \quad \forall \varepsilon \in]0, \varepsilon_0], \quad 0 < \mathcal{T}(z_0, \nu_0) \le \mathcal{T}(\varepsilon, z_0, \nu_0). \tag{2.14}$$

When dealing with (2.13), the discussion about what could happen during current times $\tau \sim 1$ is over. We would like to avoid such situations.

To obtain (2.14), supplementary conditions on A_{mf} are clearly needed. As a prerequisite we have to impose $\mathrm{S}_{\mathrm{mf}} = +\infty$ on $\mathbb{R}^n \times \mathbb{R}^n$. However, this condition may not be enough. And it is certainly not sufficient to separate the quick and rapid oscillations, and then to obtain a complete description of them. To this end, we need more restrictive conditions.

Assumption 2.4 (Complete integrability of the mean flow)**.** For each position $(z_0, \mathfrak{z}) \in \mathbb{R}^n \times \mathbb{R}^n$, the function $\Xi_{\mathrm{mf}}(z_0, \mathfrak{z}, \cdot)$ issued from (2.11) is globally defined and periodic in s of period 2π.

The above assumption implies a strong geometrical restriction concerning (2.11). Indeed, this means that the integral curves associated with (2.11) must form (by varying the initial data $\mathfrak{z} \in \mathbb{R}^n$) a foliation of $\mathbb{R}^n$ by circles. After a normal form procedure (see Remark 3.4), the equation on $\mathring{z}$ inside (2.9) can be transformed into a perturbed version of (2.11). From this perspective, Assumption 2.4 says that the dynamical system thus obtained is nearly integrable.

By construction, the function $\mathsf{A}_{\mathrm{mf}}(z_0; z; \cdot)$ is periodic (with respect to θ_τ) with period 2π, and therefore the same holds (with respect to the quick time variable s) for the source term inside (2.11). Assumption 2.4 requires also that all solutions to (2.11) share the same period 2π. This second condition is natural (but it is far from being systematically verified). Next we have a result on the uniform lifespan with respect to $\varepsilon \in]0, \varepsilon_0]$ of the flow generated by (1.12).

Theorem 2.5. *Under Assumption 2.4, the lifespan $\mathcal{T}(\varepsilon, z_0, \nu_0)$ that is associated with* (1.12) *is uniformly bounded below by some $\mathcal{T}(z_0, \nu_0) \in \mathbb{R}_+^*$, as indicated in* (2.14)*.*

From there, the issue about the oscillating structure of ${}^t(z, \nu)$ during current times $\tau \sim 1$ becomes meaningful. Now, to obtain a precise asymptotic description of the flow ${}^t(z, \nu)$, we need to impose supplementary restrictions on V_0.

Assumption 2.6 (Positivity condition on the component V_0)**.** The function V_0 is positive and it does not depend on θ_{r}. Recall that

$$\forall (z_0, z, \theta_\tau, \theta_{\mathrm{r}}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{\mathrm{r}, z_0}, \quad 0 < \mathsf{V}_0(z_0; z; \theta_\tau, \theta_{\mathrm{r}}) \equiv \mathsf{V}_0(z_0; z; \theta_\tau). \tag{2.15}$$

This simplifies the self-interactions at the level of the equation on ν. Indeed, at leading order, the source term V is not impacted by the rapid variations (in ν/ε). Next we have a WKB expansion at all orders in $\varepsilon \in]0, \varepsilon_0]$ of the flow induced by the system (1.12).

Theorem 2.7. *Under Assumptions 2.4 and 2.6, we can find profiles*

$$Z_j(z_0,\mathrm{v}_0;\tau,\theta_\tau,\theta_{\mathrm{r}}) \in C^\infty\big(\mathbb{R}^n\times\mathbb{R}\times[0,\mathcal{T}(z_0,\mathrm{v}_0)]\times\mathbb{T}\times\mathbb{T}_{\mathrm{r},z_0};\mathbb{R}^n\big),\quad j\in\mathbb{N},$$
$$\mathscr{V}_j(z_0,\mathrm{v}_0;\tau,\theta_\tau,\theta_{\mathrm{r}}) \in C^\infty\big(\mathbb{R}^n\times\mathbb{R}\times[0,\mathcal{T}(z_0,\mathrm{v}_0)]\times\mathbb{T}\times\mathbb{T}_{\mathrm{r},z_0};\mathbb{R}\big),\quad j\in\{-1\}\cup\mathbb{N},$$

which are determined through a hierarchy of well-posed modulation equations (starting from $j=-1$ up to any integer value of j), which are such that

$$\mathscr{V}_{-1}\equiv\langle\overline{\mathscr{V}}_{-1}\rangle(z_0;\tau),\quad \mathscr{V}_0\equiv\overline{\mathscr{V}}_0(z_0,\mathrm{v}_0;\tau,\theta_\tau),\quad Z_0\equiv\overline{Z}_0(z_0;\tau,\theta_\tau),\tag{2.16}$$

and which are adjusted in such a way that, in terms of the sup norm, for all $N\in\mathbb{N}^*$, we have

$$\begin{aligned} &z(\varepsilon,z_0,\mathrm{v}_0;\tau)\\ &=\sum_{j=0}^{N-2}\varepsilon^j\,Z_j\Big(z_0,\mathrm{v}_0;\tau,\frac{\tau}{\varepsilon},\frac{\langle\overline{\mathscr{V}}_{-1}\rangle(z_0;\tau)}{\varepsilon^2}+\frac{\overline{\mathscr{V}}_0(z_0,\mathrm{v}_0;\tau,\frac{\tau}{\varepsilon})}{\varepsilon}\Big)+\mathcal{O}(\varepsilon^{N-1}),\end{aligned}\tag{2.17}$$

$$\begin{aligned} &\mathrm{v}(\varepsilon,z_0,\mathrm{v}_0;\tau)\\ &=\sum_{j=-1}^{N-2}\varepsilon^j\mathscr{V}_j\Big(z_0,\mathrm{v}_0;\tau,\frac{\tau}{\varepsilon},\frac{\langle\overline{\mathscr{V}}_{-1}\rangle(z_0;\tau)}{\varepsilon^2}+\frac{\overline{\mathscr{V}}_0(z_0,\mathrm{v}_0;\tau,\frac{\tau}{\varepsilon})}{\varepsilon}\Big)+\mathcal{O}(\varepsilon^{N-1}).\end{aligned}\tag{2.18}$$

The two expansions (2.17) and (2.18) shed light on the time oscillations but also on the spatial oscillations (encoded in the variations with respect to z_0 and v_0), thus revealing *collective* aspects of the motion (which are important in many applications). This is achieved through different types of phases, including mainly:

- The *time phase* τ which is associated with quick variations.
- The *exact phase* v. It has ε^{-1} in factor at the level of the source term of (1.12). There, the scalar component v/ε comes to replace the periodic variable θ_{r} and therefore it indeed plays the role of a phase.
- The *frozen phase* v^{f} which is defined by
$$\mathrm{v}^{\mathrm{f}}(\tau)\equiv\mathrm{v}^{\mathrm{f}}(\varepsilon,z_0,\mathrm{v}_0;\tau):=\frac{1}{\varepsilon}\,\langle\overline{\mathscr{V}}_{-1}\rangle(z_0;\tau)+\overline{\mathscr{V}}_0\Big(z_0,\mathrm{v}_0;\tau,\frac{\tau}{\varepsilon}\Big).\tag{2.19}$$
It is a truncated version of v which, like v, operates with ε^{-1} in factor.
- The *rapid phase* $\langle\overline{\mathscr{V}}_{-1}\rangle$ which is associated with rapid variations (at frequencies of size ε^{-2}). As a consequence of Assumption 2.6, we will find that $\langle\overline{\mathscr{V}}_{-1}\rangle(\tau)>0$ for all $\tau>0$. Thus, the presence of rapid oscillations is sure to happen.

Remark 2.8 (About supercritical features)**.** It is worth underlying that in (2.18), the same profiles $\mathscr{V}_{-1}$ and $\mathscr{V}_0$ take part in the description of amplitudes and phases. This is typical of supercritical regimes for quasilinear equations.

Remark 2.9 (About the notion of phase)**.** Definition (ii) of a phase φ (in terms of its *bounded* aspects) is open to various interpretations. Indeed, it is based on the dependent or independent variables that are implied. For example:

- The functions $\varphi\equiv\mathrm{v}$ and $\varphi\equiv\mathrm{v}^{\mathrm{f}}$ viewed as depending on the state variable v or v^{f} and from the perspective of the profiles A and V inside (1.12) can be viewed as phases.
- The functions $\varphi\equiv\varepsilon\mathrm{v}$ and $\varphi\equiv\varepsilon\mathrm{v}^{\mathrm{f}}$ viewed this time as depending on $(\varepsilon,z_0,\mathrm{v}_0,\tau)$ and from the perspective of the profiles A and V inside (1.12) or Z_j and $\mathscr{V}_j$ inside (2.17) or (2.18) play the role of phases.

On the other hand, even if ν/ε and $\nu^{\rm f}/\varepsilon$ come to replace the periodic variable $\theta_{\rm r}$, the functions $\varphi \equiv \nu$ and $\varphi \equiv \nu^{\rm f}$ viewed as depending on $(\varepsilon, z_0, \nu_0, \tau)$ are not (strictly speaking) phases because they are obviously not uniformly bounded. It must be clear that the above names of *exact phase* and *frozen phase* are a matter of convention.

Looking at (2.17) and (2.18), at the end, we can recognize the simultaneous presence of oscillations implying the frequencies ε^{-1} and ε^{-2} with corresponding phases τ and $\langle \overline{\mathscr{V}}_{-1} \rangle$. There is also a nonlinear imbrication of oscillations carried by the (slightly unusual) expression $\overline{\mathscr{V}}_0(\cdot, \tau, \varepsilon^{-1}\tau)/\varepsilon$. The study of multiscale oscillations has been intensively developed in the past decades in various contexts including the topic of geometric optics [15, 24, 26, 27, 28], the theory of homogenization [1, 25], two-scaled Wigner measures [16, 22] or microlocal Birkhoff normal forms [22]. However, the coverage of situations which can mix oscillations of the above different types is relatively new. This is technically a difficult challenge (with potential extensions in the domain of PDEs) which apparently is not directly within reach of the aforementioned methods.

Asymptotic expansions similar to (2.17) and (2.18) already appear in the articles [7, 8]. There, they were motivated by questions arising in the study of magnetized plasmas. The actual approach is much broader than in [7, 8]. The purpose is indeed to achieve a comprehensive analysis in more general situations than before. It is to extend the preceding tools and also to explain them more briefly and clearly. We exhibit integrability and positivity conditions (Assumptions 2.4 and 2.6) allowing to progress. These conditions encompass and extend the framework of [7, 8]. They are both intrinsic, easy to test, and suitable for many applications. They are considered in Section 6 in the case of Hamilton Jacobi equations, and they are designed [9] to incorporate the influence of an external electric field (in addition to the magnetic field) on the long time dynamical behavior of charged particles. The main outcomes of our work concerning (1.12) are the following:

- A model for the leading behavior of the flow provided by

$$z(\varepsilon, z_0, \nu_0; \tau) = \overline{Z}_0\big(z_0; \tau, \frac{\tau}{\varepsilon}\big) + \varepsilon\, Z_1\Big(z_0, \nu_0; \tau, \frac{\tau}{\varepsilon}, \frac{\nu^{\rm f}(\tau)}{\varepsilon}\Big) + \mathcal{O}(\varepsilon^2), \tag{2.20}$$

$$\nu(\varepsilon, z_0, \nu_0; \tau) = \nu^{\rm f}(\varepsilon, z_0, \nu_0; \tau) + \mathcal{O}(\varepsilon). \tag{2.21}$$

- A careful analysis of the underlying stability and instability properties. This aspect is more subtle and less easy to expose since it appears throughout the text. Let us just outline some difficulties. The precise knowledge of ν (or $\nu^{\rm f}$) is essential to obtain the L^∞-precision. As a matter of fact, a perturbation of size ε^2 at the level of $\langle \overline{\mathscr{V}}_{-1} \rangle$ may have an impact of size 1 in the calculation of the $O(\varepsilon^j)$-terms of (2.17) and (2.18). This means that a very precise access to ν is crucial to govern the stability properties of the flow.

From the above perspective, our strategy is based on two main arguments:

• First, we implement a blow-up procedure. As explained in a series of remarks (see 3.4, 3.14 and 5.7), this is a kind of normal form method adapted to our context. In Section 3, this already leads to the uniform local existence of solutions (z, ν).

• Then, in Section 4, we perform a *three-scale WKB calculus* with supercritical attributes. The idea, as is typical in geometric optics [10, 13], is to replace (1.12) by profile equations. But this time, we implement the component v of the solution previously obtained as a phase (see Remarks 4.3 and 4.4 for this unusual trick). This means in particular that v is viewed as oscillating with respect to itself through the oscillating implicit relation (4.1). This yields a special notion of profile equations. This is like investigating the stability issue in a quotient space: we work modulo the determination of the *unknown* function v.

Because of its importance from an application standpoint, we focus below on the content of (2.20). The behavior of z is mainly governed by the profile $\overline{Z}_0$ which reveals some kind of (large amplitude oscillating) reduced dynamics during current times $\tau \sim 1$. The determination of $\overline{Z}_0$ may be achieved as indicated bellow.

Theorem 2.10 (Reduced equations)**.** *The function $\overline{Z}_0(\cdot)$ does not depend on v_0. It can be expressed in terms of $\Xi_{\rm mf}(\cdot)$ according to*

$$\overline{Z}_0(z_0;\tau,\theta_\tau)=\Xi_{\rm mf}\big(z_0;\langle\bar{\mathfrak{Z}}_0\rangle(z_0;\tau);\theta_\tau\big). \tag{2.22}$$

The function $\langle\bar{\mathfrak{Z}}_0\rangle$ in the right hand side of (2.22) can be determined by solving

$$\partial_\tau\langle\bar{\mathfrak{Z}}_0\rangle(z_0;\tau)=\langle\overline{\mathrm{A}}_1\rangle\big(z_0;\langle\bar{\mathfrak{Z}}_0\rangle(z_0;\tau)\big),\quad \langle\bar{\mathfrak{Z}}_0\rangle(z_0;0)=z_0, \tag{2.23}$$

where, with $\partial_{\theta_{\rm r}}^{-1}$ as in (3.24) (by identifying $\theta\equiv\theta_{\rm r}$ and $T\equiv T_{r,z_0}$), we have introduced

$$\begin{aligned}
&\mathrm{A}_1(z_0;\mathfrak{z};\theta_\tau,\theta_{\rm r})\\
&:= D_{\mathfrak{z}}\Xi_{\rm mf}(z_0;\mathfrak{z};\theta_\tau)^{-1}\\
&\quad\times\Big\{\big[\mathsf{A}_1+\mathsf{V}_0^{-1}\,\big(\partial_{\theta_{\rm r}}^{-1}\mathsf{A}_0^*\cdot\nabla_z\big)\,\mathsf{A}_0-\partial_{\theta_\tau}(\mathsf{V}_0^{-1}\partial_{\theta_{\rm r}}^{-1}\mathsf{A}_0^*)-\mathsf{V}_1\mathsf{V}_0^{-1}\mathsf{A}_0^*\\
&\quad+\big(\partial_{\theta_{\rm r}}^{-1}\mathsf{A}_0^*\cdot\nabla_z\big)\,(\mathsf{V}_0^{-1})\mathsf{A}_0^*\big]\big(z_0;\Xi_{\rm mf}(z_0;\mathfrak{z};\theta_\tau);\theta_\tau,\theta_{\rm r}\big)\Big\}
\end{aligned} \tag{2.24}$$

and where the access to the double mean value $\langle\overline{\mathrm{A}}_1\rangle$ is furnished by (2.6).

Recall that the two symbols A_1 and A_1 are different (we have $\mathrm{A}_1\not\equiv\mathsf{A}_1$). The notations A_1 and A_1 will be used with different meanings. Given the numerous difficulties of understanding the complex interplay between the various types of oscillations, Theorem 2.10 produces a rather easy and explicit way to determine what remains in a first approximation. In fact, it gives access to effective equations which are amenable to numerical computations.

2.2. **A few comments on the results.** The aim of this subsection is to help the reader understand the position, content and significance of the four preceding theorems. This is done below through a list of remarks.

Remark 2.11 (About the effective content of $\langle\bar{\mathfrak{Z}}_0\rangle$)**.** The expression $\langle\overline{\mathrm{A}}_1\rangle$ is issued from the double averaging procedure (2.6) which may go hand in hand with a number of cancellations. It follows that all the components of $\langle\bar{\mathfrak{Z}}_0\rangle$ are not necessarily activated when solving (2.23). In general, there remains a reduced number of unknowns. These are the so-called adiabatic (or guiding-center) invariants in the case of charged particles.

Remark 2.12 (About the determination of $\overline{Z}_0$)**.** The expression $\overline{Z}_0$ consists of two distinct parts: $\Xi_{\rm mf}$ and $\langle\bar{\mathfrak{Z}}_0\rangle$. As explained before, the mean flow $\Xi_{\rm mf}$ can be extracted from (2.9). In fact, this amounts to a multiplication of (1.12) by ε, and then to the extraction of a mean value involving A_0 and V_0, as in Definition 2.2. Observe that

$$\mathring{z}(\varepsilon, z_0, \mathsf{v}_0; s) = \Xi_{\mathrm{mf}}(z_0; z_0; s) + \mathcal{O}(\varepsilon). \tag{2.25}$$

Thus, in coherence with what has been said before, the mean flow does furnish the leading behavior of z during quick times s near the current time $\tau = 0$. But, near other current times $\tau \in \mathbb{R}_+^*$, the use of $\langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau)$ is needed to well describe $\overline{Z}_0(\cdot)$. The access to $\langle \bar{\mathfrak{Z}}_0 \rangle$ is much more complicated than to Ξ_{mf}. It involves the determination of A_1 which, in view of (2.24), is built with various derivatives and integrations of A_0 and V_0, as well as terms of size ε inside A, like A_1. Now, such information should be invisible (or vanishing) when performing (even multiscale) weak limits at the level of (1.12) or (2.9).

Remark 2.13 (About the geometrical interpretation of the content of $\overline{Z}_0$)**.** Under Assumption 2.4, the integral curves associated to (2.11) draw a family of circles

$$\mathcal{C}(z_0; \mathfrak{z}) := \big\{ \Xi_{\mathrm{mf}}(z_0; \mathfrak{z}; s);\ s \in \mathbb{R} \big\} \subset \mathbb{R}^n, \quad (z_0, \mathfrak{z}) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Given $z_0 \in \mathbb{R}^n$, these circles form a foliation of $\mathbb{R}^n$ parameterized by $\mathfrak{z}$. The role of $\mathfrak{z}$ is twofold. First, since $\mathfrak{z} \in \mathcal{C}(z_0; \mathfrak{z})$, the parameter $\mathfrak{z}$ points to the circle $\mathcal{C}(z_0; \mathfrak{z})$. Secondly, it specifies some origin on this circle. This allows to make sense of the numerical value θ_τ which at the level of the expression $\Xi_{\mathrm{mf}}(z_0; \mathfrak{z}; \theta_\tau)$ is comparable to the number of turns performed around $\mathcal{C}(z_0; \mathfrak{z})$, departing from the position $\mathfrak{z}$. In this context, the geometrical interpretation of (2.22) is the following. The value of $\overline{Z}_0(z_0; \tau, \theta_\tau)$ is obtained by selecting the circle $\mathcal{C}\big(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \tau)\big)$ and by carrying out on it a rotation which starts from the initial point $\langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \tau)$ and whose numerical value is θ_τ. Now, in view of (2.23), the position $\mathfrak{z} = \langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \tau)$ is in general different from z_0 as soon as $\tau \neq 0$. Thus, to obtain a representation formula of $\overline{Z}_0$ which is in line with (2.22), it does not suffice to work with $\mathfrak{z} = z_0$. This remark justifies the introduction at the level of (2.11) of the extra variable $\mathfrak{z}$. In practice, we have indeed to solve (2.11) with $\mathfrak{z}$ other than z_0.

Remark 2.14 (About the content of the leading order term)**.** At the end, the main limit behavior

$$z(\varepsilon, z_0, \mathsf{v}_0; \tau) = \Xi_{\mathrm{mf}}\Big(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau); \frac{\tau}{\varepsilon}\Big) + \mathcal{O}(\varepsilon) \tag{2.26}$$

is built with the help of $\Xi_{\mathrm{mf}}(z_0; \cdot; \cdot)$ through a *current* time variation corresponding to the passage from $\mathfrak{z} = z_0$ to $\mathfrak{z} = \langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \tau)$, combined with a *quick* time variation (made of rotations) when θ_τ is replaced by τ/ε (with a number of turns that becomes very large when $\tau \in \mathbb{R}_+^*$ is fixed and ε tends to zero). At the level of (2.26), the rapid oscillations which are associated with the phase v^{f} do not yet appear.

Remark 2.15 (About the structure of the whole asymptotic expansions)**.** Look at the contribution which at the level of (2.20) has ε in factor. In general, we have $\partial_{\theta_{\mathrm{r}}} Z_1 \not\equiv 0$. This means that the *rapid* variations are activated with a *small amplitude* ε and corresponding frequency ε^{-2}. These rapid oscillations lead to many technical problems. Physically, when dealing with magnetized plasmas, they come from a fast gyromotion. At the end, the structure of z is made of the superposition of two regimes which are built with:

- Large amplitude oscillations. We find that $\partial_{\theta_\tau} Z_0 \not\equiv 0$ when the mean flow is not constant. Then, there are quick variations (involving θ_τ) of amplitude 1;
- Strong oscillations. We find that $\partial_{\theta_{\rm r}} Z_1 \not\equiv 0$ when $\mathsf{A}_0^* \not\equiv 0$. Then, if we adopt the terminology of [11] (with ε replaced by $\sqrt{\varepsilon}$), there are *rapid* variations (involving $\theta_{\rm r}$) of amplitude ε.

Remark 2.16 (About the imbrication between the averaging procedures)**.** The extraction of (2.17) and (2.18) results from two averaging procedures. The first (in $\theta_{\rm r}$) is revealed by (2.12); the second (in θ_τ) occurs along the circles generated by $\Xi_{\rm mf}$. There is no evident order between these highly interconnected operations. In the blow-up Section 3, priority is given to θ_τ. But, in the WKB Section 4, the integration is first in $\theta_{\rm r}$ and then in θ_τ, as when passing from (2.4) to (2.6).

Remark 2.17 (About the origin of Theorem 1.5)**.** The Hamilton-Jacobi equations will be solved by the method of characteristics. When doing this, the expansion (1.10) appears through a *composition* of the oscillations involved by (2.17) or (2.18), roughly speaking by replacing (z_0, ν_0) inside (2.18) by $(z_0, \nu_0)(\varepsilon, {\rm x}^{-1})$ where ${\rm x}^{-1}$ (Lemmas 6.9 and 6.10) is the inverse of the spatial characteristic ${\rm x}$ (Lemma 6.5). This is why it is very important to keep track of the dependence on the initial data (z_0, ν_0) at the level of (2.18). This also explains how the complexity of the oscillating structures may increase. The chain rule indicates that the number of scales could become larger. In Subsections 6.3 and 6.4, we will show that this number does indeed increase.

2.3. **Plan and motivations.** The plan of the paper is as follows.

• In Section 3, we introduce a *lifting* (or *blow-up*) *procedure* which may be regarded as an adapted kind of the (more classical) *normal form* procedure. The purpose is to remove from the right hand side of (1.12) as much non-significant singular terms as possible. The idea is to absorb some artificial oscillations by changing the unknowns. This is done by (the inverse of) a nonlinear oscillating transformation. As a corollary, in Subsection 3.2.3, we can already prove Theorem 2.5.

• In Section 4, we develop a three-scale WKB analysis involving the exact phase ν. We work at the level of the profile equations (4.3) which allow to obtain rid of ν. The idea is to seek approximate solutions ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ of (4.3) in the form of expansions in powers of ε, like in (4.11) or (4.12). Formal computations lead to a hierarchy $\big($indexed by $j \in \{-1\} \cup \mathbb{N}\big)$ of well-posed equations which are highly interconnected and which allow to determine successively all the profiles $\mathfrak{Z}_j$ and $\mathcal{V}_j$ that constitute $\mathfrak{Z}^a$ and $\mathcal{V}^a$. In particular, in Paragraph 4.3.1, we identify the leading profile Z_0 which can be described as in Theorem 2.10.

• In Section 5, we justify the interest of the preceding procedure (the formal calculus) by showing that the approximate solutions ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ lead indeed to exact solutions of the system (1.12). This means passing from the $\mathfrak{Z}_j$ and $\mathcal{V}_j$ of (4.11) and (4.12) to the Z_j and $\mathscr{V}_j$ of (2.17) and (2.18). This implies the freezing of the phase ν into $\nu^{\rm f}$ (through the implicit function theorem, see Subsection 5.2) as well as a coming back to the original field (Subsection 5.3). At the end, this yields the proof of Theorem 2.7.

• In Section 6, we implement our analysis to construct classical solutions for the Cauchy problem associated with a class of oscillating Hamilton-Jacobi equations.

Of course, weak solutions may exist [14]. But the vanishing viscosity method does not furnish a precise description of their oscillating structures. By contrast, the method of characteristics does apply and it makes such accurate information available. Still, to this end, we need to implement some specific nontrivial arguments. Indeed, the differential of x is apparently highly singular which indicates that supercritical phenomena are achieved. However, this can be overcome through the Hadamard's global inverse function theorem by exploiting transparency conditions (Paragraph 6.3.2) emanating from Assumption 2.4 (or 1.3). At the end, we achieve the proof of Theorem 1.5 in Subsection 6.5.

Hamilton-Jacobi equations like (1.5), implementing a small parameter $\varepsilon \to 0$, can appear in many situations which inspire our interest in this topic, like: homogenization theory [1, 5, 6, 17, 18, 19, 25] where they are applied to traffic flows, light propagation and optics [26, 28], plasma physics [8, 9], in the presence of rough domains [21], and so on. In these references, the above different multiscale aspects are often discussed separately and partially. We provide here an extensive overview and we investigate new facets. Most importantly, we achieve a better comprehension of the nonlinear specificities induced by the influence of the oscillating term u/ε inside H. Such aspects have already been raised (for instance in [23]) but without going as far as we do.

The present approach is also motivated by the need for an accurate long time ($\tau \sim 1$) description of the dynamics of charged particles in strongly magnetized plasmas. Recall that the gyrokinetic equations [4] deal with systems of the type (1.12) during quick times, for $\tau \sim \varepsilon$ or $s \sim 1$. In fact, the mean flow can be related to the guiding center motion, while Theorem 2.7 goes far beyond this. It significantly enhances the information content of standard ray tracing methods [29] by justifying asymptotic descriptions which prevail over longer times (namely during *current times* $\tau \sim 1$) and which are valid with any order of precision (expressed in powers of ε). Given the potential implications, there is a very abundant literature (both in physics and mathematics [2, 4, 29]) related to this subject. Usual approaches are however limited because they do not capture the imbrication of oscillations revealed by (2.17) and (2.18). The introduction of the preceding three-scale framework is necessary to progress. Historically, the structure of (1.12) is already implicit in [20], and it becomes more visible in the two following contributions [7, 8] where it is studied in the purely magnetic case. The formulation and assumptions retained here are much more general, and they are designed to take into account the (potentially disruptive) influence of electric fields. But this requires a long preparatory work and a number of specific considerations. This is why this important aspect is developed in the separate contribution [9] with in perspective an analysis of the dynamical confinement properties inside fusion devices during long times.

3. Blow-up procedure

The main goal of this section is to obtain rid of the irrelevant oscillations which are put in factor of the large weight ε^{-1} in the source term of the system (1.12). As stated in Subsection 3.1, at the level of Proposition 3.2, we can exchange (1.12) with (3.3). In this procedure, the general form of the equations is not modified but (A, V) is replaced by (A, V), with A satisfying the simplified condition (3.4). When doing this, the crucial tool is a change of variables involving a map Ξ. The general structure of Ξ is specified in Subsection 3.2. The transformation of (1.12) under the blow-up procedure is detailed in Subsection 3.3, where it is explained how A and V can be deduced from A and V. By this way, it becomes possible to exhibit necessary and sufficient conditions on Ξ leading to (3.4). These conditions are the gateway to Assumption 2.4. The proof of Proposition 3.2 is achieved at the end, in Subsection 3.4.

Remark 3.1. As commented in Remark 2.1, we can always start with some $\theta_\tau \in \mathbb{T}_{\tau,z_0} := \mathbb{R}/(T_\tau(z_0)\mathbb{Z})$. It bears noting that, in such a case, Assumption 2.4 implies that the function $\Xi_{\mathrm{mf}}(z_0,\mathfrak{z},\cdot)$ issued from (2.11) should be periodic in s of period $T_\tau(z_0)$ (instead of period 2π).

3.1. **Desingularization method.** The aim of this subsection is to replace the original field z by some auxiliary field $\mathfrak{z}$. In practice, the new unknown $\mathfrak{z}$ cannot be directly expressed in terms of z. Instead, it is revealed after a *blow-up* procedure on z. The term *blow-up* must be understood here in the sense of *lifting*. More precisely, given $z(\varepsilon, z_0, \nu_0; \cdot) : \mathbb{R} \to \mathbb{R}^n$ and

$$\Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_{\mathrm{r}}) \in C^\infty([0,\varepsilon_0] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{\mathrm{r},z_0}; \mathbb{R}^n), \tag{3.1}$$

a lifting ℓ of z to $\mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{\mathrm{r},z_0}$ is an application $\ell(\varepsilon, z_0, \nu_0; \cdot) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{\mathrm{r},z_0}$

$$\ell(\varepsilon, z_0, \nu_0; \tau) = \Big(\mathfrak{z}(\tau), \frac{\tau}{\varepsilon}, \frac{\nu(\tau)}{\varepsilon}\Big)$$

leading to the commutative diagram

$$\begin{tikzcd} & \mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{\mathrm{r},z_0} \arrow[dr, "\Xi(\varepsilon,z_0;\cdot)"] & \\ \mathbb{R} \arrow[ur, "\ell(\varepsilon,z_0,\nu_0;\cdot)"] \arrow[rr, "z"] & & \mathbb{R}^n \end{tikzcd}$$

or equivalently to the formula

$$z(\tau) = \Xi\Big(\varepsilon, z_0; \mathfrak{z}(\tau); \frac{\tau}{\varepsilon}, \frac{\nu(\tau)}{\varepsilon}\Big). \tag{3.2}$$

In this process, the function ν is viewed as an input. The transformation (3.2) is driven by ν. The correspondance through (3.2) between $\mathfrak{z}$ and z (and conversely) makes sense only on condition that ν is identified and, for the moment, it is supposed to be the local solution of (1.12). Knowing this, the key tool is the map $\Xi(\cdot)$ which must be adjusted first. Then, we can pass from $\mathfrak{z}$ to z by following the two arrows at the top of the preceding diagram, or equivalently by using (3.2).

The interest of a lifting is to put (a part of) the singularities (of z) aside by raising the number of variables. Here, oscillations are put in factor inside $\Xi(\cdot)$, at the level of the periodic variables θ_τ and θ_{r}. Note again that the whole procedure is very sensitive to the choice of both ν and Ξ. The purpose is to adjust ν and Ξ in such a way that $\mathfrak{z}$ solves a system of ODEs which is inherited from (1.12) but which is less complicated than (1.12). In practice, this should manifest as a simplification of the source term A.

Proposition 3.2 (Desingularization)**.** *Under Assumption 2.4, there exists a map* $\Xi(\cdot)$ *allowing to convert* (1.12) *through* (3.2) *into the following redressed system*

$$\partial_\tau \begin{pmatrix} \mathfrak{z} \\ \nu \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} \mathrm{A} \\ \mathrm{V} \end{pmatrix} \Big(\varepsilon, z_0; \mathfrak{z}; \frac{\tau}{\varepsilon}, \frac{\nu}{\varepsilon}\Big), \quad \begin{pmatrix} \mathfrak{z} \\ \nu \end{pmatrix}(0) = \begin{pmatrix} \mathfrak{z}_0 \\ \nu_0 \end{pmatrix} \tag{3.3}$$

which takes the same form (1.12)*, with new expressions* A *and* V *satisfying (for all* $N \in \mathbb{N}$*)*

$$\begin{aligned}&C^\infty([0,\varepsilon_0]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{T}\times\mathbb{T}_{\mathrm{r},z_0};\mathbb{R}^n)\\&\ni \mathrm{A}(\varepsilon,z_0;\mathfrak{z};\theta_\tau,\theta_\mathrm{r})=\sum_{j=0}^N \varepsilon^j\mathrm{A}_j(z_0;\mathfrak{z};\theta_\tau,\theta_\mathrm{r})+\mathcal{O}(\varepsilon^{N+1}),\\&C^\infty([0,\varepsilon_0]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{T}\times\mathbb{T}_{\mathrm{r},z_0};\mathbb{R}_+^*)\\&\ni \mathrm{V}(\varepsilon,z_0;\mathfrak{z};\theta_\tau,\theta_\mathrm{r})=\sum_{j=0}^N \varepsilon^j\,\mathrm{V}_j(z_0;\mathfrak{z};\theta_\tau,\theta_\mathrm{r})+\mathcal{O}(\varepsilon^{N+1}),\end{aligned}$$

but which now involves a first component A *that is subject to the crucial property*

$$\mathrm{A}_0(z_0;\mathfrak{z};\cdot)=\mathrm{A}(0,z_0;\mathfrak{z};\cdot)\equiv 0. \tag{3.4}$$

Recall that A and V are likely distinct from A and V. These functions are defined up to $\varepsilon=0$, and they are smooth near $\varepsilon=0$. As well as for A and V from which they are issued, they do not involve v, but only $\mathfrak{z}$. This nonlinearity will be sometimes marked by the notations $\mathrm{A}(\mathfrak{z})$ and $\mathrm{V}(\mathfrak{z})$, which focus on the dependence on $\mathfrak{z}$ and simply dismiss the role of $(\varepsilon,z_0,\theta_\tau,\theta_\mathrm{r})$.

The proof of Proposition 3.2 is postponed to Subsection 3.4.

Corollary 3.3 (Uniform time of existence for (3.3))**.** *The lifespan* $\mathcal{T}(\varepsilon,\mathfrak{z}_0,\mathsf{v}_0)$ *that is associated with* (3.3) *is uniformly (in* ε *when* ε *goes to* 0*) bounded below by some* $\mathcal{T}(\mathfrak{z}_0)\in\mathbb{R}_+^*$.

In other words, we have (2.14) with z_0 replaced by $\mathfrak{z}_0$.

Proof. Taking into account (3.4), the source term $\varepsilon^{-1}\mathrm{A}$ in front of $\partial_\tau\mathfrak{z}$ at the level of (3.3) is of size $\mathrm{A}_1+O(\varepsilon)=O(1)$ instead of being of size $O(\varepsilon^{-1})$. As long as $\mathfrak{z}$ remains in a compact set, say $\mathfrak{z}\in B(0,r]$ where r is adjusted in such a way that $\mathfrak{z}_0\in B(0,r/2]$, the two expressions $\varepsilon^{-1}\mathrm{A}$ and V are bounded uniformly with respect to ε, $\mathfrak{z}$ and τ by

$$\begin{aligned}M:=&\sup_{\varepsilon\in]0,\varepsilon_0]}\ \sup_{\mathfrak{z}\in B(0,r)}\ \sup_{\theta_\tau\in\mathbb{T}}\ \sup_{\theta_\mathrm{r}\in\mathbb{T}_{\mathrm{r},z_0}}\ \max\big(\varepsilon^{-1}|\mathrm{A}(\varepsilon,z_0;\mathfrak{z};\theta_\tau,\theta_\mathrm{r})|;\,|\mathrm{V}(\varepsilon,z_0;\mathfrak{z};\theta_\tau,\theta_\mathrm{r})|\big)\\&<+\infty.\end{aligned}$$

Moreover, they are Lipschitz continuous with respect to $\mathfrak{z}$ and θ_r. By Cauchy-Lipschitz theorem, there exists a unique local solution ${}^t(\mathfrak{z},\mathsf{v})$ to (3.3). Since A and V are periodic with respect to θ_r, the explosion, if any, can only occur at the level of the component $\mathfrak{z}$. Let $\bar\tau(\varepsilon)\in\mathbb{R}_+^*$ be such that

$$\bar\tau(\varepsilon):=\sup\big\{\tilde\tau\in[0,\mathcal{T}(\varepsilon,\mathfrak{z}_0,\mathsf{v}_0)[;\ |\mathfrak{z}(\tau)|\le r,\ \forall\tau\in[0,\tilde\tau]\big\}.$$

By construction, we have $\bar\tau(\varepsilon)<\mathcal{T}(\varepsilon,\mathfrak{z}_0,\mathsf{v}_0)$. Thus, if $\bar\tau(\varepsilon)=+\infty$, there is nothing to prove. Now, assume that $\bar\tau(\varepsilon)<+\infty$. Then we should have

$$\mathfrak{z}(\bar\tau(\varepsilon))=r. \tag{3.5}$$

On the other hand, by the mean value theorem, the solution is such that

$$\begin{gathered}\forall\,\tau\in[0,\bar\tau(\varepsilon)],\quad |\mathfrak{z}(\tau)|\le|\mathfrak{z}(\tau)-\mathfrak{z}_0|+|\mathfrak{z}_0|\le M\tau+r/2,\\ \forall\,\tau\in[0,\bar\tau(\varepsilon)],\quad |\mathsf{v}(\tau)|\le|\mathsf{v}(\tau)-\mathsf{v}_0|+|\mathsf{v}_0|\le\varepsilon^{-1}M\tau+|\mathsf{v}_0|.\end{gathered}$$

It follows that for all $\tau \leq \min$
$bigl(\bar{\tau}(\varepsilon); r/4M\bigr)$,

$$|\mathfrak{z}(\tau)| \leq M\,\tau + r/2 \leq 3r/4 < r.$$

If $\bar{\tau}(\varepsilon) \leq r/4M$, we obtain $|\mathfrak{z}(\tau)| < r$ for all $\tau \in [0, \bar{\tau}(\varepsilon)]$. This is a contradiction with (3.5). Thus, we have $\bar{\tau}(\varepsilon) > r/4M$. Hence, the lifespan $\mathcal{T}(\varepsilon, \mathfrak{z}_0, \mathsf{v}_0)$ of the solution to (3.3) is uniformly (in ε) bounded below by $\mathcal{T}(\mathfrak{z}_0) := r/4M$. The scenario of Example 2.3 is avoided. □

3.2. **General structure of the lifting.** The map Ξ is built as a small perturbation (of size ε) of some map Ξ_0. More precisely

$$\Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_{\mathrm{r}}) = \Xi_0(z_0; \mathfrak{z}; \theta_\tau) + \varepsilon \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_{\mathrm{r}}), \tag{3.6}$$

with

$$\Xi_0 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}; \mathbb{R}^n), \tag{3.7}$$

$$\Xi_1 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{\mathrm{r}, z_0}; \mathbb{R}^n). \tag{3.8}$$

Observe that the form of Ξ inside (3.1) and (3.6) is the same as the one of A and V. For illustration purposes and to assist the reader in the understanding of the text, we will explain through a series of remarks what happens in the case of a standard *normal form* procedure. We start below by recalling what is meant by this.

Remark 3.4 (Normal form procedure: definition)**.** This is when
$\Xi_0(z_0; \mathfrak{z}; \theta_\tau) \equiv \Xi_{0nf}(z_0; \mathfrak{z}; \theta_\tau) := \mathfrak{z}$ and when moreover Ξ_1 does not depend on θ_τ, that is when

$$\Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_{\mathrm{r}}) \equiv \Xi_{nf}(\varepsilon, z_0; \mathfrak{z}; \theta_{\mathrm{r}}) := \mathfrak{z} + \varepsilon \Xi_{1nf}(z_0; \mathfrak{z}; \theta_{\mathrm{r}}). \tag{3.9}$$

In fact, the change (3.9) is just a small perturbation of the identity map. The formula (3.9) may suffice during quick times but certainly not (always) during longer times, see Remark 3.15.

When $\varepsilon = 0$ inside (3.6), we recover $\Xi_0(z_0; \mathfrak{z}; \theta_\tau)$. In the general case, as will be seen, we need some freedom on Ξ_0 in order to absorb singular terms. Thus, we do not take $\Xi_0 \equiv \Xi_{0nf}$. Extra admissible functions Ξ_0 are presented in Paragraph 3.2.1, while basic properties of Ξ are detailed in Paragraph 3.2.2. At the end, in Paragraph 3.2.3, we explain how to pass from the initial data z_0 to $\mathfrak{z}_0$. We also describe how to go from the field z to its corresponding lifting $\mathfrak{z}$, that is how to obtain the inverse function of (3.2).

3.2.1. *Admissible functions* Ξ_0. From now on, we suppose that $\Xi_0(z_0; .; \theta_\tau) : \mathbb{R}^n \to \mathbb{R}^n$ generates a one-to-one correspondence. Under Assumption 2.4, as a consequence of Lemma 3.10, this condition will be verified for the choice $\Xi_0 = \Xi_{\mathrm{mf}}$ that we have in mind.

Condition 3.5 (Ξ_0 is a global smooth diffeomorphism)**.** The expression Ξ_0 is subject to (3.7). Moreover, for all $(z_0, \theta_\tau) \in \mathbb{R}^n \times \mathbb{T}$, the function $\Xi_0(z_0; \cdot; \theta_\tau) : \mathbb{R}^n \to \mathbb{R}^n$,

$$\mathfrak{z} \mapsto \Xi_0(z_0; \mathfrak{z}; \theta_\tau) \tag{3.10}$$

is a diffeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$. The corresponding inverse is denoted by $\Xi_0^{-1}(z_0; \cdot; \theta_\tau)$. It is a smooth function of (z_0, z, θ_τ) on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}$.

3.2.2. *Properties of* Ξ. We look here more closely at the characteristics of the map Ξ, viewed as a perturbation of Ξ_0.

Next we have a family of diffeomorphisms indexed by $(\varepsilon, z_0, \tau, \theta_\tau, \theta_{\rm r}) \in [0,1] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{T} \times \mathbb{T}_{{\rm r},z_0}$.

Lemma 3.6. *Select two functions* Ξ_0 *and* Ξ_1 *satisfying respectively* (3.7) *and* (3.8). *Assume Condition 3.5 and define* Ξ *as it is indicated in* (3.6). *Fix a compact set* $K_0 \subset \mathbb{R}^n$ *and a positive real number* $R \in \mathbb{R}_+^*$. *Then, we can find* $\varepsilon_0 \in]0, \varepsilon_0]$ *such that, for all* $(\varepsilon, z_0, \theta_\tau, \theta_{\rm r}) \in [0, \varepsilon_0] \times K_0 \times \mathbb{T} \times \mathbb{T}_{{\rm r},z_0}$, *the map* $\Xi(\varepsilon, z_0; \cdot; \theta_\tau, \theta_{\rm r}) : B(0,R] \to \mathbb{R}^n$,

$$\mathfrak{z} \mapsto \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_{\rm r}) \tag{3.11}$$

is a diffeomorphism from $B(0,R]$ *onto its image*

$$K \equiv K(\varepsilon, z_0, R, \theta_\tau, \theta_{\rm r}) := \big\{ \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_{\rm r});\ \mathfrak{z} \in B(0,R] \big\}\,.$$

The corresponding inverse is denoted by $\Xi^{-1}(\varepsilon, z_0; \cdot; \theta_\tau, \theta_{\rm r})$. *It is a smooth function of* $(\varepsilon, z_0, z, \theta_\tau, \theta_{\rm r})$ *chosen in* $[0, \varepsilon_0] \times K_0 \times K \times \mathbb{T} \times \mathbb{T}_{{\rm r},z_0}$ *which can be expanded in powers of* ε *according to*

$$\Xi^{-1}(z_0; \mathfrak{z}, \theta_\tau, \theta_{\rm r}) = \Xi_0^{-1}(z_0; \mathfrak{z}, \theta_\tau) + \sum_{j=1}^{+\infty} \varepsilon^j \Xi_j^{-1}(z_0; \mathfrak{z}, \theta_\tau, \theta_{\rm r}) \tag{3.12}$$

where Ξ_0^{-1} *is the inverse of the map* $\mathfrak{z} \mapsto \Xi_0(z_0; \mathfrak{z}, \theta_\tau)$ *and where* Ξ_1^{-1} *is given by*

$$\Xi_1^{-1}(z_0; \mathfrak{z}, \theta_\tau, \theta_{\rm r}) := -{\rm D}_{\mathfrak{z}} \Xi_0(z_0; \mathfrak{z}, \theta_\tau)^{-1} \Xi_1(z_0; \Xi_0^{-1}(z_0; \mathfrak{z}, \theta_\tau), \theta_\tau, \theta_{\rm r}). \tag{3.13}$$

Moreover, given K_0, *by adjusting* R *large enough and* ε_0 *small enough, we can always ensure that*

$$\forall (\varepsilon, z_0, \theta_{\rm r}) \in [0, \varepsilon_0] \times K_0 \times \mathbb{T}_{{\rm r},z_0}, \quad K_0 \subset \Xi\big(\varepsilon, z_0; B(0,R]; 0, \theta_{\rm r}\big). \tag{3.14}$$

Proof. Consider the auxiliary map

$$B(0,R] \ni \mathfrak{z} \mapsto \Xi_0^{-1}\big(z_0; \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_{\rm r}); \theta_\tau\big). \tag{3.15}$$

Applying (3.6) and (3.7), the mean value theorem (in several variables) guarantees that

$$\Xi_0^{-1}\big(z_0; \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_{\rm r}); \theta_\tau\big) = \mathfrak{z} + O(\varepsilon).$$

More precisely, this is a modification of the identity map $Id_{B(0,R]}$ which is of size $O(\varepsilon)$ in terms of the C^1-norm on $B(0,R]$. Moreover, by compactness, this holds true uniformly with respect to $z_0 \in K_0$, $\theta_\tau \in \mathbb{T}$ and $\theta_{\rm r} \in \mathbb{T}_{{\rm r},z_0}$. Since the set of $\mathcal{C}^1$-diffeomorphisms on $B(0,R]$ is open, by restricting ε_0 if necessary, the map inside (3.15) is sure to be a diffeomorphism for all values of $(\varepsilon, z_0, \theta_\tau, \theta_{\rm r})$ in $[0, \varepsilon_0] \times K_0 \times \mathbb{T} \times \mathbb{T}_{{\rm r},z_0}$. Composing (3.15) by Ξ_0 on the left, we recover as expected that $\Xi(\varepsilon, z_0; .; \theta_\tau, \theta_{\rm r})$ is a diffeomorphism from $B(0,R]$ onto its image. Now, the set

$$\big\{ \Xi_0^{-1}(z_0; z_0; 0)\,;\, z_0 \in K_0 \big\}$$

is compact as the image of $K_0 \times K_0$ by the continuous function $\Xi_0^{-1}(\cdot; 0)$. Thus, for R sufficiently large, it can be included in a ball of radius R. And thereby, we have

$$\forall z_0 \in K_0, \quad z_0 = \Xi_0\big(z_0; \Xi_0^{-1}(z_0; z_0; 0); 0\big) \in \Xi_0\big(z_0; B(0,R]; 0\big).$$

This inclusion is none other than (3.14) when $\varepsilon = 0$. The general case follows by compactness and perturbative arguments, by restricting $\varepsilon_0 \in]0, \varepsilon_0]$ again if necessary. Moreover, by the definitions of Ξ and then Ξ^{-1}, we must have

$$\Xi(z_0; \Xi^{-1}, \theta_\tau, \theta_{\mathrm{r}}) = \Xi_0(z_0; \Xi^{-1}, \theta_\tau) + \varepsilon\ \Xi_1(z_0; \Xi^{-1}, \theta_\tau, \theta_{\mathrm{r}}) = \mathfrak{z}. \tag{3.16}$$

We can seek Ξ^{-1} in the form of the asymptotic expansion (3.12). Then, we can exploit the formal expansion of (3.16) in powers of ε to successively determine the Ξ_j^{-1} with $j \geq 0$. By this way, we can extract Ξ_0^{-1} (term with ε^0 in factor) and Ξ_1^{-1} (term with ε in factor) as indicated. The inverse function theorem allows to justify this calculus. □

3.2.3. *Passage from the original field z to the lifting $\mathfrak{z}$.* In practice, we fix the compact K_0, and we consider a collection of initial conditions $z_0 \in K_0$. Then, we adjust R and ε_0 to obtain (3.14). The property (3.14) is essential to guarantee that all positions in K_0 has a unique preimage inside $B(0, R]$. More precisely, for all $(\varepsilon, \mathrm{v}_0) \in [0, \varepsilon_0] \times \mathbb{R}$, we can now assert that

$$\exists!\, \mathfrak{z}_0 \equiv \mathfrak{z}_0(\varepsilon, z_0, \mathrm{v}_0) \in B(0, R]; \quad z_0 = \Xi\big(\varepsilon, z_0; \mathfrak{z}_0; 0, \frac{\mathrm{v}_0}{\varepsilon}\big) \tag{3.17}$$

or equivalently

$$\begin{aligned} \mathfrak{z}_0 \equiv \mathfrak{z}_0(\varepsilon, z_0, \mathrm{v}_0) &= \Xi^{-1}\big(\varepsilon, z_0; z_0; 0, \frac{\mathrm{v}_0}{\varepsilon}\big) \\ &= \Xi_0^{-1}(z_0; z_0; 0) + \varepsilon \Xi_1^{-1}(z_0; z_0; 0, \frac{\mathrm{v}_0}{\varepsilon}) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{3.18}$$

Note that $\mathfrak{z}_0$ does depend on ε (even if z_0 does not) and it does oscillate in ε due to the presence of v_0/ε. The change from z_0 to $\mathfrak{z}_0$ through Ξ^{-1} introduces high frequencies since in general $\partial_{\theta_{\mathrm{r}}} \Xi_1^{-1} \not\equiv 0$. However, the positions $\mathfrak{z}_0$ remain uniformly in ε in a compact neighborhood of

$$\big\{\Xi_0^{-1}(z_0; z_0; 0); z_0 \in K_0\,\big\} \subset B(0, R].$$

By the way of (3.18), all the initial data z_0 contained in K_0 can be converted into corresponding initial data $\mathfrak{z}_0 \in B(0, R]$ for the forthcoming system of ODEs on $\mathfrak{z}$. Now, let K be a compact set containing K_0 in its interior ($K_0 \subset \mathring{K} \subset K \Subset \mathbb{R}^n$). By applying Lemma 3.6 with K_0 replaced by K, we can guarantee that

$$\forall (\varepsilon, z, \theta_{\mathrm{r}}) \in [0, \tilde{\varepsilon}_0] \times K \times \mathbb{T}_{\mathrm{r}, z_0}, \quad K \subset \Xi\big(\varepsilon, z; B(0, \tilde{R}]; 0, \theta_{\mathrm{r}}\big), \tag{3.19}$$

for some $\tilde{\varepsilon}_0 \leq \varepsilon_0$ and $R \leq \tilde{R}$. By continuity, a solution $z(\varepsilon, z_0, \mathrm{v}_0; \cdot)$ of (1.12) issued from $z_0 \in K_0$ will remain in K for sufficiently small values of τ. Thus, at least locally in time, on some open time interval which may be not uniform with respect to $\varepsilon \in]0, \varepsilon_0]$, we can define

$$\mathfrak{z}(\varepsilon, z_0, \mathrm{v}_0; \tau) := \Xi^{-1}\Big(\varepsilon, z_0; z(\varepsilon, z_0, \mathrm{v}_0; \tau); \theta_\tau, \frac{\mathrm{v}(\varepsilon, z_0, \mathrm{v}_0; \tau)}{\varepsilon}\Big) \in B(0, \tilde{R}]. \tag{3.20}$$

At this stage, we have collected enough information to show Theorem 2.5.

Proof of Theorem 2.5. We can pass from the local in time solution z of (1.12) to some associated field $\mathfrak{z}$ through (3.20), with inverse formula (3.2). Now, Proposition 3.2 indicates that $\mathfrak{z}$ can be characterized by the equation (3.3), and therefore that it can be determined through (3.3) independently from (1.12). This means that any local solution to (1.12) gives rise to a local solution to (3.3), and conversely. Since by Corollary 3.3, the lifespan associated with (3.3) is uniformly bounded below (with some stability in the sup norm), the same applies concerning (1.12). $\square$

Now, the challenge is to derive the equation (3.3) on $\mathfrak{z}$, allowing to identify $\mathfrak{z}$.

3.3. **Transformation of the equations.** Given a map Ξ as in Subsection 3.2, the matter here is to show that the system (1.12) is transformed under the blow-up procedure into the system (3.3). It is also to determine how the new source terms A and V inside (3.3) can be deduced from the original A and V of (1.12).

Lemma 3.7 (Identification of the new source terms A and V from A, V and Ξ)**.** *Let ${}^t(z,\mathsf{v})(\varepsilon,z_0,\mathsf{v}_0;\cdot)$ be a local solution of* (1.12)*. Assume Condition 3.5. Define $\mathfrak{z}_0$ through* (3.17) *and $\mathfrak{z}(\varepsilon,z_0,\mathsf{v}_0;\cdot)$ locally in time through* (3.20)*. Then, the field ${}^t(\mathfrak{z},\mathsf{v})$ is the unique (local) solution of* (3.3)*, with* A *and* V *determined as indicated below:*

$$\begin{aligned} &\mathrm{A}(\varepsilon,z_0;\mathfrak{z};\theta_\tau,\theta_\mathrm{r})\\ &:= D_{\mathfrak{z}}\Xi(\varepsilon,z_0;\mathfrak{z};\theta_\tau,\theta_\mathrm{r})^{-1}\Big\{\mathsf{A}\big(\varepsilon,z_0;\Xi(\cdot);\theta_\tau,\theta_\mathrm{r}\big)-\partial_{\theta_\tau}\Xi(\varepsilon,z_0;\mathfrak{z};\theta_\tau,\theta_\mathrm{r}) \\ &\quad -\varepsilon^{-1}\mathsf{V}\big(\varepsilon,z_0;\Xi(\cdot);\theta_\tau,\theta_\mathrm{r}\big)\partial_{\theta_\mathrm{r}}\Xi(\varepsilon,z_0;\mathfrak{z};\theta_\tau,\theta_\mathrm{r})\Big\}, \end{aligned} \tag{3.21}$$

$$\mathrm{V}(\varepsilon;\mathfrak{z};\theta_\tau,\theta_\mathrm{r}) := \mathsf{V}\big(\varepsilon,z_0;\Xi(\cdot);\theta_\tau,\theta_\mathrm{r}\big), \tag{3.22}$$

where the point $\cdot$ must be replaced by $(\varepsilon,z_0;\mathfrak{z};\theta_\tau,\theta_\mathrm{r})$.

Proof. Recall that the component v remains unchanged under the blow-up procedure. It is the solution to (1.12). In view of (3.2), this forces us to define V as in (3.22). Incidentally, this means that the amplitude is maintained when transferring from V to V.

There remains to prove that $\mathfrak{z}$ is a solution to the first line of (3.3) with A adjusted as in (3.21). To this end, combine (3.2) with the first equation of (1.12) to see that we have to guarantee that

$$\partial_\tau\Big\{\Xi\big(\varepsilon,z_0;\mathfrak{z}(\tau);\frac{\tau}{\varepsilon},\frac{\mathsf{v}(\tau)}{\varepsilon}\big)\Big\} = \frac{1}{\varepsilon}\mathsf{A}\big(\varepsilon,z_0;z;\frac{\tau}{\varepsilon},\frac{\mathsf{v}(\tau)}{\varepsilon}\big).$$

In other words, exploiting again (1.12), we must impose

$$\begin{aligned} &\big(D_{\mathfrak{z}}\Xi\mathrm{A}+\partial_{\theta_\tau}\Xi+\varepsilon^{-1}\mathrm{V}\partial_{\theta_\mathrm{r}}\Xi\big)(\varepsilon,z_0;\mathfrak{z};\theta_\tau,\theta_\mathrm{r})\\ &=\mathsf{A}\big(\varepsilon,z_0;\Xi(\varepsilon,z_0;\mathfrak{z};\theta_\tau,\theta_\mathrm{r});\theta_\tau,\theta_\mathrm{r}\big). \end{aligned} \tag{3.23}$$

By inverting the matrix $D_{\mathfrak{z}}\Xi$, we recover exactly (3.21). $\square$

3.4. **Proof of Proposition 3.2.** In what follows, we need to invert the derivative ∂_{θ_i} for $i\in\{\tau,\mathrm{r}\}$. To this end, define $\mathbb{T}_T := \mathbb{R}/(T\mathbb{Z})$ and introduce the sets $L^1_*(\mathbb{T}_T)$ made of periodic functions with zero mean, namely

$$L^1_*(\mathbb{T}_T) := \Big\{\mathcal{Z}\in L^1(\mathbb{T}_T);\int_0^T\mathcal{Z}(\theta)\,d\theta=0\Big\};$$

We can define the operators $\partial_\theta^{-1}: L^1_*(\mathbb{T}_T)\to L^1_*(\mathbb{T}_T)$ according to

$$\partial_\theta^{-1}\mathcal{Z}(\theta) := \int_0^\theta\mathcal{Z}(r)\,dr-\frac{1}{T}\int_0^T\Big(\int_0^s\mathcal{Z}(r)\,dr\Big)\,ds. \tag{3.24}$$

Recall that

$$\forall \mathcal{Z} \in L^1(\mathbb{T}_T), \quad \partial_\theta^{-1} \partial_\theta \mathcal{Z} = \mathcal{Z}^* := \mathcal{Z}(\theta) - \frac{1}{T} \int_0^T \mathcal{Z}(r)\, dr, \tag{3.25a}$$

$$\forall \mathcal{Z} \in L^1_*(\mathbb{T}_T), \quad \partial_\theta \partial_\theta^{-1} \mathcal{Z} = \mathcal{Z}. \tag{3.25b}$$

Then, we apply the above arguments to define the inverse of the operators ∂_{θ_i} for $i \in \{\tau, \mathrm{r}\}$.

We seek conditions on A and V allowing to obtain rid of the problematic term of size ε^{-1} which may appear when looking at the first line of (3.3). This requires to separate inside (3.21) the leading term from the terms with ε in factor. We work at the level of (3.23). Assuming as expected that $\mathrm{A}_0 \equiv 0$, with Ξ as in (3.6) so that $\partial_{\theta_\mathrm{r}} \Xi = \varepsilon \partial_{\theta_\mathrm{r}} \Xi_1$, with A as decomposed in the beginning, we can expand (3.23) according to

$$\begin{aligned}
&\big[(D_\mathfrak{z} \Xi_0 + \varepsilon D_\mathfrak{z} \Xi_1)\big(\varepsilon \mathrm{A}_1 + O(\varepsilon^2)\big) + \partial_{\theta_\tau} \Xi_0 + \varepsilon \partial_{\theta_\tau} \Xi_1 + \mathrm{V} \partial_{\theta_\mathrm{r}} \Xi_1\big](\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_\mathrm{r}) \\
&= \mathsf{A}\big(z_0; \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_\mathrm{r}); \theta_\tau, \theta_\mathrm{r}\big).
\end{aligned}$$

On the one hand, from (3.22), we have

$$\begin{aligned}
\mathrm{V} &= \mathsf{V}(\varepsilon, z_0; \Xi_0 + \varepsilon \Xi_1; \cdot) \\
&= \mathsf{V}_0(z_0; \Xi_0 + \varepsilon \Xi_1; \cdot) + \varepsilon \mathsf{V}_1(z_0; \Xi_0 + \varepsilon \Xi_1; \cdot) + \mathcal{O}(\varepsilon^2) \\
&= \mathsf{V}_0(z_0; \Xi_0; \cdot) + \varepsilon (\Xi_1 \cdot \nabla_z) \mathsf{V}_0(z_0; \Xi_0; \cdot) + \varepsilon \mathsf{V}_1(z_0; \Xi_0; \cdot) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\mathsf{A}\big(z_0; \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_\mathrm{r}); \cdot\big) \\
&= \big(\mathsf{A}_0 + \varepsilon \mathsf{A}_1\big)\big(z_0; \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_\mathrm{r}); \cdot\big) + \mathcal{O}(\varepsilon^2) \\
&= \mathsf{A}_0(z_0; \Xi_0; \cdot) + \varepsilon (\Xi_1 \cdot \nabla_z) \mathsf{A}_0(z_0; \Xi_0; \cdot) + \varepsilon \mathsf{A}_1(z_0; \Xi_0; \cdot) + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Thus, we have to deal with the condition

$$\begin{aligned}
&\varepsilon D_\mathfrak{z} \Xi_0(z_0; \mathfrak{z}; \theta_\tau) \mathrm{A}_1(z_0; \mathfrak{z}; \theta_\tau, \theta_\mathrm{r}) + \partial_{\theta_\tau} \Xi_0(z_0; \mathfrak{z}; \theta_\tau) + \varepsilon \partial_{\theta_\tau} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_\mathrm{r}) \\
&+ \big(\mathsf{V}_0(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r}) + \varepsilon (\Xi_1 \cdot \nabla_z) \mathsf{V}_0(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r})\big) \partial_{\theta_\mathrm{r}} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_\mathrm{r}) \\
&+ \varepsilon \mathsf{V}_1(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r}) \partial_{\theta_\mathrm{r}} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_\mathrm{r}) \\
&= \mathsf{A}_0(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r}) + \varepsilon (\Xi_1 \cdot \nabla_z) \mathsf{A}_0(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r}) \\
&\quad + \varepsilon \mathsf{A}_1(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r}) + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{3.26}$$

We bring together the terms with the same power of ε in factor to obtain

$$\begin{aligned}
&\partial_{\theta_\tau} \Xi_0(z_0; \mathfrak{z}; \theta_\tau) + \mathsf{V}_0(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r}) \partial_{\theta_\mathrm{r}} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_\mathrm{r}) - \mathsf{A}_0(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r}) \\
&+ \varepsilon \big[D_\mathfrak{z} \Xi_0(z_0; \mathfrak{z}; \theta_\tau) \mathrm{A}_1(z_0; \mathfrak{z}; \theta_\tau, \theta_\mathrm{r}) + \partial_{\theta_\tau} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_\mathrm{r}) \\
&+ \big((\Xi_1 \cdot \nabla_z) \mathsf{V}_0(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r}) + \mathsf{V}_1(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r})\big) \partial_{\theta_\mathrm{r}} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_\mathrm{r}) \\
&- (\Xi_1 \cdot \nabla_z) \mathsf{A}_0(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r}) - \mathsf{A}_1(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r})\big] + \mathcal{O}(\varepsilon^2) = 0.
\end{aligned} \tag{3.27}$$

The first line must be zero, which is the same as

$$\mathsf{V}_0(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r})^{-1} \partial_{\theta_\tau} \Xi_0(z_0; \mathfrak{z}; \theta_\tau) + \partial_{\theta_\mathrm{r}} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_\mathrm{r}) = (\mathsf{V}_0^{-1} \mathsf{A}_0)(z_0; \Xi_0; \theta_\tau, \theta_\mathrm{r}).$$

First, take the mean value with respect to θ_r in order to obtain rid of Ξ_1 and to identify $\partial_{\theta_\tau} \Xi_0$. Then, subtract the result thus obtained to deduce $\partial_{\theta_\mathrm{r}} \Xi_1$. Following these lines, we can exhibit two separate conditions, namely

$$\partial_{\theta_\tau}\Xi_0 = \Big(\big(\overline{\mathsf{V}_0^{-1}}\big)^{-1}\overline{\mathsf{V}_0^{-1}\mathsf{A}_0}\Big)(z_0;\Xi_0;\theta_\tau), \tag{3.28}$$

$$\partial_{\theta_{\mathrm{r}}}\Xi_1 = \Big(\mathsf{V}_0^{-1}\mathsf{A}_0 - \mathsf{V}_0^{-1}\big(\overline{\mathsf{V}_0^{-1}}\big)^{-1}\overline{\mathsf{V}_0^{-1}\mathsf{A}_0}\Big)(z_0;\Xi_0;\theta_\tau,\theta_{\mathrm{r}}). \tag{3.29}$$

Definition 3.8 (Homological equation)**.** The nonlinear ordinary differential equation (3.28) in the variable θ_τ is called the homological equation (or sometimes the first modulation equation).

We can complete (3.28) with some initial data $\mathfrak{z} \in \mathbb{R}^n$ whose introduction has been motivated by Remark 2.13. Hence, the expression Ξ_0 becomes a function of z_0, $\mathfrak{z}$ and θ_τ, together with

$$\Xi_0(z_0;\mathfrak{z};0) = \mathfrak{z}. \tag{3.30}$$

Remark 3.9 (About the passage from z_0 to $\mathfrak{z}_0$)**.** Condition (3.30) implies that $\Xi_0^{-1}(z_0;\mathfrak{z};0) = \mathfrak{z}$. Thus, when $\Xi_1 \equiv 0$, we find that $\Xi^{-1}(\varepsilon, z_0;\mathfrak{z};0,\theta_{\mathrm{r}}) = \mathfrak{z}$. Then, in view of (3.18), we have

$$\mathfrak{z}_0 = \sum_{j=0}^{+\infty}\varepsilon^j\mathfrak{z}_{0j} = z_0 = \sum_{j=0}^{+\infty}\varepsilon^j z_{0j},$$

so that $\mathfrak{z}_{0j} = z_{0j}$ for all $j \in \mathbb{N}$.

Lemma 3.10 (Mean flow is the solution to the homological equation)**.** *We have*

$$\forall\, s \in \mathbb{R}, \quad \Xi_{\mathrm{mf}}(z_0;\mathfrak{z};s) = \Xi_0(z_0;\mathfrak{z};s). \tag{3.31}$$

As a consequence, under Assumption 2.4, the solution to the Cauchy problem (3.28)-(3.30) *is global and it is periodic with respect to θ_τ of period 2π.*

Proof. In view of Definition 2.2, the vector fields Ξ_{mf} and Ξ_0 are solutions to the same system of ODEs. By Cauchy-Lipschitz theorem, local solutions do exist and (by uniqueness) they must coincide. The content of Assumption 2.4 allows the conclusion. □

Remark 3.11 (Impact of Assumption 2.4)**.** Lemma 3.10 makes the connection between the notion of *mean flow* (Definition 2.2 appearing in the introduction after heuristic considerations) and the map Ξ_0 (derived from formal computations). The role of Assumption 2.4 is clearly to furnish global solutions to (3.28)-(3.30). It is also essential to stay in the periodic framework.

Remark 3.12 (About the verification of Condition 3.5)**.** Since the map $\Xi_0(z_0;\mathfrak{z};\cdot)$ can be viewed as a flow, Condition 3.5 is automatically verified with $\Xi_0^{-1}(z_0;\mathfrak{z};\theta_\tau) = \Xi_0(z_0;\mathfrak{z};-\theta_\tau)$.

Remark 3.13 (Some hidden constraint on Ξ_{mf} under Assumption 2.4)**.**
The right-hand side of (2.11) is periodic in s of period 2π. Thus, it can be decomposed like in (2.7) into its mean value and its quick oscillating part. It follows that the solution to (2.11) is the sum of a linear function plus a periodic function. The resulting expression may indeed be periodic only if

$$\big\langle \mathsf{A}_{\mathrm{mf}}\big(z_0;\Xi_{\mathrm{mf}}(z_0;\mathfrak{z};\cdot);\cdot\big)\big\rangle = 0, \tag{3.32}$$

which may appear as some *a posteriori* condition which must be satisfied by Ξ_{mf}.

By construction, the right-hand side of (3.29) is periodic with respect to the variable $\theta_{\rm r}$, and it is of mean zero. It can be integrated as indicated in (3.24). As a consequence, the part Ξ_1^* is completely determined from (3.29). We fix $\overline{\Xi}_1 \equiv 0$, so that

$$\begin{aligned} \Xi_1 &\equiv \Xi_1^*(z_0;\mathfrak{z};\theta_\tau,\theta_{\rm r}) \\ &:= \partial_{\theta_{\rm r}}^{-1}\Big(\frac{(\mathsf{A}_0 - (\overline{\mathsf{V}_0^{-1}})^{-1}(\overline{\mathsf{V}_0^{-1}\mathsf{A}_0})\big(z_0;\Xi_0(z_0;\mathfrak{z};\theta_\tau);\theta_\tau,\theta_{\rm r}\big)}{\mathsf{V}_0\big(z_0;\Xi_0(z_0;\mathfrak{z};\theta_\tau);\theta_\tau,\theta_{\rm r}\big)}\Big). \end{aligned} \tag{3.33}$$

At this stage, we have exhibited necessary conditions on A_0, Ξ_0, and $\Xi_1 \equiv \Xi_1^*$ to obtain $\mathrm{A}_0 \equiv 0$. We have now to show that these conditions are sufficient. By Lemma 3.6, the matrix $D_{\mathfrak{z}}\Xi$ is invertible for the data under consideration. Coming back to (3.23), we can therefore deduce the value of A, with $\mathrm{A} = \varepsilon \mathrm{A}_1 + \mathcal{O}(\varepsilon^2)$. Looking at (3.27), we find that

$$\begin{aligned} &\mathrm{A}_1(z_0;\mathfrak{z};\theta_\tau,\theta_{\rm r}) \\ &= D_{\mathfrak{z}}\Xi_0(z_0;\mathfrak{z};\theta_\tau)^{-1}\Big\{\mathsf{A}_1(z_0;\Xi_0;\theta_\tau,\theta_{\rm r}) \\ &\quad + (\Xi_1\cdot\nabla_z)\mathsf{A}_0(z_0;\Xi_0;\theta_\tau,\theta_{\rm r}) - \partial_{\theta_\tau}\Xi_1(z_0,\mathfrak{z};\theta_\tau,\theta_{\rm r}) \\ &\quad - \big[\mathsf{V}_1(z_0;\Xi_0;\theta_\tau,\theta_{\rm r}) + (\Xi_1\cdot\nabla_z)\mathsf{V}_0(z_0;\Xi_0;\theta_\tau,\theta_{\rm r})\big]\partial_{\theta_{\rm r}}\Xi_1(z_0,\mathfrak{z};\theta_\tau,\theta_{\rm r})\Big\}. \end{aligned} \tag{3.34}$$

Remark 3.14 (Limited framework of the normal form procedure)**.** The restriction on Ξ which is imposed at the level of (3.9) strongly reduces the class of systems (1.12) which can be managed by the blow up procedure. Indeed, it generates compatibility conditions on A:

- In view of (3.28), the selection of $\Xi_0(z_0;\mathfrak{z};\theta_\tau) = \mathfrak{z}$ is coherent with the homological equation if and only if $\mathsf{A}_{\rm mf} \equiv 0$. This requires that $\mathsf{V}_0^{-1}\mathsf{A}_0$ has a zero mean (in $\theta_{\rm r}$), which is very restrictive. In particular, when A_0 do not depend on $\theta_{\rm r}$, this simply means that we start already with $\mathsf{A}_0 \equiv 0$.
- In view of (3.29), the function Ξ_1 does in general depend on θ_τ when V_0 and A_0 do depend on θ_τ. Now, this would not be compatible with (3.9) which implies that $\partial_{\theta_\tau}\Xi_1 = 0$. In the normal form procedure, since $D_{\mathfrak{z}}\Xi_0 = Id$ and $\partial_{\theta_\tau}\Xi_1 = 0$, the expression leading to A_1 must be related to A_1 through
$$\begin{aligned} \mathrm{A}_1(z_0;\mathfrak{z};\theta_{\rm r}) &= \mathsf{A}_1(z_0;\mathfrak{z};\theta_{\rm r}) + (\Xi_1\cdot\nabla_z)\mathsf{A}_0(z_0;\mathfrak{z};\theta_{\rm r}) \\ &\qquad - \mathsf{V}_1(z_0;\mathfrak{z};\theta_{\rm r})\partial_{\theta_{\rm r}}\Xi_1 - (\Xi_1\cdot\nabla_z)\mathsf{V}_0(z_0;\mathfrak{z})\partial_{\theta_{\rm r}}\Xi_1, \end{aligned}$$
where Ξ_1 does not depend on θ_τ and is as in (3.33).

In this section, we have seen that the study of the system (1.12) is under Assumption 2.4 completely equivalent to the analysis of (3.3). The challenge now is to exploit (3.4) in order to derive a description of ${}^t(\mathfrak{z},\mathrm{v})$ in terms of asymptotic oscillating series in powers of $\varepsilon \in]0,\varepsilon_0]$. In fact, this requires in addition to Assumption 2.4 imposing Assumption 2.6.

Remark 3.15 (Simplified framework inherited from Assumption 2.6)**.** Under Assumption 2.6, we find that V_0 does not depend on $\theta_{\rm r}$. More precisely, we have

$$0 < \mathrm{V}_0(z_0;\mathfrak{z};\theta_\tau,\theta_{\rm r}) \equiv \mathrm{V}_0(z_0;\mathfrak{z};\theta_\tau) := \mathsf{V}_0\big(z_0;\Xi_0(z_0;\mathfrak{z};\theta_\tau);\theta_\tau\big). \tag{3.35}$$

Moreover, as seen in (2.12), the function $\mathsf{A}_{\rm mf}$ is simplified into $\overline{\mathsf{A}}_0$. On the other hand, the formulas (3.28) and (3.29) can be replaced by

$$\partial_{\theta_\tau} \Xi_0(z_0;\mathfrak{z};\theta_\tau) = \overline{\mathsf{A}}_0\big(z_0;\Xi_0(z_0;\mathfrak{z};\theta_\tau);\theta_\tau\big), \tag{3.36}$$

$$\Xi_1^*(z_0;\mathfrak{z};\theta_\tau,\theta_{\rm r}) = \mathsf{V}_0\big(z_0;\Xi_0(z_0;\mathfrak{z};\theta_\tau);\theta_\tau\big)^{-1}\partial_{\theta_{\rm r}}^{-1}\mathsf{A}_0^*\big(z_0;\Xi_0(z_0;\mathfrak{z};\theta_\tau);\theta_\tau,\theta_{\rm r}\big). \tag{3.37}$$

4. Three-scale WKB calculus

We work here under Assumptions 2.4 and 2.6. The purpose is to construct approximate solutions through formal computations. The preceding work of preparation (in Section 3) allows to formulate the problem in terms of ${}^t(\mathfrak{z},\mathsf{v})$. Thus, we can consider (3.3) and we can benefit from (3.4). We can also exploit the content of Remark 3.15. We proceed in several stages. In Subsection 4.1, we replace ${}^t(\mathfrak{z},\mathsf{v})$ by some corresponding profile ${}^t(\mathfrak{Z},\mathcal{V})$; we define a notion of profile equations with associated approximate solutions ${}^t(\mathfrak{Z}^a,\mathcal{V}^a)$; we also state the main result (Proposition 4.2) of this section. The construction of ${}^t(\mathfrak{Z}^a,\mathcal{V}^a)$ is clarified in Subsection 4.2. The proof of Proposition 4.2 is achieved in Subsection 4.3.

4.1. **Profile formulation.** The first step is to seek solutions to (3.3) in the form

$$\begin{pmatrix}\mathfrak{z}\\ \mathsf{v}\end{pmatrix}(\tau) = \begin{pmatrix}\mathfrak{Z}\\ \mathcal{V}\end{pmatrix}\Big(\tau,\frac{\tau}{\varepsilon},\frac{\mathsf{v}(\tau)}{\varepsilon}\Big), \tag{4.1}$$

where $\mathsf{v}(\tau)$ stands for the *exact solution* to (3.3). This multi-scale approach allows to separate the rapid variations (which have not yet been identified due to the presence of v) from the slower (current and quick) variations (which must be determined first and foremost). This is like knowing $\mathfrak{z}$ and v modulo the action of a one-parameter group of rotations (associated with $\theta_{\rm r}\in\mathbb{R}$) which are aimed to be ultimately specified through the the replacement of $\theta_{\rm r}$ by v/ε.

At the level of (4.1), the *profile* ${}^t(\mathfrak{Z},\mathcal{V})(\tau,\theta_\tau,\theta_{\rm r})$ may depend on the parameters ε, z_0 and v_0 (which will not be always indicated). Assuming (4.1), observe that

$$\partial_\tau\begin{pmatrix}\mathfrak{z}\\ \mathsf{v}\end{pmatrix}(\tau) = \Big[Op\big(\mathfrak{Z};\partial\big)\begin{pmatrix}\mathfrak{Z}\\ \mathcal{V}\end{pmatrix}\Big]\Big(\tau,\frac{\tau}{\varepsilon},\frac{\mathsf{v}(\tau)}{\varepsilon}\Big),$$

where we have introduced the partial differential operator

$$Op(\varepsilon,z_0;\mathfrak{Z};\theta_\tau,\theta_{\rm r}) \equiv Op(\mathfrak{Z};\partial) := \partial_\tau + \varepsilon^{-1}\partial_{\theta_\tau} + \varepsilon^{-2}\,\mathrm{V}(\varepsilon,z_0;\mathfrak{Z};\theta_\tau,\theta_{\rm r})\partial_{\theta_{\rm r}} \tag{4.2}$$

which involves the parameters $(\varepsilon,z_0)\in[0,\varepsilon_0]\times\mathbb{R}^n$, is non-linear with respect to $\mathfrak{Z}$, and implies the coefficient V which undergoes variations in $(\theta_\tau,\theta_{\rm r})\in\mathbb{T}\times\mathbb{T}_{{\rm r},z_0}$. Consider the *profile equations* which are associated to (3.3). These are the relaxed version of (3.3) made of the transport equations

$$\Big[Op(\mathfrak{Z};\partial)\begin{pmatrix}\mathfrak{Z}\\ \mathcal{V}\end{pmatrix} - \frac{1}{\varepsilon}\begin{pmatrix}\mathrm{A}\\ \mathrm{V}\end{pmatrix}(\mathfrak{Z})\Big](\varepsilon,z_0,\mathsf{v}_0;\tau,\theta_\tau,\theta_{\rm r}) = 0 \tag{4.3}$$

together with the initial data (at time $\tau=0$)

$$\begin{pmatrix}\mathfrak{Z}\\ \mathcal{V}\end{pmatrix}(\varepsilon,z_0,\mathsf{v}_0;0,\theta_\tau,\theta_{\rm r}) = \begin{pmatrix}\mathfrak{Z}_0\\ \mathcal{V}_0\end{pmatrix}(\varepsilon,z_0,\mathsf{v}_0;\theta_\tau,\theta_{\rm r}). \tag{4.4}$$

To recover the initial data of (3.3) with $\mathfrak{z}$ and v as in (4.1), in view of (3.18), we have to impose

$$\begin{pmatrix}\mathfrak{Z}\\ \mathcal{V}\end{pmatrix}\Big(\varepsilon,z_0,\mathsf{v}_0;0,0,\frac{\mathsf{v}_0}{\varepsilon}\Big) = \begin{pmatrix}\Xi^{-1}(\varepsilon,z_0;z_0,0,\frac{\mathsf{v}_0}{\varepsilon})\\ \mathsf{v}_0\end{pmatrix}. \tag{4.5}$$

To this end, it suffices to work with the (relaxed) condition

$$\begin{pmatrix}\mathfrak{Z}_0\\ \mathcal{V}_0\end{pmatrix}(\varepsilon,z_0,\mathrm{v}_0;\theta_\tau,\theta_\mathrm{r})=\begin{pmatrix}\Xi^{-1}(\varepsilon,z_0;z_0;0,\theta_\mathrm{r})\\ \mathrm{v}_0\end{pmatrix}. \tag{4.6}$$

The component $\mathfrak{Z}_0$ may be expanded in powers of ε. The same applies to Ξ^{-1}. Assuming that z_0 does not depend on ε, this yields

$$\begin{aligned}\mathfrak{Z}_0(\varepsilon,z_0,\mathrm{v}_0;\theta_\tau,\theta_\mathrm{r})&=\sum_{j=0}^{N}\varepsilon^j\,\mathfrak{Z}_{0j}(z_0;\theta_\mathrm{r})+\mathcal{O}(\varepsilon^{N+1})\\ &=\sum_{j=0}^{N}\varepsilon^j\Xi_j^{-1}(z_0;z_0,0,\theta_\mathrm{r})+\mathcal{O}(\varepsilon^{N+1}).\end{aligned}\tag{4.7}$$

In particular, taking into account (3.12) and (3.30), we find that

$$\mathfrak{Z}_{00}(z_0,\theta_\mathrm{r})=\Xi_1^{-1}(z_0;z_0,0)=z_0\equiv\mathfrak{Z}_{00}(z_0),\quad \mathfrak{Z}_{01}(z_0,\theta_\mathrm{r})=\Xi_1^{-1}(z_0;z_0,0,\theta_\mathrm{r}).$$

When z_0 depends (smoothly) on ε, we have $z_0=z_{00}+\varepsilon z_{01}+\dots$, and (4.7) can be further expanded in powers of ε to obtain

$$\mathfrak{Z}_0(\varepsilon,z_0,\mathrm{v}_0;\theta_\tau,\theta_\mathrm{r})=\sum_{j=0}^{N}\varepsilon^j\tilde{\Xi}_j^{-1}(z_{00},\dots,z_{0j},\theta_\mathrm{r})+\mathcal{O}(\varepsilon^{N+1}). \tag{4.8}$$

By the way, we can observe that the oscillations of $\mathfrak{z}_0$ are easily absorbed at time $\tau=0$ by the profile formulation, just because $\mathrm{v}(0)=\mathrm{v}_0$ and because $\mathfrak{Z}_0$ may depend on θ_r. Note also that the condition (4.4) has the effect of introducing at the level of $\mathcal{V}$ (and therefore $\mathfrak{Z}$) a dependence on v_0. Neither $\mathfrak{Z}_0$ nor $\mathcal{V}_0$ depend on θ_τ. But the variable θ_τ appears at the level of $\mathfrak{Z}$ and $\mathcal{V}$ through the variations with respect to θ_τ of the coefficient V inside (4.2).

By looking at the Cauchy problem (4.3)-(4.4), we can see that:

- The presence of v has completely disappeared;
- The equation on $\mathfrak{Z}$ is now decoupled from the one on $\mathcal{V}$.

These two properties are major assets in what follows. In fact, knowing what the component $\mathrm{v}(\tau)$ is, the solution to (3.3) can be directly deduced from (4.3) and (4.4) together with the substitution formula (4.1). Now, we would like to find good candidates for solving (4.3)-(4.4).

Definition 4.1 (Formal solutions to the profile equations)**.**
Fix $(z_0,\mathrm{v}_0)\in\mathbb{R}^n\times\mathbb{R}$. Given some time $\mathcal{T}\in\mathbb{R}_+^*$ and some integer $N\in\mathbb{N}^*$, we say that ${}^t(\mathfrak{Z}^a,\mathcal{V}^a)(\varepsilon,z_0,\mathrm{v}_0;\tau,\theta_\tau,\theta_\mathrm{r})$ is an approximate solution on $[0,\mathcal{T}]$ of order N to the Cauchy problem (4.3)-(4.4) if it satisfies (4.3)-(4.4) modulo some remainder which is of size ε^N in the supremum norm.

More precisely, Definition 4.1 means that ${}^t(\mathfrak{Z}^a,\mathcal{V}^a)(\cdot,z_0,\mathrm{v}_0;\cdot)$ is a smooth ($\mathcal{C}^\infty$) function of

$$(\varepsilon,\tau,\theta_\tau,\theta_\mathrm{r})\in]0,\varepsilon_0]\times[0,\mathcal{T}]\times\mathbb{T}\times\mathbb{T}_{\mathrm{r},z_0}$$

which is such that

$$\Big[Op(\mathfrak{Z}^a;\partial)\begin{pmatrix}\mathfrak{Z}^a\\ \mathcal{V}^a\end{pmatrix}-\frac{1}{\varepsilon}\begin{pmatrix}\mathrm{A}\\ \mathrm{V}\end{pmatrix}(\mathfrak{Z}^a)-\varepsilon^N\begin{pmatrix}\mathcal{R}_N^{\mathfrak{Z}}\\ \mathcal{R}_N^{\mathcal{V}}\end{pmatrix}\Big](\varepsilon,z_0,\mathrm{v}_0;\tau,\theta_\tau,\theta_\mathrm{r})=0 \tag{4.9}$$

together with

$$\sup_{\varepsilon\in]0,\varepsilon_0]}\ \sup_{\tau\in[0,\mathcal{T}]}\ \sup_{\theta_\tau\in\mathbb{T}}\ \sup_{\theta_{\rm r}\in\mathbb{T}_{{\rm r},z_0}} \big(|\mathcal{R}_N^{\mathfrak{Z}}|+|\mathcal{R}_N^{\mathcal{V}}|\big)(\varepsilon,z_0,\nu_0;\tau,\theta_\tau,\theta_{\rm r})<+\infty. \tag{4.10}$$

In what follows, we seek ${}^t(\mathfrak{Z}^a,\mathcal{V}^a)$ through a finite series like

$$\mathfrak{Z}^a(\varepsilon,z_0,\nu_0;\tau,\theta_\tau,\theta_{\rm r})=\sum_{j=0}^N \varepsilon^j\,\mathfrak{Z}_j(z_0,\nu_0;\tau,\theta_\tau,\theta_{\rm r}), \tag{4.11}$$

$$\mathcal{V}^a(\varepsilon,z_0,\nu_0;\tau,\theta_\tau,\theta_{\rm r})=\sum_{j=-1}^N \varepsilon^j\,\mathcal{V}_j(z_0,\nu_0;\tau,\theta_\tau,\theta_{\rm r}), \tag{4.12}$$

with

$${}^t(\mathfrak{Z}_j,\mathcal{V}_j)\in C^\infty(\mathbb{R}^n\times\mathbb{R}\times[0,\mathcal{T}]\times\mathbb{T}\times\mathbb{T}_{{\rm r},z_0};\mathbb{R}^n\times\mathbb{R}). \tag{4.13}$$

Moreover, in the same vein as (2.16), we impose

$$\mathcal{V}_{-1}\equiv\langle\overline{\mathcal{V}}_{-1}\rangle,\quad \mathcal{V}_0\equiv\overline{\mathcal{V}}_0,\quad \mathfrak{Z}_0\equiv\overline{\mathfrak{Z}}_0. \tag{4.14}$$

Proposition 4.2 (Existence of formal solutions to the profile equations)**.** *Under Assumptions 2.4 and 2.6, we can find some time $\mathcal{T}\in\mathbb{R}_+^*$ and, for all $N\in\mathbb{N}^*$, an approximate solution ${}^t(\mathfrak{Z}^a,\mathcal{V}^a)$ on $[0,\mathcal{T}]$ of order N to the Cauchy problem* (4.3)-(4.4)*, which takes the form of* (4.11)-(4.12) *together with* (4.13)-(4.14)*. The parts $\langle\overline{\mathfrak{Z}}_j\rangle$ and $\langle\overline{\mathcal{V}}_j\rangle$ are uniquely determined by a sequence of well-posed evolution equations associated with initial data issued from* (4.4)*, while the expressions $\overline{\mathfrak{Z}}_j^\star$, $\mathfrak{Z}_j^*$, $\overline{\mathcal{V}}_j^\star$ and $\mathcal{V}_j^*$ are derived from elliptic equations. We also find that $\mathfrak{Z}_0\equiv\langle\overline{\mathfrak{Z}}_0\rangle$ and $\mathfrak{Z}_1\equiv\overline{\mathfrak{Z}}_1$.*

Remark 4.3 (Meaning of the WKB hierarchy on the profiles)**.** Readers may wonder why the formal calculus is not performed directly on the system (3.3), with an expansion of ν in powers of ε. This is because a small error on the determination of ν, even of size ε, can completely shuffle (through the substitution of $\theta_{\rm r}$ for ν/ε) sequences which are expressed in powers of ε. In other words, a small change of ν at the level of (4.1) can strongly modify the asymptotic representations (4.11) and (4.12) of the profiles $\mathfrak{Z}$ and $\mathcal{V}$. It can mix the terms $\mathfrak{Z}_j$ and $\mathcal{V}_j$ and then cause intractable closure problems. This is why it is so important to work with the exact phase ν. We return to this point in Subsection 5.1.

Next we have a halfway to formal solutions of the redressed system of Proposition 3.2.

Remark 4.4. Because of the influence of ν, in the continuation of Remark 4.3, it bears noting that

$$\begin{pmatrix}\mathfrak{z}^a\\ \nu^a\end{pmatrix}(\tau):=\begin{pmatrix}\mathfrak{Z}^a\\ \mathcal{V}^a\end{pmatrix}\Big(\tau,\frac{\tau}{\varepsilon},\frac{\nu(\tau)}{\varepsilon}\Big) \tag{4.15}$$

does not provide, strictly speaking, with an approximate solution to (3.3). Indeed

$$\begin{aligned}
&\partial_\tau\mathfrak{z}^a-\frac{1}{\varepsilon}{\rm A}(\mathfrak{z}^a)\\
&=\varepsilon^N\ \mathcal{R}_N^{\mathfrak{Z}}\Big(\varepsilon,z_0,\nu_0;\tau,\frac{\tau}{\varepsilon},\frac{\nu(\tau)}{\varepsilon}\Big)\\
&\quad+\varepsilon^{-2}\Big\{\big[{\rm V}(\varepsilon,z_0;\mathfrak{z};\cdot)-{\rm V}(\varepsilon,z_0;\mathfrak{z}^a;\cdot)\big]\partial_{\theta_{\rm r}}\mathfrak{Z}^a(\varepsilon,z_0,\nu_0;\cdot)\Big\}\Big(\tau,\frac{\tau}{\varepsilon},\frac{\nu(\tau)}{\varepsilon}\Big).
\end{aligned}$$

We will see later that $\partial_{\theta_{\rm r}} \mathfrak{Z}^a = \mathcal{O}(\varepsilon^2)$. To estimate the right hand side, we need to control the difference between the exact solution $\mathfrak{z}$ and its model $\mathfrak{z}^a$. However, there is no assurance for the moment that $\mathfrak{z} - \mathfrak{z}^a$ is small, of size ε^N (or less). This issue is considered in Subsection 5.1.

4.2. **Three-scale analysis.** The expression ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ obtained through (4.11) and (4.12) is plugged into (4.3). The various contributions are ordered in increasing powers of ε. This yields in Subsection 4.2.1 a cascade of successive equations on ${}^t(\mathfrak{Z}_j, \mathcal{V}_j)$. The methodology for solving these equations is explained in Subsection 4.2.2.

4.2.1. *Formal calculus.* The matter is to list a cascade of successive equations on ${}^t(\mathfrak{Z}_j, \mathcal{V}_j)$. To this end, we perform a formal analysis at the level of (4.9) which can be expanded according to

$$\begin{aligned} &Op(\mathfrak{Z}^a;\partial) \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} (\varepsilon, z_0, \mathrm{v}_0; \tau, \theta_\tau, \theta_{\rm r}) - \frac{1}{\varepsilon} \begin{pmatrix} \mathrm{A} \\ \mathrm{V} \end{pmatrix} (\mathfrak{Z}^a) (\varepsilon, z_0, \mathrm{v}_0; \tau, \theta_\tau, \theta_{\rm r}) \\ &= \sum_{j=-2}^{N-1} \varepsilon^j \begin{pmatrix} \mathcal{L}_j \left(\overline{\mathfrak{Z}}_0, \mathfrak{Z}_1, \dots \right) \\ \mathcal{M}_j \left(\overline{\mathfrak{Z}}_0, \mathfrak{Z}_1, \dots ; \mathcal{V}_{-1}, \mathcal{V}_0, \dots \right) \end{pmatrix} + O(\varepsilon^N). \end{aligned} \tag{4.16}$$

We can resume (4.11), (4.12) and (4.14) in the form

$$\begin{aligned} &\begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} (\tau, \theta_\tau, \theta_{\rm r}) \\ &= \varepsilon^{-1} \begin{pmatrix} 0 \\ \langle \overline{\mathcal{V}}_{-1} \rangle \end{pmatrix} (\tau) + \sum_{j=0}^{N-1} \varepsilon^j \begin{pmatrix} \overline{\mathfrak{Z}}_j \\ \overline{\mathcal{V}}_j \end{pmatrix} (\tau, \theta_\tau) + \sum_{j=1}^{N} \varepsilon^j \begin{pmatrix} \mathfrak{Z}^*_j \\ \mathcal{V}^*_j \end{pmatrix} (\tau, \theta_\tau, \theta_{\rm r}) \end{aligned} \tag{4.17}$$

where the operations $\bar{\cdot}$, $\langle \cdot \rangle$ and $\cdot^*$ are furnished by (2.4)-(2.5)-(2.6). Taking into account (4.17) when dealing with the left part of (4.16), we find that

$$\begin{aligned} &Op\,(\mathfrak{Z}^a;\partial) \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} (\varepsilon, z_0, \mathrm{v}_0; \tau, \theta_\tau, \theta_{\rm r}) - \frac{1}{\varepsilon} \begin{pmatrix} \mathrm{A} \\ \mathrm{V} \end{pmatrix} (\mathfrak{Z}^a) (\varepsilon, z_0, \mathrm{v}_0; \tau, \theta_\tau, \theta_{\rm r}) \\ &= \varepsilon^{-1} \partial_\tau \begin{pmatrix} 0 \\ \langle \overline{\mathcal{V}}_{-1} \rangle \end{pmatrix} (\tau) + \sum_{j=0}^{N-1} \varepsilon^j \partial_\tau \begin{pmatrix} \overline{\mathfrak{Z}}_j \\ \overline{\mathcal{V}}_j \end{pmatrix} (\tau, \theta_\tau) \\ &\quad + \sum_{j=-1}^{N-2} \varepsilon^j \partial_{\theta_\tau} \begin{pmatrix} \overline{\mathfrak{Z}}_{j+1} \\ \overline{\mathcal{V}}_{j+1} \end{pmatrix} (\tau, \theta_\tau) + \sum_{j=1}^{N} \varepsilon^j \partial_\tau \begin{pmatrix} \mathfrak{Z}^*_j \\ \mathcal{V}^*_j \end{pmatrix} (\tau, \theta_\tau, \theta_{\rm r}) \\ &\quad + \sum_{j=0}^{N-1} \varepsilon^j \partial_{\theta_\tau} \begin{pmatrix} \mathfrak{Z}^*_{j+1} \\ \mathcal{V}^*_{j+1} \end{pmatrix} (\tau, \theta_\tau, \theta_{\rm r}) \\ &\quad + \sum_{k=-1}^{+\infty} \sum_{i=-1}^{k} \varepsilon^k \mathrm{V}_{k-i}(z_0; \mathfrak{Z}^a; \theta_\tau, \theta_{\rm r}) \partial_{\theta_{\rm r}} \begin{pmatrix} \mathfrak{Z}^*_{i+2} \\ \mathcal{V}^*_{i+2} \end{pmatrix} (\tau, \theta_\tau, \theta_{\rm r}) \\ &\quad - \sum_{k=-1}^{+\infty} \varepsilon^k \begin{pmatrix} \mathrm{A}_{k+1} \\ \mathrm{V}_{k+1} \end{pmatrix} (z_0; \mathfrak{Z}^a; \theta_\tau, \theta_{\rm r})\,. \end{aligned} \tag{4.18}$$

By definition, the expressions $\mathcal{L}_j$ and $\mathcal{M}_j$ are independent of $\varepsilon \in]0, \varepsilon_0]$. They are obtained by collecting the terms which, for $j < N$, appear in factor of ε^j. From this perspective, the right hand side of (4.18) is still not in convenient form. The difficulties come from the two last lines in (4.18) which involve in factor of ε^k expressions that still depend on ε. We seek the profile ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ to be the approximate solution to the profile equation (4.3) in the sense of preceding Definition 4.1. To this end, $\mathcal{L}_j$ and $\mathcal{M}_j$ should be zero.

By means of (3.4) and (3.35), it is easy to see that, for $j = -1$, we find

$$\mathcal{L}_{-1}(\overline{\mathfrak{Z}}_0, \mathfrak{Z}_1) = \partial_{\theta_\tau} \overline{\mathfrak{Z}}_0 + \mathrm{V}_0(z_0; \overline{\mathfrak{Z}}_0; \theta_\tau) \partial_{\theta_{\mathrm{r}}} \mathfrak{Z}_1^*. \tag{4.19}$$

Integrate this with respect to θ_{r} to deduce that $\partial_{\theta_\tau} \overline{\mathfrak{Z}}_0 = 0$, and therefore (since V_0 is positive) that $\partial_{\theta_{\mathrm{r}}} \mathfrak{Z}_1^* = 0$. This implies that $\mathfrak{Z}_1^* = 0$. In fact, we have (4.33). These relations are used below in order to exhibit the expressions of $\mathcal{L}_j$ for $j \geq 0$.

- For $j = 0$:

$$\mathcal{L}_0(\overline{\mathfrak{Z}}_0, \mathfrak{Z}_1, \mathfrak{Z}_2) = \partial_\tau \overline{\mathfrak{Z}}_0 + \partial_{\theta_\tau} \overline{\mathfrak{Z}}_1 + \mathrm{V}_0(z_0; \overline{\mathfrak{Z}}_0; \theta_\tau) \partial_{\theta_{\mathrm{r}}} \mathfrak{Z}_2^* - \mathrm{A}_1(z_0; \overline{\mathfrak{Z}}_0; \theta_\tau, \theta_{\mathrm{r}}). \tag{4.20}$$

- For $j = 1$:

$$\begin{aligned} &\mathcal{L}_1(\overline{\mathfrak{Z}}_0, \mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{Z}_3) \\ &= \partial_\tau \overline{\mathfrak{Z}}_1 + \partial_{\theta_\tau} \overline{\mathfrak{Z}}_2 + \partial_{\theta_\tau} \mathfrak{Z}_2^* \\ &\quad + [\mathrm{V}_1 + (\mathfrak{Z}_1 \cdot \nabla_{\mathfrak{z}})(\mathrm{V}_0)]\, \partial_{\theta_{\mathrm{r}}} \mathfrak{Z}_2^* + \mathrm{V}_0 \partial_{\theta_{\mathrm{r}}} \mathfrak{Z}_3^* - \mathrm{A}_2 - (\mathfrak{Z}_1 \cdot \nabla_{\mathfrak{z}})(\mathrm{A}_1). \end{aligned} \tag{4.21}$$

- For $j \geq 2$:

$$\begin{aligned} &\mathcal{L}_j(\overline{\mathfrak{Z}}_0, \dots, \mathfrak{Z}_{j+2}) \\ &= \partial_\tau \overline{\mathfrak{Z}}_j + \partial_{\theta_\tau} \overline{\mathfrak{Z}}_{j+1} + \partial_\tau \mathfrak{Z}_j^* + \partial_{\theta_\tau} \mathfrak{Z}_{j+1}^* \\ &\quad + \sum_{(i,m,k,l_1,\dots,l_k,b_1,\dots,b_k) \in S_j} \frac{1}{k!} \frac{\partial^k \mathrm{V}_m}{\partial_{\mathfrak{Z}^{b_1}} \dots \partial_{\mathfrak{Z}^{b_k}}} (z_0; \overline{\mathfrak{Z}}_0; \theta_\tau, \theta_{\mathrm{r}}) \prod_{t=1}^{k} \mathfrak{Z}_{l_t}^{b_t} \partial_{\theta_{\mathrm{r}}} \mathfrak{Z}_i^* \\ &\quad - \sum_{(m,k,l_1,\dots,l_k,b_1,\dots,b_k) \in S'_j} \frac{1}{k!} \frac{\partial^k \mathrm{A}_m}{\partial_{\mathfrak{Z}^{b_1}} \dots \partial_{\mathfrak{Z}^{b_k}}} (z_0; \overline{\mathfrak{Z}}_0; \theta_\tau, \theta_{\mathrm{r}}) \prod_{t=1}^{k} \mathfrak{Z}_{l_t}^{b_t} \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} S_j = \big\{ &(i, m, k, l_1, \dots, l_k, b_1, \dots, b_k); 2 \leq i \leq N,\ 0 \leq m < +\infty,\ k \leq j,\ 1 \leq l_t \leq N, \\ &l_1 = \dots = l_k,\ 0 \leq b_t \leq n,\ t \in \{1, \dots, k\},\ m + i + l_1 + \dots + l_k = j + 2 \big\}, \\ S'_j = \big\{ &(m, k, l_1, \dots, l_k, b_1, \dots, b_k);\ 1 \leq m < +\infty,\ k \leq j,\ 1 \leq l_t \leq N, \\ &l_1 = \dots = l_k,\ 0 \leq b_t \leq n,\ t \in \{1, \dots, k\},\ m + l_1 + \dots + l_k = j + 1 \big\}. \end{aligned}$$

The expression $\mathcal{L}_j$ can also be put in the form

$$\begin{aligned} \mathcal{L}_j(\mathfrak{Z}_0, \dots, \mathfrak{Z}_{j+2}) = \partial_\tau \overline{\mathfrak{Z}}_j &+ \partial_{\theta_\tau} \overline{\mathfrak{Z}}_{j+1} + \mathrm{V}_0 \partial_{\theta_{\mathrm{r}}} \mathfrak{Z}_{j+2}^* - (\mathfrak{Z}_j \cdot \nabla_{\mathfrak{z}})(\mathrm{A}_1) \\ &+ (\mathfrak{Z}_j \cdot \nabla_{\mathfrak{z}})(\mathrm{V}_0) \partial_{\theta_{\mathrm{r}}} \mathfrak{Z}_2^* + G_j(\overline{\mathfrak{Z}}_0, \mathfrak{Z}_1, \dots, \mathfrak{Z}_{j-1}, \mathfrak{Z}_j^*, \mathfrak{Z}_{j+1}^*), \end{aligned} \tag{4.23}$$

where the expression of G_j can be deduced from (4.22) as follows

$$
\begin{aligned}
&G_j(\overline{\mathfrak{z}}_0, \mathfrak{z}_1, \dots, \mathfrak{z}_{j-1}, \mathfrak{z}_j^*, \mathfrak{z}_{j+1}^*) \\
&= +\partial_\tau \mathfrak{z}_j^* + \partial_{\theta_\tau} \mathfrak{z}_{j+1}^* \\
&\quad + \sum_{(i,m,k,l_1,\dots,l_k,b_1,\dots,b_k)\in\mathfrak{S}_j} \frac{1}{k!} \frac{\partial^k \mathrm{V}_m}{\partial \mathfrak{z}^{b_1} \dots \partial \mathfrak{z}^{b_k}} (z_0; \overline{\mathfrak{z}}_0; \theta_\tau, \theta_\mathrm{r}) \prod_{t=1}^k \mathfrak{z}_{l_t}^{b_t} \partial_{\theta_\mathrm{r}} \mathfrak{z}_i^* \\
&\quad - \sum_{(m,k,l_1,\dots,l_k,b_1,\dots,b_k)\in\mathfrak{S}'_j} \frac{1}{k!} \frac{\partial^k \mathrm{A}_m}{\partial \mathfrak{z}^{b_1} \dots \partial \mathfrak{z}^{b_k}} (z_0; \overline{\mathfrak{z}}_0; \theta_\tau, \theta_\mathrm{r}) \prod_{t=1}^k \mathfrak{z}_{l_t}^{b_t}
\end{aligned} \tag{4.24}
$$

with

$$
\begin{gathered}
\mathfrak{S}_j = S_j \setminus \{(i,m,k) = (j+2,0,0) \text{ and } (i,m,k,l_1) = (2,0,1,j)\}, \\
\mathfrak{S}'_j = S'_j \setminus \{(m,k,l_1) = (1,1,j)\}.
\end{gathered}
$$

In a similar fashion, we can define the expressions $\mathcal{M}_j$ as indicated below.

- For $j = -1$:

$$
\begin{aligned}
&\mathcal{M}_{-1}(\overline{\mathfrak{z}}_0, \langle \overline{\mathcal{V}}_{-1} \rangle, \overline{\mathcal{V}}_0, \mathcal{V}_1) \\
&= \partial_\tau \langle \overline{\mathcal{V}}_{-1} \rangle + \partial_{\theta_\tau} \overline{\mathcal{V}}_0 + \mathrm{V}_0(z_0; \overline{\mathfrak{z}}_0; \theta_\tau) \partial_{\theta_\mathrm{r}} \mathcal{V}_1^* - \mathrm{V}_0(z_0; \overline{\mathfrak{z}}_0; \theta_\tau).
\end{aligned} \tag{4.25}
$$

- For $j \geq 0$:

$$
\begin{aligned}
&\mathcal{M}_j(\overline{\mathfrak{z}}_0, \dots, \mathfrak{z}_{j+1}, \overline{\mathcal{V}}_0, \dots, \mathcal{V}_{j+2}) \\
&= \partial_\tau \overline{\mathcal{V}}_j + \partial_{\theta_\tau} \overline{\mathcal{V}}_{j+1} + \partial_\tau \mathcal{V}_j^* + \partial_{\theta_\tau} \mathcal{V}_{j+1}^* \\
&\quad + \sum_{(i,m,k,l_1,\dots,l_k,b_1,\dots,b_k)\in S_j} \frac{1}{k!} \frac{\partial^k \mathrm{V}_m}{\partial \mathfrak{z}^{b_1} \dots \partial \mathfrak{z}^{b_k}} (z_0; \overline{\mathfrak{z}}_0; \theta_\tau, \theta_\mathrm{r}) \prod_{t=1}^k \mathfrak{z}_{l_t}^{b_t} \partial_{\theta_\mathrm{r}} \mathcal{V}_i^* \\
&\quad - \sum_{(m,k,l_1,\dots,l_k,b_1,\dots,b_k)\in S''_j} \frac{1}{k!} \frac{\partial^k \mathrm{V}_m}{\partial \mathfrak{z}^{b_1} \dots \partial \mathfrak{z}^{b_k}} (z_0; \overline{\mathfrak{z}}_0; \theta_\tau, \theta_\mathrm{r}) \prod_{t=1}^k \mathfrak{z}_{l_t}^{b_t}
\end{aligned} \tag{4.26}
$$

with

$$
\begin{aligned}
S''_j = \big\{&(m,k,l_1,\dots,l_k,b_1,\dots,b_k); 0 \leq m < +\infty,\ k \leq j+1,\ 1 \leq l_t \leq N, \\
&l_1 = \dots = l_k,\ 0 \leq b_t \leq n,\ t \in \{1,\dots,k\},\ m + l_1 + \dots + l_k = j+1\big\}.
\end{aligned}
$$

The above expression can be put in the form

$$
\begin{aligned}
&\mathcal{M}_j(\overline{\mathfrak{z}}_0, \dots, \mathfrak{z}_{j+1}, \overline{\mathcal{V}}_0, \dots, \mathcal{V}_{j+2}) \\
&= \partial_\tau \overline{\mathcal{V}}_j + \partial_{\theta_\tau} \overline{\mathcal{V}}_{j+1} + \mathrm{V}_0 \partial_{\theta_\mathrm{r}} \mathcal{V}_{j+2}^* + K_j(\mathfrak{z}_0, \dots, \mathfrak{z}_{j+1}, \mathcal{V}_1^*, \dots, \mathcal{V}_{j+1}^*),
\end{aligned}
$$

where again the expression K_j may be deduced from (4.26). Looking at (4.16), to obtain (4.9), we have to solve for $j \in \{-1, \dots, N-1\}$ the following cascade of equations

$$
\begin{pmatrix} \mathcal{L}_j(\overline{\mathfrak{z}}_0, \dots, \mathfrak{z}_{j+2}) \\ \mathcal{M}_j(\overline{\mathfrak{z}}_0, \dots, \mathfrak{z}_{j+1}, \overline{\mathcal{V}}_0, \dots, \mathcal{V}_{j+2}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.27}
$$

4.2.2. *Problem-solving strategy.* The purpose here is to explain how we can solve equations $\mathcal{L}_j \equiv 0$ and $\mathcal{M}_j \equiv 0$ to determine the profiles $\mathfrak{Z}_j$ and $\mathcal{V}_j$. This is a survey of the method that will be used (in next Subsection 4.3) to find approximate solutions to (4.3). In fact, this means manipulating the equations and expressions according to certain rules that we would like to emphasize and make explicit now.

The well-posedness of the hierarchy of equations $\mathcal{L}_j \equiv 0$ and $\mathcal{M}_j \equiv 0$ is the gateway to the existence of the profiles ${}^t(\mathfrak{Z}_j, \mathcal{V}_j)$. Because of the decoupling property which is highlighted in the previous paragraph, we can primarily determine $\mathfrak{Z}^a$. Then we can explain how to recover the remaining component $\mathcal{V}^a$.

Let us begin with the base case (initialization case) concerning $\mathcal{L}$. This means to examine the following system (S_0),

$$\begin{gathered} \mathcal{L}_{-1}(\mathfrak{Z}_0, \mathfrak{Z}_1) = 0, \\ \mathcal{L}_0(\mathfrak{Z}_0, \mathfrak{Z}_1, \mathfrak{Z}_2) = 0. \end{gathered} \tag{4.28}$$

We will see that solving this ssytem furnishes $\overline{\mathfrak{Z}}_0 \equiv \langle \overline{\mathfrak{Z}}_0 \rangle$, $\overline{\mathfrak{Z}}_1^\star$ and $\mathfrak{Z}_2^*$. Now, we address the strategy of solving every single equation $\mathcal{L}_j = 0$ for $j \geq 1$ through the following three points:

1. (1) First we take the average of the equation $\mathcal{L}_j = 0$ in both variables $\theta_\tau \in \mathbb{T}$ and $\theta_{\mathrm{r}} \in \mathbb{T}_{\mathrm{r},z_0}$ to obtain a well-posed (linear for $j \geq 1$) Cauchy differential equation on $\langle \overline{\mathfrak{Z}}_j \rangle$, where the profiles $\mathfrak{Z}_0$ up to $\mathfrak{Z}_{j-1}$, $\overline{\mathfrak{Z}}_j^\star$, $\mathfrak{Z}_j^*$ and $\mathfrak{Z}_{j+1}^*$ are viewed as input.
2. (2) Second we substitute the expression of $\partial_\tau \langle \overline{\mathfrak{Z}}_j \rangle$ obtained from 1 and take the mean value of the equation $\mathcal{L}_j = 0$ with respect to $\theta_{\mathrm{r}} \in \mathbb{T}_{\mathrm{r},z_0}$ to obtain the expression of $\overline{\mathfrak{Z}}_{j+1}^\star$.
3. (3) Third we subtract the expressions of $\partial_\tau \langle \overline{\mathfrak{Z}}_j \rangle$ and $\partial_{\theta_\tau} \overline{\mathfrak{Z}}_{j+1}^\star$ in $\mathcal{L}_j = 0$ to obtain $\mathfrak{Z}_{j+2}^*$.

Using such argument for $j \geq 1$ gives rise to $\mathfrak{Z}_j$ by combining the three equations (S_j),

$$\begin{gathered} \mathcal{L}_{j-2}(\mathfrak{Z}_0, \dots, \mathfrak{Z}_j) = 0 \\ \mathcal{L}_{j-1}(\mathfrak{Z}_0, \dots, \mathfrak{Z}_{j+1}) = 0 \\ \mathcal{L}_j(\mathfrak{Z}_0, \dots, \mathfrak{Z}_{j+2}) = 0 \end{gathered} \tag{4.29}$$

through the following steps

Step 1: Apply items (1)–(3) to the equation $\mathcal{L}_{j-2} = 0$; this leads to $\mathfrak{Z}_j^*$.

Step 2: Apply items (1) and (2) to the equation $\mathcal{L}_{j-1} = 0$; this yields $\overline{\mathfrak{Z}}_j^\star$.

Step 3: Apply item (1) to the equation $\mathcal{L}_j = 0$; this allows to identify $\langle \overline{\mathfrak{Z}}_j \rangle$.

Steps (1)–(3) allow us to determine $\mathfrak{Z}_j^*$, $\overline{\mathfrak{Z}}_j^\star$, and $\langle \overline{\mathfrak{Z}}_j \rangle$, and therefore $\mathfrak{Z}_j$. This way, it becomes possible to access all the profiles by solving successively the systems (S_j) for $j \in \{0, \dots, N-1\}$. Looking at (S_j) gives also access to $\overline{\mathfrak{Z}}_{j+1}^\star$ (through $\mathcal{L}_j$), $\mathfrak{Z}_{j+1}^*$ (through $\mathcal{L}_{j-1}$) as well as $\mathfrak{Z}_{j+2}^*$ (through again $\mathcal{L}_j$). The arising claim (for $j \geq 2$) is thus as follows: (H_j),

$$\begin{aligned} &\mathfrak{Z}_k \text{ are known on the domain } [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{\mathrm{r},z_0} \text{ for } 0 \leq k \leq j-2, \\ &\overline{\mathfrak{Z}}_{j-1}^\star,\ \mathfrak{Z}_{j-1}^*,\ \text{and } \mathfrak{Z}_j^* \text{ are known on the domain } [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{\mathrm{r},z_0}. \end{aligned} \tag{4.30}$$

This means that we argue by induction. We have first to find the constraints required to validate the starting point of the induction, which is (H_2). Then, we proceed successively. We prove that (H_{j+1}) holds true given that (H_k) is verified for $k \in \{2,\dots,j\}$. By this way, we can recover $\mathfrak{Z}_{j-1}$, $\overline{\mathfrak{Z}}_j^\star$, $\mathfrak{Z}_j^*$ and $\mathfrak{Z}_{j+1}^*$, and so on. This program is achieved in the next section.

A similar strategy is repeated concerning the profiles $\mathcal{V}_j$. We start by analyzing the basic case. Then, we propagate to the higher one on the basis of the following hypothesis $(\tilde H_j)$,

$$\begin{gathered}\mathcal{V}_k \text{ are known on the domain } [0,\mathcal{T}]\times\mathbb{T}\times\mathbb{T}_{\mathrm{r},z_0} \text{ for } -1\le k\le j-2,\\ \overline{\mathcal{V}}_{j-1}^\star, \mathcal{V}_{j-1}^* \text{ and } \mathcal{V}_j^* \text{ are known on the domain } [0,\mathcal{T}]\times\mathbb{T}\times\mathbb{T}_{\mathrm{r},z_0}.\end{gathered} \tag{4.31}$$

From the profiles $\mathfrak{Z}_j$ and $\mathcal{V}_j$ thus obtained, we can build the approximate solution ${}^t(\mathfrak{Z}^a,\mathcal{V}^a)$ as indicated in (4.11) and (4.12).

4.3. **Construction of approximate solutions.** The strategy of constructing an approximate solution was clarified in the previous section. We are concerned here with the verification of the preceding arguments. Following this line, we achieve the proof of Proposition 4.2. In Subsection 4.3.1 we explain the base case where it becomes possible to access the proof of Theorem 2.10, whereas in Subsection 4.3.2 we check the validity of hypotheses (H_j) and $(\tilde H_j)$ using the induction argument.

4.3.1. *Base case.* We want to prove here that the initialization case is well-posed. To this end, we can limit ourselves to looking at the equations $\mathcal{L}_{-1}\equiv 0$, $\mathcal{L}_0\equiv 0$ and $\mathcal{M}_{-1}\equiv 0$. As already seen, exploiting Assumption 2.6, the restriction $\mathcal{L}_{-1}\equiv 0$ reduces to

$$\mathcal{L}_{-1}(\overline{\mathfrak{Z}}_0,\mathfrak{Z}_1) = \partial_{\theta_\tau}\overline{\mathfrak{Z}}_0 + \mathrm{V}_0(z_0;\overline{\mathfrak{Z}}_0;\theta_\tau)\partial_{\theta_\mathrm{r}}\mathfrak{Z}_1^* = 0. \tag{4.32}$$

This amounts to

$$\overline{\mathfrak{Z}}_0(\tau,\theta_\tau) = \langle\overline{\mathfrak{Z}}_0\rangle(\tau), \quad \mathfrak{Z}_1(\tau,\theta_\tau,\theta_\mathrm{r}) = \overline{\mathfrak{Z}}_1(\tau,\theta_\tau). \tag{4.33}$$

The expression $\mathcal{L}_0$ from (4.20) becomes (in view of Assumption 2.6)

$$\partial_\tau\langle\overline{\mathfrak{Z}}_0\rangle + \partial_{\theta_\tau}\overline{\mathfrak{Z}}_1^\star + \mathrm{V}_0(z_0;\langle\overline{\mathfrak{Z}}_0\rangle;\theta_\tau)\partial_{\theta_\mathrm{r}}\mathfrak{Z}_2^* - \mathrm{A}_1(z_0;\langle\overline{\mathfrak{Z}}_0\rangle;\theta_\tau,\theta_\mathrm{r}) = 0. \tag{4.34}$$

At this stage, we are able to complete the proof of Theorem 2.10.

Proof of Theorem 2.10. Take the average of (4.34) in both variables θ_τ and θ_r to obtain

$$\partial_\tau\langle\overline{\mathfrak{Z}}_0\rangle(z_0;\tau) - \langle\overline{\mathrm{A}}_1\rangle(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(z_0;\tau)) = 0, \quad \langle\overline{\mathfrak{Z}}_0\rangle(z_0;0) = \Xi_0^{-1}(z_0;z_0;0) = z_0. \tag{4.35}$$

This is exactly (2.23). By restricting $\mathcal{T}$ if necessary, the Cauchy-Lipschitz theorem provides with the existence on the interval $[0,\mathcal{T}]$ of a local solution $\langle\overline{\mathfrak{Z}}_0\rangle(z_0;\cdot)$ to the above non-linear differential equation. The formula (2.24) comes from (3.34) where $\Xi_0\equiv\Xi_{\mathrm{mf}}$ and $\Xi_1\equiv\Xi_1^*$ is replaced by (3.37). On the other hand, by construction, we have (3.18) and (3.6) which lead to (2.22) after the use of the first line inside (4.17). $\square$

Formula (4.35) can be viewed as a second modulation equation. It plays an important role concerning the long time gyrokinetic equation. There are other outcomes which are issued from (4.34) and which are given by the following lemma.

Lemma 4.5 (Determination of $\overline{\mathfrak{Z}}_1^\star$ and $\mathfrak{Z}_2^*$)**.** *Under Assumption 2.6, the profiles $\overline{\mathfrak{Z}}_1^\star$ and $\mathfrak{Z}_2^*$ under the constraint $\mathcal{L}_0\equiv 0$ are given by*

$$\overline{\mathfrak{Z}}_1^\star(\tau,\theta_\tau) = (\partial_{\theta_\tau}^{-1}\overline{\mathrm{A}}_1^\star)(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau), \tag{4.36}$$

$$\mathfrak{Z}_2^*(\tau,\theta_\tau,\theta_{\mathrm{r}}) = \mathrm{V}_0(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau)^{-1}\partial_{\theta_{\mathrm{r}}}^{-1}\mathrm{A}_1^*(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau,\theta_{\mathrm{r}}). \tag{4.37}$$

Proof. We take into account Assumption 2.6 in the following analysis. First, we average the equation (4.34) with respect to θ_{r} to obtain

$$\partial_\tau\langle\overline{\mathfrak{Z}}_0\rangle(\tau) + \partial_{\theta_\tau}\overline{\mathfrak{Z}}_1^\star(\tau,\theta_\tau) - \overline{\mathrm{A}}_1(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau) = 0. \tag{4.38}$$

We also plug the ansatz (4.35) into (4.38) to obtain

$$\begin{aligned}\partial_{\theta_\tau}\overline{\mathfrak{Z}}_1^\star(\tau,\theta_\tau) &= \overline{\mathrm{A}}_1(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau) - \langle\overline{\mathrm{A}}_1\rangle(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau))\\ &= \overline{\mathrm{A}}_1^\star(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau).\end{aligned} \tag{4.39}$$

Since $\overline{\mathrm{A}}_1^\star(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\cdot) \in L^1_*(\mathbb{T})$, we can invert the operator ∂_{θ_τ} as indicated in (3.24). Thus, we obtain (4.36). Similarly to obtain $\mathfrak{Z}_2^*$, substitute (4.38) in (4.34). This gives

$$\mathrm{V}_0(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau)\partial_{\theta_{\mathrm{r}}}\mathfrak{Z}_2^*(\tau,\theta_\tau,\theta_{\mathrm{r}}) = \mathrm{A}_1^*(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau,\theta_{\mathrm{r}}). \tag{4.40}$$

Again, as $\mathrm{A}_1^*(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau,.) \in L^1_*(\mathbb{T}_{\mathrm{r},z_0})$, we can invert the operator $\partial_{\theta_{\mathrm{r}}}$ and the positive source term $\mathrm{V}_0(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau)$ to recover (4.37). □

It is obvious that to solve $\mathcal{L}_0 \equiv 0$, it is enough to adjust $\langle\overline{\mathfrak{Z}}_0\rangle$, $\overline{\mathfrak{Z}}_1^\star$ and $\mathfrak{Z}_2^*$ as indicated in (4.35), (4.36), and (4.37). By the way, this furnishes (H2).

Similarly, we can determine basic parts of the profile $\mathcal{V}^a$ through the following lemma.

Lemma 4.6 (Determination of $\langle\overline{\mathcal{V}}_{-1}\rangle$ and $\overline{\mathcal{V}}_0^\star$)**.** *Under Assumption 2.6, the expressions of the profiles $\langle\overline{\mathcal{V}}_{-1}\rangle$ and $\overline{\mathcal{V}}_0^\star$ under the constraint $\mathcal{M}_{-1} \equiv 0$ are given by*

$$\langle\overline{\mathcal{V}}_{-1}\rangle(z_0;\tau) = \int_0^\tau \langle\mathrm{V}_0\rangle(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(r))dr, \tag{4.41}$$

$$\overline{\mathcal{V}}_0^\star(z_0;\tau,\theta_\tau) = (\partial_{\theta_\tau}^{-1}\overline{\mathrm{V}}_0^\star)(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau). \tag{4.42}$$

Moreover, we have $\mathcal{V}_1^* \equiv 0$. This means that $\mathcal{V}_1 \equiv \overline{\mathcal{V}}_1$.

Proof. With the aid of expression (4.25), the restriction $\mathcal{M}_{-1} \equiv 0$ reduces to

$$\partial_\tau\langle\overline{\mathcal{V}}_{-1}\rangle - \mathrm{V}_0(z_0;\langle\overline{\mathfrak{Z}}_0\rangle;\theta_\tau) + \partial_{\theta_\tau}\overline{\mathcal{V}}_0^\star + \mathrm{V}_0(z_0;\langle\overline{\mathfrak{Z}}_0\rangle;\theta_\tau)\partial_{\theta_{\mathrm{r}}}\mathcal{V}_1^* = 0. \tag{4.43}$$

Take the average in θ_{r} to obtain rid of the last term, and then in θ_τ to suppress the penultimate term. Complete with the initial data coming from (4.4). We find that

$$\partial_\tau\langle\overline{\mathcal{V}}_{-1}\rangle(\tau) - \langle\mathrm{V}_0\rangle(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau)) = 0, \quad \langle\overline{\mathcal{V}}_{-1}\rangle(0) = 0 \tag{4.44}$$

which implies (4.41). Now take the mean value with respect to θ_{r} to obtain rid of the term $\mathrm{V}_0\partial_{\theta_{\mathrm{r}}}\mathcal{V}_1^*$ in (4.43). Then exploit (4.44) and subtract the result thus obtained to deduce

$$\begin{aligned}\partial_{\theta_\tau}\overline{\mathcal{V}}_0^\star(\tau,\theta_\tau) &= \mathrm{V}_0(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau) - \langle\mathrm{V}_0\rangle(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau))\\ &= \overline{\mathrm{V}}_0^\star(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau).\end{aligned} \tag{4.45}$$

Since $\overline{\mathrm{V}}_0^\star(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\cdot) \in L^1_*(\mathbb{T})$, we can invert the operator ∂_{θ_τ} in (4.45) to obtain (4.42). Finally, replace in (4.43) the expressions $\partial_\tau\langle\overline{\mathcal{V}}_{-1}\rangle$ and $\partial_{\theta_\tau}\overline{\mathcal{V}}_0^\star$ as indicated in (4.44) and (4.45), this gives

$$\mathrm{V}_0(z_0;\langle\overline{\mathfrak{Z}}_0\rangle(\tau);\theta_\tau)\partial_{\theta_{\mathrm{r}}}\mathcal{V}_1^*(\tau,\theta_\tau,\theta_{\mathrm{r}}) = 0.$$

Since V_0 is positive function, we must have $\mathcal{V}_1^* \equiv 0$. □

Recall that $\mathcal{V}_0^* \equiv 0$. Thus, from Lemma 4.6, we have $(\tilde{H}_1)$. Let us clarify here what happens at the initial time $\tau = 0$. From (4.7), we have to impose

$$\mathfrak{z}_j(0,0,\theta_{\rm r}) = \langle \overline{\mathfrak{z}}_j \rangle(0) + \overline{\mathfrak{z}}_j^\star(0,0) + \mathfrak{z}_j^*(0,0,\theta_{\rm r}) = \Xi_j^{-1}(z_0; z_0, 0, \theta_{\rm r}).$$

This means the cauchy problem (4.53) is accompanied with the following initial data

$$\langle \overline{\mathfrak{z}}_j \rangle(0) = \Xi_j^{-1}(z_0; z_0, 0, \theta_{\rm r}) - \overline{\mathfrak{z}}_j^\star(0,0) - \mathfrak{z}_j^*(0,0,\theta_{\rm r}). \tag{4.46}$$

Similarly for $\mathcal{V}$, we have

$$\mathcal{V}(0,0,\theta_{\rm r}) = \nu_0 = \langle \overline{\mathcal{V}} \rangle(0) + \overline{\mathcal{V}}^\star(0,0) + \mathcal{V}^*(0,0,\theta_{\rm r}). \tag{4.47}$$

4.3.2. *Induction step.* We want here to verify the validity of the hypotheses presented in Paragraph 4.2.2 concerning the induction argument. We have already checked from Lemma 4.5 and the equation (4.35) the validity of (H_2). Assume that (H_k) is valid for all $k \in \{2, \dots, j\}$. We have to prove that (H_{j+1}):

$$\begin{gathered}
\mathfrak{z}_k \text{ are known on the domain } [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{{\rm r}, z_0} \text{ for } 0 \le k \le j-1, \\
\overline{\mathfrak{z}}_j^\star,\ \mathfrak{z}_j^* \text{ and } \mathfrak{z}_{j+1}^* \text{ are known on the domain } [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{{\rm r}, z_0},
\end{gathered} \tag{4.48}$$

holds. From the validity of (H_j), the profiles $\mathfrak{z}_k$ for $0 \le k \le j-2$, $\overline{\mathfrak{z}}_{j-1}^\star$, $\mathfrak{z}_{j-1}^*$ and $\mathfrak{z}_j^*$ have been identified. We still need to determine $\mathfrak{z}_{j-1}$, $\overline{\mathfrak{z}}_j^\star$ and $\mathfrak{z}_{j+1}^*$. To this end, consider the expression $\mathcal{L}_{j-1}$ as defined in (4.23), that is

$$\begin{aligned}
&\partial_\tau \overline{\mathfrak{z}}_{j-1} + \partial_{\theta_\tau} \overline{\mathfrak{z}}_j + {\rm V}_0(z_0; \langle \overline{\mathfrak{z}}_0 \rangle; \theta_\tau) \partial_{\theta_{\rm r}} \mathfrak{z}_{j+1}^* \\
&- (\mathfrak{z}_{j-1} \cdot \nabla_{\mathfrak{z}})\, {\rm A}_1(z_0; \langle \overline{\mathfrak{z}}_0 \rangle; \theta_\tau, \theta_{\rm r}) \\
&+ (\mathfrak{z}_{j-1} \cdot \nabla_{\mathfrak{z}})\, {\rm V}_0(z_0; \langle \overline{\mathfrak{z}}_0 \rangle; \theta_\tau) \partial_{\theta_{\rm r}} \mathfrak{z}_2^* \\
&+ G_{j-1}(\overline{\mathfrak{z}}_0, \mathfrak{z}_1, \dots, \mathfrak{z}_{j-2}, \mathfrak{z}_{j-1}^*, \mathfrak{z}_j^*) = 0.
\end{aligned} \tag{4.49}$$

Substituting $\overline{\mathfrak{z}}_{j-1} = \langle \overline{\mathfrak{z}}_{j-1} \rangle + \overline{\mathfrak{z}}_{j-1}^\star$ at the level of (4.49), we obtain

$$\begin{aligned}
&\partial_\tau \langle \overline{\mathfrak{z}}_{j-1} \rangle + \partial_\tau \overline{\mathfrak{z}}_{j-1}^\star + \partial_{\theta_\tau} \overline{\mathfrak{z}}_j^\star - \big((\langle \overline{\mathfrak{z}}_{j-1} \rangle + \overline{\mathfrak{z}}_{j-1}^\star + \mathfrak{z}_{j-1}^*) \cdot \nabla_{\mathfrak{z}} \big) {\rm A}_1(z_0; \langle \overline{\mathfrak{z}}_0 \rangle; \theta_\tau, \theta_{\rm r}) \\
&+ {\rm V}_0(z_0; \langle \overline{\mathfrak{z}}_0 \rangle; \theta_\tau) \partial_{\theta_{\rm r}} \mathfrak{z}_{j+1}^* + \big((\langle \overline{\mathfrak{z}}_{j-1} \rangle + \overline{\mathfrak{z}}_{j-1}^\star + \mathfrak{z}_{j-1}^*) \cdot \nabla_{\mathfrak{z}} \big) {\rm V}_0(z_0; \langle \overline{\mathfrak{z}}_0 \rangle; \theta_\tau) \partial_{\theta_{\rm r}} \mathfrak{z}_2^* \\
&+ G_{j-1}(\overline{\mathfrak{z}}_0, \mathfrak{z}_1, \dots, \mathfrak{z}_{j-2}, \mathfrak{z}_{j-1}^*, \mathfrak{z}_j^*) = 0.
\end{aligned} \tag{4.50}$$

We obtain a linearized version of (4.34), which is

$$\begin{aligned}
&\partial_\tau \langle \overline{\mathfrak{z}}_{j-1} \rangle + \partial_{\theta_\tau} \overline{\mathfrak{z}}_j^\star + (\langle \overline{\mathfrak{z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}}) {\rm V}_0(z_0; \langle \overline{\mathfrak{z}}_0 \rangle; \theta_\tau) \partial_{\theta_{\rm r}} \mathfrak{z}_2^* \\
&+ {\rm V}_0(z_0; \langle \overline{\mathfrak{z}}_0 \rangle; \theta_\tau) \partial_{\theta_{\rm r}} \mathfrak{z}_{j+1}^* - (\langle \overline{\mathfrak{z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}}) {\rm A}_1(z_0; \langle \overline{\mathfrak{z}}_0 \rangle; \theta_\tau, \theta_{\rm r}) = \mathcal{G}_{j-1}
\end{aligned} \tag{4.51}$$

where

$$\begin{aligned}
&\mathcal{G}_{j-1}(\overline{\mathfrak{z}}_0, \mathfrak{z}_1, \dots, \mathfrak{z}_{j-2}, \overline{\mathfrak{z}}_{j-1}^\star, \mathfrak{z}_{j-1}^*, \mathfrak{z}_j^*) \\
&= -\partial_\tau \overline{\mathfrak{z}}_{j-1}^\star - G_{j-1}(\overline{\mathfrak{z}}_0, \mathfrak{z}_1, \dots, \mathfrak{z}_{j-2}, \mathfrak{z}_{j-1}^*, \mathfrak{z}_j^*) \\
&\quad + \big((\overline{\mathfrak{z}}_{j-1}^\star + \mathfrak{z}_{j-1}^*) \cdot \nabla_{\mathfrak{z}} \big) {\rm A}_1(z_0; \langle \overline{\mathfrak{z}}_0 \rangle; \theta_\tau, \theta_{\rm r}) \\
&\quad - \big((\overline{\mathfrak{z}}_{j-1}^\star + \mathfrak{z}_{j-1}^*) \cdot \nabla_{\mathfrak{z}} \big) ({\rm V}_0(z_0; \langle \overline{\mathfrak{z}}_0 \rangle; \theta_\tau)) \partial_{\theta_{\rm r}} \mathfrak{z}_2^*.
\end{aligned} \tag{4.52}$$

It is obvious that the expression $\mathcal{G}_{j-1} \equiv \mathcal{G}_{j-1}(\overline{\mathfrak{z}}_0, \mathfrak{z}_1, \dots, \mathfrak{z}_{j-2}, \overline{\mathfrak{z}}_{j-1}^\star, \mathfrak{z}_{j-1}^*, \mathfrak{z}_j^*)$ is known function, since by induction, the profiles $\mathfrak{z}_{j-1}^*$, $\mathfrak{z}_j^*$, $\overline{\mathfrak{z}}_{j-1}^\star$ and $\mathfrak{z}_k$ where

$0 \le k \le j-2$ are known functions on the domain $[0,\mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{\mathrm{r},z_0}$. With the aid of Assumption 2.6, average (4.51) in both variables θ_τ and θ_{r} to exhibit

$$\partial_\tau \langle \overline{\mathfrak{Z}}_{j-1} \rangle - (\langle \overline{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}}) \langle \overline{\mathrm{A}}_1(\langle \overline{\mathfrak{Z}}_0 \rangle) \rangle = \langle \overline{\mathcal{G}}_{j-1} \rangle. \tag{4.53}$$

This may be completed by the initial data, namely $\langle \overline{\mathfrak{Z}}_{0(j-1)} \rangle$ issued from (4.46) where, because of (H_j), the expressions $\overline{\mathfrak{Z}}^\star_{j-1}(0,0)$ and $\mathfrak{Z}^*_{j-1}(0,0,\theta_{\mathrm{r}})$ are known. The equation (4.53) is *linear*. It has therefore a solution $\langle \overline{\mathfrak{Z}}_{j-1} \rangle$ on the whole interval $[0,\mathcal{T}]$. By validity of (H_j), $\overline{\mathfrak{Z}}^\star_{j-1}$ and $\mathfrak{Z}^*_{j-1}$ are known. And thereby, in view of (2.8), this implies that the whole of $\mathfrak{Z}_{j-1}$ has been identified. Substitute $\partial_\tau \langle \overline{\mathfrak{Z}}_{j-1} \rangle$ as indicated in (4.53) inside (4.51). Then average with respect to θ_{r} to obtain

$$\partial_{\theta_\tau} \overline{\mathfrak{Z}}^\star_j = \overline{\mathcal{G}}^\star_{j-1} + (\langle \overline{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}})\ \overline{\mathrm{A}}^\star_1(\langle \overline{\mathfrak{Z}}_0 \rangle), \tag{4.54}$$

or equivalently

$$\overline{\mathfrak{Z}}^\star_j = \partial^{-1}_{\theta_\tau} \overline{\mathcal{G}}^\star_{j-1} + (\langle \overline{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}})\ \overline{\mathrm{A}}^\star_1(\langle \overline{\mathfrak{Z}}_0 \rangle). \tag{4.55}$$

Again substitute in (4.51) the expressions $\partial_\tau \langle \overline{\mathfrak{Z}}_{j-1} \rangle$ and $\partial_{\theta_\tau} \overline{\mathfrak{Z}}^\star_j$ as indicated in (4.53) and (4.54). This gives

$$\begin{aligned} \mathfrak{Z}^*_{j+1} = (\mathrm{V}_0)^{-1} \partial^{-1}_{\theta_{\mathrm{r}}} \Big[& \mathcal{G}^*_{j-1} + (\langle \overline{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}}) \mathrm{A}^*_1(\langle \overline{\mathfrak{Z}}_0 \rangle) \\ & - (\langle \overline{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}}) \mathrm{V}_0(\langle \overline{\mathfrak{Z}}_0 \rangle) \partial_{\theta_{\mathrm{r}}} \mathfrak{Z}^*_2 \Big]. \end{aligned} \tag{4.56}$$

This discussion determines (H_{j+1}).

The same strategy applies concerning the construction of the profiles $\mathcal{V}_j$, for $j \in \{-1,\dots,N+1\}$ under Assumption 2.6. Lemma 4.6 gives rise to $(\tilde{H}_1)$. Assume now that the hypotheses $(\tilde{H}_k)$ are valid for all $k \in \{1,..,j-1\}$. We have to prove that $(\tilde{H}_j)$ holds true. To this end, pick the expression $\mathcal{M}_{j-2}$ from (4.26) and solve the equation $\mathcal{M}_{j-2} \equiv 0$. We find

$$\partial_\tau \langle \overline{\mathcal{V}}_{j-2} \rangle + \partial_{\theta_\tau} \overline{\mathcal{V}}^\star_{j-1} + \mathrm{V}_0 \partial_{\theta_{\mathrm{r}}} \mathcal{V}^*_j = \mathcal{K}_{j-2} \tag{4.57}$$

where $\mathcal{K}_{j-2}$ is known function by induction. Average (4.57) with respect to θ_τ and θ_{r}, we obtain

$$\partial_\tau \langle \overline{\mathcal{V}}_{j-2} \rangle = \langle \overline{\mathcal{K}}_{j-2} \rangle. \tag{4.58}$$

The equation (4.58) together with the initial data coming from the second equation of (4.4) allows to determine $\langle \overline{\mathcal{V}}_{j-2} \rangle$ by time integration on the interval $[0,\mathcal{T}]$. Substitute (4.58) in the equation (4.57). Then, average with respect to θ_{r} to obtain

$$\overline{\mathcal{V}}^\star_{j-1} = \partial^{-1}_{\theta_\tau} \overline{\mathcal{K}}^\star_{j-2}. \tag{4.59}$$

Again, use (4.58) and (4.59) in (4.57), we obtain

$$\mathcal{V}^*_j = (\mathrm{V}_0)^{-1} \partial^{-1}_{\theta_{\mathrm{r}}} \mathcal{K}^*_{j-2}.$$

Briefly, we have just verified $(\tilde{H}_j)$.

Proof of Proposition 4.2. We select some $N \in \mathbb{N}^*$, and we build the profiles ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ as it is indicated in (4.11) and (4.12), with the aid of the profiles ${}^t(\mathfrak{Z}_j, \mathcal{V}_j)$ which have been determined in the previous Paragraphs 4.3.1 and 4.3.2. It is easy then to see that ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ is an approximate solution to the profile equation (4.3). Indeed, by construction, we have reset to zero all the terms composing the sum in (4.16). $\square$

5. Stability estimates.

Our purpose here is to show that the formal solutions to (4.3)-(4.4) can be used to approximate through (4.15) the exact solutions of the redressed system (3.3). Note that we do not compare ${}^t(\mathfrak{Z}^a,\mathcal{V}^a)$ and the solution ${}^t(\mathfrak{Z},\mathcal{V})$ to the profile equation (4.3), which would be relatively easy. Instead, we want to associate the expression ${}^t(\mathfrak{z}^a,\mathfrak{v}^a)$ of (4.15) and the solution ${}^t(\mathfrak{z},\mathfrak{v})$ of (3.3). To this end, we can always consider the expression ${}^t(r^{\mathfrak{z}},r^{\mathfrak{v}})$ defined by the weighted difference

$$\begin{pmatrix} r^{\mathfrak{z}} \\ r^{\mathfrak{v}} \end{pmatrix}(\tau) := \varepsilon^{-N}\begin{pmatrix} \mathfrak{z}-\mathfrak{z}^a \\ \mathfrak{v}-\mathfrak{v}^a \end{pmatrix}(\tau) \tag{5.1}$$

or equivalently by the relation

$$\begin{pmatrix} \mathfrak{z} \\ \mathfrak{v} \end{pmatrix}(\tau) = \varepsilon^{-1}\begin{pmatrix} 0 \\ \langle \overline{\mathcal{V}}_{-1}\rangle \end{pmatrix}(\tau) + \sum_{j=0}^{N}\varepsilon^j\begin{pmatrix} \mathfrak{Z}_j \\ \mathcal{V}_j \end{pmatrix}\Big(\tau,\frac{\tau}{\varepsilon},\frac{\mathfrak{v}(\tau)}{\varepsilon}\Big) + \varepsilon^N\begin{pmatrix} r^{\mathfrak{z}} \\ r^{\mathfrak{v}} \end{pmatrix}(\tau). \tag{5.2}$$

In Subsection 5.1, we prove that ${}^t(\varepsilon^N r^{\mathfrak{z}},\varepsilon^N r^{\mathfrak{v}})$ can indeed be viewed as a remainder, as suggested by (5.2). In Subsection 5.2, we highlight the role of the *frozen phase* $\mathfrak{v}^{\mathrm{f}}$ which is given by

$$\mathfrak{v}^{\mathrm{f}}(\tau) \equiv \mathfrak{v}^{\mathrm{f}}(\varepsilon,z_0,\mathfrak{v}_0;\tau) := \frac{1}{\varepsilon}\,\langle\overline{\mathcal{V}}_{-1}\rangle(z_0,\mathfrak{v}_0;\tau) + \overline{\mathcal{V}}_0\big(z_0,\mathfrak{v}_0;\tau,\frac{\tau}{\varepsilon}\big). \tag{5.3}$$

The frozen phase is a known quantity because it is built from $\langle\overline{\mathcal{V}}_{-1}\rangle(\tau)$ and $\overline{\mathcal{V}}_0(\tau,\theta_\tau)$ which have been already determined. It is constructed by collecting the two first terms of the expansion (5.2) of $\mathfrak{v}$. Note that the definition (5.3) coincides with (2.19) since, at the end, we will find that $\mathcal{V}_{-1}\equiv\mathscr{V}_{-1}$ and $\mathcal{V}_0\equiv\mathscr{V}_0$. By construction, we have

$$\mathfrak{v}(\tau) = \mathfrak{v}^{\mathrm{f}}(\tau) + \sum_{j=1}^{N}\varepsilon^j\mathcal{V}_j\Big(\tau,\frac{\tau}{\varepsilon},\frac{\mathfrak{v}(\tau)}{\varepsilon}\Big) + \varepsilon^N\,r^{\mathfrak{v}}(\tau). \tag{5.4}$$

For $j\geq 2$, in general, we have $\mathcal{V}_j^*\not\equiv 0$ while θ_{r} must be replaced by $\mathfrak{v}/\varepsilon$. This means that the access to $\mathfrak{v}$ is necessary to construct the right hand side of (5.4), or that the knowledge of $\mathfrak{v}$ is required to obtain a precision of size ε or more. Now, we would like to derive a self-contained representation of $\mathfrak{z}$ and $\mathfrak{v}$ (which does not call for $\mathfrak{v}$). In Subsection 5.2, we explain how $\mathfrak{v}^{\mathrm{f}}/\varepsilon$ can become a substitute for $\mathfrak{v}/\varepsilon$ in the right part of (5.2). By this way, we end up with a justified WKB expansion of the redressed field ${}^t(\mathfrak{z},\mathfrak{v})$ whose all components can be determined by formal computations. Finally, there remains to interpret this result in terms of the original field z. This is done in Subsection 5.3 where the proof of Theorem 2.7 is complete. $\square$

5.1. **Justification of the formal computations.** The aim here is to compare ${}^t(\mathfrak{z},\mathfrak{v})$ and ${}^t(\mathfrak{z}^a,\mathfrak{v}^a)$. This can be done by estimating the size of the weighted difference ${}^t(r^{\mathfrak{z}},r^{\mathfrak{v}})$. To this end, the strategy is to first exhibit a non-linear differential equation satisfied by ${}^t(r^{\mathfrak{z}},r^{\mathfrak{v}})$.

Lemma 5.1. *For all $N\in\mathbb{N}$, the expression ${}^t(r^{\mathfrak{z}},r^{\mathfrak{v}})$ issued from* (5.1) *is subject to*

$$\partial_\tau\begin{pmatrix} r^{\mathfrak{z}} \\ r^{\mathfrak{v}} \end{pmatrix}(\tau) = \begin{pmatrix} \mathcal{R}^{\mathfrak{z}} \\ \mathcal{R}^{\mathfrak{v}} \end{pmatrix}\Big(\varepsilon,z_0,\mathfrak{v}_0,r^{\mathfrak{z}};\tau,\frac{\tau}{\varepsilon},\frac{\mathfrak{v}}{\varepsilon}\Big),\qquad \begin{pmatrix} r^{\mathfrak{z}} \\ r^{\mathfrak{v}} \end{pmatrix}(0) = O(1) \tag{5.5}$$

where

$$\begin{aligned}
&\begin{pmatrix} \mathcal{R}^{\mathfrak{z}} \\ \mathcal{R}^{\nu} \end{pmatrix} (\varepsilon, z_0, \nu_0, r; \tau, \theta_\tau, \theta_{\rm r}) \\
&:= \varepsilon^{-N-2} \left[\mathrm{V}(\varepsilon, z_0; \mathfrak{Z}^a; \theta_\tau, \theta_{\rm r}) - \mathrm{V}(\varepsilon, z_0; \mathfrak{Z}^a + \varepsilon^N r; \theta_\tau, \theta_{\rm r}) \right] \partial_{\theta_{\rm r}} \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \\
&\quad + \varepsilon^{-N-1} \Big[\begin{pmatrix} \mathrm{A} \\ \mathrm{V} \end{pmatrix} (\varepsilon, z_0; \mathfrak{Z}^a + \varepsilon^N r; \theta_\tau, \theta_{\rm r}) - \begin{pmatrix} \mathrm{A} \\ \mathrm{V} \end{pmatrix} (\varepsilon, z_0; \mathfrak{Z}^a; \theta_\tau, \theta_{\rm r}) \Big] \\
&\quad - \begin{pmatrix} \mathcal{R}^{\mathfrak{Z}}_N \\ \mathcal{R}^{\mathcal{V}}_N \end{pmatrix} (\varepsilon, z_0, \nu_0; \tau, \theta_\tau, \theta_{\rm r}).
\end{aligned} \tag{5.6}$$

Proof. Taking into account (5.2), equation (3.3) can be reformulated as

$$\partial_\tau \Big[\begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \Big(\tau, \frac{\tau}{\varepsilon}, \frac{\nu(\tau)}{\varepsilon} \Big) + \varepsilon^N \begin{pmatrix} r^{\mathfrak{z}} \\ r^{\nu} \end{pmatrix} (\tau) \Big] = \frac{1}{\varepsilon} \begin{pmatrix} \mathrm{A} \\ \mathrm{V} \end{pmatrix} \Big(\varepsilon, z_0; \mathfrak{z}; \frac{\tau}{\varepsilon}, \frac{\nu(\tau)}{\varepsilon} \Big).$$

This is the same as

$$\begin{aligned}
&\partial_\tau \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \Big(\tau, \frac{\tau}{\varepsilon}, \frac{\nu(\tau)}{\varepsilon} \Big) + \varepsilon^{-1} \partial_{\theta_\tau} \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \Big(\tau, \frac{\tau}{\varepsilon}, \frac{\nu(\tau)}{\varepsilon} \Big) \\
&+ \varepsilon^{-2} \mathrm{V}\Big(\varepsilon, z_0; \mathfrak{z}; \frac{\nu(\tau)}{\varepsilon} \Big) \partial_{\theta_{\rm r}} \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \Big(\tau, \frac{\tau}{\varepsilon}, \frac{\nu(\tau)}{\varepsilon} \Big) + \varepsilon^N \partial_\tau \begin{pmatrix} r^{\mathfrak{z}} \\ r^{\nu} \end{pmatrix} (\tau) \\
&= \frac{1}{\varepsilon} \begin{pmatrix} \mathrm{A} \\ \mathrm{V} \end{pmatrix} \Big(\varepsilon, z_0; \mathfrak{z}; \frac{\tau}{\varepsilon}, \frac{\nu(\tau)}{\varepsilon} \Big)
\end{aligned} \tag{5.7}$$

or equivalently

$$\begin{aligned}
&\varepsilon^N \partial_\tau \begin{pmatrix} r^{\mathfrak{z}}_\varepsilon \\ r^{\nu}_\varepsilon \end{pmatrix} (\tau) \\
&= -Op\,(\mathfrak{Z}^a; \partial) \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \Big(\tau, \frac{\tau}{\varepsilon}, \frac{\nu}{\varepsilon} \Big) + \frac{1}{\varepsilon} \begin{pmatrix} \mathrm{A} \\ \mathrm{V} \end{pmatrix} \Big(\varepsilon, z_0; \mathfrak{Z}^a; \frac{\tau}{\varepsilon}, \frac{\nu}{\varepsilon} \Big) \\
&\quad + \varepsilon^{-1} \Big[\begin{pmatrix} \mathrm{A} \\ \mathrm{V} \end{pmatrix} \Big(\varepsilon, z_0; \mathfrak{Z}^a + \varepsilon^N r^{\mathfrak{z}}_\varepsilon; \frac{\tau}{\varepsilon}, \frac{\nu}{\varepsilon} \Big) - \begin{pmatrix} \mathrm{A} \\ \mathrm{V} \end{pmatrix} \Big(\varepsilon, z_0; \mathfrak{Z}^a; \frac{\tau}{\varepsilon}, \frac{\nu}{\varepsilon} \Big) \Big] \\
&\quad + \varepsilon^{-2} \Big[\mathrm{V}\Big(\varepsilon, z_0; \mathfrak{Z}^a; \frac{\tau}{\varepsilon}, \frac{\nu}{\varepsilon} \Big) - \mathrm{V}\Big(\varepsilon, z_0; \mathfrak{Z}^a + \varepsilon^N r^{\mathfrak{z}}_\varepsilon; \frac{\nu}{\varepsilon} \Big) \Big] \partial_{\theta_{\rm r}} \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \Big(\tau, \frac{\tau}{\varepsilon}, \frac{\nu(\tau)}{\varepsilon} \Big).
\end{aligned}$$

Using (4.9) and then dividing by ε^N we see (5.6) appear. On the other hand, at the time $t = 0$, all has been done in the WKB construction to ensure that the initial data for ${}^t(\mathfrak{z}, \nu)$ in (3.3) matches up to the order ε^N with ${}^t(\mathfrak{z}^a, \nu^a)$. This is why we have the right part of (5.5). □

Proposition 5.2 (Weighted estimates on the difference ${}^t(r^{\mathfrak{z}}, r^{\nu})$)**.** *In the sense of the sup norm, uniformly with respect to* $\varepsilon \in]0, \varepsilon_0]$ *and* $\tau \in [0, \mathcal{T}]$*, we can assert that* ${}^t(r^{\mathfrak{z}}, \varepsilon r^{\nu}) = \mathcal{O}(1)$.

Coming back to (5.2), this means that the contribution ${}^t(\varepsilon^N r^{\mathfrak{z}}, \varepsilon^N r^{\nu})$ can indeed be viewed as a remainder, namely of size ε^{N-1}.

Proof. It suffices to show that we can find two constants $C \in \mathbb{R}^*_+$ and $\tilde{C} \in \mathbb{R}^*_+$ such that

$$\sup_{\varepsilon \in]0, \varepsilon_0]} \sup_{\tau \in [0, \mathcal{T}]} |r^{\mathfrak{z}}(\tau)| \le C e^{C \mathcal{T}}, \tag{5.8}$$

$$\sup_{\varepsilon\in\,]0,\varepsilon_0]}\ \sup_{\tau\in[0,\mathcal{T}]} |\varepsilon r^{\mathrm{v}}(\tau)| \le \tilde{C}(e^{C\mathcal{T}} + \mathcal{T}). \tag{5.9}$$

We start with information which is helpful to estimate the size of (5.6). We have:

- $\partial_{\theta_{\mathrm{r}}} \mathfrak{Z}^a = \mathcal{O}(\varepsilon^2)$ since $\mathfrak{Z}_0^* = 0$ and $\mathfrak{Z}_1^* = 0$ according respectively to (4.14) and (4.33);
- The function $\mathrm{V}(\cdot)$ is locally lipschitz with respect to $\mathfrak{z}$. Thus, by the mean-value theorem and because $\mathfrak{Z}^a$, θ_τ and θ_{r} stay in compact sets, we can find some $L \in \mathbb{R}_+^*$ such that

$$\sup_{\theta_\tau\in\mathbb{T}}\ \sup_{\theta_{\mathrm{r}}\in\mathbb{T}_{\mathrm{r},z_0}} |\mathrm{V}(\varepsilon, z_0; \mathfrak{Z}^a; \theta_\tau, \theta_{\mathrm{r}}) - \mathrm{V}(\varepsilon, z_0; \mathfrak{Z}^a + \varepsilon^N r; \theta_\tau, \theta_{\mathrm{r}})| \le L\ \varepsilon^N\, |r|;$$

- For the same reasons and due to (3.4), we can find some $L' \in \mathbb{R}_+^*$ such that

$$\begin{aligned}&\sup_{\theta_\tau\in\mathbb{T}}\ \sup_{\theta_{\mathrm{r}}\in\mathbb{T}_{\mathrm{r},z_0}} |\mathrm{A}(\varepsilon, z_0; \mathfrak{Z}^a; \theta_\tau, \theta_{\mathrm{r}}) - \mathrm{A}(\varepsilon, z_0; \mathfrak{Z}^a + \varepsilon^N r; \theta_\tau, \theta_{\mathrm{r}})| \\ &\le L'\ \varepsilon^{N+1}\, |r|;\end{aligned} \tag{5.10}$$

- Recall that from (4.10) we have

$$\sup_{\varepsilon\in\,]0,\varepsilon_0]}\ \sup_{\tau\in[0,\mathcal{T}]}\ \sup_{\theta_\tau\in\mathbb{T}}\ \sup_{\theta_{\mathrm{r}}\in\mathbb{T}_{\mathrm{r},z_0}} |\mathcal{R}_N^{\mathfrak{Z}}(\varepsilon; \tau, \theta_\tau, \theta_{\mathrm{r}})| < +\infty.$$

When computing the component $\mathcal{R}^{\mathfrak{z}}$, observe in the second line of (5.6) the compensation between the loss ε^{-2} and the gain ε^2 brought by $\partial_{\theta_{\mathrm{r}}} \mathfrak{Z}^a$. The same applies in the third line of (5.6) between the loss ε^{-N-1} and the gain ε^{N+1} given by (5.10). It follows that

$$\begin{aligned}\sup_{\varepsilon\in\,]0,\varepsilon_0]}\ \sup_{\tau\in[0,\mathcal{T}]} |\partial_\tau r^{\mathfrak{z}}(\tau)| &= \sup_{\varepsilon\in\,]0,\varepsilon_0]}\ \sup_{\tau\in[0,\mathcal{T}]} \Big|\mathcal{R}^{\mathfrak{z}}\Big(\varepsilon, z_0, \mathrm{v}_0, r^{\mathfrak{z}}; \tau, \frac{\tau}{\varepsilon}, \frac{\mathrm{v}(\tau)}{\varepsilon}\Big)\Big| \\ &\le C(|r^{\mathfrak{z}}| + 1).\end{aligned}$$

On the other hand, from (5.5), we know (say with the same constant C) that

$$\sup_{\varepsilon\in\,]0,\varepsilon_0]} |r^{\mathfrak{z}}(0)| \le C.$$

Then, by Gronwall's lemma, we can recover (5.8). Now, the situation is quite different concerning the other component $\mathcal{R}^{\mathrm{v}}$. This is due to the contribution of

$$\varepsilon^{-N-1}\ \big[\mathrm{V}_0\left(z_0; \mathfrak{Z}^a + \varepsilon^N r; \theta_\tau, \theta_{\mathrm{r}}\right) - \mathrm{V}_0\left(z_0; \mathfrak{Z}^a; \theta_\tau, \theta_{\mathrm{r}}\right)\big]$$

which can actually be of large size ε^{-1}. Taking this into account, we can only assert that

$$\begin{aligned}\sup_{\varepsilon\in\,]0,\varepsilon_0]}\ \sup_{\tau\in[0,\mathcal{T}]} |\varepsilon\partial_\tau r^{\mathrm{v}}(\tau)| &\le \sup_{\varepsilon\in\,]0,\varepsilon_0]}\ \sup_{\tau\in[0,\mathcal{T}]} \Big|\varepsilon\mathcal{R}^{\mathrm{v}}\Big(\varepsilon, z_0, \mathrm{v}_0, r^{\mathfrak{z}}; \tau, \frac{\tau}{\varepsilon}, \frac{\mathrm{v}(\tau)}{\varepsilon}\Big)\Big| \\ &\le \tilde{C}(|r^{\mathfrak{z}}| + 1).\end{aligned}$$

The right-hand side does not depend on v. It can be bounded as indicated in (5.8). Then, after integration in time, this yields (5.9). $\square$

The drawback with (5.2) is the presence in the right hand side of the unknown function $\mathrm{v}(\tau)$. This is remedied in the next subsection where v is replaced by v^{f}.

5.2. **Description of the redressed field in terms of the frozen phase.** The second line of (5.2) can be interpreted as an implicit relation on ν. This is not very informative because the explicit oscillating content of $\nu(\cdot)$ remains to be clarified. Now, this may be achieved by cutting ν into the (well determined) frozen phase $\nu^{\rm f}$ introduced at the level of (5.3). Next we have a description of the exact phase ν through a WKB expansion involving only the frozen phase $\nu^{\rm f}$.

Lemma 5.3. *Fix $N \geq 2$. Then there exist profiles*

$$\mathscr{V}_j(z_0,\nu_0;\tau,\theta_\tau,\theta_{\rm r}) \in C^\infty(\mathbb{R}^n\times\mathbb{R}\times[0,\mathcal{T}]\times\mathbb{T}\times\mathbb{T}_{{\rm r},z_0};\mathbb{R}), \quad j\in\{1,\dots,N-2\}, \tag{5.11}$$

which can be computed from the $\mathcal{V}_k$ with $k \leq j$ with in particular

$$\mathscr{V}_1(\tau,\theta_\tau,\theta_{\rm r}) = \overline{\mathcal{V}}_1(\tau,\theta_\tau), \tag{5.12}$$

$$\mathscr{V}_2(\tau,\theta_\tau,\theta_{\rm r}) = \mathcal{V}_2\big(\tau,\theta_\tau,\theta_{\rm r}+\overline{\mathcal{V}}_1(\tau,\theta_\tau)\big), \tag{5.13}$$

and which are adjusted in such a way that, in terms of the sup norm, we have

$$\nu(\tau) = \nu^{\rm f}(\tau) + \sum_{j=1}^{N-2} \varepsilon^j \mathscr{V}_j\Big(z_0,\nu_0;\tau,\frac{\tau}{\varepsilon},\frac{\nu^{\rm f}(z_0,\nu_0;\tau)}{\varepsilon}\Big) + \mathcal{O}(\varepsilon^{N-1}). \tag{5.14}$$

Note a loss of precision from ε^N to ε^{N-1} when passing from the description (5.2) to (5.14). This is coherent with the loss recorded in (5.9). By convention, we set $\mathscr{V}_{-1} := \mathcal{V}_{-1}$ and $\mathscr{V}_0 := \mathcal{V}_0$. By this way, the formulation (5.14) becomes compatible with (2.18). Moreover, from (2.15) and (4.44), we can infer that the value of $\mathscr{V}_{-1}(\tau)$ is positive as soon as $\tau > 0$. Thus, frequencies of size ε^{-2} are created at time $\tau = 0$, and then they persist.

Remark 5.4 (Exact phase vs. frozen phase)**.** It is important to point out that neither ν nor $\nu^{\rm f}$ are phases in the usual sense of the term, since they both still depend on ε. The difference is that ν is (a component of) the unknown solution whereas $\nu^{\rm f}$ can be derived explicitly from the WKB calculus. At the end, there remains

$$\nu(\tau) = \nu^{\rm f}(\tau) + \sum_{j=1}^{N-2} \varepsilon^j \mathscr{V}_j\Big(z_0,\nu_0;\tau,\frac{\tau}{\varepsilon},\frac{\langle\overline{\mathcal{V}}_{-1}\rangle(\tau)}{\varepsilon^2}+\frac{\overline{\mathcal{V}}_0(\tau,\frac{\tau}{\varepsilon})}{\varepsilon}\Big) + \mathcal{O}(\varepsilon^{N-1}). \tag{5.15}$$

The sole use of the time phase τ and of the rapid phase $\langle\overline{\mathcal{V}}_{-1}\rangle(\tau)$ would not be consistent with an expansion of $\mathfrak{z}$ and ν in terms of profiles (not depending on ε) due to the presence of the extra (large) shift $\overline{\mathcal{V}}_0(\tau,\frac{\tau}{\varepsilon})/\varepsilon$. We had a choice of whether to make some phases or some profiles depend on ε. We have selected the first option.

Proof of Lemma 5.3. The idea is to seek an expression $\mathcal{V}^e(\varepsilon,\tau,\theta_\tau,\theta_{\rm r},r)$ that is adjusted in such a way that

$$\nu(\tau) = \nu^{\rm f}(\tau) + \varepsilon\mathcal{V}^e\Big(\varepsilon,\tau,\frac{\tau}{\varepsilon},\frac{\nu^{\rm f}(\tau)}{\varepsilon},\varepsilon^{N-1}r^\nu(\tau)\Big). \tag{5.16}$$

In view of (5.2), this amounts to finding $\mathcal{V}^e$ so that

$$\begin{aligned}
&\mathcal{V}^e\Big(\varepsilon,\tau,\frac{\tau}{\varepsilon},\frac{\nu^{\rm f}(\tau)}{\varepsilon},\varepsilon^{N-1}\,r^\nu(\tau)\Big) - \overline{\mathcal{V}}_1(\tau,\theta_\tau)\\
&-\sum_{j=2}^{N} \varepsilon^{j-1}\,\mathcal{V}_j\Big(\tau,\frac{\tau}{\varepsilon},\frac{\nu^{\rm f}(\tau)}{\varepsilon}+\mathcal{V}^e\big(\varepsilon,\tau,\frac{\tau}{\varepsilon},\frac{\nu^{\rm f}(\tau)}{\varepsilon},\varepsilon^{N-1}\,r^\nu(\tau)\big)\Big) - \varepsilon^{N-1}r^\nu(\tau)\\
&= 0.
\end{aligned} \tag{5.17}$$

To this end, it suffices to achieve the relaxed condition

$$\mathcal{H}\big(\varepsilon,\tau,\theta_\tau,\theta_{\rm r},r;\mathcal{V}^e(\varepsilon,\tau,\theta_\tau,\theta_{\rm r},r)\big)=0 \tag{5.18}$$

where we have introduced

$$\mathcal{H}(\varepsilon,\tau,\theta_\tau,\theta_{\rm r},r;y):=y-\overline{\mathcal{V}}_1(\tau,\theta_\tau)-\sum_{j=2}^{N}\varepsilon^{j-1}\,\mathcal{V}_j(\tau,\theta_\tau,\theta_{\rm r}+y)-r. \tag{5.19}$$

The expression $\mathcal{H}(x;y)$ may be viewed as a nonlinear functional in $y\in\mathbb{R}$ depending on the multidimensional parameter $x=(\varepsilon,\tau,\theta_\tau,\theta_{\rm r},r)$. From this perspective, the implicit relation (5.18) may define $\mathcal{V}^e$ as a function of x by applying the implicit function theorem. Here, it is possible (and more efficient to obtain global results) to work directly. Compute

$$\partial_y\mathcal{H}(\varepsilon,\tau,\theta_\tau,\theta_{\rm r},r;y)=1-\sum_{j=2}^{N}\varepsilon^{j-1}\partial_y\mathcal{V}_j(\tau,\theta_\tau,\theta_{\rm r}+y).$$

By construction, we can assert that

$$\sup_{\tau\in[0,\mathcal{T}]}\ \sup_{\theta_\tau\in\mathbb{T}}\ \sup_{\theta_{\rm r}\in\mathbb{T}_{{\rm r},z_0}}\ \sup_{y\in\mathbb{R}}\quad |\partial_y\mathcal{V}_j(\tau,\theta_\tau,\theta_{\rm r}+y)|<+\infty. \tag{5.20}$$

Note that the compactness of $[0,\mathcal{T}]$, $\mathbb{T}$ and $\mathbb{T}_{{\rm r},z_0}$, as well as the periodic behavior of $\mathcal{V}_j$ with respect to $\theta_{\rm r}$, are crucial to obtain (5.20) notwithstanding the lack of compactness concerning $y\in\mathbb{R}$. From the bound (5.20), we can deduce that for all

$$(\varepsilon,\tau,\theta_\tau,\theta_{\rm r},r,y)\in[0,\varepsilon_0]\times[0,\mathcal{T}]\times\mathbb{T}\times\mathbb{T}_{{\rm r},z_0}\times\mathbb{R}\times\mathbb{R}$$

with $\varepsilon_0\in\mathbb{R}_+^*$ small enough, we have

$$\partial_y\mathcal{H}(\varepsilon,\tau,\theta_\tau,\theta_{\rm r},r;y)>0.$$

On the other hand, it is evident that

$$\lim_{y\to\pm\infty}\quad\mathcal{H}(\varepsilon,\tau,\theta_\tau,\theta_{\rm r},r;y)=\pm\infty.$$

This means that we can find a unique position $\mathcal{V}^e(\varepsilon,\tau,\theta_\tau,\theta_{\rm r},r)\in\mathbb{R}$ leading to (5.18), with $\mathcal{V}^e$ depending smoothly on its arguments. In particular, expanding $\mathcal{V}^e$ near $r=0$ and using (5.9), we obtain from (5.16) that

$$\mathrm{v}(\tau)=\mathrm{v}^{\rm f}(\tau)+\varepsilon\mathcal{V}^e\Big(\varepsilon,\tau,\frac{\tau}{\varepsilon},\frac{\mathrm{v}^{\rm f}(\tau)}{\varepsilon},0\Big)+\mathcal{O}(\varepsilon^{N-1}). \tag{5.21}$$

Since $\mathcal{V}^e(\varepsilon,\tau,\theta_\tau,\theta_{\rm r},0)$ is also smooth in ε near $\varepsilon=0$, by expanding $\mathcal{V}^e$ in powers of ε, we can recover (5.14) in the form

$$\begin{aligned}\frac{1}{\varepsilon}\big[\mathrm{v}(\tau)-\mathrm{v}^{\rm f}(\tau)\big]&=\mathcal{V}^e\Big(\varepsilon,\tau,\frac{\tau}{\varepsilon},\frac{\mathrm{v}^{\rm f}(\tau)}{\varepsilon},0\Big)+\mathcal{O}(\varepsilon^{N-2})\\&=\sum_{j=0}^{N-2}\frac{1}{j!}\,\varepsilon^j\partial_\varepsilon^j\mathcal{V}^e\Big(0,\tau,\frac{\tau}{\varepsilon},\frac{\mathrm{v}^{\rm f}(\tau)}{\varepsilon},0\Big)+\mathcal{O}(\varepsilon^{N-2}).\end{aligned}$$

After comparison with (5.14), this furnishes

$$\mathscr{V}_j(\tau,\theta_\tau,\theta_{\rm r}):=\frac{1}{(j-1)!}\partial_\varepsilon^{j-1}\mathcal{V}^e(0,\tau,\theta_\tau,\theta_{\rm r},0),\quad\forall\,j\in\{1,\dots,N-2\}. \tag{5.22}$$

From (5.18) and (5.19) written with $r=0$, it is easy to infer that

$$\begin{aligned}&\mathcal{V}^e(\varepsilon,\tau,\theta_\tau,\theta_{\rm r},0)-\overline{\mathcal{V}}_1(\tau,\theta_\tau)\\&-\sum_{j=2}^{N}\varepsilon^{j-1}\mathcal{V}_j\big(\tau,\theta_\tau,\theta_{\rm r}+\mathcal{V}^e(\varepsilon,\tau,\theta_\tau,\theta_{\rm r},0)\big)=0.\end{aligned}\tag{5.23}$$

The term with ε^0 in factor yields $\mathcal{V}^e(0,\tau,\theta_\tau,\theta_{\rm r},0)=\overline{\mathcal{V}}_1(\tau,\theta_\tau)$. Applying (5.22) with $j=1$, we find (5.12). The next derivatives of (5.23) with respect to ε, taken at $\varepsilon=0$, allow to deduce successively how the profiles $\mathscr{V}_j$ can be expressed in terms of the $\mathcal{V}_k$ with $k\le j$, just by applying (5.22). For instance, we find (5.13) and so on. □

The preceding description (5.2) of $\mathfrak{z}$ is not fully satisfactory. Indeed, it still involves the unknown v. However, using Lemma 5.3, this difficulty can now easily be overcome.

Lemma 5.5 (Description of $\mathfrak{z}$ through the frozen phase $\mathsf{v}^{\rm f}$)**.** *Fix $N\ge 2$. There exist profiles*

$$\mathcal{Z}_j(z_0,\mathsf{v}_0;\tau,\theta_\tau,\theta_{\rm r})\in C^\infty(\mathbb{R}^n\times\mathbb{R}\times[0,\mathcal{T}]\times\mathbb{T}\times\mathbb{T}_{{\rm r},z_0};\mathbb{R}^n),\tag{5.24}$$

for $j\in\{0,\dots,N-2\}$, which can be computed from the $\mathfrak{Z}_k$ and $\mathscr{V}_k$ (or $\mathcal{V}_k$) with $k\le j$ with in particular

$$\mathcal{Z}_0(\tau,\theta_\tau,\theta_{\rm r})=\langle\overline{\mathfrak{Z}}_0\rangle(\tau),\tag{5.25}$$

$$\mathcal{Z}_1(\tau,\theta_\tau,\theta_{\rm r})=\overline{\mathfrak{Z}}_1(\tau,\theta_\tau),\tag{5.26}$$

and which are adjusted in such a way that, in terms of the supremum norm, we have

$$\mathfrak{z}(\tau)=\sum_{j=0}^{N-2}\varepsilon^j\mathcal{Z}_j\Big(z_0,\mathsf{v}_0;\tau,\frac{\tau}{\varepsilon},\frac{\mathsf{v}^{\rm f}(z_0,\mathsf{v}_0;\tau)}{\varepsilon}\Big)+\mathcal{O}(\varepsilon^{N-1}).\tag{5.27}$$

Proof. We substitute the phase v as described by (5.14) in the first component of (5.2). Since $\mathfrak{Z}_0\equiv\langle\overline{\mathfrak{Z}}_0\rangle$ and $\partial_{\theta_{\rm r}}\mathfrak{Z}_1=0$, we find that

$$\begin{aligned}\mathfrak{z}(\tau)=&\langle\overline{\mathfrak{Z}}_0\rangle(\tau)+\varepsilon\overline{\mathfrak{Z}}_1(\tau,\theta_\tau)\\&+\sum_{j=2}^{N-2}\varepsilon^j\mathfrak{Z}_j\Big(\tau,\frac{\tau}{\varepsilon},\frac{\mathsf{v}^{\rm f}(\tau)}{\varepsilon}+\sum_{i=1}^{N-2}\varepsilon^{i-1}\mathscr{V}_i\big(\tau,\frac{\tau}{\varepsilon},\frac{\mathsf{v}^{\rm f}(\tau)}{\varepsilon}\big)\Big)+\mathcal{O}(\varepsilon^{N-1}).\end{aligned}\tag{5.28}$$

Apply Taylor expansion to the function $\mathfrak{Z}_j$ with respect to the last variable to obtain

$$\begin{aligned}&\mathfrak{Z}_j\Big(\tau,\theta_\tau,\theta_{\rm r}+\overline{\mathcal{V}}_1+\sum_{i=2}^{N-2}\varepsilon^{i-1}\mathscr{V}_i\Big)\\&=\sum_{k=0}^{+\infty}\frac{1}{k!}\partial^k_{\theta_{\rm r}}\mathfrak{Z}_j(\tau,\theta_\tau,\theta_{\rm r}+\overline{\mathcal{V}}_1)\ \Big(\sum_{i=2}^{N-2}\varepsilon^{i-1}\mathscr{V}_i\Big)^k.\end{aligned}\tag{5.29}$$

Then we plug (5.29) into (5.28), and collect the terms with the same power of ε in factor. Then, compare the result with the expansion (5.27). This allows us to determine inductively the profiles $\mathcal{Z}_j$ from the $\mathfrak{Z}_k$ and $\mathscr{V}_k$ with $k\le j$. In particular, the terms which have ε^0 and ε^1 in factor furnish respectively (5.25) and (5.26). □

5.3. **Back to the original field.** The final stage is to provide a WKB expansion concerning the original field z.

Lemma 5.6 (Asymptotic description of the original field)**.** *Fix $N \geq 2$. Then there exist profiles*

$$Z_j(z_0, \nu_0; \tau, \theta_\tau, \theta_{\rm r}) \in C^\infty(\mathbb{R}^n \times \mathbb{R} \times [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{{\rm r},z_0}; \mathbb{R}^n), \tag{5.30}$$

with $j \in \{0, \dots, N-2\}$ which can be computed from the $\mathcal{Z}_k$ and $\mathscr{V}_k$ (or $\mathfrak{Z}_k$ and $\mathcal{V}_k$) with $k \leq j$ with in particular

$$Z_0(\tau, \theta_\tau) \equiv \overline{Z}_0(\tau, \theta_\tau) = \Xi_0\big(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau\big), \tag{5.31}$$

$$\begin{aligned} Z_1(\tau, \theta_\tau, \theta_{\rm r}) = {} & \big(\overline{\mathfrak{Z}}_1(\tau, \theta_\tau) \cdot \nabla_{\mathfrak{z}}\big) \Xi_0\big(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau\big) \\ & + \Xi_1\big(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau, \theta_{\rm r} + \overline{\mathcal{V}}_1(\tau, \theta_\tau)\big). \end{aligned} \tag{5.32}$$

and which are adjusted in such a way that, in terms of the sup norm, we have

$$z(\varepsilon, z_0, \nu_0; \tau) = \sum_{j=0}^{N-2} \varepsilon^j Z_j\Big(z_0, \nu_0; \tau, \frac{\tau}{\varepsilon}, \frac{\nu^{\rm f}(z_0, \nu_0; \tau)}{\varepsilon}\Big) + \mathcal{O}(\varepsilon^{N-1}). \tag{5.33}$$

Remark 5.7 (Normal form procedure: implication)**.** When Ξ is as in (3.9), we have $\partial_{\theta_\tau} \Xi_0 \equiv 0$, and therefore the dependence of $\overline{Z}_0$ on θ_τ is not activated. Then, there remains $Z_0 \equiv \langle \overline{Z}_0 \rangle$.

Proof. Using (3.2), the expression $z(\tau)$ can be recovered from Ξ, ν and $\mathfrak{z}$ which can be extracted respectively through (3.6), (5.14) and (5.27). By combining this information, we find the constraint

$$\begin{aligned} & \sum_{j=0}^{N-2} \varepsilon^j Z_j(\tau, \theta_\tau, \theta_{\rm r}) \\ & = \Xi_0\Big(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(\tau) + \sum_{j=1}^{N-2} \varepsilon^j \mathcal{Z}_j(\tau, \theta_\tau, \theta_{\rm r}); \theta_\tau\Big) + \mathcal{O}(\varepsilon^{N-1}) + \varepsilon \Xi_1\Big(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(\tau) \\ & \quad + \sum_{j=1}^{N-2} \varepsilon^j \mathcal{Z}_j(\tau, \theta_\tau, \theta_{\rm r}); \theta_\tau, \theta_{\rm r} + \overline{\mathcal{V}}_1 + \sum_{j=2}^{N-2} \varepsilon^{j-1} \mathscr{V}_j(\tau, \theta_\tau, \theta_{\rm r})\Big). \end{aligned} \tag{5.34}$$

Recall that Ξ_0 is the mean flow (Lemma 3.10) and that Ξ_1 can be deduced from (3.37). Taylor's Theorem in both variables $\mathfrak{z}$ and $\theta_{\rm r}$ can be applied to develop the right hand side of (5.34) in powers of ε. Then, by identifying the terms with the same power of ε in factor, we can obtain explicit formulas yielding the Z_j in terms of the $\mathcal{Z}_k$ and $\mathscr{V}_k$.

For ε^0, we obviously obtain (5.31). The expression having ε in factor at the level of (5.34) is composed of two contributions. The first coming from Ξ_0 yields the first line of (5.32); the second issued from Ξ_1 leads to the second line of (5.32). □

Proof of Theorem 2.7. To conclude, it suffices to compile what has been done before. The expansion (2.17) is the same as (5.33) in Lemma 5.6. On the other hand, the description of the exact phase ν is achieved in (5.14), at the le vel of Lemma 5.3. The $\mathcal{O}(\varepsilon^\infty)$ in (2.17) and (2.18) can be obtained by just varying the choice of N, with an arbitrary remainder of size ε^{N-1} which is controlled at the level of Proposition 5.2. □

6. Application to Hamilton-Jacobi equations

Let $d \in \mathbb{N}^*$. We work with the scalar function $H : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$,

$$(\tau, x, u, p) \mapsto H(\tau, x, u, p)\,.$$

Given some initial data $u_0 : \mathbb{R}^d \to \mathbb{R}$, the Hamilton-Jacobi equation built with H and u_0 takes the following form

$$\partial_\tau u + H(\tau, x, u, \nabla_x u) = 0\,, \quad u(0, x) = u_0(x). \tag{6.1}$$

The study of evolution equations like (6.1) is fundamental in classical mechanics. It is a long-standing concern [3] which has motivated many contributions. The discussion depends heavily on the functional framework retained on H and u. Roughly speaking:

- (GW) Global weak solutions u can be constructed by compactness arguments (CA), see [14] and subsequent works. But uniqueness and stability require additional constraints on both H (typically convexity conditions with respect to p) and u (entropic conditions);
- (LS) Local smooth solutions u are available by the method of characteristics (MC). But this approach can work only under minimal smoothness conditions (say H and u_0 in $\mathcal{C}^2$) and as long as the spatial projections of the (phase space) characteristics do not cross.

In Subsection 6.1, we introduce a small parameter $\varepsilon \in]0, \varepsilon_0]$ (going to zero) at the level of the system (6.1) in order to obtain a family of Cauchy problems of the type (1.5). In addition, we comment the effects of introducing such parameter in this system. In Subsection 6.2, we connect the resolution of the PDE (1.5) to the one of the ODE (1.12). Then, we show Theorem 1.5 in three steps: in Subsection 6.3, we detail the behavior of the spatial characteristic $\mathrm{x}(\varepsilon, \cdot)$; in Subsection 6.4, we construct its inverse map $\mathrm{x}^{-1}(\varepsilon, \cdot)$; and in the last Subsection 6.5, we complete the proof.

6.1. **Oscillating framework.**

To go beyond the standard results, a natural strategy is to implement a parameter (here $\varepsilon \in]0, \varepsilon_0]$) whose effect is to break (asymptotically when ε goes to zero) the usual assumptions. By this way, it can be possible to better target some underlying difficulties, and then to solve them. This is done in Paragraph 6.1.1 by introducing oscillations. In Paragraph 6.1.2, we describe the strategy which allows to achieve the proof of Theorem 1.5.

6.1.1. *Data adjustment.* Recall that we have introduced the variables $\mathfrak{u}$ and $\mathfrak{p}$ in the introduction which serve to replace respectively the terms εu and $\nabla_x u$. In addition, for $\star \in \{\tau, u\}$, we have denoted by $\theta_\star$ the periodic variable which is aimed to be replaced with the quotient $\star/\varepsilon$. We also had that the expression H introduced at the level of (1.2) is indeed a function of the variables $(\varepsilon, \theta_\tau, x, \mathfrak{u}, \mathfrak{p}, \theta_u)$, which is assumed to be smooth on its domain of definition. As indicated in (1.4), the function $\mathsf{H}(\varepsilon, \cdot)$ can be expanded in powers of ε near $\varepsilon = 0$. Now, replace H and u_0 inside (6.1) by H_ε and $u_{0\varepsilon}$ as indicated below:

- The hamiltonian $H_\varepsilon(\cdot) \equiv H(\varepsilon, \cdot)$ may depend on $\varepsilon \in]0, \varepsilon_0]$ according to

$$H_\varepsilon(\tau, x, u, p) \equiv H(\varepsilon, \tau, x, u, p) = \frac{1}{\varepsilon}\mathsf{H}\Big(\varepsilon, \frac{\tau}{\varepsilon}, x, \varepsilon u, p, \frac{u}{\varepsilon}\Big). \tag{6.2}$$

When $\mathsf{H}_0 \not\equiv 0$, the source term H_ε is of large amplitude ε^{-1}, and it does imply oscillations (in both τ and u) at frequencies of size ε^{-1}. The role of ε is precisely to impact the $\mathcal{C}^1$-estimates (on H) which become non uniform in $\varepsilon \in]0, \varepsilon_0]$;

- The initial data u_0 inside (6.1) may depend on ε. More precisely, with $\mathcal{U}_0$ smooth and as in (1.1)-(1.3), we impose $u_{0\varepsilon}(x) = \mathcal{U}_0(\varepsilon, x)$.

And thus, the initial value problems (6.1) with $H \equiv H_\varepsilon$ as in (6.2) and $u_\varepsilon(0, \cdot) = u_{0\varepsilon}$ as above is exactly the Hamilton-Jacobi equation (1.5). From now on, we consider the smooth solutions $u_\varepsilon \equiv u_\varepsilon(\tau, x) \equiv u(\varepsilon, \tau, x)$, sometimes simply denoted by u, to the the initial value problems (1.5).

Remark 6.1 (Return to a more standard regime)**.** Applying the method of characteristics, a smooth solution u_ε does exist (locally in space) on a maximal time interval $[0, \mathcal{T}_\varepsilon[$, with $\mathcal{T}_\varepsilon \in \mathbb{R}_+^*$ that may shrink to zero when ε goes to 0. Assuming that $\mathcal{U}_{00} \equiv 0$, changing τ into $\varepsilon^2 \tilde{\tau}$ and u into $\varepsilon \tilde{u}$, we obtain

$$\partial_{\tilde{\tau}} \tilde{u}_\varepsilon + \mathsf{H}(\varepsilon, \varepsilon \tilde{\tau}, x, \varepsilon^2 \tilde{u}, \varepsilon \nabla_x \tilde{u}, \tilde{u}) = 0 \,, \quad \tilde{u}_\varepsilon(0, \cdot) = \varepsilon^{-1} u_{0\varepsilon} = O(1). \tag{6.3}$$

It is clear that the lifespan associated with (6.3) is uniform in $\varepsilon \in]0, \varepsilon_0]$, and therefore that $\varepsilon^2 \lesssim \mathcal{T}_\varepsilon$. When $\mathcal{U}_{00} \not\equiv 0$, such a lower bound is no more evident. This means that, in comparison with the first well understood situation (6.3), the study of (1.5) for $\tau \sim 1$ corresponds to a very long time investigation $\tilde{\tau} \sim \varepsilon^{-2}$ for large initial data of size ε^{-1}. The latter difficulty is of course partly offset by the (nonlinear) periodic behavior of H with respect to θ_u.

6.1.2. *Strategy, intermediate results and notations.* To construct solutions u_ε (uniformly in ε) and to justify asymptotic results (when $\varepsilon \to 0$), the above-mentioned approaches (GW) and (LS) face two significant barriers:

(i) Compactness arguments (CA) are not accompanied by a (strong form of) stability allowing to compare exact and approximate solutions;
(ii) The method of characteristics (MC) may be subjected (after spatial projection) to crossing problems at times $\mathcal{T}_\varepsilon$ vanishing when ε goes to zero.

The first challenge (i) seems difficult to overcome. Indeed, the potential lack of control makes it impossible to compare the weak and approximate solutions with each other. In particular, in the continuation of the indent (i), a method relying on the absorption of small error terms cannot be implemented (with quantitative estimates). On the other hand, for reasons that have been already discussed in Remark 2.12, other more direct tools like homogenization [25] or multiscale young measures [1] are not amenable to capture the leading behavior of u_ε. And they cannot be implemented to compare the exact solution to the approximate one with a high degree of precision. To obtain accurate descriptions of the solution u_ε, we must stick to the approach (ii).

The link between (1.5) and (1.12) is achieved through Lemma 6.2 together with the representation formula (6.8). Then, we have to detail inside (6.8) the content of $\mathrm{u}(\varepsilon, \cdot)$ and $\mathrm{x}^{-1}(\varepsilon, \cdot)$. To this end, our strategy is to:

- Exhibit (Lemma 6.5) the asymptotic expansions of the spatial projections $\mathrm{x}(\varepsilon, \tau, x)$ of the (phase space) characteristics. This requires to check the validity of Assumptions 2.4 and 2.6 (of Subsection 2.1) in the contexts inherited from (1.12);

- Prove (this is done in Lemmas 6.9 and 6.10) that the map $x \mapsto \tilde{x} := \mathrm{x}(\varepsilon,\tau,x)$ is (at least for τ small enough) uniformly in $\varepsilon \in]0,\varepsilon_0]$, a local diffeomorphism and find the oscillating description of the corresponding inverse x^{-1};
- Exploit (Theorem 1.5) the formula (6.8) to reveal the final oscillating structure of u_ε.

Now, we make a compilation of some notations that have been or will be involved. It is also to provide additional clarification (to avoid confusion and misunderstandings). The symbol θ is always used for a periodic (scalar) variable. But, it may be connected to the PDE setting (like θ_τ and θ_u in Section 1) or to the ODE context (like θ_τ and θ_{r} in Section 2). The connection in the next Subsection 6.2 will imply some identifications between these variables. Keep in mind that

$$\theta_u \equiv \theta_{\mathrm{r}} \quad (\text{for } u/\varepsilon \text{ or } \mathrm{v}/\varepsilon).$$

On the other hand, the distinction between $\theta_{\mathrm{r}}^\circ$ (in Lemma 6.5) and $\hat{\theta}_{\mathrm{r}}^\circ$ (Theorem 1.5 and Lemma 6.10) comes from the fact that different phases come to replace $\theta_{\mathrm{r}}^\circ$ and $\hat{\theta}_{\mathrm{r}}^\circ$.

6.2. **Connection to differential equations.** In Paragraph 6.2.1, we will apply the method of characteristics in the context of (1.5) in order to deduce a system of ordinary differential equations which allows to solve (1.5) and which may fit with (1.12). In Paragraph 6.2.2, we provide the readers with some comments on assumptions and main result. Indeed, we comment the assumptions related to the PDE (1.5) in the context inherited from the ODE (1.12) and we give some interpretations of Theorem 1.5.

6.2.1. *Method of characteristics.* Assume that $u(\varepsilon,\cdot)$ is a local smooth solution to (1.5). Then, we consider a local solution $\mathrm{x} \equiv \mathrm{x}(\varepsilon,\tau,x)$ to the self-contained system

$$\begin{gathered}\dot{\mathrm{x}} := \frac{d\mathrm{x}}{d\tau} = \frac{1}{\varepsilon}\nabla_p \mathsf{H}\Big(\varepsilon,\frac{\tau}{\varepsilon},\mathrm{x},\varepsilon u(\varepsilon,\tau,\mathrm{x}),\nabla_x u(\varepsilon,\tau,\mathrm{x}),\frac{u(\varepsilon,\tau,\mathrm{x})}{\varepsilon}\Big),\\ \mathrm{x}(\varepsilon,0,x) = x.\end{gathered} \tag{6.4}$$

Define

$$\mathrm{p} \equiv \mathrm{p}(\varepsilon,\tau,x) := \nabla_x u\big(\varepsilon,\tau,\mathrm{x}(\varepsilon,\tau,x)\big), \quad \mathrm{u} \equiv \mathrm{u}(\varepsilon,\tau,x) := u\big(\varepsilon,\tau,\mathrm{x}(\varepsilon,\tau,x)\big). \tag{6.5}$$

Lemma 6.2 (Tracking the characteristics)**.** *The time evolution of* $(\mathrm{x},\mathrm{p},\mathrm{u}) \equiv (\mathrm{x},\mathrm{p},\mathrm{u})(\varepsilon,\tau,x)$ *is governed by the following system of coupled equations*

$$\begin{gathered}\dot{\mathrm{x}} = \frac{1}{\varepsilon}\nabla_p \mathsf{H}\Big(\varepsilon,\frac{\tau}{\varepsilon},\mathrm{x},\varepsilon\mathrm{u},\mathrm{p},\frac{\mathrm{u}}{\varepsilon}\Big),\\ \dot{\mathrm{p}} = -\big(\frac{1}{\varepsilon}\nabla_x \mathsf{H} + \partial_u \mathsf{H}\mathrm{p} + \frac{1}{\varepsilon^2}\partial_{\theta_u}\mathsf{H}\mathrm{p}\big)\Big(\varepsilon,\frac{\tau}{\varepsilon},\mathrm{x},\varepsilon\mathrm{u},\mathrm{p},\frac{\mathrm{u}}{\varepsilon}\Big),\\ \dot{\mathrm{u}} = \frac{1}{\varepsilon}\ (\nabla_p \mathsf{H}\cdot\mathrm{p} - \mathsf{H})\Big(\varepsilon,\frac{\tau}{\varepsilon},\mathrm{x},\varepsilon\mathrm{u},\mathrm{p},\frac{\mathrm{u}}{\varepsilon}\Big),\end{gathered} \tag{6.6}$$

together with

$$(\mathrm{x},\mathrm{p},\mathrm{u})(\varepsilon,0,x) = \big(x,\nabla_x\mathcal{U}_0(\varepsilon,x),\mathcal{U}_0(\varepsilon,x)\big). \tag{6.7}$$

In view of (6.7), x is the spatial foot of the characteristic emanating from $(\mathrm{x},\mathrm{p},\mathrm{u})(\varepsilon,0,x)$. The spatial projection of this characteristic is located at the time

τ at the position $\mathrm{x}(\varepsilon,\tau,x)$. When the map $x \mapsto \tilde{x} = \mathrm{x}(\varepsilon,\tau,x)$ is a local diffeomorphism, the inverse $\mathrm{x}^{-1}(\varepsilon,\tau,\tilde{x})$ does exist and it furnishes a unique feedback allowing to solve (1.5). Then, we can simply recover $u(\varepsilon,\cdot)$ through

$$u(\varepsilon,\tau,\tilde{x}) = \mathrm{u}\big(\varepsilon,\tau,\mathrm{x}^{-1}(\varepsilon,\tau,\tilde{x})\big). \tag{6.8}$$

The inversion formula (6.8) makes the transition from the Lagrangian point of view in (6.6), where u and x are functions of (τ,x), to the Eulerian perspective where u is a function measuring a quantity at the location $\tilde{x}$ through which the motion flows as time passes. In view of (6.8), to determine $u(\varepsilon,\cdot)$ at the position $(\tau,\tilde{x})$, we need to compose the (oscillating) quantity $\mathrm{u}(\varepsilon,\tau,x)$ with the (oscillating) position $x \equiv \mathrm{x}^{-1}(\varepsilon,\tau,\tilde{x})$. In doing so, we must face a *composition* of oscillations, where it is crucial to know precisely how the functions $\mathrm{u}(\cdot)$ and $\mathrm{x}^{-1}(\cdot)$ depend respectively on (ε,τ,x) and $(\varepsilon,\tau,\tilde{x})$.

Proof of Lemma 6.2. In view of (6.5), the first equation of (6.6) is just a reformulation of (6.4). Now, the definition (6.5) leads to

$$\dot{\mathrm{p}} = (\nabla_x\partial_\tau u)(\tau,\mathrm{x}) + \big((\dot{\mathrm{x}}\cdot\nabla_x)\nabla_x u\big)(\tau,\mathrm{x}), \quad \dot{\mathrm{u}} = \partial_\tau u(\tau,\mathrm{x}) + (\dot{\mathrm{x}}\cdot\nabla_x)u(\tau,\mathrm{x}). \tag{6.9}$$

Compute the spatial gradeant of (1.5) to obtain

$$\nabla_x\partial_\tau u + \frac{1}{\varepsilon}(\nabla_p\mathsf{H}\cdot\nabla_x)\nabla_x u + \frac{1}{\varepsilon}\nabla_x\mathsf{H} + \partial_u\mathsf{H}\nabla_x u + \frac{1}{\varepsilon^2}\partial_{\theta_u}\mathsf{H}\nabla_x u = 0. \tag{6.10}$$

Taking into account (6.4), the first two terms of (6.10) coincide with $\dot{\mathrm{p}}$. By this way, we can recognize the second equation inside (6.6). Finally, combining the second part of (6.9) with (1.5), (6.4) and (6.5), we find the third equation of (6.6). □

6.2.2. *Comments on assumptions and main result.* Starting from (6.6), there are different ways of falling within the context of (1.12). Indeed, the connection between (1.5) and (1.12) can be achieved through (6.6) by specifying the values of z and v in terms of x, p and u. When doing this, care must be taken to recover the special structure of (1.12). The selection of $\mathsf{v} := \mathrm{u}$ is a natural choice. Now, one is tempted to simply take $z = {}^t(\mathrm{x},\mathrm{p})$. But, to ensure that the profiles A and V do not depend on $\mathsf{v} \equiv \mathrm{u}$ as required in (1.12), we must incorporate $\varepsilon\mathrm{u}$ as a component of z. For this reason, we work with

$$\begin{gathered} z = {}^t(z_x,z_p,z_u) := {}^t(\mathrm{x},\mathrm{p},\varepsilon\mathrm{u}) \in \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R} \equiv \mathbb{R}^n, \\ \mathsf{v} := \mathrm{u}, \quad n = 2d+1. \end{gathered} \tag{6.11}$$

We want to be consistent with the notation used in Sections 3, 4, and 5. To this end, we compare the system on (z,v) that is issued from (6.6)-(6.11) with (1.12). With $\mathsf{A} = {}^t(\mathsf{A}_x,\mathsf{A}_p,\mathsf{A}_u)$ as well as $\theta_u \equiv \theta_r$, these two systems can be identified on condition that

$$\begin{aligned} \mathsf{A}_x(\varepsilon;z;\theta_\tau,\theta_{\mathrm{r}}) &:= \nabla_p\mathsf{H}(\varepsilon,\theta_\tau,z_x,z_u,z_p,\theta_{\mathrm{r}}), \\ \mathsf{A}_p(\varepsilon;z;\theta_\tau,\theta_{\mathrm{r}}) &:= -(\nabla_x\mathsf{H} + \varepsilon\partial_u\mathsf{H}z_p + \varepsilon^{-1}\partial_{\theta_u}\mathsf{H}z_p)(\varepsilon,\theta_\tau,z_x,z_u,z_p,\theta_{\mathrm{r}}), \\ \mathsf{A}_u(\varepsilon;z;\theta_\tau,\theta_{\mathrm{r}}) &:= \varepsilon\left[(z_p\cdot\nabla_p)\mathsf{H} - \mathsf{H}\right](\varepsilon,\theta_\tau,z_x,z_u,z_p,\theta_{\mathrm{r}}), \\ \mathsf{V}(\varepsilon;z;\theta_\tau,\theta_{\mathrm{r}}) &:= \left[(z_p\cdot\nabla_p)\mathsf{H} - \mathsf{H}\right](\varepsilon,\theta_\tau,z_x,z_u,z_p,\theta_{\mathrm{r}}). \end{aligned} \tag{6.12}$$

The function A_p must be smooth near $\varepsilon = 0$. This could be inconsistent with the weight ε^{-1} remaining in the second line of (6.12). On the other hand, in coherence with Assumption 2.6, the expression V_0 must be positive. These considerations lead to Assumptions 1.1 and 1.2. Then, with the convention $\mathsf{H}_{-1} \equiv 0$ and $\mathsf{A}_j = {}^t(\mathsf{A}_{jx}, \mathsf{A}_{jp}, \mathsf{A}_{ju})$, for all $j \in \mathbb{N}$, we find that

$$\begin{gathered} \mathsf{A}_{jx} := \nabla_p \mathsf{H}_j, \\ \mathsf{A}_{jp} := -\nabla_x \mathsf{H}_j - \partial_u \mathsf{H}_{j-1} z_p - \partial_{\theta_u} \mathsf{H}_{j+1} z_p, \\ \mathsf{A}_{ju} := (z_p \cdot \nabla_p) \mathsf{H}_{j-1} - \mathsf{H}_{j-1}, \\ \mathsf{V}_j := (z_p \cdot \nabla_p) \mathsf{H}_j - \mathsf{H}_j. \end{gathered} \tag{6.13}$$

In particular $\mathsf{A}_0 = {}^t(\nabla_p \mathsf{H}_0, -\nabla_x \mathsf{H}_0 - \partial_{\theta_u} \mathsf{H}_1 z_p, 0)$. Since V_0 is defined in terms of H_0, in view of Assumption 1.1, the function V_0 does not depend on θ_{r}. Thus, we can apply (2.12) to see that

$$\mathsf{A}_{\mathrm{mf}} = \overline{\mathsf{A}}_0 = {}^t(\nabla_p \mathsf{H}_0, -\nabla_x \mathsf{H}_0 - \overline{\partial_{\theta_u} \mathsf{H}_1} z_p, 0) = {}^t(\nabla_p \mathsf{H}_0, -\nabla_x \mathsf{H}_0, 0).$$

This implies that the mean flow (Definition 2.2) which is denoted by $\Xi_{\mathrm{mf}} \equiv \Xi_0(\mathfrak{z}; s) = {}^t(\Xi_{0x}, \Xi_{0p}, \Xi_{0u})$ is such that

$$\Xi_0(\mathfrak{z}; s) = {}^t(\Xi_{0x}, \Xi_{0p}, \mathfrak{z}_u), \quad \forall\, \mathfrak{z} = {}^t(\mathfrak{z}_x, \mathfrak{z}_p, \mathfrak{z}_u) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}, \tag{6.14}$$

where ${}^t(\Xi_{0x}, \Xi_{0p})$ satisfies (with $\theta_\tau \equiv s$ as in the introduction) the Hamiltonian system (1.8). Then, Assumption 2.4 amounts to the same thing as Assumption 1.3.

Remark 6.3 (Common situations leading to Assumption 1.3)**.** In general, it is not easy to test the periodic condition presented in Assumption 1.3. We furnish below a list of situations where $\Xi_0(\mathfrak{z}; \cdot)$ is indeed periodic.

Suppose that the function H_0 does not depend on x. Then, the mean flow is given by

$$\Xi_0(\mathfrak{z}; s) = {}^t\big(\Xi_{0x}(\mathfrak{z}; s), \mathfrak{z}_p, \mathfrak{z}_u\big), \quad \Xi_{0x}(\mathfrak{z}; s) = \mathfrak{z}_x + \int_0^s \nabla_p \mathsf{H}_0(\tilde{s}, \mathfrak{z}_u, \mathfrak{z}_p)\, d\tilde{s}. \tag{6.15}$$

When $\langle \nabla_p \mathsf{H}_0 \rangle \equiv 0$, it is obvious that the mean flow Ξ_0 is periodic in s of period 2π. When moreover $\nabla_p \mathsf{H}_0^\star \not\equiv 0$, the function Ξ_0 is non constant (the dynamic is nontrivial).

For similar reasons, if we assume that H_0 does not depend on p and is such that $\langle \nabla_x \mathsf{H}_0 \rangle \equiv 0$, the function $\Xi_0(\mathfrak{z}; \cdot)$ is periodic in s of period 2π.

When $\partial_s \mathsf{H}_0 \equiv 0$, the system (1.8) is autonomous. Then, Assumption 1.3 is satisfied on condition that the level curves of $\mathsf{H}_0(\cdot, \mathfrak{z}_u, \cdot)$ are (locally) diffeomorphic to a family of circles (existence of Liouville torus and thereby of action-angle variables).

In line with (3.37), we find that the lifting Ξ is given by

$$\Xi(\mathfrak{z}; \theta_\tau, \theta_{\mathrm{r}}) = \Xi_0(\mathfrak{z}; \theta_\tau) + \varepsilon \Xi_1^*(\mathfrak{z}; \theta_\tau, \theta_{\mathrm{r}}), \quad \Xi_1^* = {}^t(\Xi_{1x}^*, \Xi_{1p}^*, \Xi_{1u}^*)$$

together with

$$\begin{gathered} \Xi_{1x}^* \equiv 0, \quad \Xi_{1u}^* \equiv 0, \\ \Xi_{1p}^* = -\mathsf{V}_0(\Xi_0(\mathfrak{z}; \theta_\tau); \theta_\tau)^{-1} \mathsf{H}_1^*(\theta_\tau, \Xi_{0x}, \mathfrak{z}_u, \Xi_{0p}, \theta_{\mathrm{r}})\ \Xi_{0p}(\mathfrak{z}; \theta_\tau). \end{gathered} \tag{6.16}$$

From definition (6.11) of z and v together with (6.5) and the initial data of (6.4), at time $\tau = 0$, we must start with

$$(z_0, \mathsf{v}_0)(\varepsilon, x) = \big(x, \nabla_x \mathcal{U}_0(\varepsilon, x), \varepsilon \mathcal{U}_0(\varepsilon, x), \mathcal{U}_0(\varepsilon, x)\big) = \sum_{j=0}^{+\infty} \varepsilon^j (z_{0j}, \mathsf{v}_{0j})(x). \tag{6.17}$$

Observe in particular that

$$\begin{aligned} z_{00}(x) &= (x, \nabla_x \mathcal{U}_{00}(x), 0), \\ z_{01}(x) &= \big(0, \nabla_x \mathcal{U}_{01}(x), \mathcal{U}_{00}(x)\big), \\ \mathrm{v}_{00}(x) &= \mathcal{U}_{00}(x). \end{aligned} \tag{6.18}$$

From now on, we select x in a ball $B(0,R]$ for some $R \in \mathbb{R}_+^*$. Knowing what A, V and $\Xi_0 \equiv \Xi_{\mathrm{mf}}$ (Lemma 3.10) are, we can deduce the value of A_1 through (3.34), and then we have access to $\langle \overline{\mathfrak{Z}}_0 \rangle$ through (4.34). Now, consider

$$K := \Big\{ \Xi_0 \big(\langle \overline{\mathfrak{Z}}_0 \rangle \big(z_{00}(x); \tau \big); r \big); x \in B(0,R],\ \tau \in [0,1],\ r \in \mathbb{R} \Big\} \subset \mathbb{R}^n. \tag{6.19}$$

Since $\Xi_0(\mathfrak{z}; \cdot)$ is periodic, the set K (which is presented in Assumption 1.4) is compact.

The main purpose of Section 6 is to prove Theorem 1.5 which is an important consequence of Theorem 2.7. Looking at the asymptotic description (1.10) of the solution u_ε, it bears noting that:

- The function ψ_ε is a phase in the sense (ii) of Section 1: it is smooth scalar function and its first derivatives are uniformly bounded of size at most $\mathcal{O}(1)$;
- The rapid variable $\hat{\theta}_{\mathrm{r}}^0$ is activated at the level of the profiles $\mathscr{U}_j$ as soon as $j \geq 1$;
- Recall that $\mathscr{V}_{-1} \equiv \mathcal{V}_{-1}$ and look at (4.44). Since V_0 is a positive function, in view of (6.23), (6.45) and (6.61), we can assert that $\overline{\mathscr{U}}_{-1}$ is not zero for $\tau > 0$. Thus (time) oscillations at frequency ε^{-3} do occur inside (1.10);
- Recall that $\mathscr{V}_0 \equiv \mathcal{V}_0 = \langle \overline{\mathscr{V}}_0 \rangle(\tau) + \overline{\mathscr{V}}_0^{\star}(\tau, \theta_\tau)$. At time $\tau = 0$, taking into account (4.47), this is just $\mathrm{v}_{00} = \mathcal{U}_{00}$ which may be chosen non-zero. The same remains true for $\tau \in \mathbb{R}_+^*$ (small enough). In view of (6.24), (6.46) and (6.62), we find in general that $\overline{\mathscr{U}}_0 \not\equiv 0$. This means that the $O(\varepsilon^2)$ terms inside (1.11) is also essential.

The construction of the phase ψ_ε appearing in (1.11) is explained in what follows. In the next Section 6.3, we start the proof of Theorem 1.5 by looking at the component $\mathrm{x}(\varepsilon, \cdot)$ of (6.6), which is the spatial projection of z.

6.3. **Spatial component of the characteristics** x. The first thing to check is the uniform local existence of x. Below, we prove that the map $\mathrm{x}(\varepsilon; \cdot)$ exists locally uniformly in ε.

Lemma 6.4. *Select any $R \in \mathbb{R}_+^*$. Under Assumptions 1.1, 1.2 and 1.3, we can find some $\mathcal{T} \in \mathbb{R}_+^*$ such that the solution $(z, \mathrm{v})(\varepsilon, \tau, x)$ to (1.12) with A and V as in (6.12) and initial data z_0 and v_0 as in (6.17) is, for all $\varepsilon \in]0, \varepsilon_0]$, defined on $[0, \mathcal{T}] \times B(0,R]$. In particular, the two components $\mathrm{x}(\varepsilon, \tau, x)$ and $\mathrm{u}(\varepsilon, \tau, x)$ exist on a uniform domain.*

Proof. Recall that we have selected positions x inside $B(0,R]$. From (6.17), we find that (z_0, v_0) stays in the compact set $B(0,R] \times B(0, M_1] \times B(0, M_0]^2$ with

$$M_j := \|\mathcal{U}_0\|_{W^{j,\infty}([0,\varepsilon_0] \times B(0,R]; \mathbb{R})} < +\infty, \quad j \in \{0,1\}. \tag{6.20}$$

This means that the initial data coming from (6.17) remain uniformly in a compact set. On the other hand, we have seen that Assumptions 1.1, 1.2 and 1.3 imply

Assumption 2.4 when dealing with the system on (z, ν) which is issued from (6.6)-(6.11). Thus, it suffices to apply Theorem 2.5. $\square$

From there, the aspects about the oscillating structure of x make sense. To this end, we clarify the asymptotic expansion of x in Paragraph 6.3.1. Then, in Paragraph 6.3.2, we explore the effect of a transparency condition emanating from Assumption 1.3 on the differential of x: $\mathrm{D}_x \mathrm{x}(\cdot)$. Indeed, the latter condition furnishes a control on the size of the Jacobian matrix $\mathrm{D}_x \mathrm{x}(\cdot)$.

6.3.1. *Asymptotic expansion of* x. Note that Assumptions 1.1 and 1.2 also imply Assumption 2.6. We can apply Theorems 2.7 to obtain a description of (z, ν) which is coherent with (2.17) and (2.18). We find that $\mathrm{x} \equiv \mathrm{x}(\varepsilon, \tau, x)$ is given by

$$\begin{aligned} \mathrm{x} = \sum_{j=0}^{N-2} \varepsilon^j Z_{jx} \Big((z_0, \nu_0)(\varepsilon, x); \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathscr{V}}_{-1} \rangle \big(z_0(\varepsilon, x); \tau \big)}{\varepsilon^2} \\ + \frac{\overline{\mathscr{V}}_0 \big((z_0, \nu_0)(\varepsilon, x); \tau, \frac{\tau}{\varepsilon} \big)}{\varepsilon} \Big) + \mathcal{O}(\varepsilon^{N-1}) \end{aligned} \tag{6.21}$$

where the Z_{jx}, $\langle \overline{\mathscr{V}}_{-1} \rangle$ and $\overline{\mathscr{V}}_0$ are issued from the procedure of Section 5. Before proceeding, the expansion of $\mathrm{x}(\varepsilon, \tau, x)$ must be further simplified. Next we a result on the asymptotic oscillating description of the spatial component of the characteristics.

Lemma 6.5. *Under Assumptions 1.1, 1.2, and 1.3, the map* $(\tau, x) \mapsto \mathrm{x}(\varepsilon, \tau, x)$ *can be expressed according to the following asymptotic expansion (which is valid for all* $N \in \mathbb{N}$ *with* $N \geq 2$*)*

$$\begin{aligned} &\mathrm{x}(\varepsilon, \tau, x) \\ &= Z^\circ_{0x}(x; \tau, \frac{\tau}{\varepsilon}) + \varepsilon^1 Z^\circ_{1x}(x; \tau, \frac{\tau}{\varepsilon}) \\ &\quad + \sum_{j=2}^{N} \varepsilon^j Z^\circ_{jx} \Big(x; \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathscr{V}}^\circ_{-1} \rangle (x; \tau)}{\varepsilon^2} + \frac{\overline{\mathscr{V}}^\circ_0 \big(x; \tau, \frac{\tau}{\varepsilon} \big)}{\varepsilon} \Big) + \mathcal{O}(\varepsilon^{N+1}), \end{aligned} \tag{6.22}$$

where, with z_{00}, z_{01} *and* ν_{00} *as in* (6.18)*, we have introduced*

$$\langle \overline{\mathscr{V}}^\circ_{-1} \rangle (x; \tau) := \langle \overline{\mathscr{V}}_{-1} \rangle \big(z_{00}(x); \tau \big), \tag{6.23}$$

$$\overline{\mathscr{V}}^\circ_0 (x; \tau, \theta_\tau) := \big(z_{01}(x) \cdot \nabla_{z_0} \big) \langle \overline{\mathscr{V}}_{-1} \rangle \big(z_{00}(x); \tau \big) + \overline{\mathscr{V}}_0 \big(z_{00}(x), \nu_{00}(x); \tau, \theta_\tau \big), \tag{6.24}$$

and where, for $j \geq 0$*, the profiles* $Z^\circ_{jx}(x; \tau, \theta_\tau, \theta^\circ_{\mathrm{r}})$ *are periodic with respect to the two last variables* $\theta_\tau \in \mathbb{T}$ *and* $\theta^\circ_{\mathrm{r}} \in \mathbb{T}$.

Recall that the three couples of phases

$$\langle \overline{\mathscr{V}}_{-1} \rangle, \ \overline{\mathscr{V}}_0 \text{ in (6.21)}, \quad \langle \overline{\mathscr{V}}^\circ_{-1} \rangle, \ \overline{\mathscr{V}}^\circ_0 \text{ in (6.22)}, \quad \overline{\hat{\mathscr{V}}}^\circ_{-1},], \overline{\hat{\mathscr{V}}}^\circ_0 \text{ in (6.44)}$$

are (in general) distinct from one another. This is why the symbols θ_{r}, $\theta^\circ_{\mathrm{r}}$, and $\hat{\theta}^\circ_{\mathrm{r}}$ are not the same. This is aimed at highlighting the difference between the various phases that are involved.

Proof of Lemma 6.5. Comparing (6.21) with (6.22), there are two improvements:

(A) The first two terms of the expansion - that is the first line of (6.22) have been clarified;

(B) The structure of the phase - that is what comes to replace $\theta_{\rm r}^\circ$ in the second line of (6.22) has been reduced.

We consider one item at the time.

(A) Recall (5.34) and (6.16). Since $\Xi_{1x}^* \equiv 0$, there remains

$$\begin{aligned}
&\sum_{j=0}^{N-2} \varepsilon^j \, Z_{jx}\big((z_0,\mathrm{v}_0)(\varepsilon,x);\tau,\theta_\tau,\theta_{\rm r}\big) \\
&= \Xi_{0x}\Big(\sum_{j=0}^{N-2} \varepsilon^j \mathcal{Z}_j\big((z_0,\mathrm{v}_0)(\varepsilon,x);\tau,\theta_\tau,\theta_{\rm r}\big);\theta_\tau\Big) + \mathcal{O}(\varepsilon^{N-1}).
\end{aligned} \tag{6.25}$$

Keep in mind that θ_τ and $\theta_{\rm r}$ must be replaced as indicated in (6.21). After this substitution, we can assert that

$$\begin{aligned}
&\sum_{j=0}^{N-2} \varepsilon^j \, \mathcal{Z}_j\Big((z_0,\mathrm{v}_0)(\varepsilon,x);\tau,\frac{\tau}{\varepsilon},\frac{\langle\overline{\mathscr{V}}_{-1}\rangle\big(z_0(\varepsilon,x);\tau\big)}{\varepsilon^2} \\
&\quad + \frac{\overline{\mathscr{V}}_0\big((z_0,\mathrm{v}_0)(\varepsilon,x);\tau,\frac{\tau}{\varepsilon}\big)}{\varepsilon}\Big) \\
&= \sum_{j=0}^{N-2} \varepsilon^j \, \tilde{\mathcal{Z}}_j\Big(x;\tau,\frac{\tau}{\varepsilon},\frac{\langle\overline{\mathscr{V}}_{-1}^\circ\rangle(x;\tau)}{\varepsilon^2} + \frac{\overline{\mathscr{V}}_0^\circ\big(x;\tau,\frac{\tau}{\varepsilon}\big)}{\varepsilon}\Big) + \mathcal{O}(\varepsilon^{N-1}).
\end{aligned} \tag{6.26}$$

To elucidate the origin of (6.26), we expand $\langle\overline{\mathscr{V}}_{-1}\rangle(\cdot;\tau)$ and $\overline{\mathscr{V}}_0(\cdot;\tau,\theta_\tau)$ composed with $(z_0,\mathrm{v}_0)(\varepsilon,x)$ in powers of ε. Knowing (2.16), this yields (6.23) and (6.24) as well as

$$\begin{aligned}
&\langle\overline{\mathscr{V}}_{-1}\rangle\big(z_0(\varepsilon,x);\tau\big) + \varepsilon\overline{\mathscr{V}}_0\big((z_0,\mathrm{v}_0)(\varepsilon,x);\tau,\frac{\tau}{\varepsilon}\big) \\
&= \langle\overline{\mathscr{V}}_{-1}^\circ\rangle(x;\tau) + \varepsilon\overline{\mathscr{V}}_0^\circ\big(x;\tau,\frac{\tau}{\varepsilon}\big) + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{6.27}$$

Then, we stick to the following strategy in order to recover the profiles $\tilde{\mathcal{Z}}_j$ introduced in (6.26):

(A.1) Use (6.27) to localize the oscillations at frequencies ε^{-2} at the position $\langle\overline{\mathscr{V}}_{-1}^\circ\rangle + \varepsilon\overline{\mathscr{V}}_0^\circ$ and incorporate the $O(1)$-remainder inside a $O(1)$-shift in $\theta_{\rm r}$ of the $\mathcal{Z}_j\big((z_0,\mathrm{v}_0)(\varepsilon,x);\tau,\theta_\tau,\cdot\big)$. Note that this shift depends smoothly on x, τ and τ/ε, and therefore it can be incorporated inside the $\mathcal{Z}_j$ by modifying their description;

(A.2) Expand the (new) preceding profiles $\mathcal{Z}_j(\cdot;\tau,\theta_\tau,\theta_{\rm r})$ composed with $(z_0,\mathrm{v}_0)(\varepsilon,x)$ in powers of ε by using (6.17);

(A.3) Gather the ε^j-terms coming from the left-hand side of (6.26) after applying the above two steps to recover the final expressions $\tilde{\mathcal{Z}}_j$.

In particular, we can consider (5.25), (5.26), (5.28), and (5.29) to see that

$$\begin{aligned}
\mathcal{Z}_0(z_0,\mathrm{v}_0;\tau,\theta_\tau,\theta_{\rm r}) &= \langle\overline{\mathfrak{Z}}_0\rangle(z_0;\tau), \\
\mathcal{Z}_1(z_0,\mathrm{v}_0;\tau,\theta_\tau,\theta_{\rm r}) &= \overline{\mathfrak{Z}}_1(z_0;\tau,\theta_\tau), \\
\mathcal{Z}_2(z_0,\mathrm{v}_0;\tau,\theta_\tau,\theta_{\rm r}) &= \mathfrak{Z}_2\big(z_0,\mathrm{v}_0;\tau,\theta_\tau,\theta_{\rm r}+\overline{\mathcal{V}}_1(z_0,\mathrm{v}_0;\tau,\theta_\tau)\big).
\end{aligned} \tag{6.28}$$

Following the methodology of the above explanations, (6.28) reveals that the first three terms of the right hand side of (6.26) are defined as follows:

$$\tilde{\mathcal{Z}}_0(x;\tau) := \langle\overline{\mathfrak{Z}}_0\rangle(z_{00}(x);\tau) \tag{6.29}$$

$$\begin{aligned}\tilde{\mathcal{Z}}_1(x;\tau,\theta_\tau) &:= (z_{01}(x)\cdot\nabla_{z_0})\langle\overline{\mathfrak{Z}}_0\rangle\big(z_{00}(x);\tau\big) + \overline{\mathfrak{Z}}_1\big(z_{00}(x);\tau,\theta_\tau\big) \qquad (6.30)\\ \tilde{\mathcal{Z}}_2(x;\tau,\theta_\tau,\theta_{\rm r}^\circ) &:= \mathfrak{z}_2\big(z_{00}(x),\nu_{00}(x);\tau,\theta_\tau,\theta_{\rm r}^\circ + b(x;\tau,\theta_\tau)\big)\\ &\quad + (z_{01}(x)\cdot\nabla_{z_0})\overline{\mathfrak{Z}}_1\big(z_{00}(x);\tau,\theta_\tau\big) + (z_{02}(x)\cdot\nabla_{z_0})\langle\overline{\mathfrak{Z}}_0\rangle(z_{00}(x);\tau)\\ &\quad + \frac{1}{2}\,\mathrm{D}^2_{z_0}\langle\overline{\mathfrak{Z}}_0\rangle(z_{00}(x);\tau)\big(z_{01}(x),z_{01}(x)\big)\end{aligned} \tag{6.31}$$

where

$$\begin{aligned}&b(x;\tau,\theta_\tau)\\ &:= \overline{\mathcal{V}}_1(z_{00}(x),\nu_{00}(x);\tau,\theta_\tau) + \big[\big(z_{01}(x)\cdot\nabla_{z_0}\big) + \nu_{01}(x)\partial_{\nu_0}\big]\overline{\mathscr{V}}_0(z_{00}(x),\nu_{00}(x);\tau,\theta_\tau)\\ &\quad + \big(z_{02}(x)\cdot\nabla_{z_0}\big)\langle\overline{\mathscr{V}}_{-1}\rangle(z_{00}(x);\tau) + \frac{1}{2}\mathrm{D}^2_{z_0}\langle\overline{\mathscr{V}}_{-1}\rangle(z_{00}(x);\tau)\big(z_{01}(x),z_{01}(x)\big).\end{aligned}$$

By using (5.31) together with (6.29), the leading term issued (after expansion in powers of ε) from the right hand side of (6.25) is

$$Z^\circ_{0x}(x;\tau,\theta_\tau) := \Xi_{0x}\big(\tilde{\mathcal{Z}}_0(x;\tau);\theta_\tau\big) = \Xi_{0x}\big(\langle\overline{\mathfrak{Z}}_0\rangle(z_{00}(x);\tau);\theta_\tau\big). \tag{6.32}$$

Remark 6.6 (About the content of Z°_{0x})**.** The expression $\langle\overline{\mathfrak{Z}}_0\rangle$ is given by (4.35) with A_1 as in (3.34). Since $\Xi_{0u}(\mathfrak{z};\theta_\tau) = \mathfrak{z}_u$ and because $\Xi^*_{1u}\equiv 0$, we have

$$\langle\overline{\mathrm{A}}_{1u}\rangle = \langle\overline{\mathsf{A}}_{1u}\rangle = \big[(\mathfrak{z}_p\cdot\nabla_p)\langle\mathsf{H}_0\rangle - \langle\mathsf{H}_0\rangle\big](\mathfrak{z}).$$

Because of (1.7) we know that $\langle\overline{\mathrm{A}}_{1u}\rangle > 0$. The component $\langle\overline{\mathfrak{Z}}_{0u}\rangle$ is therefore strictly increasing (in τ), and these variations can affect (by a coupling effect) the two components $\langle\overline{\mathfrak{Z}}_{0x}\rangle$ and $\langle\overline{\mathfrak{Z}}_{0p}\rangle$. On the other hand, in view of Remark 6.3, the map Ξ_{0x} can depend on $s\equiv\theta_\tau$. In general, the function Z°_{0x} is subject to variations in both τ and θ_τ.

By applying (5.32) and (6.30) to the component Z_{1x}, since $\Xi_{1x}\equiv\Xi^*_{1x}\equiv 0$, we can collect the ε-terms from (6.25) after expansion to see that

$$Z^\circ_{1x}(x;\tau,\theta_\tau) := \big[\tilde{\mathcal{Z}}_1(x;\tau,\theta_\tau)\cdot\nabla_{\mathfrak{z}}\big]\Xi_{0x}\big(\langle\overline{\mathfrak{Z}}_0\rangle(z_{00}(x);\tau);\theta_\tau\big). \tag{6.33}$$

In the above process, the variable $\theta_{\rm r}$ is not requested, as it could be. This is the first important simplification.

(B) Let us now consider the second improvement. To elucidate what happens for $j\geq 2$, in view of (6.25), it suffices to compose $\Xi_{0x}(\cdot;\theta_\tau)$ with (6.26). The main things have already been said at the level of A.1). There remains the form exhibited in (6.22) with in particular:

$$\begin{aligned}Z^\circ_{2x}(x;\tau,\theta_\tau,\theta^0_{\rm r}) &:= \big(\tilde{\mathcal{Z}}_2(x;\tau,\theta_\tau,\theta^\circ_{\rm r})\cdot\nabla_{\mathfrak{z}}\big)\Xi_{0x}\big(\langle\overline{\mathfrak{Z}}_0\rangle(z_{00}(x);\tau);\theta_\tau\big)\\ &\quad + \frac{1}{2}\mathrm{D}^2_{\mathfrak{z}}\Xi_{0x}\big(\langle\overline{\mathfrak{Z}}_0\rangle(z_{00}(x);\tau);\theta_\tau\big)\big(\tilde{\mathcal{Z}}_1(x;\tau,\theta_\tau),\tilde{\mathcal{Z}}_1(x;\tau,\theta_\tau)\big).\end{aligned} \tag{6.34}$$

□

Remark 6.7 (About the matching of initial data)**.** It is instructive to compare the value $\mathrm{x}(\varepsilon,0,x)$ given by (6.17), that is $z_{0x}(\varepsilon,x)\equiv x$ with the formula (6.22) at time $\tau=0$. Since $\mathscr{V}_{-1}\equiv\mathcal{V}_{-1}$, taking into account (4.44), (6.18), and (6.23), we find $\langle\overline{\mathscr{V}}^\circ_{-1}\rangle(x;0) = \langle\overline{\mathcal{V}}_{-1}\rangle(x,\nabla_x\mathcal{U}_{00}(x),0;0) = 0$. On the other hand, from (6.24) and (4.47), we have

$$\overline{\mathscr{V}}^\circ_0(x;0,0) = \overline{\mathscr{V}}_0\big(x,\nabla_x\mathcal{U}_{00}(x),0,\mathcal{U}_{00}(x);0,0\big) = \nu_{00}(x)\equiv\mathcal{U}_{00}(x).$$

Then, from (6.22), we can infer that

$$\mathrm{x}(\varepsilon,0,x)=\sum_{j=0}^{N}\varepsilon^j Z^{\circ}_{jx}\Big(x;0,0,\frac{\nu_{00}(x)}{\varepsilon}\Big)+\mathcal{O}(\varepsilon^{N+1}).$$

But, in line with Remark 3.9, since $\Xi^*_{1x}\equiv 0$, the initial data $\mathfrak{z}_{0x}$ must reduce to

$$\mathfrak{z}_{0x}(\varepsilon,x)=\Xi^{-1}_{0x}\big(z_0(\varepsilon,x),0\big)=z_{0x}(\varepsilon,x)=x.$$

This means that $Z^{\circ}_{0x}(x,0,0)\equiv x$ and $Z^{\circ}_{jx}(x,0,0,\theta^{\circ}_{\mathrm{r}})\equiv 0$ for all $j\geq 1$. These properties could be deduced from the preceding construction of the Z°_{jx}. The absence of the variable $\theta^{\circ}_{\mathrm{r}}$ is specific to $\mathfrak{z}_x$ and to $\tau=0$. Since $\Xi^*_{1p}\not\equiv 0$, the component $\mathfrak{z}_{0p}(\varepsilon,x)$ can indeed oscillate with respect to $\nu_{00}(x)/\varepsilon$. Moreover, due to coupling effects, the expressions Z°_{jx} can depend on $\theta^{\circ}_{\mathrm{r}}$ for $\tau>0$.

6.3.2. *Role of transparency conditions.* The aim of this paragraph is to compute $\mathrm{D}_x\mathrm{x}$. It is also to show how some kind of transparency conditions (emanating from Assumption 1.3) leads to have a control on the size of this differential. In view of the expansion (6.22), we can compute the Jacobian matrix $\mathrm{D}_x\mathrm{x}(\varepsilon,\tau,x)$ according to

$$\begin{aligned}
&\mathrm{D}_x\mathrm{x}(\varepsilon,\tau,x)\\
&=\mathrm{D}_x Z^{\circ}_{0x}(x;\tau,\frac{\tau}{\varepsilon})\\
&\quad+\partial_{\theta^{\circ}_{\mathrm{r}}}Z^{\circ}_{2x}\Big(x;\tau,\frac{\tau}{\varepsilon},\frac{\langle\overline{\mathscr{V}}^{\circ}_{-1}\rangle(x;\tau)}{\varepsilon^2}+\frac{\overline{\mathscr{V}}^{\circ}_0(x;\tau,\frac{\tau}{\varepsilon})}{\varepsilon}\Big)\otimes\nabla_x\langle\overline{\mathscr{V}}^{\circ}_{-1}\rangle(x;\tau)+\mathcal{O}(\varepsilon).
\end{aligned}\tag{6.35}$$

Recalling (1.8) and (6.32) we see that

$$\begin{aligned}
Z^{\circ}_{0x}(x;\tau,s)&=\Xi_{0x}\big(\langle\overline{\mathfrak{Z}}_0\rangle(z_{00}(x);\tau);s\big)\\
&=\langle\overline{\mathfrak{Z}}_{0x}\rangle(z_{00}(x);\tau)+\int_0^s\nabla_p\mathsf{H}_0\Big(r,\Xi_{0x}(\langle\overline{\mathfrak{Z}}_0\rangle(z_{00}(x);\tau);r),\\
&\qquad\langle\overline{\mathfrak{Z}}_{0u}\rangle(z_{00}(x);\tau),\Xi_{0p}(\langle\overline{\mathfrak{Z}}_0\rangle(z_{00}(x);\tau);r)\Big)\,dr.
\end{aligned}\tag{6.36}$$

In view of (6.35), the expression $\mathrm{D}_xZ^{\circ}_{0x}(x;\tau,s)$ must be computed at the position $s=\tau/\varepsilon$. This means that s must be replaced by τ/ε in (6.36). At first sight, the integral (in r) from 0 up to τ/ε should furnish a contribution of size ε^{-1}, which would indicate that $\mathrm{D}_x\mathrm{x}(\varepsilon,\tau,x)$ is very large (and therefore out of control). However, as mentioned in Remark 3.13, Assumption 1.3 implies that

$$\langle\nabla_p\mathsf{H}_0(\cdot,\Xi_{0x}(\mathfrak{z};\cdot),\mathfrak{z}_u,\Xi_{0p}(\mathfrak{z};\cdot))\rangle\equiv 0.\tag{6.37}$$

This identity can be viewed as a *transparency condition*: nonlinear terms that should contribute (without Assumption 1.3) disappear in practice. As a matter of fact, denoting by $\lfloor s\rfloor$ the integer part of the real number s, we can assert that

$$\begin{aligned}
&Z^{\circ}_{0x}(x;\tau,s)\\
&=\langle\overline{\mathfrak{Z}}_{0x}\rangle\big(z_{00}(x);\tau\big)+\int_{2\pi\lfloor s/2\pi\rfloor}^{s}\nabla_p\mathsf{H}_0\Big(r,\Xi_{0x}(\langle\overline{\mathfrak{Z}}_0\rangle(z_{00}(x);\tau);r),\\
&\qquad\langle\overline{\mathfrak{Z}}_{0u}\rangle(z_{00}(x);\tau),\Xi_{0p}(\langle\overline{\mathfrak{Z}}_0\rangle(z_{00}(x);\tau);r)\Big)\,dr.
\end{aligned}\tag{6.38}$$

We see here why the cancelation property (6.37) is crucial. It allows to reduce the long time integration (when s is replaced by τ/ε with $\tau>0$ and $\varepsilon\ll 1$) in the second line of (6.36) to an integration over some interval of uniformly bounded size (in s), as indicated in (6.38). We have the following interesting expression of the differential $\mathrm{D}_x \mathrm{x}$.

Lemma 6.8 (Differential of the characteristic x)**.** *Select $R \in \mathbb{R}_+^*$. Under Assumption 1.3, for all $\varepsilon \in [0, \varepsilon_0]$, for all $(\tau, x) \in [0, \mathcal{T}] \times B(0, R]$, we have*

$$\mathrm{D}_x \mathrm{x}(\varepsilon, \tau, x) = \mathrm{Id} + \mathcal{O}(\tau) + \mathcal{O}(\| D_{x,p,u} \nabla_p \mathsf{H}_0 \|) + \mathcal{O}(\varepsilon). \tag{6.39}$$

Formula (6.39) with Assumption 1.4 are the gate to prove that the map $x \mapsto \mathrm{x}(\varepsilon; \cdot)$ is uniformly invertible. We will prove this fact in the next Section 6.4.

Proof of Lemma reflem6.4. From (4.35) with (6.18) we have

$$\langle \overline{\mathfrak{Z}}_{0x} \rangle \big(z_{00}(x); \tau \big) = x + \int_0^\tau \langle \overline{\mathrm{A}}_{1x} \rangle \big(\langle \overline{\mathfrak{Z}}_0 \rangle (x, \nabla_x \mathcal{U}_{00}(x), 0; r) \big) \, dr. \tag{6.40}$$

From (6.38) and (6.40), since $\Xi_0(\mathfrak{z}; \cdot)$ is globally bounded (since it is a periodic function), with $\|\!| \cdot |\!\|$ as in (1.9), we can already infer that

$$\mathrm{D}_x Z_{0x}^\circ (x; \tau, s) = \mathrm{Id} + \mathcal{O}(\tau) + \mathcal{O}(|\!\| D_{x,p,u} \nabla_p \mathsf{H}_0 |\!\|). \tag{6.41}$$

On the other hand, we can exploit (4.41) together with $\mathscr{V}_{-1} \equiv \mathcal{V}_{-1}$, as well as (6.18), (6.13) and (6.23) to deduce that

$$\begin{aligned} \langle \overline{\mathscr{V}}_{-1}^\circ \rangle (x; \tau) = \int_0^\tau & \big[(\langle \overline{\mathfrak{Z}}_{0p} \rangle (x, \nabla_x \mathcal{U}_{00}(x), 0; r) \cdot \nabla_p) \langle \mathsf{H}_0 \rangle \\ & - \langle \mathsf{H}_0 \rangle \big] \big(\langle \overline{\mathfrak{Z}}_0 \rangle (x, \nabla_x \mathcal{U}_{00}(x), 0; r) \big) dr. \end{aligned} \tag{6.42}$$

In view of (6.35) and taking into account (6.38), with $\langle \overline{\mathfrak{Z}}_0 \rangle \equiv \langle \overline{\mathfrak{Z}}_0 \rangle (z_{00}(x); \tau)$, we obtain that

$$\begin{aligned} & \mathrm{D}_x \mathrm{x}(\varepsilon, \tau, x) \\ & = \mathrm{Id} + \int_0^\tau \mathrm{D}_x \big[\langle \overline{\mathrm{A}}_{1x} \rangle \big(\langle \overline{\mathfrak{Z}}_0 \rangle (x, \nabla_x \mathcal{U}_{00}(x), 0; r) \big) \big] \, dr \\ & \quad + \int_{2\pi \lfloor \tau / 2\pi\varepsilon \rfloor}^{\tau/\varepsilon} \mathrm{D}_x \big[\nabla_p \mathsf{H}_0 \big(r, \Xi_{0x} (\langle \overline{\mathfrak{Z}}_0 \rangle; r), \langle \overline{\mathfrak{Z}}_{0u} \rangle (z_{00}(x); \tau), \Xi_{0p} (\langle \overline{\mathfrak{Z}}_0 \rangle; r) \big) \big] dr \\ & \quad + \partial_{\theta_{\mathrm{r}}^\circ} Z_{2x}^\circ \Big(x; \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathscr{V}}_{-1}^\circ \rangle (x; \tau)}{\varepsilon^2} + \frac{\overline{\mathscr{V}}_0^\circ \big(x; \tau, \frac{\tau}{\varepsilon} \big)}{\varepsilon} \Big) \otimes \nabla_x \langle \overline{\mathscr{V}}_{-1}^\circ \rangle (x; \tau) + \mathcal{O}(\varepsilon). \end{aligned} \tag{6.43}$$

Using (6.42), we can control the forth line of (6.43) by some $\mathcal{O}(\tau)$. On the other hand, we have $0 \le \tau/\varepsilon - 2\pi \lfloor \tau / 2\pi\varepsilon \rfloor \le 2\pi$ in such a way that $\mathrm{D}_x \mathrm{x}(\varepsilon, \tau, x)$ is as in (6.39). ☐

6.4. **Inverse map x^{-1}.** The inverse of the spatial characteristic $\mathrm{x}(\varepsilon, \tau, x)$ is denoted by $\mathrm{x}^{-1}(\varepsilon, \tau, \tilde{x})$. The existence part of Theorem 1.5 relies on (6.8). To this end, it is important to show that $\mathrm{x}^{-1}(\varepsilon, \tau, \tilde{x})$ is uniformly defined. Below, we present the statement of the uniform local existence of x^{-1}.

Lemma 6.9 (Uniform local existence of x^{-1})**.** *Select $R \in \mathbb{R}_+^*$. Under Assumptions 1.3 and 1.4, for $|\tau|$ small enough, the map $x \in B(0, R] \mapsto \tilde{x} = \mathrm{x}(\varepsilon, \tau, x)$ is for all $\varepsilon \in]0, \varepsilon_0]$ (by restricting ε_0 if necessary) locally uniformly (in ε) invertible.*

Proof. The proof is based on the expansion (6.39). We take $\tau \le \mathcal{T}$ and $\varepsilon \le \varepsilon_0$ with $\mathcal{T}$ and ε_0 small enough. We work under Assumption 1.4 with δ small enough. In view of (6.41) and (6.39), both $\mathrm{D}_x Z_{0x}^\circ (x; \tau, s)$ and $\mathrm{D}_x \mathrm{x}(\varepsilon, \tau, x)$ are of the form $\mathrm{Id} + \mathrm{B}$ with $\|\mathrm{B}\| < 1$. Thus, for all $\tau \in [0, \mathcal{T}]$, the maps $x \mapsto Z_{0x}^\circ (x; \tau, s)$ and $x \mapsto \mathrm{x}(\varepsilon, \tau, x)$

are locally invertible (uniformly in $s \equiv \theta_\tau$ for the first map and in $\varepsilon \in]0, \varepsilon_0]$ for the second). $\square$

The next step is to find the asymptotic expansion of the inverse x^{-1}. We have the following asymptotic description of the inverse map of the spatial component of the characteristics.

Lemma 6.10. *Under Assumptions 1.1, 1.2, 1.3, and 1.4, for $|\tau|$ small enough, for all $\varepsilon \in]0, \varepsilon_0]$ and for all $N \geq 2$, the inverse map $\mathrm{x}^{-1}(\varepsilon, \tau, \tilde{x})$ can be expanded according to*

$$\begin{aligned} &\mathrm{x}^{-1}(\varepsilon, \tau, \tilde{x}) \\ &= \hat{Z}^\circ_{0x}\Big(\tilde{x}; \tau, \frac{\tau}{\varepsilon}\Big) + \varepsilon\, \hat{Z}^\circ_{1x}\Big(\tilde{x}; \tau, \frac{\tau}{\varepsilon}\Big) \\ &\quad + \sum_{j=2}^{N} \varepsilon^j\, \hat{Z}^\circ_{jx}\Big(\tilde{x}; \tau, \frac{\tau}{\varepsilon}, \frac{\overline{\hat{\mathscr{V}}}{}^\circ_{-1}(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon^2} + \frac{\overline{\hat{\mathscr{V}}}{}^\circ_{0}(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon}\Big) + \mathcal{O}(\varepsilon^{N+1}), \end{aligned} \tag{6.44}$$

where $\hat{Z}^\circ_{0x} \equiv (Z^\circ_{0x})^{-1}$ is the local inverse of the map $x \mapsto Z^\circ_{0x}(x; \tau, \theta_\tau)$, where we have introduced

$$\overline{\hat{\mathscr{V}}}{}^\circ_{-1}(\tilde{x}; \tau, \theta_\tau) := \langle \overline{\mathscr{V}}{}^\circ_{-1} \rangle \big(\hat{Z}^\circ_{0x}(\tilde{x}, \tau, \theta_\tau); \tau\big), \tag{6.45}$$

$$\begin{aligned} &\overline{\hat{\mathscr{V}}}{}^\circ_{0}(\tilde{x}; \tau, \theta_\tau) \\ &:= \big(\hat{Z}^\circ_{1x}(\tilde{x}; \tau, \theta_\tau) \cdot \nabla_x\big) \langle \overline{\mathscr{V}}{}^\circ_{-1} \rangle \big(\hat{Z}^\circ_{0x}(\tilde{x}, \tau, \theta_\tau); \tau\big) + \overline{\mathscr{V}}{}^\circ_{0}\big(\hat{Z}^\circ_{0x}(\tilde{x}, \tau, \theta_\tau); \tau, \theta_\tau\big), \end{aligned} \tag{6.46}$$

and where, for $j \geq 0$, the profiles $\hat{Z}^\circ_{jx}(\tilde{x}; \tau, \theta_\tau, \hat{\theta}^\circ_{\mathrm{r}})$ are periodic with respect to the two last variables $\theta_\tau \in \mathbb{T}$ and $\hat{\theta}^\circ_{\mathrm{r}} \in \mathbb{T}$.

The rest of this section is devoted to the proof of Lemma 6.10. The proof is achieved in three steps: in Paragraph 6.4.1, we give the formal expansion of the inverse x^{-1}; at ε^2-order, we face a strong nonlinearity which is overcome by implementing the Hadamard's global inverse function theorem in Paragraph 6.4.2; finally, in Paragraph 6.4.3, we complete the proof of Lemma 6.10 by justifying the formal WKB expansion.

6.4.1. *Formal equations for* x^{-1}. We turn now to the proof of (6.44). By the definition of x^{-1}, we have

$$\mathrm{x}\big(\varepsilon, \tau, \mathrm{x}^{-1}(\varepsilon, \tau, \tilde{x})\big) = \tilde{x}\,. \tag{6.47}$$

We can seek x^{-1} in the form of an asymptotic expansion similar to (6.22), like in (6.44), that is with a hat "$\hat{\ }$" on each expression to make the distinction. In other words, we can postulate (6.44) and use (6.47) to check that (6.44) is indeed convenient. This means to deal with

$$\sum_{j=0}^{N} \varepsilon^j\, Z^\circ_{jx}\Big(\mathrm{x}^{-1}; \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathscr{V}}{}^\circ_{-1} \rangle (\mathrm{x}^{-1}; \tau)}{\varepsilon^2} + \frac{\overline{\mathscr{V}}{}^\circ_{0}(\mathrm{x}^{-1}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon}\Big) + \mathcal{O}(\varepsilon^{N+1}) = \tilde{x}\,. \tag{6.48}$$

The leading order term inside (6.48) gives rise to $Z^\circ_{0x}(\hat{Z}^\circ_{0x}; \tau, \theta_\tau) = \tilde{x}$. This relation can be achieved with $\hat{Z}^\circ_{0x} \equiv (Z^\circ_{0x})^{-1}$. The next term, the one which has ε in factor inside (6.48), leads to

$$\hat{Z}^\circ_{1x} \equiv \hat{Z}^\circ_{1x}(\tilde{x}; \tau, \theta_\tau) := -\,\mathrm{D}_x Z^\circ_{0x}\big(\hat{Z}^\circ_{0x}(\tilde{x}; \tau, \theta_\tau); \tau, \theta_\tau\big)^{-1}\; Z^\circ_{1x}\big(\hat{Z}^\circ_{0x}(\tilde{x}; \tau, \theta_\tau); \tau, \theta_\tau\big)\,.$$

As foreseen, the rapid variable $\hat{\theta}^0_{\rm r}$ can be activated at the level of the profiles $\hat{Z}^\circ_{jx}$ only for $j \geq 2$. Now, in coherence with (6.45) and (6.46), we can assert that

$$\begin{aligned}&\frac{1}{\varepsilon^2}\,\langle\overline{\mathscr{V}}^\circ_{-1}\rangle(\mathrm{x}^{-1}(\varepsilon,\tau,\tilde{x});\tau)+\frac{1}{\varepsilon}\,\overline{\mathscr{V}}^\circ_0(\mathrm{x}^{-1}(\varepsilon,\tau,\tilde{x});\tau,\theta_\tau)\\&=\frac{1}{\varepsilon^2}\,\overline{\hat{\mathscr{V}}}^\circ_{-1}(\tilde{x};\tau,\theta_\tau)+\frac{1}{\varepsilon}\,\overline{\hat{\mathscr{V}}}^\circ_0(\tilde{x};\tau,\theta_\tau)\\&\quad+\big(\hat{Z}^\circ_{2x}\cdot\nabla_x\big)\langle\overline{\mathscr{V}}^\circ_{-1}\rangle\big(\hat{Z}^\circ_{0x}(\tilde{x},\tau,\theta_\tau);\tau\big)+\frac{1}{2}\,D^2_x\langle\overline{\mathscr{V}}^\circ_{-1}\rangle(\hat{Z}^\circ_{1x},\hat{Z}^\circ_{1x})\\&\quad+\big(\hat{Z}^\circ_{1x}\cdot\nabla_x\big)\overline{\mathscr{V}}^\circ_0\big(\hat{Z}^\circ_{0x}(\tilde{x},\tau,\theta_\tau);\tau,\theta_\tau\big)+\mathcal{O}(\varepsilon)\,.\end{aligned}$$

We may see (6.48) as a consequence of a relaxed condition involving τ, $\tilde{x}$, θ_τ and $\hat{\theta}^\circ_{\rm r}$. Then, we can work with $\tilde{x}$, θ_τ and $\hat{\theta}^\circ_{\rm r}$ fixed in compact sets. In what follows, these variables are mentioned only when it is necessary to avoid confusion. In this perspective, the contribution which has ε^2 in factor inside (6.48) can be written

$$\hat{Z}^\circ_{2x}+\mathcal{F}(\tau,\hat{Z}^\circ_{2x})=0 \tag{6.49}$$

with by construction

$$\begin{aligned}&\mathcal{F}(\tau,\hat{Z}^\circ_{2x})\\&:=\mathrm{D}_xZ^\circ_{0x}(\hat{Z}^\circ_{0x};\tau,\theta_\tau)^{-1}Z^\circ_{2x}\big(\hat{Z}^\circ_{0x};\tau,\theta_\tau,\hat{\theta}^\circ_{\rm r}+(\hat{Z}^\circ_{2x}\cdot\nabla_x)\langle\overline{\mathscr{V}}^\circ_{-1}\rangle(\hat{Z}^\circ_{0x};\tau)+\mathcal{D}^0\big)+\mathcal{D}^1,\end{aligned}$$

where $\mathcal{D}^0$ and $\mathcal{D}^1$ are entirely determined (since they depend on the already known functions $\hat{Z}^\circ_{0x}$ and $\hat{Z}^\circ_{1x}$). Remark that (6.49) is a *nonlinear* equation. This means that the actual asymptotic calculus is *critical*. We come back to this point in Paragraph 6.4.2.

For $j \geq 3$, the situation is easier since we have to deal with a linearized version of (6.49), which looks like

$$\begin{aligned}&\hat{Z}^\circ_{jx}+(\hat{Z}^\circ_{jx}\cdot\nabla_x)\langle\overline{\mathscr{V}}^\circ_{-1}\rangle(\hat{Z}^\circ_{0x};\tau)\mathrm{D}_xZ^\circ_{0x}(\hat{Z}^\circ_{0x};\tau,\theta_\tau)^{-1}\partial_{\theta^0_{\rm r}}Z^\circ_{2x}\big(\hat{Z}^\circ_{0x};\tau,\theta_\tau,\hat{\theta}^\circ_{\rm r}\\&+\tilde{\mathcal{D}}^0\big)+\mathcal{D}^j=0,\end{aligned}\tag{6.50}$$

where $\tilde{\mathcal{D}}^0$ and $\mathcal{D}^j$ are known functions since they depend on the $\hat{Z}^\circ_{kx}$ with $k<j$. Due again to (6.42), this may be formulated as $\big(\mathrm{Id}+\mathcal{O}(\tau)\big)\hat{Z}^\circ_{jx}+\mathcal{D}^j=0$ which has obviously a unique solution for $|\tau|$ small enough.

6.4.2. *Nonlinear modulation equation.* We now come back to solve the nonlinearity inherited from (6.49). Since Z°_{2x} is periodic in $\theta^\circ_{\rm r}$, the function $\mathcal{F}$ is (locally in time) uniformly bounded (say by some $\mathrm{R}\in\mathbb{R}^*_+$) with respect to the variable $\hat{Z}^\circ_{2x}\in\mathbb{R}^d$. Let $\chi\in\mathcal{C}^\infty(\mathbb{R};\mathbb{R})$ with $\chi\equiv 1$ for $|r|\leq 1$ and $\chi\equiv 0$ for $2\leq r$. We define $\chi_{\mathrm{R}}(r):=\chi(r/\mathrm{R})$. Let $\eta\in\mathbb{R}^*_+$, introduce the following auxiliary expression (in the new unknown $\tilde{Z}^\circ_{2x}$)

$$f_{\eta,\mathrm{R}}(\tau,\tilde{Z}^\circ_{2x}):=\tilde{Z}^\circ_{2x}+\mathcal{F}\big(\chi_\eta(\tau)\,\tau,\chi_{\mathrm{R}^2}(|\tilde{Z}^\circ_{2x}|^2)\,\tilde{Z}^\circ_{2x}\big). \tag{6.51}$$

Then, consider the smooth map $F_{\eta,\mathrm{R}}:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}\times\mathbb{R}^d$,

$$(\tau,\tilde{Z}^\circ_{2x})\mapsto\big(\tau,f_{\eta,\mathrm{R}}(\tau,\tilde{Z}^\circ_{2x})-\mathcal{F}(0,0)\big).$$

We want to apply the Hadamard's global inverse function theorem to the $\mathcal{C}^2$ mapping $F_{\eta,\mathrm{R}}$. To this end, we have to check the needed assumptions:

- find $F_{\eta,\mathrm{R}}(0,0)=\big(0,f_{\eta,\mathrm{R}}(0,0)-\mathcal{F}(0,0)\big)=\big(0,\mathcal{F}(0,0)-\mathcal{F}(0,0)\big)=(0,0)$;

• The limit of $|F_{\eta,\mathrm{R}}(\tau,\tilde Z^\circ_{2x})|$ is $+\infty$ when $|\tau|$ goes to $+\infty$ is infinite because the first component of $F_{\eta,\mathrm{R}}$ is just τ. The limit of $|F_{\eta,\mathrm{R}}(\tau,\tilde Z^\circ_{2x})|$ is also $+\infty$ when $|\tilde Z^\circ_{2x}|$ goes to $+\infty$ because, because of the cutoff χ_{R^2}, it is bounded below by the limit of $|\tilde Z^\circ_{2x}+\mathcal F(\chi_\eta(\tau)\,\tau,0)|$ when $|\tilde Z^\circ_{2x}|$ goes to $+\infty$. The map F is proper;

• Let us study the structure of the Jacobian matrix of $F_{\eta,\mathrm{R}}$. To this end, we have to control $\partial_\tau F_{\eta,\mathrm{R}}=(1,\partial_\tau f_{\eta,\mathrm{R}})$ and $D_{\tilde Z^\circ_{2x}}F_{\eta,\mathrm{R}}=(0,D_{\tilde Z^\circ_{2x}}f_{\eta,\mathrm{R}})$. We start by looking at the region of $\mathbb R\times\mathbb R^d$ where $\sqrt2\,\mathrm{R}\le|\tilde Z^\circ_{2x}|$. Then, we consider the ball $|\tilde Z^\circ_{2x}|\le\sqrt2\,\mathrm{R}$ with first $2\eta\le|\tau|$ and finally $|\tau|\le2\eta$.

For $\sqrt2\,\mathrm{R}\le|\tilde Z^\circ_{2x}|$, we have just to deal with

$$f_{\eta,\mathrm{R}}(\tau,\tilde Z^\circ_{2x}):=\tilde Z^\circ_{2x}+\mathcal F\big(\chi_\eta(\tau)\tau,0\big)$$

so that $\partial_\tau f_{\eta,\mathrm{R}}=\mathcal O(1)$ and $D_{\tilde Z^\circ_{2x}}f_{\eta,\mathrm{R}}=\mathrm{Id}$.

For $|\tilde Z^\circ_{2x}|\le\sqrt2\,\mathrm{R}$, we find that

$$\begin{aligned}&|\partial_\tau f_{\eta,\mathrm{R}}(\tau,\tilde Z^\circ_{2x})|\\&\le(\|\tau\chi'(\tau)\|_\infty+\|\chi\|_\infty)\max_{|\tau|\le2\eta,\|\hat Z^\circ_{2x}\|\le\sqrt2\,\mathrm{R}}\|\partial_\tau\mathcal F(\tau,\hat Z^\circ_{2x})\|_\infty<+\infty.\end{aligned}$$

Now, we look at $D_{\tilde Z^\circ_{2x}}f_{\eta,\mathrm{R}}$. For $2\eta\le|\tau|$, since $\langle\overline{\mathscr V}^\circ_{-1}\rangle\big|_{\tau=0}\equiv0$, there remains

$$\begin{aligned}&f_{\eta,\mathrm{R}}(\tau,\tilde Z^\circ_{2x})\\&=\tilde Z^\circ_{2x}+\mathrm D_xZ^\circ_{0x}(\hat Z^\circ_{0x}\big|_{\tau=0};0,\theta_\tau)^{-1}Z^\circ_{2x}\big(\hat Z^\circ_{0x}\big|_{\tau=0};0,\theta_\tau,\hat\theta^\circ_{\mathrm r}+\mathcal D^0\big|_{\tau=0}\big)+\mathcal D^1\big|_{\tau=0},\end{aligned}$$

so that $D_{\tilde Z^\circ_{2x}}f_{\eta,\mathrm{R}}(\tau,\tilde Z^\circ_{2x})=\mathrm{Id}$. In the same vein, for $|\tau|\le2\eta$, exploiting the structure of $\mathcal F$ and (6.42) again, we find that

$$\begin{aligned}D_{\tilde Z^\circ_{2x}}f_{\eta,\mathrm{R}}(\tau,\tilde Z^\circ_{2x})&=\mathrm{Id}+(D_{\hat Z^\circ_{2x}}\mathcal F)\big(\chi_\eta(\tau)\tau,\chi_{\mathrm R^2}(|\tilde Z^\circ_{2x}|^2)\tilde Z^\circ_{2x}\big)D_{\tilde Z^\circ_{2x}}\big(\chi_{\mathrm R^2}(|\tilde Z^\circ_{2x}|^2)\tilde Z^\circ_{2x}\big)\\&=\mathrm{Id}+\mathcal O(1)\,|\nabla_x\langle\overline{\mathscr V}^\circ_{-1}\rangle(\hat Z^\circ_{0x};\tau)|\\&=\mathrm{Id}+\mathcal O(\eta).\end{aligned}$$

The final outcome is

$$D_{\tau,\tilde Z^\circ_{2x}}F_{\eta,\mathrm{R}}(\tau,\tilde Z^\circ_{2x})=\begin{pmatrix}1&0\\\mathcal O(1)&\mathrm{Id}+\mathcal O(\eta)\end{pmatrix}.$$

For $|\eta|$ chosen small enough, the Jacobian matrix of $F_{\eta,\mathrm{R}}$ is bounded and the corresponding Jacobian determinant is nonzero at each point.

Thus, we can assert that $F_{\eta,\mathrm{R}}$ is one-to-one and onto. In particular, the position $(\tau,-\mathcal F(0,0))$ has a unique preimage. This furnishes some $\tilde Z^\circ_{2x}$ such that $f_{\eta,\mathrm{R}}(\tau,\tilde Z^\circ_{2x})=0$. In view of (6.51), this can be achieved only by some $\tilde Z^\circ_{2x}$ satisfying $|\tilde Z^\circ_{2x}|\le\mathrm{R}$. Now, for $|\tau|\le\eta$, knowing that $|\tilde Z^\circ_{2x}|\le\mathrm{R}$, the relation (6.49) is satisfied, since it is exactly the same as $F_{\eta,\mathrm{R}}(\tau,\tilde Z^\circ_{2x})=(\tau,-\mathcal F(0,0))$. At the end, we can say that the solution $\hat Z^\circ_{2x}$ to (6.49) exists and it is just the restriction of $\tilde Z^\circ_{2x}$ for $|\tau|\le\eta$.

6.4.3. *Proof of Lemma 6.10.* Select some $N \in \mathbb{N}^*$ with $N \geq 2$. Define the formal approximate solution x_a^{-1} as follows

$$\begin{aligned}
&\mathrm{x}_a^{-1}(\varepsilon;\tau,\tilde{x},\theta_\tau,\hat{\theta}_\mathrm{r}^\circ) \\
&:= \hat{Z}_{0x}^\circ\big(\tilde{x};\tau,\theta_\tau\big) + \varepsilon \hat{Z}_{1x}^\circ\big(\tilde{x};\tau,\theta_\tau\big) + \sum_{j=2}^{N} \varepsilon^j \hat{Z}_{jx}^\circ\big(\tilde{x};\tau,\theta_\tau,\hat{\theta}_\mathrm{r}^\circ\big),
\end{aligned} \tag{6.52}$$

where the profiles $\hat{Z}_{jx}^\circ$ are the ones constructed in the previous Paragraphs 6.4.1 and 6.4.2. To summarize, we have to show that this formal solution can be exploited to approximate the exact solution x^{-1} of (6.47). Indeed, we consider the error term R_{inv} (corresponds to the inverse) defined through the relation

$$\mathrm{x}^{-1}(\varepsilon;\tau,\tilde{x}) = \mathrm{x}_a^{-1}(\varepsilon;\tau,\tilde{x},\frac{\tau}{\varepsilon},\frac{\overline{\hat{\mathscr{V}}}{}^\circ_{-1}(\tilde{x};\tau,\frac{\tau}{\varepsilon})}{\varepsilon^2} + \frac{\overline{\hat{\mathscr{V}}}{}^\circ_{0}(\tilde{x};\tau,\frac{\tau}{\varepsilon})}{\varepsilon}) + \varepsilon^{N+1} R_{\mathrm{inv}}.$$

It is sufficient then to prove that R_{inv} can be viewed as remainder. Now, since θ_τ and $\hat{\theta}_\mathrm{r}^\circ$ belong to compact sets (torus), as well as $\tilde{x}$ (we work locally in space, with $\tilde{x}$ in a compact set), the preceding arguments (in Paragraphs 6.4.1 and 6.4.2) can be applied uniformly with respect to these variables, yielding

- the determination of the profiles $\hat{Z}_{0x}^\circ$, $\hat{Z}_{1x}^\circ$ and $\hat{Z}_{jx}^\circ$ for all $j \geq 3$ and the specification of the nonlinear equation (6.49) at ε^2-order in Paragraph 6.4.1;
- the determination of some $\hat{Z}_{2x}^\circ$ as a solution to the above mentioned nonlinear modulation equation (6.49) in Paragraph 6.4.2.

Moreover, the linearized version of (6.47) along this approximate solution furnishes for the error term R_{inv} an equation similar to (6.50). By this way, we can construct an approximate solution to (6.47) which takes indeed the form of (6.44) and which inherits a precision at any order (in terms of powers of ε). Hence, we obtain the stability and thereby (6.44) is proved. □

6.5. **Proof of Theorem 1.5.** With $\mathrm{u} \equiv \mathrm{v}$ as in Lemma 6.4 and x^{-1} as in Lemma 6.9, the formula (6.8) can be applied to recover the existence part of Theorem 1.5. We can now turn to the proof of (1.10). Knowing that $\mathrm{u} \equiv \mathrm{v}$ is given by an expansion similar to (2.18) and that x^{-1} is as in (6.44), the formula (6.8) reveals that u_ε can be obtained through a composition of three-scale oscillations. More precisely, at the level of (2.18), the values of z_0 and v_0 must be computed as indicated in (6.17) as functions of x, and then x must be replaced by the expression x^{-1} of (6.44). In other words, with $\mathrm{x}^{-1} \equiv \mathrm{x}^{-1}(\varepsilon,\tau,\tilde{x})$ as in (6.44), we have

$$\begin{aligned}
u(\varepsilon,\tau,\tilde{x}) = &\frac{1}{\varepsilon}\,\langle\overline{\mathscr{V}}_{-1}\rangle\big(z_0(\varepsilon,\mathrm{x}^{-1});\tau\big) + \overline{\mathscr{V}}_0\big((z_0,\mathrm{v}_0)(\varepsilon,\mathrm{x}^{-1});\tau,\frac{\tau}{\varepsilon}\big) \\
&+ \mathcal{O}(\varepsilon^{N+1}) + \sum_{j=1}^{N} \varepsilon^j \mathscr{V}_j\Big((z_0,\mathrm{v}_0)(\varepsilon,\mathrm{x}^{-1});\tau,\frac{\tau}{\varepsilon}, \\
&\frac{\langle\overline{\mathscr{V}}_{-1}\rangle(z_0(\varepsilon,\mathrm{x}^{-1});\tau)}{\varepsilon^2} + \frac{\overline{\mathscr{V}}_0\big((z_0,\mathrm{v}_0)(\varepsilon,\mathrm{x}^{-1});\tau,\frac{\tau}{\varepsilon}\big)}{\varepsilon}\Big).
\end{aligned} \tag{6.53}$$

This asymptotic description of u_ε is not yet in suitable form. We can further improve it in order to recover the oscillatory structure (1.10). Before proceeding, we have to consider the following preliminary steps:

(i) Expand the initial data $(z_0,\mathrm{v}_0)(\varepsilon;\cdot)$ composed with x^{-1} in powers of ε;

(ii) Clarify the structure of the phase that comes to replace $\hat{\theta}_{\mathrm{r}}^{\circ}$ in (1.10). More precisely, explain how $\hat{\theta}_{\mathrm{r}}^{\circ}$ can become a substitute for θ_r in the right hand side of (6.53).

(i) The initial data $(z_0, \nu_0)(\varepsilon; \cdot)$ composed with $\mathrm{x}^{-1} \equiv \mathrm{x}^{-1}(\varepsilon, \tau, \tilde{x})$ can be expanded in powers of ε according to

$$\begin{aligned}
&(z_0, \nu_0)(\varepsilon; \mathrm{x}^{-1}) \\
&= \sum_{j=0}^{N} \varepsilon^j \ (\hat{z}_{0j}, \hat{\nu}_{0j})\Big(\tilde{x}; \tau, \frac{\tau}{\varepsilon}, \frac{\overline{\hat{\mathscr{V}}}^{\circ}_{-1}(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon^2} + \frac{\overline{\hat{\mathscr{V}}}^{\circ}_{0}(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon}\Big) + \mathcal{O}(\varepsilon^{N+1}).
\end{aligned} \tag{6.54}$$

In particular, taking into account (6.17), (6.18), and (6.44), we find that

$$\hat{z}_{00} \equiv \overline{\hat{z}}_{00}(\tilde{x}; \tau, \theta_\tau) := z_{00} \circ \hat{Z}^{\circ}_{0x}(\tilde{x}; \tau, \theta_\tau)\,; \tag{6.55}$$

$$\hat{z}_{01} \equiv \overline{\hat{z}}_{01}(\tilde{x}; \tau, \theta_\tau) := \big((\hat{Z}^{\circ}_{1x}(\tilde{x}; \tau, \theta_\tau) \cdot \nabla_x) z_{00} + z_{01}\big) \circ \hat{Z}^{\circ}_{0x}(\tilde{x}; \tau, \theta_\tau); \tag{6.56}$$

$$\hat{\nu}_{00} \equiv \overline{\hat{\nu}}_{00}(\tilde{x}; \tau, \theta_\tau) := \nu_{00} \circ \hat{Z}^{\circ}_{0x}(\tilde{x}; \tau, \theta_\tau). \tag{6.57}$$

(ii) We need now to clarify the expansion of the part which comes to replace θ_r in (6.53). Exploit (6.55), (6.56) and (6.57) in order to expand the parts involving $\langle \overline{\mathscr{V}}_{-1} \rangle$ and $\overline{\mathscr{V}}_0$ according to

$$\begin{aligned}
&\frac{\langle \overline{\mathscr{V}}_{-1} \rangle\big(z_0(\varepsilon, \mathrm{x}^{-1}); \tau\big)}{\varepsilon^2} + \frac{\overline{\mathscr{V}}_0\big((z_0, \nu_0)(\varepsilon, \mathrm{x}^{-1}); \tau, \frac{\tau}{\varepsilon}\big)}{\varepsilon} \\
&= \frac{1}{\varepsilon^2} \langle \overline{\mathscr{V}}_{-1} \rangle \big(\overline{\hat{z}}_{00}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}); \tau\big) + \frac{1}{\varepsilon}\Big\{ \overline{\mathscr{V}}_0 \big(\overline{\hat{z}}_{00}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}), \overline{\hat{\nu}}_{00}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}); \tau, \frac{\tau}{\varepsilon}\big) \\
&\quad + \big[\overline{\hat{z}}_{01}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}) \cdot \nabla_{z_0}\big] \langle \overline{\mathscr{V}}_{-1} \rangle \big(\overline{\hat{z}}_{00}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}); \tau\big) \Big\} \\
&\quad + \sum_{j=0}^{N} \varepsilon^j \, \mathscr{U}_j^{\circ} \Big(\tilde{x}; \tau, \frac{\tau}{\varepsilon}, \frac{\overline{\hat{\mathscr{V}}}^{\circ}_{-1}(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon^2} + \frac{\overline{\hat{\mathscr{V}}}^{\circ}_{0}(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon}\Big) + \mathcal{O}(\varepsilon^{N+1}).
\end{aligned} \tag{6.58}$$

In view of (6.23), we have

$$\mathrm{D}_x \langle \overline{\mathscr{V}}^{\circ}_{-1} \rangle (x; \tau) = \mathrm{D}_x z_{00}(x) \ \nabla_{z_0} \langle \overline{\mathscr{V}}_{-1} \rangle \big(z_{00}(x); \tau\big). \tag{6.59}$$

Consider then (6.23), (6.24), (6.45), and plug (6.59) in (6.46), the expression (6.58) becomes

$$\begin{aligned}
&\frac{\langle \overline{\mathscr{V}}_{-1} \rangle\big(z_0(\varepsilon, \mathrm{x}^{-1}); \tau\big)}{\varepsilon^2} + \frac{\overline{\mathscr{V}}_0\big((z_0, \nu_0)(\varepsilon, \mathrm{x}^{-1}); \tau, \frac{\tau}{\varepsilon}\big)}{\varepsilon} \\
&= \frac{\overline{\hat{\mathscr{V}}}^{\circ}_{-1}(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon^2} + \frac{\overline{\hat{\mathscr{V}}}^{\circ}_{0}(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon} \\
&\quad + \sum_{j=0}^{N} \varepsilon^j \, \mathscr{U}_j^{\circ} \Big(\tilde{x}; \tau, \frac{\tau}{\varepsilon}, \frac{\overline{\hat{\mathscr{V}}}^{\circ}_{-1}(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon^2} + \frac{\overline{\hat{\mathscr{V}}}^{\circ}_{0}(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon}\Big) + \mathcal{O}(\varepsilon^{N+1}).
\end{aligned} \tag{6.60}$$

In view of (6.53) and (6.60), we can assert that the first two terms of the expansion (1.10) are identified as follows

$$\overline{\mathscr{U}}_{-1}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}) := \overline{\hat{\mathscr{V}}}^{\circ}_{-1}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}); \tag{6.61}$$

$$\overline{\mathscr{U}}_{0}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}) := \overline{\hat{\mathscr{V}}}^{\circ}_{0}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}). \tag{6.62}$$

This explains how the phase $\psi_\varepsilon = \varepsilon\overline{\mathscr{U}}_{-1} + \varepsilon^2\overline{\mathscr{U}}_0$ does appear inside (1.10). The strategy of recovering the remaining oscillatory structure (1.10) is as follows:

(1) Use (6.60) to localize the oscillations at frequencies ε^{-3} at the position $\varepsilon\overline{\mathscr{U}}_{-1} + \varepsilon^2\overline{\mathscr{U}}_0$ and incorporate the $\mathcal{O}(1)$-remainder inside $\mathcal{O}(1)$-shift in θ_{r} of the profiles $\mathscr{V}_j$ in (6.53);
(2) Expand the new profiles $\mathscr{V}_j(\cdot;\tau,\theta_\tau,\theta_{\mathrm{r}})$ thus obtained after step (1) composed with $(z_0,\mathrm{v}_0)(\varepsilon,\mathrm{x}^{-1})$ in powers of ε using the expansion (6.54);
(3) Gather the ε^j-terms coming from (6.53) after applying the above two steps to recover the expressions $\mathscr{U}_j$ for $j \geq 1$.

Of course, the rapid oscillations (implying $\overline{\hat{\mathscr{V}}}{}^\circ_{-1}$ and $\overline{\hat{\mathscr{V}}}{}^\circ_0$) involved by $(z_0,\mathrm{v}_0)(\varepsilon;\mathrm{x}^{-1})$ at the level of (6.54) and those appearing in the sum inside (6.58) are still present. But they can be incorporated inside the profiles $\mathscr{U}_j$ with $j \geq 1$. $\square$

Remark 6.11 (About the matching of initial data)**.** We have $\mathrm{x}(\varepsilon,0,x) = x$ and $\mathrm{x}^{-1}(\varepsilon,0,x) = x$. As prescribed by (1.3) and the initial data inside (1.5), we find that

$$\begin{aligned} u(\varepsilon,0,x) &= \mathrm{u}(\varepsilon,0,\mathrm{x}^{-1}(\varepsilon,0,x)) = \mathrm{u}(\varepsilon,0,x) = \mathrm{v}_0(\varepsilon,x) \\ &= (\mathcal{U}_{00} + \varepsilon\mathcal{U}_{01} + \dots + \varepsilon^N\mathcal{U}_{0N})(x) + \mathcal{O}(\varepsilon^{N+1}). \end{aligned}$$

Compare this with (1.10) at time $\tau = 0$. The above line implies that

$$\overline{\mathscr{U}}_{-1}(x;0,0) = 0, \quad \mathscr{U}_j(x;0,0,\hat{\theta}^\circ_{\mathrm{r}}) = \mathrm{v}_{0j}(x) = \mathcal{U}_{0j}(x), \quad \forall j \in \mathbb{N}.$$

In view of (6.61), knowing that $\mathscr{V}_{-1|\tau=0} \equiv \mathcal{V}_{-1|\tau=0} \equiv 0$, this is consistent. The same applies for $\overline{\mathscr{U}}_0$ and so on.

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