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INTEGRAL REPRESENTATION OF A SOLUTION OF HEUN'S GENERAL EQUATION

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Abstract. We establish an integral representation for the Frobenius solution with an exponent zero at $z = 0$ of the general Heun equation. First we present an extension of Mellin's lemma which provides a powerful method that takes into account differential equations which are not of the form studied by Mellin. That is the case for equations of Heun's type. It is that aspect which makes our work different from Valent's work. The method is powerful because it allows obtaining directly the nucleus equation of the representation. The integral representation formula obtained with this method leads quickly and naturally to already known results in the case of hypergeometric functions. The generalisation of this method gives a type of differential equations which form is a novelty and deserves to be studied further.

1. Introduction

This problem already was studied by Sleeman in [9] and by Valent in [10]. Sleeman first sought to solve the three-termed recurrence relation associated to the Heun equation using the Laplace transform method. This resolution allowed him to obtain a Barnes type integral representation similar to the well-known Gauss hypergeometric one.

Valents approach is different from Sleeman's one; ours as well. Although we use the relation $L_x K = M_t K$ ([5] and [1]) as does G. Valent, however our work is completely different. To establish our integral representation we preferred to develop an extension of Mellin lemma to take into account the differential equations that respond to a form satisfied by equations of Heun type.

Thus, after extension, the Mellin method easily gives us the nucleus equation $K(xt)$ occurring in the integral representation.

2. Reminder of some Heun equation's properties

The canonical form of the general Heun equation is [8]:

$$\frac{d^2y}{dz^2} + \big(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\big)\frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}y = 0. \tag{2.1}$$

Where

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- $y(z)$ is a function of the complex variable z;
- $(\alpha, \beta, \gamma, \delta, q, a)$ are complex or real parameters with $a \neq 0, 1$;
- q is an accessory parameter that allows to characterise Heun's equation solutions.

The five parameters are linked with the relation

$$\gamma + \delta + \epsilon = \alpha + \beta + 1 \tag{2.2}$$

Heun equation is of the Fushsian type, with four regular singularities $(0, 1, a, \infty)$ of which three $(0, 1, a)$ are at finite distance. The four singularities' exponents are the following:

$$\{0, 1-\gamma\}; \quad \{0, 1-\delta\}; \quad \{0, 1-\epsilon\}; \quad \{\alpha, \beta\} \tag{2.3}$$

Equation (2.2) has a Riemann scheme with a P-symbol in the form

$$P \begin{pmatrix} 0 & 1 & a & \infty & & \\ 0 & 0 & 0 & \alpha & z & q \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & & \end{pmatrix} \tag{2.4}$$

This equation noted $(0, 4, 0)$ in the Klein-Blôcher-Ince classification, was developed by Heun [4] in 1889 to generalise the Gauss hypergeometric equation.

Our aim is to find an integral representation of the Heun function being a Frobenius' solution of the Heun equation given in another form as follows [8]:

$$\begin{aligned} &z(z-1)(z-a)y''(z) + [\gamma(z-1)(z-a) + \delta z(z-a) + \epsilon z(z-1)]\, y'(z) \\ &+ (\alpha\beta z - q)y(z) = 0 \end{aligned} \tag{2.5}$$

The Frobenius' solution, noted $H_l(a, q; \alpha, \beta, \gamma, \delta; z)$ is the entire solution defined for the exponent zero at the point $z = 0$. It admits the power series expansion

$$H_l(a, q; \alpha, \beta, \gamma, \delta; z) \equiv \sum_{n=0}^{\infty} c_n z^n \quad |z| < 1 \tag{2.6}$$

with

$$c_0 = 1 \quad c_1 = \frac{q}{\gamma a} \quad \gamma \neq 0, -1, -2, \dots \tag{2.7}$$

and

$$\begin{aligned} &a(n+2)(n+1+\gamma)c_{n+2} \\ &= \left[q + (n+1)(\alpha + \beta - \delta + (\gamma + \delta - 1)a) + (n+1)^2(a+1)\right] c_{n+1} \\ &\quad - (n+\alpha)(n+\beta)c_n \quad n \geq 0\,. \end{aligned} \tag{2.8}$$

The function $H_l(a, q; \alpha, \beta, \gamma, \delta; z)$ is normalised with the relation

$$H_l(a, q; \alpha, \beta, \gamma, \delta; 0) = 1\,. \tag{2.9}$$

It admits the following important particular cases ([8], p9, formula (1.3.9)):

$$\begin{gathered} H_l(1, \alpha\beta; \alpha, \beta, \gamma, \delta; z) = {}_2F_1(\alpha, \beta, \gamma; z) \quad \forall \delta \in \mathbb{C} \\ H_l(0, 0; \alpha, \beta, \gamma, \delta; z) = {}_2F_1(\alpha, \beta, \alpha + \beta - \delta + 1; z) \quad \forall \gamma \in \mathbb{C} \\ H_l(a, a\alpha\beta; \alpha, \beta, \gamma, \alpha + \beta - \gamma + 1; z) = {}_2F_1(\alpha, \beta, \gamma; z), \end{gathered} \tag{2.10}$$

where ${}_2F_1(\alpha, \beta, \gamma; z)$ is the usual notation of the Gauss hypergeometric function, also noted $F(\alpha, \beta, \gamma; z)$.

On another hand we know that ([7, p.258, sec.9.8]) and [2, p.68, theorem 2.2.5]: (Euler)

$$
{}_2F_1(\alpha,\beta,\gamma;z) = (1-z)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\gamma-\beta,\gamma;z) \quad \text{with } |\arg(1-z)| < \pi
$$
$$
{}_2F_1(\alpha,\beta,\beta;z) = (1-z)^{-\alpha} \quad \text{with } |\arg(1-z)| < \pi \quad \forall \beta \in \mathbb{C}\,.
$$

Hence

$$
H_l(1,\alpha\beta;\alpha,\beta,\beta,\delta;z) = (1-z)^{-\alpha} \quad \text{with } |\arg(1-z)| < \pi; \quad \forall \beta,\ \delta \in \mathbb{C},
$$
$$
H_l(a,a\alpha\beta;\alpha,\beta,\beta,\alpha+1;z) = (1-z)^{-\alpha} \quad \text{with } |\arg(1-z)| < \pi \quad \forall \beta \in \mathbb{C}
$$

3. Integral representation for $H_l(a,q;\alpha,\beta,\gamma,\delta;z)$ function

To find an integral representation of the Heun function $H_l(a,q;\alpha,\beta,\gamma,\delta;z)$ solution of equation (2.5), let's consider the second order differential operator L_x. The representation of the function of interest $y(x)$ is taken as

$$
y(x) = \int_C K(x,t)\,\mathtt{v}(t)dt\,. \tag{3.1}
$$

To verify the differential equation

$$
L_x(y) = 0, \tag{3.2}
$$

the researched representation will derive from the following lemma that we consider as lemma of general principle [5].

Lemma 3.1 (Lemma of General Principle)**.**

- *If it is possible to verify that the nucleus $K(x,t)$ satisfies the partial differential equation in the form*
$$
L_x(K) = M_t(K), \tag{3.3}
$$
where M_t is a differential operator involving only t and $\frac{\partial}{\partial t}$.
- *If $\mathtt{v}(t)$ is a solution of the differential equation*
$$
\overline{M_t}(\mathtt{v}) = 0, \tag{3.4}
$$
where $\overline{M_t}$ is the operator adjoint to M_t.
- *And if t_1 and t_2 are two extremities of the curve C so that*
$$
[K\mathtt{v}]_{t_1}^{t_2} = 0 \tag{3.5}
$$

then we have $L_x(y) = 0$. This means the function $y(x)$ represented by (3.1) is indeed a solution of the differential equation (3.2).

Proof. We consider $y(x)$ given by (3.1). Let us apply the operator L_x; we formally have

$$
L_x(y) = \int_C L_x(K)\,\mathtt{v}(t)dt\,. \tag{3.6}
$$

If the nucleus $K(xt)$ verifies the relation (3.3) then we get

$$
L_x(y) = \int_C L_x(K)\mathtt{v}(t)dt = \int_C M_t(K)\mathtt{v}(t)dt\,. \tag{3.7}
$$

Using the Lagrange's Identity, which has the form [5]

$$
\mathtt{v}(t)M_t\{K(xt)\} - K(xt)\overline{M_t}\{\mathtt{v}(t)\} = \frac{\partial}{\partial t}\{K\mathtt{v}\}, \tag{3.8}
$$

we obtain

$$L_x(y) = \int_C K(xt)\overline{M_t}\{\mathtt{v}(t)\}dt + [K\mathtt{v}]_C \tag{3.9}$$

And if $\overline{M_t}\{\mathtt{v}(t)\} = 0$ and $[K\mathtt{v}]_{t_1}^{t_2} = 0$ then we get the expected result,

$$L_x(y) = 0. \tag{3.10}$$

The lemma 3.1 is proved. □

To establish our principal theorem, we need few lemmas, particularly Mellin's lemma as presented in Ince [5, p.195, sect.8.4].

Lemma 3.2 (Mellin's Lemma). *Consider a differential equation in the form*

$$L_x(y) \equiv x^n F\big(x\frac{d}{dx}\big)y + G\big(x\frac{d}{dx}\big)y = 0 \tag{3.11}$$

Let H be any one-variable polynomial and $K(z)$ any solution of the ordinary differential equation

$$z^n F\big(z\frac{d}{dz}\big) - H\big(z\frac{d}{dz}\big)K = 0 \tag{3.12}$$

Then $K(xt)$ satisfies the partial differential equation

$$\{x^n F\big(x\frac{d}{dx}\big) + G\big(x\frac{d}{dx}\big)\}K = \{G\big(t\frac{d}{dt}\big) + t^{-n}H\big(t\frac{d}{dt}\big)\}K, \tag{3.13}$$

or

$$L_x(K) = L_t(K). \tag{3.14}$$

Thus the integral

$$y(x) = \int_C K(xt)\mathtt{v}(t)dt \tag{3.15}$$

is a solution of (3.11)*, provided that $\mathtt{v}(t)$ is solution of the ordinary differential equation*

$$\overline{M_t}(\mathtt{v}) = 0, \tag{3.16}$$

where $\overline{M_t}$ is the operator adjoint to M_t and provided that the extremities (t_1, t_2) on the curve C are appropriately selected.

For the proof of the above lemma, we refer the reader to Ince [5], Mellin [6] and Bateman [1, p.184].

The Mellin's lemma as establised here does apply only to the differential equation in the form (3.11), which excludes equations of Heun's type, which have a different form. Therefore we will adapt it so that equations of Heun type are taken into account. For that purpose we need the following lemma.

Lemma 3.3 (Our Extension of Mellin's Lemma). *Consider a differential equation in the form*

$$L_x(y) \equiv x^2 F\big(x\frac{d}{dx}\big)y + xG\big(x\frac{d}{dx}\big)y + \widetilde{G}\big(x\frac{d}{dx}\big)y = 0\,. \tag{3.17}$$

Let H be any one-variable polynomial and $K(z)$ any solution of the ordinary differential equation

$$\big\{z^2 F\big(z\frac{d}{dz}\big) + zG\big(z\frac{d}{dz}\big)\big\}K(z) - H\big(z\frac{d}{dz}\big)K(z) = 0\,. \tag{3.18}$$

Then $K(xt)$ satisfies the partial differential equation

$$\big\{x^2F\big(x\frac{d}{dx}\big)+xG\big(x\frac{d}{dx}\big)+\widetilde{G}\big(x\frac{d}{dx}\big)\big\}K=\big\{\widetilde{G}\big(t\frac{d}{dt}\big)+t^{-1}H\big(t\frac{d}{dt}\big)\big\}K, \tag{3.19}$$

or

$$L_x(K)=M_t(K). \tag{3.20}$$

Thus the integral

$$y(x)=\int_C K(xt)\,\mathtt{v}(t)dt \tag{3.21}$$

is a solution of (3.17) *provided that* $\mathtt{v}(t)$ *is solution of the ordinary differential equation*

$$\overline{M_t}(\mathtt{v})=0, \tag{3.22}$$

where $\overline{M_t}$ *is the operator adjoint to* M_t *and provided the end-points* (t_1,t_2) *on the curve* C *are appropriately selected.*

Proof. First, let us note [1, p.184] that if $w=K(xt)=K(Z)$ then $F(x\frac{d}{dx})w=F(t\frac{d}{dt})w=F(z\frac{d}{dz})w$.

Let us consider the partial differential equation

$$\big\{x^2F\big(x\frac{d}{dx}\big)+xG\big(x\frac{d}{dx}\big)+\widetilde{G}\big(x\frac{d}{dx}\big)\big\}w=\big\{\widetilde{G}\big(t\frac{d}{dt}\big)+t^{-1}H\big(t\frac{d}{dt}\big)\big\}w \tag{3.23}$$

which also can be expressed as

$$\big\{x\big\{xF\big(x\frac{d}{dx}\big)+xG\big(x\frac{d}{dx}\big)\big\}+\widetilde{G}\big(x\frac{d}{dx}\big)\big\}w=\big\{\widetilde{G}\big(t\frac{d}{dt}\big)+t^{-1}H\big(t\frac{d}{dt}\big)\big\}w \tag{3.24}$$

By posing

$$F_1\big(x,\frac{d}{dx}\big)=xF\big(x\frac{d}{dx}\big)+G\big(x\frac{d}{dx}\big), \tag{3.25}$$

Equation (3.24) becomes

$$\big\{xF_1\big(x,\frac{d}{dx}\big)+\widetilde{G}\big(x\frac{d}{dx}\big)\big\}w=\big\{\widetilde{G}\big(t\frac{d}{dt}\big)+t^{-1}H\big(t\frac{d}{dt}\big)\big\}w\,. \tag{3.26}$$

Let us consider first the second member of (3.25), to which we apply Mellin's lemma; this leads to consider the partial differential equation

$$\big\{xF\big(x\frac{d}{dx}\big)+G\big(x\frac{d}{dx}\big)\big\}w=\big\{G\big(t\frac{d}{dt}\big)+t^{-1}H_0\big(t\frac{d}{dt}\big)\big\}w \tag{3.27}$$

It is satisfied by $w=K(xt)=K(z)$ if we have

$$\{zF\big(z\frac{d}{dz}\big)-H_0\big(z\frac{d}{dz}\big)\}K(z)=0\,. \tag{3.28}$$

Which gives

$$xF\big(x\frac{d}{dx}\big)=H_0\big(x\frac{d}{dx}\big)\,. \tag{3.29}$$

By reporting this result into (3.25) we get

$$F_1\big(x,\frac{d}{dx}\big)=H_0\big(x\frac{d}{dx}\big)+G\big(x\frac{d}{dx}\big)\,. \tag{3.30}$$

Then reporting this result into (3.26) we get

$$\{x\{H_0\big(x\frac{d}{dx}\big)+G\big(x\frac{d}{dx}\big)\}+\widetilde{G}\big(x\frac{d}{dx}\big)\}w=\{\widetilde{G}\big(t\frac{d}{dt}\big)+t^{-1}H\big(t\frac{d}{dt}\big)\}w\,. \tag{3.31}$$

Which is an equation of Mellin's type. It is satisfied by $w = K(xt) = K(z)$ if we have

$$\big[z\{H_0\big(z\frac{d}{dz}\big) + G\big(z\frac{d}{dz}\big)\}\big]K(z) - H\big(z\frac{d}{dz}\big)K(z) = 0 \tag{3.32}$$

However, from (3.28) we have $H_0(z\frac{d}{dz})K(z) = zF(z\frac{d}{dz})K(z)$. Thus, by reporting this result into (3.32) we obtain

$$\big[z\{zF\big(z\frac{d}{dz}\big) + G\big(z\frac{d}{dz}\big)\}\big]K(z) - H\big(z\frac{d}{dz}\big)K(z) = 0\,. \tag{3.33}$$

Finally we get the following equation verified by the nucleus of the searched integral representation

$$\big[z^2F\big(z\frac{d}{dz}\big) + zF\big(z\frac{d}{dz}\big) - H\big(z\frac{d}{dz}\big)\big]K(z) = 0 \tag{3.34}$$

Thus the first part of our lemma is established. The second part of our lemma derives directly from lemma 1, so called "lemma of general principle". With both of these parts our lemma is proved. □

Lemma 3.4. *The Heun equation (2.5) satisfies the form* (3.17) *in the lemma 3.3.*

Proof. Consider equation (2.5), which after multiplication by x and sign change, can be written as

$$L_x(y) \equiv x^2F(x\frac{d}{dx})y + xG(x\frac{d}{dx})y + \widetilde{G}(x\frac{d}{dx})y \tag{3.35}$$

$$F\big(x\frac{d}{dx}\big) = -\{\big(x\frac{d}{dx}\big)^2 + (\gamma+\delta+\epsilon)\big(x\frac{d}{dx}\big) + \alpha\beta\} \tag{3.36}$$

$$G\big(x\frac{d}{dx}\big) = \{(a+1)\big(x\frac{d}{dx}\big)^2 + \big[(a+1)\gamma + a\delta + \epsilon\big]\big(x\frac{d}{dx}\big) + q\} \tag{3.37}$$

$$\widetilde{G}\big(x\frac{d}{dx}\big) = -a\{\big(x\frac{d}{dx}\big)^2 + \gamma\big(x\frac{d}{dx}\big)\} \tag{3.38}$$

This indeed is the form (3.17) of the lemma 3.3. □

Prior to discuss applications, let us determine the operator M_t to be associated with the operator L_x so that lemma 3.3 applies.

4. Determination of operator M_t

Consider the operator

$$\hat{M}_t = t(t-1)\frac{d}{dt} + \{\sigma - (\rho+\sigma)t\}\,. \tag{4.1}$$

It corresponds to the solution

$$\mathtt{v}(t) = t^{\sigma}(1-t)^{\rho}\,. \tag{4.2}$$

Thus, taking the adjoint to $\widehat{M_t}$, which is notated $\overline{\widehat{M_t}}$, gives

$$\begin{aligned}\overline{\hat{M}_t} &= -\frac{d}{dt}[(t^2-t)\mathtt{v}] + \{\sigma - (\rho+\sigma)t\}\mathtt{v} \\ &= -(t^2-t)\frac{d\mathtt{v}}{dt} - (2t-1)\mathtt{v} + \{\sigma - (\rho+\sigma)t\}\mathtt{v},\end{aligned} \tag{4.3}$$

$$\overline{\hat{M}_t} = t(1-t)\frac{d\mathtt{v}}{dt} + \{(\sigma+1) - (\rho+\sigma+2)t\}\mathtt{v}\,. \tag{4.4}$$

To apply lemma 3.3, the operator $\widehat{\widehat{M_t}}D_t$ is defined as follows ([5, p.195, sec.8.41], [1, p184]):

$$\widehat{\widehat{M_t}}D_t = t(1-t)\frac{d^2}{dt^2} + \{(\sigma+1) - (\rho+\sigma+2)t\}\frac{d}{dt} \quad \text{with } D_t = \frac{d}{dt}\,. \tag{4.5}$$

Decomposing the operator $\widehat{\widehat{M_t}}D_t$ and regrouping gives

$$\widehat{\widehat{M_t}}D_t = -\big[\big(t\frac{d}{dt}\big)^2 + (\rho+\sigma+2)\big(t\frac{d}{dt}\big)\big] + t^{-1}\big[\big(t\frac{d}{dt}\big)^2 + (\sigma+1)\big(t\frac{d}{dt}\big)\big]. \tag{4.6}$$

Multiplying $\widehat{\widehat{M_t}}D_t$ by $a \neq 0$, the operator M_t necessary to apply lemma 3.3 is defined as

$$\begin{aligned} M_t &= a\widehat{\widehat{M_t}}D_t \\ &= -a\big[\big(t\frac{d}{dt}\big)^2 + (\rho+\sigma+2)\,\big(t\frac{d}{dt}\big)\big] + t^{-1}\{a\big[\big(t\frac{d}{dt}\big)^2 + (\sigma+1)\big(t\frac{d}{dt}\big)\big]\}\,. \end{aligned} \tag{4.7}$$

Comparing the expression $\widetilde{G}(x\frac{d}{dx})$ in (3.38) with the corresponding expression of M_t in (4.7) provides the important relation

$$-\gamma + (\rho+\sigma+2) = 0\,. \tag{4.8}$$

Hence

$$\rho = \gamma - \sigma - 2\,. \tag{4.9}$$

And if one takes

$$\sigma = c - 1, \tag{4.10}$$

then

$$\rho = \gamma - (c-1) - 2 = \gamma - c + 1 - 2 = \gamma - c - 1\,. \tag{4.11}$$

So that

$$\sigma = c - 1 \Rightarrow \rho = \gamma - c - 1\,. \tag{4.12}$$

In this case,

$$\mathtt{v}(t) = t^{c-1}(1-t)^{\gamma-c-1}\,. \tag{4.13}$$

5. Differential Equation verified by nucleus $K(xt)$

From (4.7) the polynomial $H(z\frac{d}{dz})$ involved in the definition of the nucleus is given by the coefficient of t^{-1} as follows:

$$H\big(t\frac{d}{dt}\big) = a\left[\big(t\frac{d}{dt}\big)^2 + (\sigma+1)\big(t\frac{d}{dt}\big)\right]. \tag{5.1}$$

Since σ is defined by $\sigma = c-1$, hence

$$H\big(z\frac{d}{dz}\big) = a\left[\big(z\frac{d}{dz}\big)^2 + c\big(z\frac{d}{dz}\big)\right] \tag{5.2}$$

The nucleus equation as given by lemma 3.3 therefore can be written

$$z^2F\big(z\frac{d}{dz}\big)K(z) + zG\big(z\frac{d}{dz}\big)K(z) - H\big(z\frac{d}{dz}\big)K(z) = 0\,. \tag{5.3}$$

When taking into account (5.2), it gives

$$\{z^2F\big(z\frac{d}{dz}\big) + zG\big(z\frac{d}{dz}\big)\}K(z) - a\{\big(z\frac{d}{dz}\big)^2 + c\big(z\frac{d}{dz}\big)\}K(z) = 0\,. \tag{5.4}$$

Let us explicit equation (5.4); for that purpose refer to (3.36) and (3.37) for a definition of $F(z\frac{d}{dz})$ and $G(z\frac{d}{dz})$; thus

$$\begin{aligned} &z^2\big\{-\big[\big(z\frac{d}{dz}\big)^2+(\gamma+\delta+\epsilon)\big(z\frac{d}{dz}\big)+\alpha\beta\big]\big\}K(z)\\ &+z\big\{(a+1)\big(z\frac{d}{dz}\big)^2+[(a+1)\gamma+a\delta+\epsilon]\left(z\frac{d}{dz}\right)+q\big\}K(z)\\ &-a\big\{\big(z\frac{d}{dz}\big)^2+c\big(z\frac{d}{dz}\big)\big\}K(z)=0 \end{aligned} \tag{5.5}$$

Comparing this equation with (3.35), (3.36), (3.37) and (3.38) shows that it will be of Heun's type by taking $\gamma = c$ everywhere in (5.5); then simplifying by z and changing sign gives if setting $u = K(z)$

$$z(z-1)(z-a)\frac{d^2u}{dz^2}+[c(z-1)(z-a)+\delta z(z-a)+\epsilon z(z-1)]\,\frac{du}{dz}+(\alpha\beta z-q)u=0 \tag{5.6}$$

This is the equation verified by the nucleus $K(xt)$ of the representation.

Hence the solution of this equation of Frobenius' type of exponent 0 at point $z = 0$ is

$$H_l(a,q;\alpha,\beta,c,\delta;xt)\,. \tag{5.7}$$

6. Integral relation obtained

The following relation can be formally established from the previous section,

$$y(x)=\int_c H_l(a,q;\alpha,\beta,c,\delta;xt)t^{c-1}(1-t)^{\gamma-c-1}dt \quad \text{with } \Re\gamma>\Re c>0\,. \tag{6.1}$$

On the curve C the following extremities are selected for the integration $t_1 = 0$ and $t_2 = 1$. With that choice of extremities and the one of function $\mathtt{v}(t) = t^{c-1}(1-t)^{\gamma-c-1}$, conditions (3.4) and (3.5) of lemma 3.1 are verified, hence $y(x)$ is a solution of equation (2.5) and satisfies the form (3.17) of lemma 3.3. Now let us examine the initial conditions at the origin to determine the unique solution. Thus

$$y(0)=\int_0^1 t^{c-1}(1-t)^{\gamma-c-1}dt=\frac{\Gamma(c)\Gamma(\gamma-c)}{\Gamma(\gamma)}, \tag{6.2}$$

$$\begin{aligned} y'(0)&=\frac{q}{ac}\int_0^1 t^{c}(1-t)^{\gamma-c-1}dt=\frac{q}{ac}\frac{\Gamma(c+1)\Gamma(\gamma-c)}{\Gamma(\gamma+1)}\\ &=\frac{q}{ac}\frac{c\Gamma(c)\Gamma(\gamma-c)}{\gamma\Gamma(\gamma)}=\frac{q}{a\gamma}\frac{\Gamma(c)\Gamma(\gamma-c)}{\Gamma(\gamma)} \end{aligned} \tag{6.3}$$

These two initial conditions determine the unique solution

$$\frac{\Gamma(c)\Gamma(\gamma-c)}{\Gamma(\gamma)}H_l(a,q;\alpha,\beta,\gamma,\delta;x) \tag{6.4}$$

Consequently (6.1) becomes

$$\frac{\Gamma(c)\Gamma(\gamma-c)}{\Gamma(\gamma)}H_l(a,q;\alpha,\beta,\gamma,\delta;x)=\int_0^1 H_l(a,q;\alpha,\beta,c,\delta;xt)t^{c-1}(1-t)^{\gamma-c-1}dt \tag{6.5}$$

with $\Re\gamma > \Re c > 0$ and $|x| < \min(|a|, 1)$. The representation obtained is therefore given by the following theorem.

Theorem 6.1. *The Heun function $H_l(a,q;\alpha,\beta,\gamma,\delta;x)$ has the integral representation*

$$H_l(a,q;\alpha,\beta,\gamma,\delta;x)=\frac{\Gamma(\gamma)}{\Gamma(c)\Gamma(\gamma-c)}\int_0^1 H_l(a,q;\alpha,\beta,c,\delta;xt)t^{c-1}(1-t)^{\gamma-c-1}dt \tag{6.6}$$

with $\Re\gamma>\Re c>0$ and $|x|<\min(|a|,1)$.

Corollary 6.2. *The Heun function $H_l(a,q;\alpha,\beta,\gamma,\delta;x)$ also admits the representation*

$$H_l(a,q;\alpha,\beta,\gamma+1,\delta;x)=\gamma\int_0^1 H_l(a,q;\alpha,\beta,\gamma,\delta;xt)t^{\gamma-1}dt \tag{6.7}$$

with $\Re\gamma>0$ and $|x|<\min(|a|,1)$.

Proof. In (6.7), first replace γ by $\gamma+1$. Then replace c by γ and get the announced result. □

7. Analytic continuation

Consider the Heun function $H_l(a,q;\alpha,\beta,\gamma,\delta;x)$ defined in (6.6) by the integral representation and let find its analytic continuation [7], [2], [11] and [3, p.62, Theorem 9]. For that purpose let us show that it is an analytic function for each of their variables $\alpha,\beta,\gamma,\delta;x$. To start with, let us define a function $Q(t)$ as follows:

$$t^{c-1}(1-c)^{\gamma-c-1}H_l(a,q;\alpha,\beta,c,\delta;xt)=t^{\sigma-1}(1-t)^{\tau-1}Q(t) \tag{7.1}$$

where

$$Q(t)=t^{c-\sigma}(1-c)^{\gamma-c-\tau}H_l(a,q;\alpha,\beta,c,\delta;xt).$$

Let us also define the following closed domain

$$\begin{gathered}S_r\equiv\Big\{0\le t\le 1;\sigma\le\Re c\le N;\tau\le\Re(\gamma-c)\le N;|\beta|\le N;\\ |\delta|\le N;|q|\le N;|x|\le N;|r|\le N;|\arg(r-\sigma-x)|\le\pi-\sigma|\Big\},\end{gathered} \tag{7.2}$$

where

$$\begin{gathered}N>0 \text{ and } N \text{ is arbitrarily large}\\ \sigma>0 \text{ and } \tau>0 \text{ are arbitrarily small}\\ r>0 \text{ with } r=\min(|a|,1)\end{gathered} \tag{7.3}$$

Hence, function $Q(t)$ is continuous for all its variables in the closed domain S_r. Therefore, it will be bounded in that domain; thus

$$|t^{c-\sigma}(1-c)^{\gamma-c-\tau}H_l(a,q;\alpha,\beta,c,\delta;xt)|\le C \tag{7.4}$$

where C is a constant.

It results that in the domain under study,

$$|t^{c-1}(1-c)^{\gamma-c-1}H_l(a,q;\alpha,\beta,c,\delta;xt)|\le C\,t^{\sigma-1}(1-c)^{\tau-1} \tag{7.5}$$

Since the integral $\int_0^1 t^{\sigma-1}(1-t)^{\tau-1}dt$ is convergent, integral (6.6) defining the Heun function $H_l(a,q;\alpha,\beta,\gamma,\delta;x)$ is uniformly convergent in the domain of interest and therefore represents an analytic function for each of its variables.

Since constants σ, τ and N are arbitrary, the Heun function $H_l(a,q;\alpha,\beta,\gamma,\delta;x)$ is an analytic function for each of their variables in the domain $\Re\gamma>\Re c>0$ and $|\arg(r-x)|<\pi$ in the plane of variable x cut along the real axis for $\Re x\ge 1$. Thus the following result has been obtained.

Theorem 7.1. *The Heun function $H_l(a,q;\alpha,\beta,\gamma,\delta;x)$ has the analytic continuation*

$$H_l(a,q;\alpha,\beta,\gamma,\delta;x)=\frac{\Gamma(\gamma)}{\Gamma(c)\Gamma(\gamma-c)}\int_0^1 H_l(a,q;\alpha,\beta,c,\delta;xt)t^{c-1}(1-t)^{\gamma-c-1}dt \tag{7.6}$$

with $\Re\gamma>\Re c>0$ and $|\arg(r-x)|<\pi$ with $r=\min(|a|,1)$

Corollary 7.2. *The Heun function $H_l(a,q;\alpha,\beta,\gamma,\delta;x)$ also admits the analytical continuation*

$$H_l(a,q;\alpha,\beta,\gamma+1,\delta;x)=\gamma\int_0^1 H_l(a,q;\alpha,\beta,\gamma,\delta;xt)t^{\gamma-1}dt \tag{7.7}$$

with $\Re\gamma>0$ and $|\arg(r-x)|<\pi$ with $r=\min(|a|,1)$.

The proof of the above corollary is identical to the one for corollary 6.2.

8. Important particular cases

Particularising parameters a and q gives, using the theorem 7.1 and corollary 7.2, the following results.

Theorem 8.1. *The hypergeometric function $F(\alpha,\beta,\gamma;x)$ admits the analytical continuation*

$$F(\alpha,\beta,\gamma;x)=\frac{\Gamma(\gamma)}{\Gamma(c)\Gamma(\gamma-c)}\int_0^1 F(\alpha,\beta,c;xt)t^{c-1}(1-t)^{\gamma-c-1}dt \tag{8.1}$$

with $\Re\gamma>\Re c>0$ and $|\arg(1-x)|<\pi$.

Proof. Taking $a=1$ and $q=\alpha\beta$ in theorem 7.1 and referring to point (2.10), results of the announced results is obtained. □

Corollary 8.2. *The hypergeometric function $F(\alpha,\beta,\gamma;x)$ admits the analytical continuation*

$$F(\alpha,\beta,\gamma+1;x)=\gamma\int_0^1 F(\alpha,\beta,\gamma;xt)t^{\gamma-1}dt \tag{8.2}$$

with $\Re\gamma>0$ and $|\arg(1-x)|<\pi$.

Proof. Taking $a=1$ and $q=\alpha\beta$ in theorem 7.1 and referring to point 2.10, results of the announced results is obtained. □

Theorem 8.1 and corollary 8.2, which are particular cases of our theorem7.1 and our corollary 7.2 are exactly identical to formulas given by Lebedev [7, p.277, exercise 6] and [2, p.68, theorem 2.2.4].

9. Conclusions

With our extension of Mellin's lemma we have obtained an integral representation of a power series solution of the general Heun equation, thus showing the full strength of Mellin's lemma itself. We think the method could give also an integral representation of the power series solutions of differential equations satisfying the following novel general form

$$L_x(y)\equiv\sum_{i=1}^{n}x^iF_i\big(x\frac{d}{dx}\big)y+\widetilde{G}\big(x\frac{d}{dx}\big)y=0 \tag{9.1}$$

Equation of Heun's type are those which are obtained when $n = 2$; and when $n = 1$, the equations satisfying the form (3.11) of the equations studied by Mellin ; in that class are the hypergeometric equations. Case $n \geq 3$ remained to be studied.

One can notice that the integral representation obtained for the Frobenius solution of exponent zero at the origin leads exactly, as a particular case, to an integral representation formula that is well-known for the hypergeometric functions.

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