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QUASILINEARIZATION AND BOUNDARY VALUE PROBLEMS FOR RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We apply the quasilinearization method to a Dirichlet boundary value problem and to a right focal boundary value problem for a Riemann-Liouville fractional differential equation. First, we sue the method of upper and lower solutions to obtain the uniqueness of solutions of the Dirichlet boundary value problem. Next, we apply a suitable fixed point theorem to establish the existence of solutions. We develop a quasilinearization algorithm and construct sequences of approximate solutions that converge monotonically and quadratically to the unique solution of the boundary value problem. Two examples are exhibited to illustrate the main result for the Dirichlet boundary value problem.

1. Introduction

The method of quasilinearization was introduced by Bellman [5, 6] in the 1960s; the method produces a numerical algorithm that generates approximate solutions of nonlinear problems with sequences of solutions of linear problems. Under suitable hypotheses, the sequences of approximate solutions converge monotonically and quadratically. In the case of boundary value problems for ordinary differential equations, under modest hypotheses, the sequences of approximate solutions converge to a unique solution.

Initially, quasilinearization proved to be useful in the study of initial value problems for ordinary differential equations and we cite as examples [16, 17, 18, 23]. There are many applications of quasilinearization to boundary value problems for ordinary differential equations and we cite [1, 2, 11, 12, 14, 21]. More recently, quasilinearization has become a useful tool in the study of initial value problems for fractional differential equations; see [7, 8, 9, 19, 20, 24, 26], for example. Khan [13] has applied the quasilinearization method to a nonlocal boundary value problem for fractional differential equations of Caputo type. To our knowledge, the quasilinearization method has received little attention for boundary value problems for fractional differential equations of Riemann-Liouville type. In this article, we consider two boundary value problems for fractional differential equations of Riemann-Liouville type and apply the method of quasilinearization. Specifically, we

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consider a Dirichlet boundary value problem and we consider a right focal boundary value value problem.

In Section 2 we provide preliminary definitions and we provide known results of functions satisfying fractional differential inequalities at absolute extreme points. In Section 3, we employ the method of upper and lower solutions and obtain the uniqueness of solutions of a two-point Dirichlet fractional boundary value problem for a Riemann-Liouville fractional differential equation of order $1 < \alpha < 2$ under suitable hypotheses. Then, we apply a suitable fixed point theorem and obtain the existence of a solution. In Section 4, we construct a sequence of upper solutions and a sequence of lower solutions, each of which converge monotonically to the unique solution, and obtain a quadratic rate of convergence. In Section 5, we outline the application of the quasilinearization method to a two-point right focal boundary value problem. In Section 6, we exhibit two specific examples to illustrate the application of Theorem 4.2, the main result of Section 4.

2. Preliminaries

We refer the reader to [10, 15, 22] for thorough presentations on the theory of fractional differential equations.

Definition 2.1 ([15])**.** Let $0 < \alpha$ and $a \in \mathbb{R}$. The α^{th}-order Riemann-Liouville fractional integral of a function y is defined by

$$I_a^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)ds, \quad a \le t, \tag{2.1}$$

provided the right-hand side exists. For $\alpha = 0$, define I_a^α to be the identity map. Moreover, let n denote a positive integer and assume $n-1 < \alpha \le n$. The α^{th}-order Riemann-Liouville fractional derivative is defined as

$$D_a^\alpha y(t) = D^n I_a^{n-\alpha} y(t), \quad a \le t, \tag{2.2}$$

where D^n denotes the classical n^{th}-order derivative, if the right-hand side exists.

Definition 2.2 ([15])**.** We denote by $C[0,1]$ the space of continuous functions y on $[0,1]$ with the norm

$$\|y\|_C = \max_{t\in[0,1]} |y(t)|.$$

The following theorems are stated and proved (for a minimum value) in [3] and [25]. They are important results for the application of upper and lower solutions to fractional differential equations.

Theorem 2.3 ([3])**.** *Assume $y \in C^2[0,1]$ attains its maximum value at $t_0 \in (0,1)$. Then, for all $1 < \alpha < 2$,*

$$D_0^\alpha y(t_0) \le -\frac{(\alpha-1)}{\Gamma(2-\alpha)} t_0^{-\alpha} y(t_0).$$

Moreover, if $y(t_0) \ge 0$, then $D_0^\alpha y(t_0) \le 0$.

Theorem 2.3 will not apply to the boundary value problems we consider because the condition $y \in C^2[0,1]$ is too strong. The following result provides the same differential inequality under weaker conditions and is suitable for the application to the boundary value problems we consider.

Theorem 2.4 ([25])**.** *Assume that $y \in C[0,1]$ satisfies the following conditions:*

(i) $D_0^\alpha y \in C[0,1]$ *for* $1<\alpha<2$*;*
(ii) y *attains its global maximum at* $t_0 \in (0,1)$.

Then

$$D_0^\alpha y(t_0) \leq -\frac{(\alpha-1)}{\Gamma(2-\alpha)} t_0^{-\alpha} y(t_0).$$

Moreover, if $y(t_0) \geq 0$*, then* $D_0^\alpha y(t_0) \leq 0$.

Theorem 2.5 ([25])**.** *Assume that* $y \in C(0,1]$ *satisfies the following conditions:*

(i) $D_0^\nu u \in C[0,1]$ *for* $0<\nu<1$*;*
(ii) y *attains its global maximum at* $t_0 \in (0,1]$.

Then,

$$D_0^\nu y(t_0) \geq \frac{1}{\Gamma(1-\nu)} t_0^{-\nu} y(t_0).$$

Moreover, if $y(t_0) \geq 0$*, then* $D_0^\nu y(t_0) \geq 0$.

3. Existence and uniqueness of solutions

Let $1<\alpha<2$ and assume throughout that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous. Initially, we shall consider the two point Dirichlet boundary value problem for a Riemann-Liouville fractional differential equation

$$D_0^\alpha y(t) = f(t, y(t)), \quad 0 \leq t \leq 1, \tag{3.1}$$
$$y(0)=0, \quad y(1)=0. \tag{3.2}$$

We begin with the assumption that f is increasing as a function of the second component and obtain results for the uniqueness of solutions. In the case of second order ordinary differential equations, this is a standard assumption to obtain uniqueness of solutions.

Theorem 3.1. *Assume* $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ *is continuous, assume* $\frac{\partial f}{\partial y} = f_y : [0,1]\times\mathbb{R}\to\mathbb{R}$ *is continuous and assume* $f_y>0$ *on* $[0,1]\times\mathbb{R}$*. Then a continuous solution of the fractional boundary value problem* (3.1)-(3.2) *is unique if it exists.*

Proof. Assume for the sake of contradiction that y_1 and y_2 denote two distinct continuous solutions of (3.1)-(3.2) in $C[0,1]$. Let $u=y_1-y_2$. Then $u\in C[0,1]$ and $D_0^\alpha u \in C[0,1]$. Without loss of generality assume that $u(t)$ has a positive maximum at $t_0\in[0,1]$.

Since $u(0)=u(1)=0$, u does not have a positive maximum at $t_0=0$ or $t_0=1$. Now, assume $t_0\in(0,1)$. Then, $u(t_0)>0$. Apply Theorem 2.4 to obtain

$$D_0^\alpha u(t_0) < 0.$$

However, y_1 and y_2 satisfy (3.1), and so

$$D_0^\alpha u(t_0) = f(t_0, y_1(t_0)) - f(t_0, y_2(t_0)) > 0,$$

since f is increasing in y. Thus, $(y_1-y_2)(t)$ does not have a positive maximum at $t_0\in[0,1]$.

Similarly, y_2-y_1 does not have a positive maximum at $t_0\in[0,1]$. Thus, a continuous solution of (3.1)-(3.2) is unique if it exists. $\square$

Definition 3.2. We say $w \in C[0,1]$ is a lower solution of the fractional boundary value problem (3.1)-(3.2) if $w(0) = w(1) = 0$, $D_0^\alpha w \in C[0,1]$, and

$$D_0^\alpha w(t) \geq f(t, w(t)), \quad 0 \leq t \leq 1.$$

We say $v \in C[0,1]$ is an upper solution of the fractional boundary value problem (3.1)-(3.2) if $v(0) = v(1) = 0$, $D_0^\alpha v \in C[0,1]$, and

$$D_0^\alpha v(t) \leq f(t, v(t)), \quad 0 \leq t \leq 1.$$

Theorem 3.3. *Assume $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, assume $\frac{\partial f}{\partial y} = f_y : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and assume $f_y > 0$ on $[0,1] \times \mathbb{R}$. Also, assume w and v are lower and upper solutions of the fractional boundary value problem* (3.1)-(3.2)*, respectively. Then,*

$$w(t) \leq v(t), \quad 0 \leq t \leq 1.$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.1. Assume w is a lower solution and v is an upper solution of the fractional boundary value problem (3.1)-(3.2), respectively. Assume for the sake of contradiction that $w \leq v$ is false. Assume that $(w - v)(t)$ has a positive maximum at $t_0 \in [0,1]$.

Since $(w - v)(0) = (w - v)(1) = 0$, $(w - v)$ does not have a positive maximum either at 0 or at 1. Now, assume $t_0 \in (0,1)$. Then $(w - v)(t_0) > 0$. Apply Theorem 2.4 to obtain

$$D_0^\alpha (w - v)(t_0) < 0.$$

However, w and v are lower and upper solutions of the fractional boundary value problem (3.1)-(3.2), respectively, and so

$$D_0^\alpha (w - v)(t_0) \geq f(t_0, w(t_0)) - f(t_0, v(t_0)) > 0,$$

since f is increasing in the second variable. Thus, $(w - v)(t)$ does not have a positive maximum at $t_0 \in [0,1]$. □

Remark 3.4. We chose to prove both Theorems 3.1 and 3.3. It is the case that Theorem 3.1 is an immediate corollary of Theorem 3.3 since a continuous solution of (3.1)-(3.2) is also a lower solution and an upper solution of (3.1)-(3.2).

We now turn to the question of existence of solutions of the fractional boundary value problem (3.1)-(3.2). Bai and Lü [4] derived the Green's function corresponding to the fractional boundary value problem (3.1)-(3.2) as follows:

$$G(t,s) = \begin{cases} G_1(t,s), & 0 \leq t \leq s \leq 1, \\ G_2(t,s), & 0 \leq s \leq t \leq 1, \end{cases} \tag{3.3}$$

where

$$G_1(t,s) = -\frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)},$$
$$G_2(t,s) = \frac{(t-s)^{\alpha-1} - [t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}.$$

Consequently, the fractional boundary value problem (3.1)-(3.2) is equivalent to a Fredholm integral equation of the second kind

$$y(t) = \int_0^1 G(t,s) f(s, y(s)) ds$$

in the sense that $y \in C[0,1]$ and $y(t) = \int_0^1 G(t,s) f(s,y(s))ds$ if, and only if, $D_0^\alpha y \in C[0,1]$ and y is a continuous solution of the fractional boundary value problem (3.1)-(3.2). Further, Bai and Lü [4] obtained the following two properties of the Green's function:

- $G(t,s) < 0, \quad (t,s) \in [0,1] \times [0,1]$.
- $\max_{0 \le t \le 1} |G(t,s)| = |G(s,s)| = \frac{[s(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, \quad s \in (0,1)$.

Clearly, we have

$$\max_{t\in[0,1]} \int_0^1 |G(t,s)| ds \le \frac{1}{\Gamma(\alpha)} \int_0^1 [s(1-s)]^{\alpha-1} ds = \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}.$$

Theorem 3.5. *Assume $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded. Then there exists $y \in C[0,1]$ satisfying*

$$D_0^\alpha y(t) = g(t, y(t)), \quad 0 \le t \le 1,$$

and the boundary conditions (3.2).

Proof. Define the completely continuous operator $T : C[0,1] \to C[0,1]$ by

$$Ty(t) = \int_0^1 G(t,s) g(s,y(s)) ds,$$

where $G(t,s)$ is given by (3.3). The complete continuity of T is proved in [4], for example.

Note that $g(t,y(t)) \in C[0,1]$ for any $y \in C[0,1]$. So, an application of the Schauder fixed point theorem implies that the fractional boundary value problem (3.1)-(3.2) has a continuous solution. To see this, let

$$M = \sup\big\{ \big|g(t,y)\big| : 0 \le t \le 1, \ y \in \mathbb{R} \big\}$$

and let

$$\Omega = \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}.$$

Then

$$\begin{aligned}
\|Ty\|_C &= \max_{t\in[0,1]} \big|Ty(t)\big| \\
&= \max_{t\in[0,1]} \Big| \int_0^1 G(t,s) g(s,y(s)) ds \Big| \\
&\le \max_{t\in[0,1]} \int_0^1 |G(t,s)| \big|g(s,y(s))\big| ds \le M\Omega.
\end{aligned}$$

Define

$$\mathcal{U} = \{ y \in C[0,1] : \|y\|_C \le M\Omega \}.$$

Then $\mathcal{U}$ is a closed convex subset of $C[0,1]$ and $T : \mathcal{U} \to \mathcal{U}$. Thus, the Schauder fixed point theorem implies there exists a fixed point, $y \in \mathcal{U}$ of the operator T and the theorem is proved. □

Theorem 3.6. *Assume $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume w and v are lower and upper solutions of the fractional boundary value problem* (3.1)-(3.2)*, respectively, and assume*

$$w(t) \le v(t), \quad 0 \le t \le 1.$$

Then there exists a continuous solution $y \in C[0,1]$ of (3.1)-(3.2) *satisfying*

$$w(t) \le y(t) \le v(t), \quad 0 \le t \le 1.$$

Proof. Define a truncation of $f(t,y)$ by

$$g(t,y(t)) = \begin{cases} f(t,v(t)) + \frac{y(t)-v(t)}{1+(y(t)-v(t))}, & \text{if } y(t) > v(t), \\ f(t,y(t)), & \text{if } w(t) \le y(t) \le v(t), \\ f(t,w(t)) + \frac{y(t)-w(t)}{1+(w(t)-y(t))}, & \text{if } y(t) < w(t). \end{cases}$$

Define an operator $T : C[0,1] \to C[0,1]$ by

$$Ty(t) = \int_0^1 G(t,s)g(s,y(s))ds.$$

Note that the truncation g is bounded and continuous on $[0,1] \times \mathbb{R}$ and so by Theorem 3.5 there exists $y \in C[0,1]$, a fixed point of T, satisfying

$$D_0^\alpha y(t) = g(t,y(t)), \quad 0 \le t \le 1,$$

and the boundary conditions (3.2).

Let y denote a continuous fixed point of the operator T. To complete the proof, we only show

$$w(t) \le y(t) \le v(t), \quad 0 \le t \le 1.$$

Then by the definition of the truncation g it follows that y is a continuous solution of the original boundary value problem (3.1)-(3.2).

We show the details that $y - v$ does not have a positive maximum at $t_0 \in [0,1]$. Assume for the sake of contradiction that $y-v$ has a positive maximum at $t_0 \in [0,1]$. Because of the boundary conditions (3.2), $t_0 \in (0,1)$. Then

$$D_0^\alpha y(t_0) = f(t_0,v(t_0)) + \frac{y(t_0)-v(t_0)}{1+(y(t_0)-v(t_0))}.$$

Since v is an upper solution of (3.1)-(3.2), it follows that

$$D_0^\alpha (y-v)(t_0) \ge \frac{y(t_0)-v(t_0)}{1+(y(t_0)-v(t_0))} > 0.$$

Theorem 2.4 applies to $y - v$ so, $D_0^\alpha (y-v)(t_0) < 0$. Thus, $y - v$ does not have a positive maximum at $t_0 \in (0,1)$. Since, $(y-v)(0) = (y-v)(1) = 0$.

$$y(t) \le v(t), \quad 0 \le t \le 1.$$

The argument to show $y(t) \ge w(t)$, $0 \le t \le 1$, is completely analogous and so the theorem is proved. □

The following result is an immediate corollary of Theorems 3.3 and 3.6.

Theorem 3.7. *Assume $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, assume $\frac{\partial f}{\partial y} = f_y : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and assume $f_y > 0$ on $[0,1] \times \mathbb{R}$. Also, assume w and v are lower and upper solutions of the fractional boundary value problem* (3.1)-(3.2)*, respectively. Then, there exists a unique continuous solution $y \in C[0,1]$ of* (3.1)-(3.2) *satisfying*

$$w(t) \le y(t) \le v(t), \quad 0 \le t \le 1.$$

4. The monotone method and quadratic convergence

In this section, we describe the monotone method and obtain a quadratic rate of convergence. Since the uniqueness and existence results have been obtained in Section 3, the details presented in this sections are completely standard; see, [12] or [16]. Thus, we outline the construction. In the first theorem of this section, the monotone iterates are constructed.

Theorem 4.1. *Assume $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, assume $\frac{\partial f}{\partial y} = f_y : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and assume $f_y > 0$ on $[0,1] \times \mathbb{R}$. Also, assume w_0 and v_0 are lower and upper solutions of the fractional* (3.1)-(3.2)*, respectively. Then, there exists a unique continuous solution $y \in C[0,1]$ of* (3.1)-(3.2) *satisfying*

$$w_0(t) \le y(t) \le v_0(t), \quad 0 \le t \le 1.$$

Moreover, there exist sequences $\{w_n\}$, $\{v_n\}$ of lower and upper solutions of the fractional boundary value problem (3.1)-(3.2)*, respectively, each of which converges monotonically to the unique continuous solution y of the fractional boundary value problem* (3.1)-(3.2) *and satisfy*

$$w_n(t) \le w_{n+1}(t) \le y(t) \le v_{n+1}(t) \le v_n(t), \quad 0 \le t \le 1, \quad n = 0, 1, \dots.$$

Proof. Let w_0, v_0 denote a lower and an upper solution of (3.1)-(3.2), respectively. Theorem 3.7 applies and there exists a unique continuous solution $y \in C[0,1]$ of (3.1)-(3.2) satisfying

$$w_0(t) \le y(t) \le v_0(t), \quad 0 \le t \le 1.$$

Define the function $h(w_0, v_0; t, y)$ on $[0,1] \times \mathbb{R}$ by

$$h(w_0, v_0; t, y) = f(t, w_0(t)) + f_y(t, v_0(t))(y - w_0(t))$$

and consider the boundary value problem for the linear non-homogeneous fractional differential equation

$$D_0^\alpha y(t) = h(w_0, v_0; t, y(t)), \quad 0 \le t \le 1, \quad y(0) = y(1) = 0. \tag{4.1}$$

Note that

$$h(w_0, v_0; t, w_0(t)) = f(t, w_0(t)), \quad 0 \le t \le 1,$$

and so,

$$D_0^\alpha w_0(t) \ge f(t, w_0(t)) = h(w_0, v_0; t, w_0(t)), \quad 0 \le t \le 1. \tag{4.2}$$

Moreover, there exists $c(t)$ satisfying $w_0(t) \le c(t) \le v_0(t)$ such that

$$f(t, v_0(t)) = f(t, w_0(t)) + f_y(t, c(t))(v_0 - w_0)(t).$$

Thus,

$$\begin{aligned} f(t, v_0(t)) &= f(t, w_0(t)) + f_y(t, c(t))(v_0 - w_0)(t) \\ &\le f(t, w_0(t)) + f_y(t, v_0(t))(v_0 - w_0)(t) \\ &= h(w_0, v_0; t, v_0(t)) \quad 0 \le t \le 1, \end{aligned}$$

since f_y is increasing in y for each $t \in [0,1]$. Thus,

$$h(w_0, v_0; t, v_0(t)) \ge f(t, v_0(t)) \ge D_0^\alpha v_0(t), \quad 0 \le t \le 1. \tag{4.3}$$

In particular, (4.2) and (4.3) imply w_0 and v_0 are lower and upper solutions of (4.1) respectively as well. Since, h satisfies the hypotheses of Theorem 3.6, there exists a continuous solution, $w_1(t)$, of (4.1) satisfying

$$w_0(t) \le w_1(t) \le v_0(t), \quad 0 \le t \le 1.$$

Note that there exists $w_0(t) \le c(t) \le w_1(t) \le v_0(t)$ such that

$$f(t, w_1(t)) - f(t, w_0(t)) = f_y(t, c(t))(w_1(t) - w_0(t)) \le f_y(t, v_0(t))(w_1(t) - w_0(t))$$

and so,

$$D_0^\alpha w_1(t) = h(w_0, v_0; t, w_1(t)) \ge f(t, w_1(t)), \quad 0 \le t \le 1.$$

In particular, w_1 is a lower solution of (3.1)-(3.2) since $w_1 \in C[0, 1]$.

Now define the function $k(v_0; t, y)$ on $[0, 1] \times \mathbb{R}$ by

$$k(v_0; t, y) = f(t, v_0(t)) + f_y(t, v_0(t))(y - v_0(t))$$

and consider the boundary value problem for the linear nonhomogeneous fractional differential equation

$$D_0^\alpha y(t) = k(v_0; t, y(t)), \quad 0 \le t \le 1, \quad y(0) = y(1) = 0. \tag{4.4}$$

Note that

$$k(v_0; t, v_0(t)) = f(t, v_0(t)), \quad 0 \le t \le 1,$$

and

$$D_0^\alpha v_0(t) \le f(t, v_0(t)) = k(v_0; t, v_0(t)), \quad 0 \le t \le 1.$$

Thus, v_0 is an upper solution of (4.4). Note that there exists $c(t)$ satisfying $w_0(t) \le c(t) \le v_0(t)$ such that

$$\begin{aligned} D_0^\alpha w_0(t) &\ge f(t, w_0(t)) = f(t, v_0(t)) + f_y(t, c(t))(w_0(t) - v_0(t)) \\ &\ge f(t, v_0(t)) + f_y(t, v_0(t))(w_0(t) - v_0(t)) \\ &= k(v_0(t); t, w_0(t)), \quad 0 \le t \le 1, \end{aligned}$$

and so, w_0 is a lower solution of (4.4). Since k satisfies the hypotheses of Theorem 3.6 there exists a continuous solution, $v_1(t)$, of (4.4) satisfying

$$w_0(t) \le v_1(t) \le v_0(t), \quad 0 \le t \le 1.$$

An application of the mean value theorem again will give

$$k(v_0; t, v_1(t)) \le f(t, v_1(t)), \quad 0 \le t \le 1.$$

To see this, for some $v_1(t) \le c(t) \le v_0(t)$,

$$\begin{aligned} f(t, v_1(t)) &= f(t, v_0(t)) + f_y(t, c(t))(v_1(t) - v_0(t)) \\ &\ge f(t, v_0(t)) + f_y(t, v_0(t))(v_1(t) - v_0(t)). \end{aligned}$$

Thus,

$$D_0^\alpha v_1(t) = k(v_0; t, v_1(t)) \le f(t, v_1(t)), \quad 0 \le t \le 1,$$

and v_1 is an upper solution of (3.1)-(3.2) since $v_1 \in C[0, 1]$.

Finally, apply Theorem 3.3 to obtain

$$w_1(t) \le v_1(t), \quad 0 \le t \le 1.$$

Apply Theorem 3.6 with lower and upper solutions, w_1 and v_1, respectively, keeping in mind that the continuous solution y obtained in Theorem 3.6 is unique, to obtain

$$w_0(t) \le w_1(t) \le y(t) \le v_1(t) \le v_0(t), \quad 0 \le t \le 1,$$

where y is the unique solution of the fractional boundary value problem, (3.1)-(3.2).

The construction of the sequences of lower and upper solutions proceeds by induction. Assume the sequences $\{w_k\}_{k=0}^n$ and $\{v_k\}_{k=0}^n$ have been constructed inductively such that for each k,

$$h(w_k, v_k; t, y) = f(t, w_k(t)) + f_y(t, v_k(t))(y - w_k(t)),$$
$$k(v_k; t, y) = f(t, v_k(t)) + f_y(t, v_k(t))(y - v_k(t)),$$

where w_k is a continuous solution of the fractional boundary value problem

$$D_0^\alpha y(t) = h(w_{k-1}, v_{k-1}; t, y(t)), \quad 0 \le t \le 1, \quad y(0) = y(1) = 0,$$

v_k is a continuous solution of the fractional boundary value problem

$$D_0^\alpha y(t) = k(v_{k-1}; t, y(t)), \quad 0 \le t \le 1, \quad y(0) = y(1) = 0,$$

and

$$w_{k-1}(t) \le w_k(t) \le y(t) \le v_k(t) \le v_{k-1}(t), \quad 0 \le t \le 1, \quad k = 1, \dots, n.$$

Here w_k, v_k, $k = 0, \dots, n$ denote a lower solution and an upper solution, respectively, of (3.1)-(3.2), and y is the unique continuous solution of (3.1)-(3.2).

To complete the induction argument, consider the boundary value problem for the linear nonhomogeneous fractional differential equation

$$D_0^\alpha y(t) = h(w_n, v_n; t, y(t)), \quad 0 \le t \le 1, \quad y(0) = y(1) = 0. \tag{4.5}$$

Note that

$$h(w_n, v_n; t, w_n(t)) = f(t, w_n(t)), \quad 0 \le t \le 1,$$
$$h(w_n, v_n; t, v_n(t)) \ge f(t, v_n(t)), \quad 0 \le t \le 1.$$

So, w_n, v_n denote a lower and an upper solution, respectively, of (4.5) as well.

The arguments above to show the existence of a lower solution, $w_1(t)$, and an upper solution, $v_1(t)$, and the inequalities

$$w_0(t) \le w_1(t) \le y(t) \le v_1(t) \le v_0(t), \quad 0 \le t \le 1,$$

are readily adapted to show the existence of a lower solution, $w_{n+1}(t)$, and an upper solution, $v_{n+1}(t)$, and the inequalities

$$w_n(t) \le w_{n+1}(t) \le y(t) \le v_{n+1}(t) \le v_n(t), \quad 0 \le t \le 1.$$

To complete the proof, $\{w_n\}$ and $\{v_n\}$ are monotone sequences of continuous functions bounded above and below, respectively, on a compact domain. So by Dini's theorem, each converges uniformly to w and to v respectively on $[0, 1]$.

$$k(v_n; t, v_{n+1}(t)) = f(t, v_n(t)) + f_y(t, v_n(t))(v_{n+1} - v_n)(t) \to f(t, v(t)) \text{ as } n \to \infty,$$

where the convergence is uniform on $[0, 1]$. So $v = y$ is the unique continuous solution of (3.1)-(3.2). Similarly,

$$h(w_n, v_n; t, w_{n+1}(t)) = f(t, w_n(t)) + f_y(t, v_n(t))(w_{n+1} - w_n)(t) \to f(t, w(t)) \text{ as } n \to \infty$$

uniformly on $[0, 1]$ and so, $w = y$ is also the unique continuous solution of (3.1)-(3.2). $\square$

We now obtain an estimate on the error bound. To obtain the quadratic convergence, assume one further condition on f, that f_{yy} exists and $f_{yy} \ge 0$.

For each n, define the error e_n as follows:

$$e_n(t) = v_n(t) - w_n(t), \quad 0 \le t \le 1.$$

So, $0 \le e_n(t)$ for $0 \le t \le 1$. Denote by $\|e_n\|_C$ the error bound

$$\|e_n\|_C = \max_{t\in[0,1]} \big|e_n(t)\big|.$$

Theorem 4.2. *Assume $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous, assume $\frac{\partial f}{\partial y} = f_y : [0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and assume $f_y > 0$ on $[0,1]\times\mathbb{R}$. Assume in addition that f_{yy} exists and $f_{yy} \ge 0$ on $[0,1]\times\mathbb{R}$. Assume w_0 and v_0 are lower and upper solutions of the fractional boundary value problem* (3.1)-(3.2)*, respectively. Then, there exists a unique solution $y \in C[0,1]$ of* (3.1)-(3.2) *satisfying*

$$w_0(t) \le y(t) \le v_0(t), \quad 0 \le t \le 1.$$

Moreover, there exist sequences $\{w_n\}$, $\{v_n\}$ of lower and upper solutions of the fractional boundary value problem (3.1)-(3.2)*, respectively, each of which converges monotonically and quadratically to the unique solution y of the fractional boundary value problem* (3.1)-(3.2) *and satisfy*

$$w_n(t) \le w_{n+1}(t) \le y(t) \le v_{n+1}(t) \le v_n(t), \quad 0 \le t \le 1, \quad n = 0,1,\dots.$$

Proof. Employ the construction in the proof of Theorem 4.1 and recall

$$D_0^\alpha w_{n+1}(t) = h(w_n, v_n; t, w_{n+1}(t)) = f(t, w_n(t)) + f_y(t, v_n(t))(w_{n+1}(t) - w_n(t)),$$
$$D_0^\alpha v_{n+1}(t) = k(v_n; t, v_{n+1}(t)) = f(t, v_n(t)) + f_y(t, v_n(t))(v_{n+1}(t) - v_n(t)).$$

Then

$$\begin{aligned}
&D_0^\alpha e_n(t)\\
&= D_0^\alpha v_{n+1}(t) - D_0^\alpha w_{n+1}(t)\\
&= [f(t, v_n(t)) - f(t, w_n(t))] + f_y(t, v_n(t))[v_{n+1}(t) - v_n(t) - w_{n+1}(t) + w_n(t)]\\
&= [f(t, v_n(t)) - f(t, w_n(t))] + f_y(t, v_n(t))[e_{n+1}(t) - e_n(t)].
\end{aligned}$$

By the mean value theorem, there exists $c_n(t)$ satisfying $w_n(t) < c_n(t) < v_n(t)$ such that

$$f(t, v_n(t)) - f(t, w_n(t)) = f_y(t, c_n(t))e_n(t).$$

Thus,

$$\begin{aligned}
D_0^\alpha e_{n+1}(t) &= f_y(t, c_n(t))e_n(t) + f_y(t, v_n(t))e_{n+1}(t) - f_y(t, v_n(t))e_n(t)\\
&= f_y(t, v_n(t))e_{n+1}(t) + [f_y(t, c_n(t)) - f_y(t, v_n(t))]e_n(t).
\end{aligned}$$

Employ the mean value theorem again for $f_y(t, c_n(t)) - f_y(t, v_n(t))$ and there exists $\hat{c}_n(t)$ satisfying

$$c_n(t) < \hat{c}_n(t) < v_n(t)$$

such that

$$f_y(t, c_n(t)) - f_y(t, v_n(t)) = f_{yy}(t, \hat{c}_n(t))(c_n(t) - v_n(t)).$$

Then

$$D_0^\alpha e_{n+1}(t) = f_y(t, v_n(t))e_{n+1}(t) + f_{yy}(t, \hat{c}_n(t))(c_n(t) - v_n(t))e_n(t).$$

Note that e_{n+1} satisfies the boundary conditions (3.2) and employ the Green's function (3.3). Then

$$\begin{aligned}
&0 \le e_{n+1}(t)\\
&= \int_0^1 G(t,s)\left[f_y(s, v_n(s))e_{n+1}(s) + f_{yy}(s, \hat{c}_n(s))(c_n(s) - v_n(s))e_n(s)\right] ds
\end{aligned}$$

$$\begin{aligned}
&\le \int_0^1 G(t,s)\,[f_{yy}(s,\hat{c}_n(s))(c_n(s)-v_n(s))e_n(s)]\,ds \\
&\le \int_0^1 |G(t,s)|\big|f_{yy}(s,\hat{c}_n(s))(c_n(s)-v_n(s))e_n(s)\big|ds \\
&\le \int_0^1 |G(t,s)|\big|f_{yy}(s,\hat{c}_n(s))\big|\big|(c_n(s)-v_n(s))\big|\big|e_n(s)\big|ds \\
&\le M\Omega\big\|e_n\big\|_C^2,
\end{aligned}$$

where

$$M = \max\{|f_{yy}(t,y(t))|, \quad w_0(t) \le y(t) \le v_0(t), \quad 0 \le t \le 1\},$$

$$\max_{t\in[0,1]} \int_0^1 |G(t,s)|ds \le \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} = \Omega.$$

Thus, we have

$$\|e_{n+1}\|_C \le M\Omega\|e_n\|_C^2$$

and hence, the rate of convergence is quadratic. □

5. The right focal problem

In this section we shall discuss briefly a fractional boundary value problem

$$D_0^\alpha y(t) = f(t,y(t)), \quad 0 \le t \le 1, \tag{5.1}$$

$$y(0) = 0, \quad D_0^{\alpha-1} y(1) = 0. \tag{5.2}$$

We shall refer to the fractional boundary value problem (5.1)-(5.2) as a right focal boundary value problem.

To obtain the uniqueness, existence and comparison results of Section 3, Theorem 2.4 played the key role. Similar results can be obtained for the right focal problem and now Theorem 2.4 shares the key role with Theorem 2.5.

Definition 5.1. We say $w \in C[0,1]$ is a lower solution of the fractional boundary value problem (5.1)-(5.2) if $w(0) = D_0^{\alpha-1}w(1) = 0$, $D_0^\alpha w \in C[0,1]$, and

$$D_0^\alpha w(t) \ge f(t,w(t)), \quad 0 \le t \le 1.$$

We say $v \in C[0,1]$ is an upper solution of (5.1)-(5.2) if $v(0) = D_0^{\alpha-1}v(1) = 0$, $D_0^\alpha v \in C[0,1]$, and

$$D_0^\alpha v(t) \le f(t,v(t)), \quad 0 \le t \le 1.$$

Theorem 5.2. *Assume $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous, assume $\frac{\partial f}{\partial y} = f_y : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous and assume $f_y > 0$ on $[0,1]\times\mathbb{R}$. Then a continuous solution of* (5.1)-(5.2) *is unique if it exists.*

Proof. Assume for the sake of contradiction that y_1 and y_2 denote two distinct continuous solutions of (5.1)-(5.2) in $C[0,1]$. Let $u = y_1 - y_2$. Then, $u \in C[0,1]$ and $D_0^\alpha u \in C[0,1]$. Without loss of generality assume that $u(t)$ has a positive maximum at $t_0 \in [0,1]$.

First, we assume $t_0 = 0$. Since $u(0) = 0$, u does not have a positive maximum at $t_0 = 0$. Next, we assume $t_0 = 1$. Then, $u(1) > 0$. Using Theorem 2.5, we have

$$D_0^{\alpha-1}u(1) > 0.$$

This is a contradiction to the boundary conditions, (5.2). Finally, assume $t_0 \in (0,1)$. Then, $u(t_0) > 0$. Using Theorem 2.4, we have

$$D_0^\alpha u(t_0) < 0.$$

However, y_1 and y_2 satisfy (3.1), and so

$$D_0^\alpha u(t_0) = f(t_0, y_1(t_0)) - f(t_0, y_2(t_0)) > 0,$$

since f is increasing in y. Thus, $(y_1 - y_2)(t)$ does not have a positive maximum at $t_0 \in [0,1]$. Similarly, $(y_1 - y_2)(t)$ does not have a positive maximum at $t_0 \in [0,1]$. $\square$

The analogue of Theorem 3.3 is obtained similarly we state the analogous theorem without proof.

Theorem 5.3. *Assume $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, assume $\frac{\partial f}{\partial y} = f_y : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and assume $f_y > 0$ on $[0,1] \times \mathbb{R}$. Also, assume w and v are lower and upper solutions of the fractional boundary value problem* (5.1)-(5.2), *respectively. Then*

$$w(t) \le v(t), \quad 0 \le t \le 1.$$

Before proceeding we construct a Green's function and fixed point operator. Consider

$$D_0^\alpha y(t) = h(t), \quad y(0) = 0, \quad D_0^{\alpha-1} y(1) = 0,$$

and assume $h \in C[0,1]$. Then

$$y(t) = c_1 t^{\alpha-2} + c_2 t^{\alpha-1} + I_0^\alpha h(t) = c_1 t^{\alpha-2} + c_2 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds. \tag{5.3}$$

The condition $y(0) = 0$ implies $c_1 = 0$. Now apply the condition $D_0^{\alpha-1} y(1) = 0$. Apply the operator $D_0^{\alpha-1}$ to (5.3) and

$$\begin{aligned} D_0^{\alpha-1} y(t) &= c_2 \Gamma(\alpha) + D_0^{\alpha-1} I_0^\alpha h(t) \\ &= c_2 \Gamma(\alpha) + D I_0^{2-\alpha} I_0^\alpha h(t) \\ &= c_2 \Gamma(\alpha) + I^1 h(t). \end{aligned}$$

Since $D_0^{\alpha-1} y(1) = 0$, we have

$$c_2 = \frac{-\int_0^1 h(s) ds}{\Gamma(\alpha)}.$$

Thus,

$$y(t) = -\frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 h(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds$$

and the right focal Green's function denoted by $G(\alpha-1; t, s)$ has the form

$$G(\alpha-1; t, s) = \begin{cases} G_1(\alpha-1; t, s), & 0 \le t \le s \le 1, \\ G_2(\alpha-1; t, s), & 0 \le s \le t \le 1, \end{cases}$$

where

$$G_1(\alpha-1; t, s) = -\frac{t^{\alpha-1}}{\Gamma(\alpha)},$$

$$G_2(\alpha-1; t, s) = \frac{(t-s)^{\alpha-1} - t^{\alpha-1}}{\Gamma(\alpha)}.$$

The fractional boundary value problem (5.1)-(5.2) is equivalent to a Fredholm integral equation of the second kind

$$y(t) = \int_0^1 G(\alpha - 1; t, s) f(s, y(s)) ds$$

in the sense that $y \in C[0,1]$ and $y(t) = \int_0^1 G(\alpha - 1; t, s) f(s, y(s)) ds$ if, and only if, $D_0^\alpha y \in C[0,1]$ and y is a continuous solution of the fractional boundary value problem (5.1)-(5.2). It is another straightforward argument to show the fixed point operator

$$Ty(t) = \int_0^1 G(\alpha - 1; t, s) f(s, y(s)) ds$$

is a completely continuous map under the assumption that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

The following properties are easy to observe.

- $G(t,s) < 0, \quad (t,s) \in [0,1] \times [0,1]$.
- $\max_{0 \le t \le 1} |G(t,s)| = |G(s,s)| = \frac{s^{\alpha-1}}{\Gamma(\alpha)}, \quad s \in (0,1)$.
- $$\max_{t \in [0,1]} \int_0^1 |G(t,s)| ds \le \frac{1}{\Gamma(\alpha)} \int_0^1 s^{\alpha-1} ds = \frac{1}{\Gamma(\alpha+1)}.$$

From here, the construction of the quasilinearization method follows as in the construction for the Dirichlet boundary value problem.

Theorem 5.4. *Assume $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume w and v are lower and upper solutions of (5.1)-(5.2), respectively, and assume*

$$w(t) \le v(t), \quad 0 \le t \le 1.$$

Then, there exists a continuous solution $y \in C[0,1]$ of (5.1)-(5.2) satisfying

$$w(t) \le y(t) \le v(t), \quad 0 \le t \le 1.$$

Theorem 5.5. *Assume $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, assume $\frac{\partial f}{\partial y} = f_y : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and assume $f_y > 0$ on $[0,1] \times \mathbb{R}$. Assume in addition that f_{yy} exists and $f_{yy} \ge 0$ on $[0,1] \times \mathbb{R}$. Assume w_0 and v_0 are lower and upper solutions of (5.1)-(5.2), respectively. Then there exists a unique solution $y \in C[0,1]$ of (5.1)-(5.2) satisfying*

$$w_0(t) \le y(t) \le v_0(t), \quad 0 \le t \le 1.$$

Moreover, there exist sequences $\{w_n\}$, $\{v_n\}$ of lower and upper solutions of (5.1)-(5.2), respectively, each of which converges monotonically and quadratically to the unique solution y of (5.1)-(5.2) and satisfy

$$w_n(t) \le w_{n+1}(t) \le y(t) \le v_{n+1}(t) \le v_n(t), \quad 0 \le t \le 1, \quad n = 0, 1, \dots.$$

6. Examples

We close with two specific examples illustrating the application of Theorem 4.2.

Example 6.1. Consider the fractional boundary value problem

$$D_0^\alpha y(t) = e^y, \quad 0 \le t \le 1, \tag{6.1}$$

with the Dirichlet boundary conditions (3.2). So $f(t,y) = e^y$ satisfies the hypotheses of Theorem 4.2. Set $v_0(t) = 0$, $0 \le t \le 1$ and set $w_0(t) = t^{\alpha-1}(t-1) = t^\alpha - t^{\alpha-1}$,

$0 \le t \le 1$. Clearly, $v_0(t)$ is an upper solution of (6.1), (3.2). As for w_0, note that $w_0(t) \le 0$, $0 \le t \le 1$, and so, $e^{w_0(t)} \le 1$. $D_0^\alpha w_0(t) = \Gamma(\alpha+1) > 1$. Hence, $w_0(t)$ is a lower solution of (6.1), (3.2) and Theorem 4.2 applies to (6.1), (3.2).

Before exhibiting a second example, we point out that the conclusions of Theorem 4.2 remain valid if the condition $f_{yy} \ge 0$ on $[0,1] \times \mathbb{R}$ is replaced by the condition

$$f_{yy}(t,y) \ge 0 \quad \text{if } w_0(t) \le y \le v_0(t), \quad 0 \le t \le 1.$$

Example 6.2. Consider the fractional boundary value problem

$$D_0^\alpha y(t) = y^3 - 1, \quad 0 \le t \le 1, \tag{6.2}$$

with the Dirichlet boundary conditions (3.2). So now $f_y \ge 0$ on $[0,1] \times \mathbb{R}$. Set $v_0(t) = t^{\alpha-1}(1-t)$ and $w_0(t) = 0$, $0 \le t \le 1$. Then

$$f_{yy}(t,y) = 6y \ge 0 \quad \text{if } w_0(t) \le y \le v_0(t), \quad 0 \le t \le 1.$$

For this example, it is clear that w_0 is a lower solution. To see that v_0 is an upper solution, note that $v_0(t) \ge 0$, $0 \le t \le 1$ and so,

$$D_0^\alpha v_0(t) = -\Gamma(\alpha+1) < -1 \le v_0^3(t) - 1, \quad 0 \le t \le 1.$$

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