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AN EXISTENCE RESULT FOR HEMIVARIATIONAL INEQUALITIES

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Abstract. We present a general method for obtaining solutions for an abstract class of hemivariational inequalities. This result extends many results to the nonsmooth case. Our proof is based on a nonsmooth version of the Mountain Pass Theorem with Palais-Smale or with Cerami compactness condition. We also use the Principle of Symmetric Criticality for locally Lipschitz functions.

1. Introduction

Let $(X, \|\cdot\|)$ be a real, separable, reflexive Banach space, and let $(X^\star, \|\cdot\|_\star)$ be its dual. Also assume that the inclusion $X \hookrightarrow L^l(\mathbb{R}^N)$ is continuous with the embedding constants $C(l)$, where $l \in [p, p^\star]$ $(p \geq 2, p^\star = \frac{Np}{N-p})$. Let us denote by $\|\cdot\|_l$ the norm of $L^l(\mathbb{R}^N)$. Let $A : X \to X^\star$ be a potential operator with the potential $a : X \to \mathbb{R}$, i.e. a is Gâteaux differentiable and

$$\lim_{t\to 0} \frac{a(u+tv)-a(u)}{t} = \langle A(u), v\rangle,$$

for every $u, v \in X$. Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $X^\star$ and X. For a potential we always assume that $a(0) = 0$. We suppose that $A : X \to X^\star$ satisfies the following properties:

- A is hemicontinuous, i.e. A is continuous on line segments in X and $X^\star$ equipped with the weak topology.
- A is homogeneous of degree $p-1$, i.e. for every $u \in X$ and $t > 0$ we have $A(tu) = t^{p-1}A(u)$. Consequently, for a homogeneous hemicontinuous operator of degree $p-1$, we have $a(u) = \frac{1}{p}\langle A(u), u\rangle$.
- $A : X \to X^\star$ is a strongly monotone operator, i.e. there exists a function $\kappa : [0,\infty) \to [0,\infty)$ which is positive on $(0,\infty)$ and $\lim_{t\to\infty}\kappa(t) = \infty$ and such that for all $u, v \in X$,

$$\langle A(u) - A(v), u - v\rangle \geq \kappa(\|u-v\|)\|u-v\|\,.$$

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In this paper we suppose that the operator $A : X \to X^\star$ is a potential, hemicontinuous, strongly monotone operator, homogeneous of degree $p-1$.

Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a measurable function which satisfies the following growth condition:

(F1) $|f(x,s)| \le c(|s|^{p-1} + |s|^{r-1})$, for a.e. $x \in \mathbb{R}^N$, for all $s \in \mathbb{R}$

(F1') The embedings $X \hookrightarrow L^r(\mathbb{R}^n)$ are compact ($p < r < p^\star$).

Let $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$F(x,u) = \int_0^u f(x,s)ds, \quad \text{for a.e. } x \in \mathbb{R}^N,\ \forall s \in \mathbb{R}. \tag{1.1}$$

For a.e. $x \in \mathbb{R}^N$ and for every $u, v \in \mathbb{R}$, we have:

$$|F(x,u) - F(x,v)| \le c_1|u-v|\left(|u|^{p-1} + |v|^{p-1} + |u|^{r-1} + |v|^{r-1}\right), \tag{1.2}$$

where c_1 is a constant which depends only of u and v. Therefore, the function $F(x,\cdot)$ is locally Lipschitz and we can define the partial Clarke derivative, i.e.

$$F_2^0(x,u;w) = \limsup_{y\to u,\ t\to 0^+} \frac{F(x,y+tw) - F(x,y)}{t}, \tag{1.3}$$

for every $u, w \in \mathbb{R}$ and for a.e. $x \in \mathbb{R}$.

Now, we formulate the hemivariational inequality problem that will be studied in this paper:

Find $u \in X$ such that

$$\langle Au, v\rangle + \int_{\mathbb{R}^N} F_2^0(x,u(x);-v(x))dx \ge 0, \quad \forall\, v \in X. \tag{1.4}$$

When the function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous, the problem (1.4) is reduced to the problem:

Find $u \in X$ such that

$$\langle Au, v\rangle = \int_{\mathbb{R}^N} f(x,u(x))v(x)dx, \quad \forall\, v \in X. \tag{1.5}$$

Such problems have been studied by many authors, see [1, 3, 4, 5, 9, 10, 19, 20].

To study the existence of solutions of the problem (1.4) we introduce the functional $\Psi : X \to \mathbb{R}$ defined by $\Psi(u) = a(u) - \Phi(u)$, where $a(u) = \frac{1}{p}\langle A(u), u\rangle$ and $\Phi(u) = \int_{\mathbb{R}^N} F(x,u(x))dx$. From Proposition 5.1 we will see that the critical points of the functional Ψ are the solutions of the problem (1.4). Therefore it is enough to study the existence of critical points of the functional Ψ. Considering such a problem is motivated by the works of Clarke [8], D. Motreanu and P.D. Panagiotopoulos [22] and by the recent book of D. Motreanu and V. Rădulescu [23] , where several applications are given.

To study the existence of the critical point of the function Ψ is necessary to impose some condition on function f:

(F2) There exists $\alpha > p$, $\lambda \in [0, \frac{\kappa(1)(\alpha-p)}{C^p(p)}[$ and a continuous function $g : \mathbb{R} \to \mathbb{R}_+$, such that for a.e. $x \in \mathbb{R}^N$ and for all $u \in \mathbb{R}$ we have

$$\alpha F(x,u) + F_2^0(x,u;-u) \le g(u), \tag{1.6}$$

where $\lim_{|u|\to\infty} g(u)/|u|^p = \lambda$.

(F2') There exists $\alpha \in (\max\{p, p^\star \frac{r-p}{p^\star - p}\}, p^\star)$ and a constant $C > 0$ such that for a.e. $x \in \mathbb{R}^N$ and for all $u \in \mathbb{R}$ we have
$$-C|u|^\alpha \geq F(x,u) + \frac{1}{p} F_2^0(x,u;-u). \tag{1.7}$$

Next, we impose further assumptions on f. First we define two functions by
$$\underline{f}(x,s) = \lim_{\delta \to 0^+} \operatorname{essinf}\{f(x,t) : |t-s| < \delta\},$$
$$\overline{f}(x,s) = \lim_{\delta \to 0^+} \operatorname{esssup}\{f(x,t) : |t-s| < \delta\},$$
for every $s \in \mathbb{R}$ and for a.e. $x \in \mathbb{R}^N$. It is clear that the function $\underline{f}(x,\cdot)$ is lower semicontinuous and $\overline{f}(x,\cdot)$ is upper semicontinuous. The following hypothesis on f was introduced by Chang [7].

(F3) The functions $\underline{f}, \overline{f}$ are N-measurable, i.e. for every measurable function $u : \mathbb{R}^N \to \mathbb{R}$ the functions $x \mapsto \underline{f}(x,u(x)), x \mapsto \overline{f}(x,u(x))$ are measurable.

(F4) For every $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that for a.e. $x \in \mathbb{R}^N$ and for every $s \in \mathbb{R}$ we have
$$|f(x,s)| \leq \varepsilon |s|^{p-1} + c(\varepsilon)|s|^{r-1}.$$

(F5) For the $\alpha \in (p, p^\star)$ from condition (F2), there exists a $c^\star > 0$ such that for a.e. $x \in \mathbb{R}^N$ and for all $s \in \mathbb{R}$ we have
$$F(x,u) \geq c^\star(|u|^\alpha - |u|^p).$$

Remark 1.1. We observe that if we impose the following condition on f,

(F4') $\lim_{\varepsilon \to 0^+} \operatorname{esssup}\{\frac{|f(x,s)|}{|s|^p} : (x,s) \in \mathbb{R}^N \times (-\varepsilon, \varepsilon)\} = 0$,

then this condition with (F1) imply (F4).

The main result of this paper can be formulated in the following manner.

Theorem 1.2. (1) *If conditions (F1), (F1'), and (F2)–(F5) hold, then problem* (1.4) *has a nontrivial solution.*

(2) *If conditions (F1), (F1'), (F2'), (F3), and (F4) hold, then problem* (1.4) *has a nontrivial solution.*

Let G be the compact topological group $O(N)$ or a subgroup of $O(N)$. We suppose that G acts continuously and linear isometric on the Banach space X. We denote by
$$X^G = \{u \in H : gx = x \text{ for all } g \in G\}$$
the fixed point set of the action G on X. It is well known that X^G is a closed subspace of X. We suppose that the potential $a : X \to \mathbb{R}$ of the operator $A : X \to X^\star$ is G-invariant and the next condition for the function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ holds:

(F6) For a.e. $x \in \mathbb{R}^N$ and for every $g \in G, s \in \mathbb{R}$ we have $f(gx,s) = f(x,s)$.

In several applications the condition (F1') is replaced by the condition

(F1") The embeddings $X^G \hookrightarrow L^r(\mathbb{R}^N)$ are compact ($p < r < p^\star$).

Now, using the Principle of Symmetric Criticality for locally Lipschitz functions, proved by Krawciewicz and Marzantovicz [14], from the above theorem we obtain the following corollary, which is very useful in the applications.

Corollary 1.3. *We suppose that the potential $a : X \to \mathbb{R}$ is G-invariant and (F6) is satisfied. Then the following assertions hold.*

(a) *If (F1), (F1”), and (F2)–(F5) are fulfilled, then problem* (1.4) *has a nontrivial solution.*
(b) *If (F1), (F1’), (F2’), F3), and (F4) are fulfilled, then problem* (1.4) *has a nontrivial solution.*

Next, we give an example of a discontinuous function f for which the problem (1.4) has a nontrivial solution.

Example. Let $(a_n) \subset \mathbb{R}$ be a sequence with $a_0 = 0, a_n > 0, n \in \mathbb{N}^\star$ such that the series $\sum_{n=0}^{\infty} a_n$ is convergent and $\sum_{n=0}^{\infty} a_n > 1$. We introduce the following notation

$$A_n := \sum_{k=0}^{n} a_k, A := \sum_{k=0}^{\infty} a_k.$$

With these notations we have $A > 1$ and $A_n = A_{n-1} + a_n$ for every $n \in \mathbb{N}^\star$. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = s|s|^{p-2}\left(|s|^{r-p} + A_n\right)$, for all $s \in (-n-1, -n] \cup [n, n+1), n \in \mathbb{N}$ and $r, s \in \mathbb{R}$ with $r > p > 2$. The function f defined above satisfies the properties (F1), (F2’), (F3), and (F4). The discontinuity set of f is $\mathcal{D}_f = \mathbb{Z}^\star = \mathbb{Z} \setminus \{0\}$. It is easy to see that the function f satisfies the conditions (F1) and (F4’), therefore (F4). Let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by $F(u) = \int_0^u f(s)ds$ with $u \in [n, n+1)$, when $n \geq 1$. Because $F(u) = F(-u)$, it is sufficient to consider the case $u > 0$. We have $F(u) = \sum_{k=0}^{n-1} \int_k^{k+1} f(s)ds + \int_n^u f(s)ds$. Therefore, for $F(u) = \frac{1}{r}u^r + \frac{1}{p}A_n u^p - \frac{1}{p}\sum_{k=0}^{n} a_k k^p$, for every $u \in [n, n+1]$. It is easy to see that $F^0(u; -u) = -uf(u)$ for every $u \in (n, n+1]$. i.e. $F^0(u, -u) = -u^r - A_n u^p$. Thus,

$$F(u) + \frac{1}{p}F^0(u, -u) = -(\frac{1}{p} - \frac{1}{r})u^r - \frac{1}{p}\sum_{k=0}^{n} a_k k^p \leq -(\frac{1}{p} - \frac{1}{r})u^r.$$

If we choose $C = \frac{1}{p} - \frac{1}{r}, \alpha = r > 2$, the condition (F2’) is fulfilled.

This paper is organized as follows: In Section 2, some facts about locally Lipschitz functions are given; In Section 3 a key inequality is proved; in Section 4 the Palais-Smale and Cerami condition is verified for the function Ψ; in Section 5 we prove Theorem 2 and in the last section we give some concrete applications.

2. Preliminaries and preparatory results

Let $(X, \|\cdot\|)$ be a real Banach space and $(X^\star, \|\cdot\|_\star)$ its dual. Let $U \subset X$ be an open set. A function $\Psi : U \to \mathbb{R}$ is called locally Lipschitz function if each point $u \in U$ possesses a neighborhood N_u of u and a constant $K > 0$ which depends on N_u such that

$$|f(u_1) - f(u_2)| \leq K\|u_1 - u_2\|, \quad \forall\, u_1,\ u_2 \in N_u.$$

The generalized directional derivative of a locally Lipschitz function $\Psi : X \to \mathbb{R}$ in $u \in U$ in the direction $v \in X$ is defined by

$$\Psi^0(u; v) = \limsup_{w \to u\ t \searrow 0} \frac{1}{t}(\Psi(w + tv) - \Psi(w)).$$

It is easy to verify that $\Psi^0(u; -v) = (-\Psi)^0(u; v)$ for every $u \in U$ and $v \in X$.

The generalized gradient of Ψ in $u \in X$ is defined as being the subset of $X^\star$ such that

$$\partial\Psi(u) = \{z \in X^* : \langle z,\ v\rangle \leq \Psi^0(u; v),\ \forall\, v \in X\},$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $X^\star$ and X. The subset $\partial\Psi(u) \subset X^\star$ is nonempty, convex and $w^\star$-compact and we have

$$\Psi^0(u;v) = \max\{\langle z, v\rangle : z \in \partial\Psi(u)\},\ \forall\, v \in X.$$

If $\Psi_1, \Psi_2 : U \to \mathbb{R}$ are two locally Lipschitz functions, then

$$(\Psi_1 + \Psi_2)^0(u;\ v) \le \Psi_1^0(u;\ v) + \Psi_2^0(u;\ v)$$

for every $u \in U$ and $v \in X$. We define the function $\lambda_\Psi(u) = \inf\{\|x^\star\|_\star : x^\star \in \Psi(u)\}$. This function is lower semicontinuous and this infimum is attained, because $\partial\Psi(u)$ is $w^\star$-compact. A point $u \in X$ is a critical point of Ψ, if $\lambda_\Psi(u) = 0$, which is equivalent with $\Psi^0(u;v) \ge 0$ for every $v \in X$. For a real number $c \in \mathbb{R}$ we denote by

$$K_c = \{u \in X : \lambda_\Psi(u) = 0,\ \Psi(u) = c\}.$$

Remark 2.1. If $\Psi : X \to \mathbb{R}$ is locally Lipschitz and we take $u \in X$ and $\mu > 0$, the next two assertions are equivalent:

(a) $\Psi^0(u, v) + \mu\|v\| \ge 0$, for all $v \in X$;
(b) $\lambda_\Psi(u) \le \mu$.

Now, we define the following terms.

(i) Ψ satisfies the (PS)-condition at level c (in short, $(PS)_c$) if every sequence $\{x_n\} \subset X$ such that $\Psi(x_n) \to c$ and $\lambda_\Psi(x_n) \to 0$ has a convergent subsequence.
(ii) Ψ satisfies the (CPS)-condition at level c (in short, $(CPS)_c$) if every sequence $\{x_n\} \subset X$ such that $\Psi(x_n) \to c$ and $(1 + \|x_n\|)\lambda_\Psi(x_n) \to 0$ has a convergent subsequence.

It is clear that $(PS)_c$ implies $(CPS)_c$.

Now, we consider a globally Lipschitz function $\varphi : X \to \mathbb{R}$ such that $\varphi(x) \ge 1$, for all $x \in X$ (or, generally, $\varphi(x) \ge \alpha$, $\alpha > 0$). We say that

(iii) Ψ satisfies the $(\varphi - PS)$-condition at level c (in short, $(\varphi - PS)_c$) if every sequence $\{x_n\} \subset X$ such that $\Psi(x_n) \to c$ and $\varphi(x_n)\lambda_\Psi(x_n) \to 0$ has a convergent subsequence.

The compactness $(\varphi - PS)_c$-condition in (iii) contains the assertions (i) and (ii) in the sense that if $\varphi \equiv 1$ we get the $(PS)_c$-condition and if $\varphi(x) = 1 + \|x\|$ we have the $(C)_c$-condition.

In the next we use the following version of the Mountain Pass Theorem, see Kristály-Motreanu-Varga [17], which contains the classical result of Chang [7] and Kourogenis-Papageorgiu [16].

Proposition 2.2 (Mountain Pass Theorem)**.** *Let X be a Banach space, $\Psi : X \to R$ a locally Lipschitz function with $\Psi(0) \le 0$ and $\varphi : X \to R$ a globally Lipschitz function such that $\varphi(x) \ge 1$, $\forall x \in X$. Suppose that there exists a point $x_1 \in X$ and constants $\rho, \alpha > 0$ such that*

(i) $\Psi(x) \ge \alpha$, $\forall x \in X$ *with* $\|x\| = \rho$
(ii) $\|x_1\| > \rho$ *and* $\Psi(x_1) < \alpha$
(iii) *The function Ψ satisfies the $(\varphi - PS)_c$-condition, where*

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Psi(\gamma(t)),$$

with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = x_1\}$.

Then the minimax value c in (iii) is a critical value of Ψ, i.e. K_c is nonempty, and, in addition, $c \geq \alpha$.

Let G be a compact topological group which acts linear isometrically on the real Banach space X, i.e. the action $G \times X \to X$ is continuous and for every $g \in G$, $g : X \to X$ is a linear isometry. The action on X induces an action of the same type on the dual space X^* defined by $(gx^*)(x) = x^*(gx)$, for all $g \in G$, $x \in X$ and $x^* \in X^*$. Since

$$\|gx^*\|_\star = \sup_{\|x\|=1} |(gx^*)(x)| = \sup_{\|x\|=1} |x^*(gx)|,$$

the isometry assumption for the action of G implies

$$\|gx^*\|_\star = \sup_{\|x\|=1} |x^*(x)| = \|x^*\|_\star, \ \forall\, x^* \in X^*, \ g \in G.$$

We suppose that $\Psi : X \to \mathbb{R}$ is a locally Lipschitz and G-invariant function, i.e., $\Psi(gx) = \Psi(x)$ for every $g \in G$ and $x \in X$. From Krawcewicz-Marzantowicz [10] we have the relation

$$g\partial\Psi(x) = \partial\Psi(gx) = \partial\Psi(x), \text{ for every } g \in G \text{ and } x \in X.$$

Therefore, the subset $\partial\Psi(x) \subset X^*$ is G-invariant, so the function $\lambda_\Psi(x) = \inf_{w \in \partial\Psi(x)} \|w\|_\star$, $x \in X$, is G-invariant. The fixed points set of the action G, i.e. $X^G = \{x \in X \mid gx = x \, \forall\, g \in G\}$ is a closed linear subspace of X.

We conclude this section with the Principle of Symmetric Criticality, first proved by Palais [24] for differentiable functions and for locally Lipschitz proved by Krawciewicz and Marzantovicz [14].

Theorem 2.3. *Let $\Psi : X \to \mathbb{R}$ be a G-invariant locally Lipschitz function and $u \in X^G$ a fixed point. Then $u \in X^G$ is a critical point of Ψ if and only if u is a critical point of $\Psi^G = \psi|_{X^G} : X^G \to \mathbb{R}$.*

3. Some basic lemmas

Define the function $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \int_{\mathbb{R}^N} F(x, u(x))dx, \quad \forall\, u \in X, \tag{3.1}$$

where the function F is defined in (1.1).

Remark 3.1. The following two results are true for the general growth condition (f_1), but it is sufficient to prove them in the case when the function f satisfies the growth condition $|f(x,s)| \leq c|u|^{p-1}$ for a.e. $x \in \mathbb{R}^N, \forall\, s \in \mathbb{R}$. For simplicity we denote $h(u) = c|u|^{p-1}$ and in the next two results we use only that the function h is monotone increasing, convex and $h(0) = 0$.

Proposition 3.2. *The function $\Phi : X \to \mathbb{R}$, defined by $\Phi(u) = \int_{\mathbb{R}^N} F(x, u(x))dx$ is locally Lipschitz on bounded sets of X.*

Proof. For every $u, v \in X$, with $\|u\|, \|v\| < r$, we have

$$\begin{aligned}
&\|\Phi(u) - \Phi(v)\| \\
&\le \int_{\mathbb{R}^N} |F(x, u(x)) - F(x, v(x))| dx \\
&\le c_1 \int_{\mathbb{R}^N} |u(x) - v(x)| [h(|u(x)|) + h(|v(x)|)] \\
&\le c_2 \big(\int_{\mathbb{R}^N} |u(x) - v(x)|^p \big)^{1/p} \big[\big(\int_{\mathbb{R}^N} (h(|u(x)|)^{p'} dx \big)^{1/p'} + \big(\int_{\mathbb{R}^N} (h(|v(x)|)^{p'} dx \big)^{1/p'} \big] \\
&\le c_2 \|u - v\|_p [\|h(|u|)\|_{p'} + \|h(|v|)\|_{p'}) \\
&\le C(u, v) \|u - v\|,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and we used the Hölder inequality, the subadditivity of the norm $\|\cdot\|_{p'}$ and the fact that the inclusion $X \hookrightarrow L^p(\mathbb{R}^N)$ is continuous. We observe that C(u, v) is a constant which depends only of u and v. ☐

Proposition 3.3. *If condition (F1) holds, then for every $u, v \in X$, then*

$$\Phi^0(u; v) \le \int_{\mathbb{R}^N} F_2^0(x, u(x); v(x)) dx. \tag{3.2}$$

Proof. It is sufficient to prove the proposition for the function f, which satisfies only the growth condition $|f(x, s)| \le c|u|^{p-1}$ from Remark 3.1. Let us fix the elements $u, v \in X$. The function $F(x, \cdot)$ is locally Lipschitz and therefore continuous. Thus $F_2^0(x, u(x); v(x))$ can be expressed as the upper limit of $\big(F(x, y+tv(x)) - F(x, y)\big)/t$, where $t \to 0^+$ takes rational values and $y \to u(x)$ takes values in a countable subset of $\mathbb{R}$. Therefore, the map $x \to F_2^0(x, u(x); v(x))$ is measurable as the "countable limsup" of measurable functions in x. From condition (F1) we get that the function $x \to F_2^0(x, u(x); v(x))$ is from $L^1(\mathbb{R}^N)$.

Using the fact that the Banach space X is separable, there exists a sequence $w_n \in X$ with $\|w_n - u\| \to 0$ and a real number sequence $t_n \to 0^+$, such that

$$\Phi^0(u, v) = \lim_{n \to \infty} \frac{\Phi(w_n + t_n v) - \Phi(w_n)}{t_n}. \tag{3.3}$$

Since the inclusion $X \hookrightarrow L^p(\mathbb{R}^N)$ is continuous, we get $\|w_n - u\|_p \to 0$. Using [6, Theorem IV.9], there exists a subsequence of (w_n) denoted in the same way, such that $w_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$. Now, let $\varphi_n : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be the function defined by

$$\begin{aligned}
\varphi_n(x) = &- \frac{F(x, w_n(x) + t_n v(x)) - F(x, w_n(x))}{t_n} \\
&+ c_1 |v(x)| [h(|w_n(x) + t_n v(x)|) + h(|w_n(x)|)].
\end{aligned}$$

We see that the the functions φ_n are measurable and non-negative. If we apply Fatou's lemma, we get

$$\int_{\mathbb{R}^N} \liminf_{n \to \infty} \varphi_n(x) dx \le \liminf_{n \to \infty} \int_{\mathbb{R}^N} \varphi_n(x) dx.$$

This inequality is equivalent to

$$\int_{\mathbb{R}^N} \limsup_{n \to \infty} [-\varphi_n(x)] dx \ge \limsup_{n \to \infty} \int_{\mathbb{R}^N} [-\varphi_n(x)] dx. \tag{3.4}$$

For simplicity in the calculus we introduce the following notation:

(i) $\varphi_n^1(x) = \frac{F(x,w_n(x)+t_nv(x))-F(x,w_n(x))}{t_n}$;
(ii) $\varphi_n^2(x) = c_1|v(x)|[h(|w_n(x)+t_nv(x)|)+h(|w_n(x)|)]$.

With these notation, we have $\varphi_n(x) = -\varphi_n^1(x) + \varphi_n^2(x)$.

Now we prove the existence of limit $b = \lim_{n\to\infty}\int_{\mathbb{R}^N}\varphi_n^2(x)dx$. Using the facts that the inclusion $X \hookrightarrow L^p(\mathbb{R}^N)$ is continuous and $\|w_n - u\| \to 0$, we get $\|w_n - u\|_p \to 0$. Using [6, Theorem IV.9], there exist a positive function $g \in L^p(\mathbb{R}^N)$, such that $|w_n(x)| \le g(x)$ a.e. $x \in \mathbb{R}^N$. Considering that the function h is monotone increasing, we get

$$|\varphi_n^2(x)| \le c_1|v(x)|[h(g(x)+|v(x)|)+h(g(x))], \quad \text{a.e. } x \in \mathbb{R}^N.$$

Moreover, $\varphi_n^2(x) \to 2c_1|v(x)|h(|u(x)|)$ for a.e. $x \in \mathbb{R}^N$. Thus, using the Lebesque dominated convergence theorem, we have

$$b = \lim_{n\to\infty}\int_{\mathbb{R}^N}\varphi_n^2(x)dx = \int_{\mathbb{R}^N}2c_1|v(x)|h(|u(x)|)dx. \tag{3.5}$$

If we denote by $I_1 = \limsup_{n\to\infty}\int_{\mathbb{R}^N}[-\varphi_n(x)]dx$, then using (3.3) and (3.5), we have

$$I_1 = \limsup_{n\to\infty}\int_{\mathbb{R}^N}[-\varphi_n(x)]dx = \Phi^0(u;v) - b. \tag{3.6}$$

Next we estimate the expression $I_2 = \int_{\mathbb{R}^N}\limsup_{n\to\infty}[-\varphi_n(x)]dx$. We have the inequality

$$\int_{\mathbb{R}^N}\limsup_{n\to\infty}[\varphi_n^1(x)]dx - \int_{\mathbb{R}^N}\lim_{n\to\infty}\varphi_n^2(x)dx \ge I_2. \tag{3.7}$$

Using the fact that $w_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$ and $t_n \to 0^+$, we get

$$\int_{\mathbb{R}^N}\lim_{n\to\infty}\varphi_n^2(x)dx = 2c_1\int_{\mathbb{R}^N}|v(x)|h(|u(x)|)dx.$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^N}\limsup_{n\to\infty}\varphi_n^1(x)dx &\le \int_{\mathbb{R}^N}\limsup_{y\to u(x),\, t\to 0^+}\frac{F(x,y+tv(x))-F(x,y)}{t}dx\\ &= \int_{\mathbb{R}^N}F_2^0(x,u(x);v(x))dx.\end{aligned}$$

Using relations (3.4), (3.6), (3.7) and the above estimates, we obtain the desired result. □

4. The Palais-Smale and Cerami compactness condition

In this section we study the situation when the function Ψ satisfies the $(PS)_c$ and $(CPS)_c$ conditions. We have the following result.

Proposition 4.1. *Let $(u_n) \subset X$ be a $(PS)_c$ sequence for the function $\Psi : X \to \mathbb{R}$. If the conditions (F1) and (F2) are fulfilled, then the sequence (u_n) is bounded in X.*

Proof. Because $(u_n) \subset X$ is a $(PS)_c$ sequence for the function Ψ, we have $\Psi(u_n) \to c$ and $\lambda_\Psi(u_n) \to 0$. From the condition $\Psi(u_n) \to c$ we get $c+1 \ge \Psi(u_n)$ for sufficiently large $n \in \mathbb{N}$.

Because $\lambda_\Psi(u_n) \to 0$, $\|u_n\| \ge \|u_n\|\lambda_\Psi(u_n)$ for every sufficiently large $n \in \mathbb{N}$. From the definition of $\lambda_\Psi(u_n)$ results the existence of an element $z^\star_{u_n} \in \partial\Psi(u_n)$, such that $\lambda_\Psi(u_n) = \|z^\star_{u_n}\|_\star$. For every $v \in X$, we have $|z^\star_{u_n}(v)| \le \|z^\star_{u_n}\|_\star \|v\|$, therefore $\|z^\star_{u_n}\|_\star\|v\| \ge -z^\star_{u_n}(v)$. If we take $v = u_n$, then $\|z^\star_{u_n}\|_\star \|u_n\| \ge -z^\star_{u_n}(u_n)$.

Using the properties $\Psi^0(u, v) = \max\{z^\star(v) : z^\star \in \partial\Psi(u)\ \}$ for every $v \in X$, we have $-z^\star(v) \ge -\Psi^0(u, v)$ for all $z^\star \in \partial\Psi(u)$ and $v \in X$. If we take $u = v = u_n$ and $z^\star = z^\star_{u_n}$, we get $-z^\star_{u_n}(u_n) \ge -\Psi^0(u_n, u_n)$. Therefore, for every $\alpha > 0$, we have

$$\frac{1}{\alpha}\|u_n\| \ge \frac{1}{\alpha}\|z^\star_{u_n}\|_\star\|u_n\| \ge -\frac{1}{\alpha}\Psi^0(u_n, u_n).$$

When we add the above inequality with $c + 1 \ge \Psi(u_n)$, we obtain

$$c + 1 + \frac{1}{\alpha}\|u_n\| \ge \Psi(u_n) - \frac{1}{\alpha}\Psi^0(u_n; u_n).$$

Using the above inequality, $\Psi^0(u, v) \le \langle A(u), v\rangle + \Phi^0(u, -v)$, and Proposition 3.3 we get

$$\begin{aligned}
&c + 1 + \frac{1}{\alpha}\|u_n\| \\
&\ge \Psi(u_n) - \frac{1}{\alpha}\Psi^0(u_n; u_n) \\
&= \frac{1}{p}\langle A(u_n), u_n\rangle - \Phi(u_n) - \frac{1}{\alpha}\left(\langle A(u_n), u_n\rangle + \Phi^0(u_n; -u_n)\right) \\
&\ge (\frac{1}{p} - \frac{1}{\alpha})\langle A(u_n), u_n\rangle - \int_{\mathbb{R}^N}\left[F(x, u_n(x)) + \frac{1}{\alpha}F_2^0(x, u_n(x); -u_n(x))\right]dx \\
&\ge (\frac{1}{p} - \frac{1}{\alpha})\langle A(u_n), u_n\rangle - \frac{1}{\alpha}\int_{\mathbb{R}^N} g(u_n(x))dx.
\end{aligned}$$

The relation $\lim_{|u|\to\infty} \frac{g(u)}{|u|^p} = \lambda$ assures the existence of a constant M, such that $\int_{\mathbb{R}^N} g(u_n(x))dx \le M + \lambda\int_{\mathbb{R}^N} |u_n(x)|^p dx$. We use again that the inclusion $X \hookrightarrow L^p(\mathbb{R}^N)$ is continuous, that $a(u) = \frac{1}{p}\langle A(u), u\rangle$ and that

$$a(u) = \|u\|^p\langle A(\frac{u}{\|u\|}), \frac{u}{\|u\|}\rangle \ge \kappa(1)\|u\|^p,$$

to obtain

$$\begin{aligned}
c + 1 + \|u_n\| &\ge (\frac{1}{p} - \frac{1}{\alpha})\langle A(u_n), u_n\rangle - \frac{\lambda C^p(p)}{\alpha}\|u_n\|^p - \frac{M}{\alpha} \\
&\ge \frac{\kappa(1)(\alpha - p) - \lambda C^p(p)}{\alpha}\|u_n\|^p - \frac{M}{\alpha}.
\end{aligned}$$

From the above inequality, it results that the sequence (u_n) is bounded. $\square$

Proposition 4.2. *If conditions (F1), (F2') and (F4) hold, then every $(CPS)_c(c > 0)$ sequence $(u_n) \subset X$ for the function $\Psi : X \to \mathbb{R}$ is bounded in X.*

Proof. Let $(u_n) \subset X$ be a $(CPS)_c$ $(c > 0)$ sequence for the function Ψ, i.e. $\Psi(u_n) \to c$ and $(1+\|u_n\|)\lambda_\Psi(u_n) \to 0$. From $(1+\|u_n\|)\lambda_\Psi(u_n) \to 0$, we get $\|u_n\|\lambda_\Psi(u_n) \to 0$ and $\lambda_\Psi(u_n) \to 0$. As in Proposition 4.1, there exists $z^\star_{u_n} \in \partial\Psi(u_n)$ such that

$$\frac{1}{p}\|z^\star_{u_n}\|_\star\|u_n\| \ge -\Psi^0(u_n; \frac{1}{p}u_n).$$

From this inequality, Proposition 3.3, condition (F2') and the property $\Psi^0(u;v) \le \langle Au, v\rangle + \Phi^0(u;-v)$ we get

$$\begin{aligned}
c+1 &\ge \Psi(u_n) - \frac{1}{p}\Psi^0(u_n;u_n) \\
&\ge a(u_n) - \Phi(u_n) - \frac{1}{p}\left[\langle Au_n, u_n\rangle + \Phi^0(u_n;-u_n)\right] \\
&\ge -\int_{\mathbb{R}^N}\left[F(x,u_n(x)) + \frac{1}{p}F_2^0(x,u_n(x);-u_n(x))\right]dx \\
&\ge C\|u_n\|_\alpha^\alpha.
\end{aligned}$$

Therefore, the sequence (u_n) is bounded in $L^\alpha(\mathbb{R}^N)$. From the condition (F4) follows that, for every $\varepsilon > 0$, there exists $c(\varepsilon) > 0$, such that for a.e. $x \in \mathbb{R}^N$,

$$F(x,u(x)) \le \frac{\varepsilon}{p}|u(x)|^p + \frac{c(\varepsilon)}{r}|u(x)|^r.$$

After integration, we obtain

$$\Phi(u) \le \frac{\varepsilon}{p}\|u\|_p^p + \frac{c(\varepsilon)}{r}\|u\|_r^r.$$

Using the above inequality, the expression of Ψ, and $\|u\|_p \le C(p)\|u\|$, we obtain

$$\frac{\kappa(1) - \varepsilon C^p(p)}{p}\|u\|^p \le \Psi(u) + \frac{c(\varepsilon)}{r}\|u\|_r^r \le c + 1 + \|u\|_r^r.$$

Now, we study the behaviour of the sequence $(\|u_n\|_r)$. We have the following two cases:

(i) If $r = \alpha$, then it is easy to see that the sequence $(\|u_n\|_r)$ is bounded in $\mathbb{R}$.

(ii) If $r \in (\alpha, p^\star)$ and $\alpha > p^\star\frac{r-p}{p^\star - p}$, then we have

$$\|u\|_r^r \le \|u\|_\alpha^{(1-s)\alpha} \cdot \|u\|_{p^\star}^{sp^\star},$$

where $r = (1-s)\alpha + sp^\star, s \in (0,1)$.

Using the inequality $\|u\|_{p^\star}^{sp^\star} \le C^{sp^\star}(p)\|u\|^{sp^\star}$, we obtain

$$\frac{\kappa(1) - \varepsilon C^p(p)}{p}\|u\|^p \le c + 1 + \frac{c(\varepsilon)}{r}\|u\|_\alpha^{(1-s)\alpha}\|u\|^{sp^\star}. \tag{4.1}$$

When in the inequality (4.1) we take $\varepsilon \in \left(0, \frac{\kappa(1)}{C^p(p)}\right)$ and use b), we obtain that the sequence (u_n) is bounded in X. □

The main result of this section is as follows.

Theorem 4.3. (1) *If conditions (F1), (F1'), and (F2)–(F4) hold, then Ψ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$.*

(2) *If conditions (F1), (F1'), (F2'), (F3), and (F4) hold, then Ψ satisfies the $(CPS)_c$ condition for every $c > 0$.*

Proof. Let $(u_n) \subset X$ be a $(PS)_c(c \in \mathbb{R})$ or a $(CPS)_c(c > 0)$ sequence for the function $\Psi(u_n)$. Using Propositions 4.1 4.2, it follows that (u_n) is a bounded sequence in X. As X is reflexive Banach space, the existence of an element $u \in X$ results, such that $u_n \rightharpoonup u$ weakly in X. Because the inclusions $X \hookrightarrow L^r(\mathbb{R}^N)$ is compact, we have that $u_n \to u$ strongly in $L^r(\mathbb{R}^N)$.

Next we estimate the expressions $I_n^1 = \Psi^0(u_n; u_n - u)$ and $I_n^2 = \Psi^0(u; u - u_n)$. First we estimate the expression $I_n^2 = \Psi^0(u; u - u_n)$. We know that $\Psi^0(u; v) = \max\{z^\star(v) : z^\star \in \partial\Psi(u)\}$, $\forall\, v \in X$. Therefore, there exists $z_u^\star \in \partial\Psi(u)$, such that $\Psi^0(u; v) = z_u^\star(v)$ for all $v \in X$. From the above relation and from the fact that $u_n \rightharpoonup u$ weakly in X, we get $\Psi^0(u; u - u_n) = z_u^\star(u - u_n) \to 0$.

Now, we estimate the expression $I_n^1 = \Psi^0(u_n; u_n - u)$. From $\lambda_\Psi(u_n) \to 0$ follows the existence of a positive real numbers sequence $\mu_n \to 0$, such that $\lambda_\Psi(u_n) \le \mu_n$. If we use the Remark 2.1, we get $\Psi^0(u_n, u_n - u) + \mu_n \|u_n - u\| \ge 0$.

Now, we estimate the expression $I_n = \Phi^0(u_n; u - u_n) + \Phi(u; u - u_n)$. For the simplicity in calculus we introduce the notations $h_1(s) = |s|^{p-1}$ and $h_2(s) = |s|^r$. For this we observe that if we use the continuity of the functions h_1 and h_2, the condition (F4) implies that for every $\varepsilon > 0$, there exists a $c(\varepsilon) > 0$ such that

$$\max\left\{|\underline{f}(x,s)|, |\overline{f}(x,s)|\right\} \le \varepsilon h_1(s) + c(\varepsilon) h_2(s), \tag{4.2}$$

for a.e. $x \in \mathbb{R}^N$ and for all $s \in \mathbb{R}$. Using this relation and Proposition 3.3, we have

$$\begin{aligned}
I_n &= \Phi^0(u_n; u - u_n) + \Phi(u; u - u_n)\\
&\le \int_{\mathbb{R}^N} \left[F_2^0(x, u_n(x); u_n(x) - u(x)) + F_2^0(x, u(x); u(x) - u_n(x))\right] dx\\
&\le \int_{\mathbb{R}^N} \left[\underline{f}(x, u_n(x))(u_n(x) - u(x)) + \overline{f}(x, u(x))(u(x) - u_n(x))\right] dx\\
&\le 2\varepsilon \int_{\mathbb{R}^N} [h_1(u(x)) + h_1(u_n(x))]\, |u_n(x) - u(x)| dx\\
&\quad + 2c_\varepsilon \int_{\mathbb{R}^N} [(h_2(u(x)) + h_2(u_n(x))]\, |u_n(x) - u(x)| dx.
\end{aligned}$$

Using Hölder inequality and that the inclusion $X \hookrightarrow L^p(\mathbb{R}^N)$ is continuous, we get

$$\begin{aligned}
I_n &\le 2\varepsilon C(p) \|u_n - u\| (\|h_1(u)\|_{p'} + \|h_1(u_n)\|_{p'})\\
&\quad + 2c(\varepsilon) \|u_n - u\|_r (\|h_2(u)\|_{r'} + \|h_2(u_n)\|_{r'}),
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Using the fact that the inclusion $X \hookrightarrow L^r(\mathbb{R}^N)$ is compact, we get that $\|u_n - u\|_r \to 0$ as $n \to \infty$. For $\varepsilon \to 0^+$ and $n \to \infty$ we obtain that $I_n \to 0$.

Finally, we use the inequality $\Psi^0(u; v) \le \langle A(u), v\rangle + \Phi^0(u; -v)$. If we replace v with $-v$, we get $\Psi^0(u, -v) \le -\langle A(u), v\rangle + \Phi^0(u; v)$, therefore $\langle A(u), v\rangle \le \Phi^0(u; v) - \Psi^0(u, -v)$.

In the above inequality we replace u and v by $u = u_n, v = u - u_n$ then $u = u, v = u_n - u$ and we get

$$\begin{aligned}
\langle A(u_n), u - u_n\rangle &\le \Phi^0(u_n, u - u_n) - \Psi^0(u_n; u_n - u),\\
\langle A(u), u_n - u\rangle &\le \Phi^0(u, u_n - u) - \Psi^0(u, u - u_n).
\end{aligned}$$

Adding these relations, we have the following key inequality:

$$\begin{aligned}
&\|u_n - u\| \kappa(u_n - u)\\
&\le \langle A(u_n - u), u_n - u\rangle\\
&\le \left[\Phi^0(u_n; u - u_n) + \Phi(u; u - u_n)\right] - \Psi^0(u_n; u_n - u) - \Psi^0(u; u - u_n)\\
&= I_n - I_n^1 - I_n^2.
\end{aligned}$$

Using the above relation and the estimations of I_n, I_n^1 and I_n^2, we obtain

$$\|u_n - u\|\kappa(u_n - u) \le I_n + \mu_n\|u_n - u\| - z_u^\star(u_n - u).$$

If $n \to \infty$, from the above inequality we obtain the assertion of the theorem. $\square$

Remark 4.4. It is important to observe then the above results remain true if we replace the Banach space X with every closed subspace Y of X.

5. Proof of Theorem 1.2

In this section we prove the main result of this paper, whihc is a result of Mountain Pass type. First we prove that the critical points of the function $\Psi : X \to \mathbb{R}$ defined by $\Psi(u) = a(u) - \Phi(u)$ are solutions of problem (1.4).

Proposition 5.1. *If $0 \in \partial\Psi(u)$, then u solves the problem* (1.4).

Proof. Because $0 \in \partial\Psi(u)$, we have $\Psi^0(u; v) \ge 0$ for every $v \in X$. Using the Proposition 3.3 and a property of Clarke derivative we obtain

$$\begin{aligned} 0 \le \Psi^0(u; v) &\le \langle u, v\rangle + (-\Phi)^0(u; v) \\ &= \langle A(u), v\rangle + \Phi^0(u; -v) \\ &\le \langle A(u), v\rangle + \int_{\mathbb{R}^N} F_2^0(x, u(x), -v(x))dx, \end{aligned}$$

for every $v \in X$. $\square$

Proof of Theorem 1.2. Using (1) in Theorem 4.3, and conditions (F1)–(F4), it follows that the functional $\Psi(u) = \frac{1}{p}\langle A(u), u\rangle - \Phi(u)$ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$. From Proposition 2.2 we verify the following geometric hypotheses:

$$\exists\, \alpha, \rho > 0, \quad \text{such that } \Psi(u) \ge \beta \text{ on } B_\rho(0) = \{u \in X : \|u\| = \rho\}, \tag{5.1}$$

$$\Psi(0) = 0 \quad \text{and there exists } v \in H \setminus B_\rho(0) \text{ such that } \Psi(v) \le 0. \tag{5.2}$$

For the proof of relation (5.1), we use the relation (F4), i.e. $|f(x, s)| \le \varepsilon|s|^{p-1} + c(\varepsilon)|s|^{r-1}$. Integrating this inequality and using that the inclusions $X \hookrightarrow L^p(\mathbb{R}^N)$, $X \hookrightarrow L^r(\mathbb{R}^N)$ are continuous, we get that

$$\begin{aligned} \Psi(u) &\ge \frac{\kappa(1) - \varepsilon C(p)}{p}\langle A(u), u\rangle - \frac{1}{r}c(\varepsilon)C(r)\|u\|_r^r \\ &\ge \frac{\kappa(1) - \varepsilon C(p)}{p}\|u\|^p - \frac{1}{r}c(\varepsilon)C(r)\|u\|^r. \end{aligned}$$

The right member of the inequality is a function $\chi : \mathbb{R}_+ \to \mathbb{R}$ of the form $\chi(t) = At^p - Bt^r$, where $A = \frac{\kappa(1) - \varepsilon C(p)}{p}$, $B = \frac{1}{r}c(\varepsilon)C(r)$. The function χ attains its global maximum in the point $t_M = (\frac{pA}{rB})^{\frac{1}{r-p}}$. When we take $\rho = t_M$ and $\beta \in]0, \chi(t_M)]$, it is easy to see that the condition (5.1) is fulfilled.

From (F5) we have $\Psi(u) \le \frac{1}{p}\langle A(u), u\rangle + c^\star\|u\|_p^p - c^\star\|u\|_\alpha^\alpha$. If we fix an element $v \in H \setminus \{0\}$ and in place of u we put tv, then we have

$$\Psi(tv) \le (\frac{1}{p}\langle A(v), v\rangle + c^\star\|v\|_p^p)t^p - c^\star t^\alpha\|v\|_\alpha^\alpha.$$

From this we see that if t is large enough, $tv \notin B_\rho(0)$ and $\Psi(tv) < 0$. So, the condition (5.2) is satisfied and Proposition 2.2 assures the existence of a nontrivial critical point of Ψ.

Now when we use (2) in Theorem 4.3, from conditions (F1), (F2'), (F3), and (F4), we get that the function Ψ satisfies the condition $(CPS)_c$ for every $c > 0$. We use again the Proposition 2.2, which assures the existence of a nontrivial critical point for the function Ψ. It is sufficient to prove only the relation (5.2), because (5.1) is proved in the same way.

To prove the relation (5.2) we fix an element $u \in X$ and we define the function $h : (0, +\infty) \to \mathbb{R}$ by $h(t) = \frac{1}{t} F(x, t^{1/p}u) - C\frac{p}{\alpha-p} t^{\frac{\alpha}{p}-1}|u|^\alpha$. The function h is locally Lipschitz. We fix a number $t > 1$, and from the Lebourg's main value theorem follows the existence of an element $\tau \in (1, t)$ such that

$$h(t) - h(1) \in \partial_t h(\tau)(t - 1),$$

where ∂_t denotes the generalized gradient of Clarke with respect to $t \in \mathbb{R}$. From the Chain Rules we have

$$\partial_t F(x, t^{1/p}u) \subset \frac{1}{p} \partial F(x, t^{1/p}u) t^{\frac{1}{p}-1} u.$$

Also we have

$$\partial_t h(t) \subset -\frac{1}{t^2} F(x, t^{1/p}u) + \frac{1}{t} \partial F(x, t^{1/p}u) t^{\frac{1}{p}-1} u - C t^{\frac{\alpha}{p}-2} |u|^\alpha.$$

Therefore,

$$\begin{aligned} h(t) - h(1) &\subset \partial_t h(\tau)(t - 1) \\ &\subset -\frac{1}{t^2} \Big[F(x, t^{1/p}u) - t^{1/p} u \partial F(x, t^{1/p}u) + C|t^{1/p}u|^\alpha \Big] (t - 1). \end{aligned}$$

Using the relation (F2'), we obtain that $h(t) \geq h(1)$; therefore,

$$\frac{1}{t} F(x, t^{1/p}u) - C\frac{p}{\alpha - p} t^{\frac{\alpha}{p}-1} |u|^\alpha \geq F(x, u) - C\frac{p}{\alpha - p}|u|^\alpha.$$

From this inequality, we get

$$F(x, t^{1/p}) \geq tF(x, u) + C\frac{p}{\alpha - p}[t^{\alpha/p} - t]|u|^\alpha, \tag{5.3}$$

for every $t > 1$ and $u \in \mathbb{R}$. Let us fix an element $u_0 \in X \setminus \{0\}$; then for every $t > 1$, we have

$$\begin{aligned} \Psi(t^{1/p}u_0) &= \frac{1}{p} \langle A(t^{1/p}u_0), t^{1/p}u_0 \rangle - \int_{\mathbb{R}^N} F(x, t^{1/p}u_0(x))dx \\ &\leq \frac{t}{p} \langle Au_0, u_0 \rangle - t \int_{\mathbb{R}^N} F(x, u_0(x))dx - C\frac{p}{\alpha - p}[t^{\alpha/p} - t] \|u_0\|_\alpha^\alpha. \end{aligned}$$

If t is sufficiently large, then for $v_0 = t^{1/p}u_0$ we have $\Psi(v_0) \leq 0$. This completes the proof. □

6. Applications

In the first two examples we suppose that X is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$.

Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a measurable function as in the introduction of this paper.

Application 6.1. We consider the function $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ which satisfies the following conditions:

(a) $V(x) > 0$ for all $x \in \mathbb{R}^N$

(b) $V(x) \to +\infty$ as $|x| \to +\infty$.

Let X be the Hilbert space defined by

$$X = \{u \in H^1(\mathbb{R}^N) : \int (|\nabla u(x)|^2 + V(x)|u(x)|^2)dx < \infty\},$$

with the inner product

$$\langle u, v\rangle = \int (\nabla u \nabla v + V(x)uv)dx.$$

It is well known that if the conditions (a) and (b) are fulfilled then the inclusion $X \hookrightarrow L^2(\mathbb{R}^N)$ is compact [11], therefore the condition (F1') is satisfied.

Now we formulate the problem.

Find a positive $u \in X$ such that for every $v \in X$ we have

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx + \int_{\mathbb{R}^N} F_2^0(x, u(x); -v(x))dx \geq 0. \tag{6.1}$$

We have the following result.

Corollary 6.2. *If conditions (F1), (F2'), (F3), (F4), and (a), (b) hold, the problem 6.1 has a nontrivial positive solution.*

Proof. We replace the function f by $f_+ : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ defined by

$$f_+(x, u) = \begin{cases} f(x, u) & \text{if } u \geq 0; \\ 0, & \text{if } u < 0 \end{cases} \tag{6.2}$$

and use (2) in Theorem 1.2. □

Remark 6.3. The above result improves a result in Gazolla-Rădulescu [10].

Application 6.4. Now, we consider $Au := - \triangle u + |x|^2 u$ for $u \in D(A)$, where

$$D(A) := \{u \in L^2(\mathbb{R}^N) : Au \in L^2(\mathbb{R}^N)\}.$$

Here $|\cdot|$ denotes the Euclidian norm of $\mathbb{R}^N$. In this case the Hilbert space X is defined by

$$X = \{\ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + |x|^2 u^2)dx < \infty\},$$

with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + |x|^2 uv)dx.$$

The inclusion $X \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $s \in [2, \frac{2N}{N-2})$, see Kavian [12, Exercise 20, pp. 278]. Therefore, the condition (F1') is satisfied.

Now, we formulate the next problem.

Find a positive $u \in X$ such that for every $v \in X$ we have

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + |x|^2 uv)dx + \int_{\mathbb{R}^N} F_2^0(x, u(x); -v(x))dx \geq 0. \tag{6.3}$$

Corollary 6.5. *If (F1), (F2), (F3), and (F4) hold, then problem* (6.3) *has a positive solution.*

The proof of this corollary is similar to that of Corollary 6.2.

Remark 6.6. This result improves a result from Varga [28], where the condition (F5) was used.

Application 6.7. In this example we suppose that G is a subgroup of the group $O(N)$. Let Ω be an unbounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and the elements of G leave Ω invariant, i.e. $g(\Omega) = \Omega$ for every $g \in G$. We suppose that Ω is compatible with G, see the book of Willem [29] Definition 1.22. The action of G on $X = W_0^{1,p}$ is defined by

$$gu(x) := u(g^{-1}x).$$

The subspace of invariant function X^G is defined by

$$X^G := \{u \in X : gu = u, \ \forall g \in G \}.$$

The norm on X is defined by

$$\|u\| = \Big(\int_\Omega (|\nabla u|^p + |u|^p) dx \Big)^{1/p}.$$

If Ω is compatible with G, then the embeddings $X \hookrightarrow L^s(\Omega)$, with $p < s < p^\star$ are compact, see the paper of Kobayashi and Otani [13]. Therefore the condition (F2") is satisfied.

We consider the potential $a : X \to \mathbb{R}$ defined by $a(u) = \frac{1}{p}\|u\|^p$. This function is G-invariant because the action of G is isometric on X. The Gateaux differential $A : X \to X^\star$ of the function $a : X \to \mathbb{R}$ is given by

$$\langle Au, v\rangle = \int_\Omega \big(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\big)\, dx.$$

The operator A is homogeneous of degree $p - 1$ and strongly monotone, because $p \geq 2$.

Now, we formulate the following problem.

Find $u \in X \setminus \{0\}$ such that for every $v \in X$ we have

$$\int_\Omega \big(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\big)\, dx + \int_\Omega F_2^0(x, u(x); -v(x))dx \geq 0. \tag{6.4}$$

We have the following result.

Corollary 6.8. *If we suppose that the condition (F6) is true, then the following assertions hold.*

- (a) *If conditions (F1)–(F5) are fulfilled, then problem* (1.4) *has a nontrivial solution.*
- (b) *If conditions (F1), (F2'), (F3), and (F4) are fulfilled, then problem* (1.4) *has a nontrivial symmetric solution.*

Remark 6.9. The result (a) from Corollary 6.8 is similar to the a result obtained by Kobayashi, Ôtani [13], but the difference is that in the paper [13] the "Principle of Symmetric Criticality" was used for Szulkin type functional, see [27].

Application 6.10. In this case we consider $\Omega = \tilde{\Omega} \times \mathbb{R}^N, N - m \geq 2, \tilde{\Omega} \subset \mathbb{R}^m (m \geq 1)$ is open bounded and $2 \leq p \leq N$. We consider the Banach space $X = W_0^{1,p}(\Omega)$ with the norm $\|u\| = (\int_\Omega |\nabla u|^p)^{1/p}$. Let G be a subgroup of $O(N)$ defined by $G = id^m \times O(N - m)$. The action of G on X is defined by $gu(x_1, x_2) = u(x_1, g_1x_2)$

for every $(x_1, x_2) \in \tilde{\Omega} \times \mathbb{R}^{N-m}$ and $g = id^m \times g_1 \in G$. The subspace of invariant function is defined by

$$X^G = W^{1,p}_{0,G} = \{u \in X : gu = u, \ \forall g \in G\}.$$

The action of G on X is isometric, that is

$$\|gu\| = \|u\|, \ \forall g \in G.$$

If $2 \leq p \leq N$, from a result of Lions [18] follows that the embeddings $X \hookrightarrow L^s(\Omega), p < s < p^\star$ are compact. Therefore the condition (f_2'') is true. In this case condition (F6) will be replaced by

(F6') $f(x, y_1, u) = f(x, y_2, u)$ for every $y_1, y_2 \in \mathbb{R}^{N-m}$ $(N - m \geq 2)$, $|y_1| = |y_2|$; i.e., the function $f(x, \cdot, u)$ is spherically symmetric on $\mathbb{R}^{N-m}$.

We consider the potential $a : X \to \mathbb{R}$ defined by $a(u) = \frac{1}{p}\|u\|^p$. This functional is G-invariant because the action of G is isometric on X. The Gateaux differential $A : X \to X^\star$ of the functional $a : X \to \mathbb{R}$ is given by

$$\langle Au, v\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx.$$

The operator A is homogeneous of degree $p - 1$ and strongly monotone, because $p \geq 2$.

Now, we formulate the following problem.

Find $u \in X \setminus \{0\}$ such that for every $v \in X$ we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx + \int_\Omega F_2^0(x, u(x); -v(x)) dx \geq 0. \tag{6.5}$$

We have the following result.

Corollary 6.11. (a) *If conditions (F1)–(F5), and (F6) hold, then problem* (6.5) *has a nontrivial solution.*

(b) *If conditions (F1), (F2'), (F3), (F4), and (F6') hold, then problem* (6.5) *has a nontrivial solution.*

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