Electronic Journal of Differential Equations, Vol. 2021 (2021), No. 27, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

HARNACK'S INEQUALITY FOR QUASILINEAR ELLIPTIC EQUATIONS WITH GENERALIZED ORLICZ GROWTH

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Abstract. We prove Harnack's inequality for bounded weak solutions to quasilinear second order elliptic equations with generalized Orlicz growth conditions. Our approach covers new cases of variable exponent and (p,q) growth conditions.

1. Introduction and main results

This article concerns quasilinear elliptic equations of the form

$$\operatorname{div}\Big(g(x,|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big)=0,\quad x\in\Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n\geq 2$.

Throughout this article we assume that the function $g(x,\mathrm{v}):\Omega\times\mathbb{R}_+\to\mathbb{R}_+$, $\mathbb{R}_+=[0,+\infty)$ satisfies the following assumptions:

(A1) $g(\cdot,\mathrm{v})\in L^1(\Omega)$ for all $\mathrm{v}\in\mathbb{R}_+$, $g(x,\cdot)$ is continuous and non-decreasing for almost all $x\in\Omega$, $\lim_{\mathrm{v}\to+0}g(x,\mathrm{v})=0$ and $\lim_{\mathrm{v}\to+\infty}g(x,\mathrm{v})=+\infty$;

(A2) there exist $c_1>0$, $q>1$ and $b_0\geq 0$ such that

$$\frac{g(x,\mathrm{w})}{g(x,\mathrm{v})}\leq c_1\Big(\frac{\mathrm{w}}{\mathrm{v}}\Big)^{q-1} \tag{1.2}$$

for all $x\in\Omega$ and for all $\mathrm{w}\geq\mathrm{v}>b_0$;

(A3) there exists $p>1$ such that

$$\frac{g(x,\mathrm{w})}{g(x,\mathrm{v})}\geq\Big(\frac{\mathrm{w}}{\mathrm{v}}\Big)^{p-1} \tag{1.3}$$

for all $x\in\Omega$ and for all $\mathrm{w}\geq\mathrm{v}>0$;

(A4) for any $K>0$ and for any ball $B_{8r}(x_0)\subset\Omega$ there exists $c_2(K)>0$ such that

$$g(x_1,\mathrm{v}/r)\leq c_2(K)\,e^{\lambda(r)}g(x_2,\mathrm{v}/r)$$

for all x_1, $x_2\in B_r(x_0)$ and for all $r\leq\mathrm{v}\leq K$. Here $\lambda(r):(0,r_*)\to\mathbb{R}_+$ is a continuous, non-increasing function, satisfying the conditions described below.

2010 *Mathematics Subject Classification.* 35B65, 35D30, 35J60, 49N60.
Key words and phrases. Elliptic equation; non-standard growth; non-log condition; bounded solution; Harnack inequality.

Submitted September 7, 2020. Published April 7, 2021.

The following functions defined on $\Omega \times \mathbb{R}_+$ satisfy assumptions (A1)–(A4):

$$g(x,\mathrm{v}) = \mathrm{v}^{p(x)-1} + \mathrm{v}^{q(x)-1},$$
$$g(x,\mathrm{v}) = \mathrm{v}^{p(x)-1}\big(1+\ln(1+\mathrm{v})\big),$$
$$g(x,\mathrm{v}) = \mathrm{v}^{p-1} + a(x)\mathrm{v}^{q-1}, \quad a(x) \ge 0,$$
$$g(x,\mathrm{v}) = \mathrm{v}^{p-1}\big(1+b(x)\ln(1+\mathrm{v})\big), \quad b(x) \ge 0,$$

where the exponents p, q, $p(\cdot)$, $q(\cdot)$, and the coefficients $a(\cdot)$ and $b(\cdot)$ satisfy the following conditions:

(i) $1 < p < p(x) \le q(x) < q < +\infty$ for all $x \in \Omega$;

(ii)
$$|p(x)-p(y)| + |q(x)-q(y)| \le \frac{\lambda(|x-y|)}{\big|\ln|x-y|\big|}, \quad x,y \in \Omega,\ x \ne y, \tag{1.4}$$
the function $\lambda(r)/|\ln r|$ is non-decreasing on $(0,r_*)$, $\lim_{r\to 0}\lambda(r)/|\ln r| = 0$;

(iii)
$$|a(x)-a(y)| \le A|x-y|^{\alpha} e^{\lambda(|x-y|)}, \quad x,y \in \Omega,\ x \ne y, \tag{1.5}$$
$A > 0$, $0 < q-p \le \alpha \le 1$, the function $r^{\alpha}e^{\lambda(r)}$ is non-decreasing on $(0,r_*)$, $\lim_{r\to 0} r^{\alpha}e^{\lambda(r)} = 0$;

(iv)
$$|b(x)-b(y)| \le \frac{B\, e^{\lambda(|x-y|)}}{\big|\ln|x-y|\big|}, \qquad x,y \in \Omega,\ x \ne y,\ B > 0, \tag{1.6}$$
the function $e^{\lambda(r)}/|\ln r|$ is non-decreasing on $(0,r_*)$, $\lim_{r\to 0} e^{\lambda(r)}/|\ln r| = 0$.

The study of regularity of minima of functionals with non-standard growth of (p,q)-type was initiated by Zhikov [35, 36, 37, 38, 40], Marcellini [22, 23] and Lieberman [21]. In the last thirty years, the qualitative theory of second order equations with so-called “log-condition” (i.e. if $0 \le \lambda(r) \le L < +\infty$) has been actively developed; see, for instance, [1, 2, 4, 5, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19, 25, 33]. These classes of equations have numerous applications in physics and have been attracted attention for several decades; see [7, 28, 34] and references therein.

The case when conditions (1.4), (1.5), (1.6) hold differs substantially from the log-case. To the best of our knowledge there are only a few results in this direction. Zhikov [39] obtained a generalization of the logarithmic condition which guarantees the denseness of smooth functions in a Sobolev space $W^{1,p(x)}(\Omega)$. Particularly, this result holds if $1 < p \le p(x)$ and

$$|p(x)-p(y)| \le L\,\frac{|\ln|\ln|x-y|\,||}{|\ln|x-y|\,|}, \quad x,y \in \Omega,\ x \ne y,\ L < p/n.$$

In the case when the variable exponent $p(x)$ satisfies the condition

$$\begin{gathered}|p(x)-p(x_0)| \le L\frac{\ln\ln\ln|x-x_0|^{-1}}{\ln|x-x_0|^{-1}},\\ 0 < L < p/(n+1), \quad x,x_0 \in \Omega,\ |x-x_0| < 1/27,\end{gathered} \tag{1.7}$$

Alkhutov and Krasheninnikova [3] proved the continuity of solutions to the $p(x)$-Laplace equation at the point x_0, and Surnachev [31] established the Harnack inequality for solutions. The continuity of solutions to the $p(x)$-Laplace equation up to the boundary were proved by Alkhutov and Surnachev [6] under the additional condition

$$\int_0 \exp\Big(- C \exp\big(\beta\lambda(r)\big)\Big) \frac{dr}{r} = +\infty, \tag{1.8}$$

where C and β are some positive constants, depending only upon the data. We note that the function $\lambda(r) = L \ln\ln\ln r^{-1}$, $r \in (0, e^{-e})$, $L\beta < 1$, satisfies condition (1.8).

In [30], we attempted to systematize and unify the approach to establish the local regularity of bounded solutions of elliptic and parabolic equations with non-standard growth. For this, we have introduced elliptic and parabolic $\mathcal{B}_1$ classes, which generalize the well-known $\mathfrak{B}_p$ classes ($p > 1$) of De Giorgi, Ladyzhenskaya, Ural'tseva [20] and cover their other numerous and scattered analogues (see references in [30]). It was proved in [30] that functions from the $\mathcal{B}_{1,g,\lambda}(\Omega)$ class are continuous if conditions (A2), (A4) and (1.8) are fulfilled. In addition, if condition (A3) is fulfilled, then the solutions of (1.1) belong to the $\mathcal{B}_{1,g,\lambda}(\Omega)$ class. At the same time, we do not use the specific properties of the generalized Orlicz and Sobolev-Orlicz spaces, as was done, for example, in the papers of Harjulehto, Hästö et al [16, 17, 18, 19]. Although it should be noted that in the case when $0 \le \lambda(r) < L < +\infty$, the assumptions (A2)–(A4) are almost equivalent to the conditions $(\mathrm{aDec})^{\infty}_q$, (A1-n), $(\mathrm{aInc})_p$ from their papers.

Returning to our paper [30], we note that there are no Harnack-type theorems in it. Although, such type results were obtained in [1, 2, 4, 5, 27] in the log-case and in [31] under condition (1.7). Therefore, it is natural to conjecture that the Harnack inequality holds for bounded solutions of (1.1) under the conditions (A1)–(A4). In this paper, we give a positive answer to this hypothesis. This also encompasses the classic results of Moser [26], Serrin [29], Trudinger [32] and Di Benedetto & Trudinger [15] for bounded solutions in the standard growth case, and of course, we use some of the ideas of Moser and Trudinger in our proofs.

Before formulating the main results, let us remind the reader the definition of a weak solution to (1.1). Moreover, throughout the article, we use the well-known notation for sets in $\mathbb{R}^n$, spaces of functions and their elements, etc. (see, e.g. [20]). In particular, we will use the notation $\fint_E f\,dx = |E|^{-1}\int_E f\,dx$ for any measurable set $E \subset \mathbb{R}^n$ with $|E| \neq 0$ and $f \in L^1(E)$, where $|E|$ denotes the n-dimensional Lebesgue measure of E. We set

$$G(x, \mathrm{v}) = g(x, \mathrm{v})\mathrm{v} \quad \text{for } x \in \Omega,\ \mathrm{v} \ge 0 \tag{1.9}$$

and write $u \in W^{1,G}(\Omega)$ if $u \in W^{1,1}(\Omega)$ and $\int_\Omega G(x, |\nabla u|)\,dx < +\infty$; $u \in W^{1,G}_{\rm loc}(\Omega)$ if $u \in W^{1,G}(E)$ for any open set E compactly embedding in Ω. We denote by $W^{1,G}_0(\Omega)$ the set of all functions $u \in W^{1,G}(\Omega)$ which have a compact support in Ω.

Definition 1.1. We say that a function $u : \Omega \to \mathbb{R}$ is a bounded weak solution (subsolution, supersolution) to (1.1) if $u \in W^{1,G}_{\rm loc}(\Omega) \cap L^\infty(\Omega)$ and the integral equality (inequality)

$$\int_\Omega g(x, |\nabla u|)\frac{\nabla u}{|\nabla u|}\,\nabla\varphi\,dx = (\le, \ge)\,0 \tag{1.10}$$

holds for any $\varphi \in W^{1,G}_0(\Omega)$ (for subsolutions and supersolutions, we require $\varphi \ge 0$).

We refer to the parameters $M = \operatorname{ess\,sup}_\Omega |u|$, n, p, q, c_1, $c_2(M)$ as our structural data, and we write γ if it can be quantitatively determined a priori only in terms of the above quantities. The generic constant γ may vary from line to line. The main result of this paper reads as follows.

Theorem 1.2 (weak Harnack inequality). *Fix a point $x_0 \in \Omega$ and consider the ball $B_{8\rho}(x_0) \subset \Omega$. Let u be a non-negative bounded weak supersolution to* (1.1) *under conditions* (A1)–(A4). *Then for any $0 < s < n/(n-1)$ it holds:*

$$\begin{aligned} &\Big(\fint_{B_{5\rho/4}(x_0)} g^s\Big(x_0, \frac{u + 2(1+b_0)\rho}{\rho}\Big)dx\Big)^{1/s} \\ &\leq \Lambda(\gamma, 3n, \rho)\, g\Big(x_0, \frac{m(\rho) + 2(1+b_0)\rho}{\rho}\Big), \end{aligned} \tag{1.11}$$

where $m(\rho) = \operatorname{ess\,inf}_{B_\rho(x_0)} u$ and $\Lambda(c, \beta, \rho) = \exp\big(c \exp\big(\beta\lambda(\rho)\big)\big)$ for any c, $\beta \in \mathbb{R}$ and $\rho \in (0, r_*)$.

Corollary 1.3. *Let u be a non-negative bounded weak solution to* (1.1) *under conditions* (A1)–(A4), *and let ρ_0 be a sufficiently small positive number such that $B_{8\rho_0}(x_0) \subset \Omega$. There exist positive numbers c, β depending only on the data such that if $\Lambda(c, \beta, r) \leq \frac{3}{2}\Lambda(c, \beta, 2r)$ for all $0 < r \leq \rho/2 < \rho_0/2$, and additionally*

$$\int_0 \Lambda(-c, \beta, r)\frac{dr}{r} = +\infty \quad \textit{and} \quad \lim_{r\to 0} r\Lambda(c, \beta, r) = 0,$$

then the solution u is continuous at x_0. Particularly, the function $\lambda(r) = L \ln\ln\ln r^{-1}$, $r \in (0, e^{-e})$, satisfies the above conditions if $0 < L < 1/\beta$.

Theorem 1.4 (Moser-type sup-estimate of solutions). *Fix a point $x_0 \in \Omega$ and consider the ball $B_{8\rho}(x_0) \subset \Omega$. Let conditions* (A1)–(A4) *be fulfilled, and let u be a non-negative bounded weak solution to* (1.1), *$M(\rho) = \operatorname{ess\,sup}_{B_\rho(x_0)} u$. Then*

$$g\Big(x_0, \frac{M(\rho) + 2(1+b_0)\rho}{\rho}\Big) \leq \gamma e^{2n\lambda(\rho)} \fint_{B_{5\rho/4}(x_0)} g\Big(x_0, \frac{u + 2(1+b_0)\rho}{\rho}\Big)dx. \tag{1.12}$$

From Theorems 1.2 and 1.4 we arrive at the following theorem.

Theorem 1.5 (Harnack inequality). *Let all the assumptions of Theorems 1.2, 1.4 be fulfilled. Then there exist positive constants C, c, β depending only on the data, such that*

$$\operatorname{ess\,sup}_{B_\rho(x_0)} u \leq C\Lambda(c, \beta, \rho)\Big(\operatorname{ess\,inf}_{B_\rho(x_0)} u + (1+b_0)\rho\Big), \tag{1.13}$$

where $\Lambda(c, \beta, \rho)$ was defined in Theorem 1.2.

The rest of this article contains the proofs of Theorems 1.2 and 1.4.

2. Proof of Theorem 1.2 (weak Harnack inequality)

For proving Theorem 1.2, we need some inequalities and several lemmas. First, we note simple analogues of Young's inequality:

$$g(x, a)b \leq \varepsilon g(x, a)a + g(x, b/\varepsilon)b \quad \text{if } \varepsilon, a, b > 0,\ x \in \Omega; \tag{2.1}$$

$$g(x, a)b \leq \frac{1}{\varepsilon} g(x, a)a + \varepsilon^{p-1} g(x, b)b \quad \text{if } \varepsilon \in (0, 1],\ a, b > 0,\ x \in \Omega. \tag{2.2}$$

In fact, if $b \leq \varepsilon a$, then $g(x, a)\, b \leq \varepsilon g(x, a)a$, and if $b > \varepsilon a$, then since the function $\mathrm{v} \to g(x, \mathrm{v})$ is increasing we have that $g(x, a)b \leq g(x, b/\varepsilon)b$, which proves inequality (2.1). Using assumption (A3) by similar arguments we arrive at inequality (2.2).

Next, we set

$$\mathcal{G}(x,\mathrm{w}) = \int_0^{\mathrm{w}} g(x,\mathrm{v})\,d\mathrm{v} \quad \text{for } x\in\Omega,\ \mathrm{w}>0. \tag{2.3}$$

Then the following inequalities hold:

$$\mathcal{G}(x,\mathrm{w}) \ge \gamma\, G(x,\mathrm{w}) \quad \text{for all } x\in\Omega,\ \mathrm{w}\ge 2(1+b_0), \tag{2.4}$$

$$G(x,\mathrm{w}) \ge p\mathcal{G}(x,\mathrm{w}) \quad \text{for all } x\in\Omega,\ \mathrm{w}>0. \tag{2.5}$$

Indeed, if $x\in\Omega$ and $\mathrm{w}\ge 2(1+b_0)$ then by (1.2), (1.9), and (2.3), we have

$$\mathcal{G}(x,\mathrm{w}) = \int_0^{\mathrm{w}} g(x,\mathrm{v})\,d\mathrm{v} \ge \int_{b_0}^{\mathrm{w}} g(x,\mathrm{v})\,d\mathrm{v} \ge \frac{g(x,\mathrm{w})}{c_1\mathrm{w}^{q-1}}\int_{b_0}^{\mathrm{w}} \mathrm{v}^{q-1}d\mathrm{v} \ge \frac{1-2^{-q}}{c_1 q}\,G(x,\mathrm{w}),$$

which implies (2.4). Now, let $x\in\Omega$ and $\mathrm{w}>0$ be arbitrary, then by (1.3), (1.9) and (2.3) we obtain

$$\mathcal{G}(x,\mathrm{w}) = \int_0^{\mathrm{w}} g(x,\mathrm{v})\,d\mathrm{v} \le \frac{g(x,\mathrm{w})}{\mathrm{w}^{p-1}}\int_0^{\mathrm{w}} \mathrm{v}^{p-1}\,d\mathrm{v} = \frac{1}{p}\,g(x,\mathrm{w})\mathrm{w} = \frac{1}{p}\,G(x,\mathrm{w}),$$

which yields (2.5).

The rest of the lemmas in this section are successive stages in the proof of Theorem 1.2. The proof follows Trudinger's strategy [32], which we adapted to equation (1.1) under conditions (A1)–(A4).

Lemma 2.1. *Let all the assumptions of Theorem 1.2 be fulfilled. Then there exists positive constant γ depending only on the known data such that*

$$\exp\Big(\fint_{B_{2\rho}(x_0)} \ln\big(u+2(1+b_0)\rho\big)\,dx\Big) \le \Lambda(\gamma,3n,\rho)\big[m(\rho)+2(1+b_0)\rho\big]. \tag{2.6}$$

Proof. We fix $\sigma\in(0,1)$, for any $\rho\le r<r(1+\sigma)\le 2\rho$, we take a function $\zeta\in C_0^\infty(B_{r(1+\sigma)}(x_0))$, $0\le\zeta\le 1$, $\zeta=1$ in $B_r(x_0)$ and $|\nabla\zeta|\le(\sigma r)^{-1}$. Let

$$w=\ln\frac{\kappa}{\overline{u}}, \quad \overline{u}=u+2(1+b_0)\rho, \tag{2.7}$$

where the constant κ is defined by the condition $(w)_{x_0,2\rho}=\fint_{B_{2\rho}(x_0)} w\,dx=0$, i.e.

$$\kappa=\exp\Big(\fint_{B_{2\rho}(x_0)}\ln\overline{u}\,dx\Big). \tag{2.8}$$

We test (1.10) by $\varphi=\frac{\overline{u}\,(w-k)_+}{\mathcal{G}(x_0,\overline{u}/\rho)}\zeta^q$, $(w-k)_+=\max\{0,\,w-k\}$, $k>0$. Since we are dealing with bounded and non-negative solutions (supersolutions), then this and all other test functions used in the paper belong to $W_0^{1,G}(\Omega)$. This is a consequence of conditions (A1) and (A2) and the result of Marcus and Mizel [24, Theorem 2]. So, we have

$$\begin{aligned}
&\int_{A_{k,r(1+\sigma)}} \frac{G(x,|\nabla u|)}{\mathcal{G}(x_0,\overline{u}/\rho)}\zeta^q\,dx \\
&+\int_{A_{k,r(1+\sigma)}} \frac{G(x,|\nabla u|)}{\mathcal{G}(x_0,\overline{u}/\rho)}\Big\{\frac{G(x_0,\overline{u}/\rho)}{\mathcal{G}(x_0,\overline{u}/\rho)}-1\Big\}(w-k)_+\zeta^q\,dx \\
&\le \frac{\gamma}{\sigma}\int_{A_{k,r(1+\sigma)}} \frac{g(x,|\nabla u|)}{\mathcal{G}(x_0,\overline{u}/\rho)}\,\frac{\overline{u}}{\rho}\,(w-k)_+\zeta^{q-1}\,dx,
\end{aligned}$$

where $A_{k,r} = B_r(x_0) \cap \{w > k\}$. By (2.5), the value in curly brackets is estimated from below as follows:

$$\frac{G(x_0, \overline{u}/\rho)}{\mathcal{G}(x_0, \overline{u}/\rho)} - 1 \geq p - 1, \tag{2.9}$$

and therefore

$$\begin{aligned} &\int_{A_{k,r(1+\sigma)}} \frac{G(x, |\nabla u|)}{\mathcal{G}(x_0, \overline{u}/\rho)} \zeta^q \, dx + (p-1) \int_{A_{k,r(1+\sigma)}} \frac{G(x, |\nabla u|)}{\mathcal{G}(x_0, \overline{u}/\rho)} (w-k)_+ \zeta^q \, dx \\ &\leq \gamma \int_{A_{k,r(1+\sigma)}} \frac{g(x, |\nabla u|)}{\mathcal{G}(x_0, \overline{u}/\rho)} \frac{\overline{u}}{\sigma\rho\zeta} (w-k)_+ \zeta^q \, dx. \end{aligned} \tag{2.10}$$

We use inequality (2.1) with $a = |\nabla u|$, $b = \frac{\overline{u}}{\sigma\rho\zeta}$ and sufficiently small $\varepsilon > 0$, and then (2.4) with $\mathrm{w} = \overline{u}/\rho$, to estimate from above the right-hand side of (2.10):

$$\begin{aligned} &\gamma \int_{A_{k,r(1+\sigma)}} \frac{g(x, |\nabla u|)}{\mathcal{G}(x_0, \overline{u}/\rho)} \, \frac{\overline{u}}{\sigma\rho\zeta} \, (w-k)_+ \zeta^q \, dx \\ &\leq \frac{p-1}{2} \int_{A_{k,r(1+\sigma)}} \frac{G(x, |\nabla u|)}{\mathcal{G}(x_0, \overline{u}/\rho)} (w-k)_+ \zeta^q \, dx \\ &\quad + \frac{\gamma}{\sigma} \int_{A_{k,r(1+\sigma)}} \frac{g\big(x, \frac{\gamma\overline{u}}{\sigma\rho\zeta}\big)}{g(x_0, \overline{u}/\rho)} (w-k)_+ \zeta^{q-1} \, dx. \end{aligned}$$

Combining this inequality and (2.10), we obtain

$$\int_{A_{k,r(1+\sigma)}} \frac{G(x, |\nabla u|)}{\mathcal{G}(x_0, \overline{u}/\rho)} \zeta^q \, dx \leq \frac{\gamma}{\sigma} \int_{A_{k,r(1+\sigma)}} \frac{g\big(x, \frac{\gamma\,\overline{u}}{\sigma\rho\zeta}\big)}{g(x_0, \overline{u}/\rho)} \, (w-k)_+ \zeta^{q-1} \, dx. \tag{2.11}$$

Since $\frac{\gamma\overline{u}}{\sigma\rho\zeta} \geq \frac{\overline{u}}{\rho} \geq b_0$ and $|x - x_0| < r(1+\sigma) \leq 2\rho$ for $x \in A_{k,r(1+\sigma)}$, then using conditions (A2) and (A4), we get that for all $x \in A_{k,r(1+\sigma)}$, it holds

$$g\Big(x, \frac{\gamma\,\overline{u}}{\sigma\rho\zeta}\Big) \leq \gamma\,(\sigma\zeta)^{1-q}\, g(x, \overline{u}/\rho) \leq \gamma\,(\sigma\zeta)^{1-q}\, e^{\lambda(\rho)} g\,(x_0, \overline{u}/\rho)\,.$$

So, from (2.11) we obtain

$$\int_{A_{k,r(1+\sigma)}} \frac{G(x, |\nabla u|)}{\mathcal{G}(x_0, \overline{u}/\rho)} \zeta^q \, dx \leq \gamma\, \sigma^{-q}\, e^{\lambda(\rho)} \int_{A_{k,r(1+\sigma)}} (w-k)_+ \, dx. \tag{2.12}$$

To estimate the term on the left-hand side of (2.12), we use (2.1) with $\varepsilon = 1$, $a = \overline{u}/\rho$, $b = |\nabla u|$, assumption (A4), the definitions of the functions G, $\mathcal{G}$, w (see equalities (1.9), (2.3) and (2.7), respectively) and (2.5):

$$\begin{aligned} \int_{A_{k,r(1+\sigma)}} |\nabla w| \zeta^q \, dx &= \int_{A_{k,r(1+\sigma)}} \frac{|\nabla u|}{\overline{u}} \, \frac{g(x, \overline{u}/\rho)}{g(x, \overline{u}/\rho)} \zeta^q \, dx \\ &\leq \frac{1}{\rho} \, |A_{k,r(1+\sigma)}| + \frac{1}{\rho} \int_{A_{k,r(1+\sigma)}} \frac{G(x, |\nabla u|)}{G\,(x, \overline{u}/\rho)} \zeta^q \, dx \\ &\leq \frac{1}{\rho} \, |A_{k,r(1+\sigma)}| + \gamma \frac{e^{\lambda(\rho)}}{\rho} \int_{A_{k,r(1+\sigma)}} \frac{G(x, |\nabla u|)}{\mathcal{G}(x_0, \overline{u}/\rho)} \zeta^q \, dx. \end{aligned} \tag{2.13}$$

Collecting (2.12) and (2.13), we obtain

$$\int_{A_{k,r(1+\sigma)}} |\nabla w| \zeta^q \, dx \leq \frac{\gamma\, e^{2\lambda(\rho)}}{\sigma^q \rho} \Big(|A_{k,r(1+\sigma)}| + \int_{A_{k,r(1+\sigma)}} (w-k)_+ \, dx \Big).$$

From this, using Sobolev's embedding theorem and standard iteration arguments (see, for instance [20, Section 2, Theorem 5.3]), and choosing k from the condition

$$k \geq \gamma\, e^{2n\lambda(\rho)}\Big(\fint_{B_{2\rho}(x_0)} |w|^{\frac{n}{n-1}}\, dx\Big)^{\frac{n-1}{n}} + 1,$$

we obtain that

$$\operatorname{ess\,sup}_{B_\rho(x_0)} w \leq \gamma\, e^{2n\lambda(\rho)}\Big(\fint_{B_{2\rho}(x_0)} |w|^{\frac{n}{n-1}}\, dx\Big)^{\frac{n-1}{n}} + 1. \tag{2.14}$$

To estimate the right-hand side of (2.14) we use the Poincaré inequality by our choice of κ (see (2.8)) we have

$$\begin{aligned}\Big(\fint_{B_{2\rho}(x_0)} |w|^{\frac{n}{n-1}}\, dx\Big)^{\frac{n-1}{n}} &= \Big(\fint_{B_{2\rho}(x_0)} |w-(w)_{x_0,2\rho}|^{\frac{n}{n-1}}\, dx\Big)^{\frac{n-1}{n}} \\ &\leq \gamma\, \rho^{1-n}\int_{B_{2\rho}(x_0)} |\nabla w|\, dx.\end{aligned} \tag{2.15}$$

Next, similarly to (2.13), we have

$$\begin{aligned}\int_{B_{2\rho}(x_0)} |\nabla w|\, dx &\leq \int_{B_{4\rho}(x_0)} |\nabla w|\zeta^q\, dx \\ &\leq \gamma\rho^{n-1} + \gamma\frac{e^{\lambda(\rho)}}{\rho}\int_{B_{4\rho}(x_0)} \frac{G(x,|\nabla u|)}{\mathcal{G}(x_0,\overline{u}/\rho)}\zeta^q\, dx,\end{aligned} \tag{2.16}$$

here we have $\zeta \in C_0^\infty(B_{4\rho}(x_0))$, $0 \leq \zeta \leq 1$, $\zeta = 1$ in $B_{2\rho}(x_0)$, and $|\nabla\zeta| \leq 2/\rho$. In addition, testing (1.10) by $\varphi = \frac{\overline{u}\zeta^q}{\mathcal{G}(x_0,\overline{u}/\rho)}$, similarly to (2.12), we obtain

$$\int_{B_{4\rho}(x_0)} \frac{G(x,|\nabla u|)}{\mathcal{G}(x_0,\overline{u}/\rho)}\zeta^q\, dx \leq \gamma\rho^n e^{\lambda(\rho)}. \tag{2.17}$$

Now, collecting (2.14)–(2.17) and taking into account (2.7) and (2.8), we arrive at the required inequality (2.6). The proof is complete. □

Lemma 2.2. *Under the assumptions of Theorem 1.2 there exists $\delta_0 = \delta_0(\rho) > 0$ depending only on the data and ρ, such that*

$$\begin{aligned}&\Big(\fint_{B_{3\rho/2}(x_0)} \big(u + 2(1+b_0)\rho\big)^{\delta_0} dx\Big)^{1/\delta_0} \\ &\leq \Lambda(\gamma, 2n, \rho)\exp\Big(\fint_{B_{2\rho}(x_0)} \ln\big(u + 2(1+b_0)\rho\big)\, dx\Big).\end{aligned} \tag{2.18}$$

Proof. Let us fix $\sigma \in (0,1)$ and for any $3\rho/2 \leq r < r(1+\sigma) \leq 2\rho$ consider the function $\zeta \in C_0^\infty\big(B_{r(1+\sigma)}(x_0)\big)$, $0 \leq \zeta \leq 1$, $\zeta = 1$ in $B_r(x_0)$, $|\nabla\zeta| \leq (\sigma r)^{-1}$. We define

$$v = \ln\frac{u + 2(1+b_0)\rho}{\kappa} = \ln\frac{\overline{u}}{\kappa}, \quad v_\mu = \max\{v,\mu\}, \quad \mu > 0.$$

Testing (1.10) by $\varphi = \frac{v_\mu^{s-1}\overline{u}\zeta^l}{\mathcal{G}(x_0,\overline{u}/\rho)}$, $s \geq 1$, $l \geq q$, and using (2.9), we have

$$\begin{aligned}&(p-1)\int_{B_{r(1+\sigma)}(x_0)} \frac{G(x,|\nabla u|)}{\mathcal{G}(x_0,\overline{u}/\rho)}\, v_\mu^{s-1}\zeta^l\, dx \\ &\leq (s-1)\int_{B_{r(1+\sigma)}(x_0)\cap\{v>\mu\}} \frac{G(x,|\nabla u|)}{\mathcal{G}(x_0,\overline{u}/\rho)}\, v_\mu^{s-2}\zeta^l\, dx\end{aligned}$$

$$+\gamma\, l \int_{B_{r(1+\sigma)}(x_0)} \frac{g(x,|\nabla u|)}{\mathcal{G}(x_0,\overline{u}/\rho)}\,\frac{\overline{u}}{\sigma\rho\zeta}\, v_\mu^{s-1}\zeta^l\,dx.$$

Choosing μ from the condition $\frac{s}{\mu}=\frac{p-1}{2}$ and using inequalities (2.1), (2.4) and conditions (A2) and (A4) similarly to the derivation of (2.12), from the previous we obtain

$$\int_{B_{r(1+\sigma)}(x_0)} \frac{G(x,|\nabla u|)}{\mathcal{G}(x_0,\overline{u}/\rho)} v_\mu^{s-1}\zeta^l\,dx \le \frac{\gamma\, l^\gamma e^{\lambda(\rho)}}{\sigma^q}\int_{B_{r(1+\sigma)}(x_0)} v_\mu^{s-1}\zeta^{l-q}\,dx. \tag{2.19}$$

Estimating the term on the left-hand side of (2.19), similarly to (2.13), we obtain

$$\begin{aligned}\int_{B_{r(1+\sigma)}(x_0)} |\nabla v_\mu|\, v_\mu^{s-1}\zeta^l\,dx &\le \int_{B_{r(1+\sigma)}(x_0)} \frac{|\nabla u|}{\overline{u}}\, v_\mu^{s-1}\zeta^l\,dx\\ &\le \frac{\gamma\, l^\gamma}{\sigma^q}\,\frac{e^{2\lambda(\rho)}}{\rho}\int_{B_{r(1+\sigma)}(x_0)} v_\mu^{s-1}\zeta^{l-q}\,dx\\ &\le \frac{\gamma\, l^\gamma}{\sigma^q}\,\frac{e^{2\lambda(\rho)}}{\rho}\int_{B_{r(1+\sigma)}(x_0)} v_\mu^{s}\zeta^{l-q}\,dx.\end{aligned}$$

Using Sobolev's embedding theorem, from this we have

$$\fint_{B_r(x_0)} v_\mu^{\frac{sn}{n-1}}\,dx \le \Big(\frac{\gamma s\, e^{2\lambda(\rho)}}{\sigma^q}\fint_{B_{r(1+\sigma)}(x_0)} v_\mu^s\,dx\Big)^{\frac{n}{n-1}}. \tag{2.20}$$

For $j=0,1,2,\dots$, we define the sequences

$$\begin{gathered} r_j=\frac{\rho}{2}(3+2^{-j}),\quad B_j=B_{r_j}(x_0),\\ s_j=\Big(\frac{n}{n-1}\Big)^{j+1},\quad \mu_j=\frac{2s_j}{p-1},\quad y_j=\fint_{B_j} v_{\mu_j}^{s_j}\,dx.\end{gathered}$$

Then inequality (2.20) can be rewritten in the form

$$y_{j+1}^{1/s_{j+1}} \le \big(\gamma 2^{jq} s_j\, e^{2\lambda(\rho)}\big)^{1/s_j} y_j^{1/s_j},\quad j=0,1,2,\dots, \tag{2.21}$$

and by Sobolev's inequality and (2.17), we have

$$y_0\le \gamma\exp\big(\frac{2n\lambda(\rho)}{n-1}\big). \tag{2.22}$$

From this by iteration, for $j=0,1,2,\dots$, we have

$$\begin{aligned} y_{j+1}^{1/s_{j+1}} &\le \gamma^{\sum_{i=0}^j \frac{1}{s_i}} 2^{q\sum_{i=1}^j \frac{i}{s_i}}\Big(\frac{n}{n-1}\Big)^{\sum_{i=0}^j \frac{i+1}{s_i}}\exp\Big(2\lambda(\rho)\sum_{i=0}^j\frac{1}{s_i}\Big) y_0^{\frac{n-1}{n}}\\ &\le \gamma e^{2n\lambda(\rho)}.\end{aligned} \tag{2.23}$$

Let $m\in\mathbb{N}$ be arbitrary, then there exists $j\ge 1$ such that $s_{j-1}<m\le s_j$. Using Hölder's inequality, from (2.23) we obtain

$$\fint_{B_{3\rho/2}(x_0)} \frac{v_+^m}{m!}\,dx \le \fint_{B_{3\rho/2}(x_0)} \frac{v_{\mu_j}^m}{m!}\,dx \le \frac{\gamma\, y_j^{m/s_j}}{m!} \le \frac{\gamma^{m+1}}{m!}\,e^{2nm\lambda(\rho)} \le \gamma^{m+1}e^{2nm\lambda(\rho)}.$$

Choosing $\delta_0=\delta_0(\rho)$ from the condition

$$\delta_0=\frac{1}{2\gamma}\,e^{-2n\lambda(\rho)}, \tag{2.24}$$

from the previous we have

$$\fint_{B_{3\rho/2}(x_0)} \frac{(\delta_0 v_+)^m}{m\,!}\,dx \le \gamma\, 2^{-m},$$

which implies

$$\fint_{B_{3\rho/2}(x_0)} e^{\delta_0 v}\,dx \le \fint_{B_{3\rho/2}(x_0)} e^{\delta_0 v_+}\,dx \le \sum_{m=0}^{\infty} \fint_{B_{3\rho/2}(x_0)} \frac{(\delta_0 v_+)^m}{m!}\,dx \le 2\gamma.$$

From this, since $e^{\delta_0 v} = (\overline{u}/\kappa)^{\delta_0}$ we have

$$\Big(\fint_{B_{3\rho/2}(x_0)} \overline{u}^{\delta_0}\,dx\Big)^{1/\delta_0} \le (2\gamma)^{1/\delta_0}\kappa \le \Lambda(\gamma, 2n, \rho)\,\kappa,$$

that together with (2.8) yields the desired inequality (2.18). This completes the proof. $\square$

The next lemma is a simple consequence of Lemmas 2.1 and 2.2.

Lemma 2.3. *Let all the assumptions of Lemma 2.2 be fulfilled, and set*

$$\delta_1 = \delta_0/(q-1), \tag{2.25}$$

where δ_0 is defined by (2.24)*. Then*

$$\begin{aligned} &\Big(\fint_{B_{3\rho/2}(x_0)} g^{\delta_1}\Big(x_0, \frac{u + 2(1+b_0)\rho}{\rho}\Big)\,dx\Big)^{1/\delta_1} \\ &\le \Lambda(\gamma, 3n, \rho)\, g\Big(x_0, \frac{m(\rho) + 2(1+b_0)\rho}{\rho}\Big). \end{aligned} \tag{2.26}$$

Proof. By condition (A2) we have

$$\begin{aligned} &\fint_{B_{3\rho/2}(x_0)} \frac{g^{\delta_1}\big(x_0, \frac{u+2(1+b_0)\rho}{\rho}\big)}{g^{\delta_1}\big(x_0, \frac{m(\rho)+2(1+b_0)\rho}{\rho}\big)}\,dx \\ &\le 1 + c_1^{\delta_1} \fint_{B_{3\rho/2}(x_0)\cap\{u>m(\rho)\}} \Big(\frac{u + 2(1+b_0)\rho}{m(\rho) + 2(1+b_0)\rho}\Big)^{\delta_0}\,dx. \end{aligned}$$

By Lemmas 2.1 and 2.2 the second term on the right-hand side of this inequality is estimated from above as follows:

$$\fint_{B_{3\rho/2}(x_0)} \Big(\frac{u + 2(1+b_0)\rho}{m(\rho) + 2(1+b_0)\rho}\Big)^{\delta_0}\,dx \le \Lambda(\gamma, 3n, \rho),$$

which completes the proof. $\square$

To complete the proof of Theorem 1.2 we need the following lemma.

Lemma 2.4 (Inverse Hölder inequality)**.** *Let the assumptions of Theorem 1.2 be fulfilled, then for all $\delta_1 \le s < n/(n-1)$ we have*

$$\begin{aligned} &\Big(\fint_{B_{5\rho/4}(x_0)} g^{s}\Big(x_0, \frac{u + 2(1+b_0)\rho}{\rho}\Big)dx\Big)^{1/s} \\ &\le \Lambda(\gamma, 2n+1, \rho)\Big(\fint_{B_{3\rho/2}(x_0)} g^{\delta_1}\Big(x_0, \frac{u + 2(1+b_0)\rho}{\rho}\Big)dx\Big)^{1/\delta_1}. \end{aligned} \tag{2.27}$$

Proof. We set $\psi(x,\mathrm{w}) = \frac{\mathcal{G}(x,\mathrm{w})}{\mathrm{w}}$ for $x \in \Omega$, $\mathrm{w} > 0$, and note that by (2.4) and (2.5), we have

$$g(x,\mathrm{w}) \le \gamma \psi(x,\mathrm{w}) \quad \text{for all } x \in \Omega,\ \mathrm{w} \ge 2(1+b_0), \tag{2.28}$$

$$\psi(x,\mathrm{w}) \le \frac{1}{p} g(x,\mathrm{w}) \quad \text{for all } x \in \Omega,\ \mathrm{w} > 0, \tag{2.29}$$

which gives

$$\psi'_{\mathrm{w}}(x,\mathrm{w}) \le \gamma \frac{\psi(x,\mathrm{w})}{\mathrm{w}} \quad \text{for all } x \in \Omega,\ \mathrm{w} \ge 2(1+b_0), \tag{2.30}$$

$$\psi'_{\mathrm{w}}(x,\mathrm{w}) = \frac{g(x,\mathrm{w}) - \psi(x,\mathrm{w})}{\mathrm{w}} \ge (p-1)\, \frac{\psi(x,\mathrm{w})}{\mathrm{w}} \quad \text{for all } x \in \Omega,\ \mathrm{w} > 0. \tag{2.31}$$

We need a Cacciopoli-type inequality for negative powers of $\psi(x_0, \overline{u}/\rho)$. To establish it, we fix $\sigma \in (0,1)$ and $r > 0$ such that $5\rho/4 \le r < r(1+\sigma) \le 3\rho/2$, and take a function $\zeta \in C_0^\infty\big(B_{r(1+\sigma)}(x_0)\big)$, $0 \le \zeta \le 1$, $\zeta = 1$ in $B_r(x_0)$, $|\nabla \zeta| \le (\sigma r)^{-1}$. Testing (1.10) by $\varphi = \psi^{-\tau}(x_0, \overline{u}/\rho)\zeta^\theta$, $0 < \tau < 1$, $\theta \ge q$, and using (2.31), we obtain

$$\begin{aligned}
&(p-1)\tau \int_{B_{r(1+\sigma)}(x_0)} \psi^{-\tau}(x_0, \overline{u}/\rho) \frac{G(x, |\nabla u|)}{\overline{u}} \zeta^\theta \, dx \\
&\le \frac{\gamma\, \theta}{\sigma \rho} \int_{B_{r(1+\sigma)}(x_0)} \psi^{-\tau}(x_0, \overline{u}/\rho) g(x, |\nabla u|) \zeta^{\theta - 1} \, dx,
\end{aligned}$$

which by (2.1), (A2), (A4) and (2.28) implies

$$\begin{aligned}
&\int_{B_{r(1+\sigma)}(x_0)} \psi^{-\tau}(x_0, \overline{u}/\rho) \frac{G(x, |\nabla u|)}{\overline{u}} \zeta^\theta \, dx \\
&\le \frac{\gamma\, \theta^q}{(\sigma\tau)^q}\, \frac{e^{\lambda(\rho)}}{\rho} \int_{B_{r(1+\sigma)}(x_0)} \psi^{1-\tau}(x_0, \overline{u}/\rho) \zeta^{\theta - q} \, dx.
\end{aligned} \tag{2.32}$$

Based on inequality (2.32), we organize Moser-type iterations for the function $\psi(x_0, \overline{u}/\rho)$. To do this, we fix $0 < t < n/(n-1)$ and $l \ge nq/(n-1)$, then by the Sobolev inequality and by (2.30) and (2.29), we obtain

$$\begin{aligned}
&\Big(\int_{B_{r(1+\sigma)}(x_0)} \psi^t(x_0, \overline{u}/\rho) \zeta^l \, dx \Big)^{\frac{n-1}{n}} \\
&\le \gamma \int_{B_{r(1+\sigma)}(x_0)} \big| \nabla \big[\psi^{\frac{t(n-1)}{n}}(x_0, \overline{u}/\rho) \zeta^{\frac{l(n-1)}{n}} \big] \big| \, dx \\
&\le \gamma t \int_{B_{r(1+\sigma)}(x_0)} \psi^{\frac{t(n-1)}{n} - 1}(x_0, \overline{u}/\rho) \frac{g(x_0, \overline{u}/\rho)}{\overline{u}} \, |\nabla u| \zeta^{\frac{l(n-1)}{n}} \, dx \\
&\quad + \frac{\gamma\, l}{\sigma \rho} \int_{B_{r(1+\sigma)}(x_0)} \psi^{\frac{t(n-1)}{n}}(x_0, \overline{u}/\rho) \zeta^{\frac{l(n-1)}{n} - 1} \, dx.
\end{aligned} \tag{2.33}$$

Using (2.1), (A4), (2.28) and (2.32) with $\tau = 1 - t(n-1)/n$ and $\theta = l(n-1)/n$, we estimate the first term on the right-hand side of (2.33) as follows:

$$\begin{aligned}
&\int_{B_{r(1+\sigma)}(x_0)} \psi^{\frac{t(n-1)}{n}-1}(x_0,\overline{u}/\rho)\frac{g(x_0,\overline{u}/\rho)}{\overline{u}}\,|\nabla u|\zeta^{\frac{l(n-1)}{n}}\,dx \\
&\le \gamma e^{\lambda(\rho)}\int_{B_{r(1+\sigma)}(x_0)} \psi^{\frac{t(n-1)}{n}-1}(x_0,\overline{u}/\rho)\frac{g(x,\overline{u}/\rho)}{\overline{u}}\,|\nabla u|\zeta^{\frac{l(n-1)}{n}}\,dx \\
&\le \gamma e^{\lambda(\rho)}\int_{B_{r(1+\sigma)}(x_0)} \psi^{\frac{t(n-1)}{n}-1}(x_0,\overline{u}/\rho)\frac{G\left(x,|\nabla u|\right)}{\overline{u}}\zeta^{\frac{l(n-1)}{n}}\,dx \\
&+ \gamma\frac{e^{\lambda(\rho)}}{\rho}\int_{B_{r(1+\sigma)}(x_0)} \psi^{\frac{t(n-1)}{n}-1}(x_0,\overline{u}/\rho)g(x,\overline{u}/\rho)\zeta^{\frac{l(n-1)}{n}}\,dx \\
&\le \frac{\gamma\, l^q}{\sigma^q}\big[1-\frac{t(n-1)}{n}\big]^{-q}\frac{e^{2\lambda(\rho)}}{\rho}\int_{B_{r(1+\sigma)}(x_0)} \psi^{\frac{t(n-1)}{n}}(x_0,\overline{u}/\rho)\zeta^{\frac{l(n-1)}{n}-q}\,dx.
\end{aligned}\tag{2.34}$$

Combining (2.33), (2.34), we arrive at

$$\begin{aligned}
&\Big(\fint_{B_r(x_0)} \psi^t(x_0,\overline{u}/\rho)dx\Big)^{\frac{n-1}{n}} \\
&\le \frac{\gamma\, l^q}{\sigma^q}\Big(1-\frac{t(n-1)}{n}\Big)^{-q} e^{2\lambda(\rho)}\fint_{B_{r(1+\sigma)}(x_0)} \psi^{\frac{t(n-1)}{n}}(x_0,\overline{u}/\rho)dx,
\end{aligned}\tag{2.35}$$

for $0 < t < \frac{n}{n-1}$ and $l \ge \frac{nq}{n-1}$.

Now, let $\delta_1 \le s < n/(n-1)$, and let j be a non-negative integer such that

$$s\big(\frac{n-1}{n}\big)^{j+1} \le \delta_1 \le s\big(\frac{n-1}{n}\big)^{j}.\tag{2.36}$$

Setting in (2.35) $l = nq$, $r = r_i = \frac{\rho}{4}(6-2^{-i})$, $r(1+\sigma) = r_{i+1}$, $B_i = B_{r_i}(x_0)$ and $t = t_i = s\left(\frac{n-1}{n}\right)^i$ for $i = 0, 1, \dots, j+1$, we have

$$\begin{aligned}
&\Big(\fint_{B_i} \psi^{t_i}(x_0,\overline{u}/\rho)dx\Big)^{1/t_i} \\
&\le \big[\gamma 2^{iq}\Big(1-\frac{n-1}{n}s\Big)^{-q} e^{2\lambda(\rho)}\big]^{1/t_{i+1}}\Big(\fint_{B_{i+1}} \psi^{t_{i+1}}(x_0,\overline{u}/\rho)dx\Big)^{1/t_{i+1}}.
\end{aligned}$$

Iterating this relation and using Hölder's inequality, we obtain

$$\begin{aligned}
&\Big(\fint_{B_{5\rho/4}(x_0)} \psi^s(x_0,\overline{u}/\rho)dx\Big)^{1/s} \\
&= \Big(\fint_{B_0} \psi^{t_0}(x_0,\overline{u}/\rho)dx\Big)^{1/t_0} \\
&\le \prod_{i=0}^{j}\Big[\gamma\, 2^{iq}e^{2\lambda(\rho)}\Big(1-\frac{n-1}{n}s\Big)^{-q}\Big]^{1/t_{i+1}}\Big(\fint_{B_{j+1}} \psi^{t_{j+1}}(x_0,\overline{u}/\rho)dx\Big)^{1/t_{j+1}} \\
&\le 2^{q\sum_{i=0}^{j} i/t_{i+1}}\Big[\gamma e^{2\lambda(\rho)}\Big(1-\frac{n-1}{n}s\Big)^{-q}\Big]^{\sum_{i=0}^{j} 1/t_{i+1}}\Big(\gamma\fint_{B_{3\rho/2}(x_0)} \psi^{\delta_1}(x_0,\overline{u}/\rho)dx\Big)^{1/\delta_1},
\end{aligned}$$

and by (2.36), (2.25) and (2.24) we have

$$\sum_{i=0}^{j}\frac{1}{t_{i+1}}\le\frac{1}{\delta_1}\,\frac{n}{n-1}\sum_{i=0}^{\infty}\big(\frac{n-1}{n}\big)^i=\frac{n^2}{\delta_1(n-1)},$$
$$\sum_{i=0}^{j}\frac{i}{t_{i+1}}\le j\sum_{i=0}^{j}\frac{1}{t_{i+1}}\le\frac{\gamma(\lambda(\rho)+1)}{\delta_1}.$$

From this, and recalling the definition of δ_1 (see again (2.25) and (2.24)), we arrive at the required inequality (2.27). This completes the proof. ◻

Combining Lemmas 2.3 and 2.4, we obtain that

$$\Big(\fint_{B_{5\rho/4}(x_0)} g^s\Big(x_0,\frac{u+2(1+b_0)\rho}{\rho}\Big)dx\Big)^{1/s}\le\Lambda(\gamma,3n,\rho)g\Big(x_0,\frac{m(\rho)+2(1+b_0)\rho}{\rho}\Big),$$

which proves Theorem 1.2.

3. Proof of Theorem 1.4 (sup-estimate of solutions)

Let us fix σ, $\sigma_1\in(0,1)$, $\rho\le r<r(1+\sigma\sigma_1)<r(1+\sigma)\le 5\rho/4$, and consider a function $\zeta\in C_0^\infty\big(B_{r(1+\sigma\sigma_1)}(x_0)\big)$ such that $0\le\zeta\le1$, $\zeta=1$ in $B_r(x_0)$ and $|\nabla\zeta|\le(\sigma\sigma_1 r)^{-1}$. Testing (1.10) by $\varphi=\overline{u}\mathcal{G}^{s-1}(x_0,\overline{u}/\rho)\zeta^l$, $s\ge1$, $l\ge\max\{q,s/2\}$, and using (2.5), we have

$$\begin{aligned}&s\int_{B_{r(1+\sigma\sigma_1)}(x_0)}G(x,|\nabla u|)\,\mathcal{G}^{s-1}(x_0,\overline{u}/\rho)\zeta^l\,dx\\&\le l\int_{B_{r(1+\sigma\sigma_1)}(x_0)}g(x,|\nabla u|)\frac{\overline{u}}{\sigma\sigma_1\rho\zeta}\,\mathcal{G}^{s-1}(x_0,\overline{u}/\rho)\zeta^l\,dx.\end{aligned}\tag{3.1}$$

Using (2.1) with $\varepsilon=\frac{s}{2l}$, $a=|\nabla u|$, $b=\frac{\overline{u}}{\sigma\sigma_1\rho\zeta}$, we estimate the right-hand side of (3.1) from above as follows:

$$\begin{aligned}&l\int_{B_{r(1+\sigma\sigma_1)}(x_0)}g(x,|\nabla u|)\frac{\overline{u}}{\sigma\sigma_1\rho\zeta}\,\mathcal{G}^{s-1}(x_0,\overline{u}/\rho)\zeta^l\,dx\\&\le\frac{s}{2}\int_{B_{r(1+\sigma\sigma_1)}(x_0)}G(x,|\nabla u|)\,\mathcal{G}^{s-1}(x_0,\overline{u}/\rho)\zeta^l\,dx\\&\quad+l\int_{B_{r(1+\sigma\sigma_1)}(x_0)}g\Big(x,\frac{\overline{u}}{\varepsilon\sigma\sigma_1\rho\zeta}\Big)\frac{\overline{u}}{\sigma\sigma_1\rho\zeta}\,\mathcal{G}^{s-1}(x_0,\overline{u}/\rho)\zeta^l\,dx,\end{aligned}\tag{3.2}$$

moreover, since $\frac{\overline{u}}{\varepsilon\sigma\sigma_1\rho\zeta}\ge\frac{\overline{u}}{\rho}\ge2(1+b_0)$, conditions (A2), (A4), inequality (2.4) and $\varepsilon=s/(2l)$ give the estimate

$$\begin{aligned}&l\int_{B_{r(1+\sigma\sigma_1)}(x_0)}g\Big(x,\frac{\overline{u}}{\varepsilon\sigma\sigma_1\rho\zeta}\Big)\frac{\overline{u}}{\sigma\sigma_1\rho\zeta}\,\mathcal{G}^{s-1}(x_0,\overline{u}/\rho)\zeta^l\,dx\\&\le\frac{c_1 l}{\varepsilon^{q-1}}\,\frac{1}{(\sigma\sigma_1)^q}\int_{B_{r(1+\sigma\sigma_1)}(x_0)}g\Big(x,\frac{\overline{u}}{\rho}\Big)\frac{\overline{u}}{\rho}\,\mathcal{G}^{s-1}(x_0,\overline{u}/\rho)\zeta^{l-q}\,dx\\&\le\frac{\gamma\,l^q e^{\lambda(\rho)}}{(\sigma\sigma_1)^q}\int_{B_{r(1+\sigma\sigma_1)}(x_0)}\mathcal{G}^s(x_0,\overline{u}/\rho)\zeta^{l-q}\,dx.\end{aligned}\tag{3.3}$$

Combining (3.1), (3.2), (3.3), we obtain

$$\begin{aligned} &s\int_{B_{r(1+\sigma\sigma_1)}(x_0)} G(x,|\nabla u|)\,\mathcal{G}^{s-1}(x_0,\overline{u}/\rho)\zeta^l\,dx \\ &\le \frac{\gamma l^q e^{\lambda(\rho)}}{(\sigma\sigma_1)^q}\int_{B_{r(1+\sigma\sigma_1)}(x_0)} \mathcal{G}^s(x_0,\overline{u}/\rho)\zeta^{l-q}\,dx. \end{aligned}$$

In turn, using this inequality, as well as (A4), (2.1) and (2.4), we deduce that

$$\begin{aligned} &\int_{B_{r(1+\sigma\sigma_1)}(x_0)} \big|\nabla\big[\mathcal{G}^s(x_0,\overline{u}/\rho)\zeta^l\big]\big|\,dx \\ &\le \frac{s}{\rho}\int_{B_{r(1+\sigma\sigma_1)}(x_0)} \mathcal{G}^{s-1}(x_0,\overline{u}/\rho)g(x_0,\overline{u}/\rho)|\nabla u|\,\zeta^l\,dx \\ &\quad + \frac{l}{\sigma\sigma_1\rho}\int_{B_{r(1+\sigma\sigma_1)}(x_0)} \mathcal{G}^s(x_0,\overline{u}/\rho)\zeta^{l-1}\,dx \\ &\le \gamma s\frac{e^{\lambda(\rho)}}{\rho}\int_{B_{r(1+\sigma\sigma_1)}(x_0)} G(x,|\nabla u|)\,\mathcal{G}^{s-1}(x_0,\overline{u}/\rho)\zeta^l\,dx \\ &\quad + \frac{\gamma s\,l}{\sigma\sigma_1\rho}\int_{B_{r(1+\sigma\sigma_1)}(x_0)} \mathcal{G}^s(x_0,\overline{u}/\rho)\zeta^{l-1}\,dx \\ &\le \frac{\gamma s\,l^q}{(\sigma\sigma_1)^q}\frac{e^{2\lambda(\rho)}}{\rho}\int_{B_{r(1+\sigma\sigma_1)}(x_0)} \mathcal{G}^s(x_0,\overline{u}/\rho)\zeta^{l-q}\,dx. \end{aligned}$$

Combining this and Sobolev's inequality, we arrive to

$$\begin{aligned} \Big(\int_{B_r(x_0)} \mathcal{G}^{\frac{sn}{n-1}}(x_0,\overline{u}/\rho)dx\Big)^{\frac{n-1}{n}} &\le \int_{B_{r(1+\sigma\sigma_1)}(x_0)} \big|\nabla\big[\mathcal{G}^s(x_0,\overline{u}/\rho)\zeta^l\,\big]\big|dx \\ &\le \frac{\gamma s\,l^q}{(\sigma\sigma_1)^q}\frac{e^{2\lambda(\rho)}}{\rho}\int_{B_{r(1+\sigma\sigma_1)}(x_0)} \mathcal{G}^s(x_0,\overline{u}/\rho)dx. \end{aligned} \tag{3.4}$$

Now, for i, $j=0,1,2,\dots$, we define the sequences

$$r_{i,j}=\frac{\rho}{4}(5-2^{-i})+\frac{\rho}{8}\,2^{-i-j},\quad s_j=\big(\frac{n}{n-1}\big)^j,\quad l_j=q\big(\frac{n}{n-1}\big)^j.$$

Let $\zeta_{i,j}\in C_0^\infty\big(B_{r_{i,j}}(x_0)\big)$, $0\le\zeta_{i,j}\le 1$, $\zeta_{i,j}=1$ in $B_{r_{i,j+1}}(x_0)$, $|\nabla\zeta_{i,j}|\le\frac{2^{i+j+4}}{\rho}$. For i, $j=0,1,2,\dots$, we also set $r_i=r_{i,\infty}$, $M_i=\operatorname{ess\,sup}_{B_{r_i}(x_0)}u$ and

$$y_{i,j}=\Big(\fint_{B_{r_{i,j}}(x_0)} \mathcal{G}^{s_j}(x_0,\overline{u}/\rho)dx\Big)^{1/s_j}.$$

From (3.4) we obtain

$$y_{i,j+1}\le\Big(\gamma 2^{(i+j)q}e^{2\lambda(\rho)}\Big)^{1/s_j}y_{i,j},\quad i,j=0,1,2,\dots. \tag{3.5}$$

We iterate inequality (3.5) with respect to j and use the fact that $r_{i+1}=r_{i,0}$ to obtain

$$\begin{aligned} \mathcal{G}\Big(x_0,\frac{M_i+2(1+b_0)\rho}{\rho}\Big) &\le \gamma\,2^{i\gamma}e^{2n\lambda(\rho)}\fint_{B_{r_{i+1}}(x_0)} \mathcal{G}(x_0,\overline{u}/\rho)dx \\ &\le \gamma 2^{i\gamma}e^{2n\lambda(\rho)}\frac{M_{i+1}+2(1+b_0)\rho}{\rho}\fint_{B_{r_{i+1}}(x_0)} g(x_0,\overline{u}/\rho)dx. \end{aligned}$$

This inequality, (2.2), and (2.4) imply that, for any $\varepsilon \in (0,1)$ and $i = 0,1,2,\dots$, the following inequalities hold:

$$\begin{aligned}
&g\Big(x_0, \frac{M_i + 2(1+b_0)\rho}{\rho}\Big) \\
&\le \varepsilon^{p-1} g\Big(x_0, \frac{M_{i+1} + 2(1+b_0)\rho}{\rho}\Big) \\
&\quad + \frac{1}{\varepsilon} g\Big(x_0, \frac{M_i + 2(1+b_0)\rho}{\rho}\Big) \frac{M_i + 2(1+b_0)\rho}{M_{i+1} + 2(1+b_0)\rho} \\
&\le \varepsilon^{p-1} g\Big(x_0, \frac{M_{i+1} + 2(1+b_0)\rho}{\rho}\Big) + \frac{\gamma\, 2^{i\gamma}}{\varepsilon} e^{2n\lambda(\rho)} \fint_{B_{5\rho/4}(x_0)} g(x_0, \overline{u}/\rho)dx.
\end{aligned}$$

Iterating the resulting inequality, for each $i \ge 1$ we have

$$\begin{aligned}
g\Big(x_0, \frac{M(\rho) + 2(1+b_0)\rho}{\rho}\Big) &= g\Big(x_0, \frac{M_0 + 2(1+b_0)\rho}{\rho}\Big) \\
&\le \varepsilon^{i(p-1)} g\Big(x_0, \frac{M_i + 2(1+b_0)\rho}{\rho}\Big) \\
&\quad + \gamma \varepsilon^{-1} e^{2n\lambda(\rho)} \sum_{j=0}^{i-1} (\varepsilon^{p-1} 2^{\gamma})^j \fint_{B_{5\rho/4}(x_0)} g(x_0, \overline{u}/\rho)dx.
\end{aligned}$$

Finally, choosing ε from the condition $\varepsilon^{p-1} 2^{\gamma} = 1/2$ and passing i to infinity, we arrive at

$$g\Big(x_0, \frac{M(\rho) + 2(1+b_0)\rho}{\rho}\Big) \le \gamma e^{2n\lambda(\rho)} \fint_{B_{5\rho/4}(x_0)} g\Big(x_0, \frac{u + 2(1+b_0)\rho}{\rho}\Big) dx.$$

This completes the proof of Theorem 1.4.

Acknowledgments. This work was supported by Ministry of Education and Science of Ukraine (project numbers 0121U109525 and 0119U100421), and by the Volkswagen Foundation project "From Modeling and Analysis to Approximation".

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