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VANISHING OF SOLUTIONS OF DIFFUSION EQUATION WITH CONVECTION AND ABSORPTION

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Abstract. We study the vanishing of solutions of the Cauchy problem for the equation

$$u_t = \sum_{i,j=1}^{N} a_{ij}(u^m)_{x_i x_j} + \sum_{i=1}^{N} b_i(u^n)_{x_i} - cu^p, \quad (x,t) \in S = \mathbb{R}^N \times (0,+\infty).$$

Obtained results depend on relations of parameters of the problem and growth of initial data at infinity.

1. Introduction

We consider the Cauchy problem for the equation

$$u_t = \sum_{i,j=1}^{N} a_{ij}(u^m)_{x_i x_j} + \sum_{i=1}^{N} b_i(u^n)_{x_i} - cu^p, \quad (x,t) \in S = \mathbb{R}^N \times (0,+\infty) \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $m > 1 > p > 0$, $n \geq 1$, a_{ij}, b_i $(i,j = 1,\dots,N)$, c are real numbers, $a_{ij} = a_{ji}$, $\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j > 0$ for $\sum_{i=1}^{N} \xi_i^2 > 0$ $(\xi_i \in \mathbb{R}, i = 1,\dots,N)$, $c > 0$, $u_0(x)$ is a nonnegative continuous function which can be increasing at infinity. Equation (1.1) is encountered, for example, when simulating a process of diffusion or heat propagation accompanied by convection and absorption. It is parabolic for $u > 0$ and degenerates into a first-order equation for $u = 0$. Due to degeneracy the Cauchy problem (1.1), (1.2) can have not a classical solution even when initial data are smooth.

Put $B_h = \{x \in \mathbb{R}^N : |x| < h\}$ $(0 < h < +\infty)$. By $\vec{\nu} = (\nu_1,\dots,\nu_N)$ we denote the outward unit normal to the boundary of a considered domain.

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Definition. The function $u(x,t)$ which is nonnegative and continuous in $\overline{S}$ we call a generalized solution of the equation (1.1) in S if $u(x,t)$ satisfies the integral identity

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{B_R}\{uf_t+u^m\sum_{i,j=1}^{N}a_{ij}f_{x_ix_j}-u^n\sum_{i=1}^{N}b_if_{x_i}-cu^pf\}dx\,dt-\int_{B_R}uf\big|_{t_1}^{t_2}dx\\&-\int_{t_1}^{t_2}\int_{\partial B_R}u^m\sum_{i,j=1}^{N}a_{ij}f_{x_j}\nu_i ds\,dt=0\end{aligned}\tag{1.3}$$

for all $R>0$, $0<t_1<t_2<+\infty$ and any nonnegative function $f(x,t)\in C^{2,1}_{x,t}(\overline{B}_R\times[t_1,t_2])$ such that $f(x,t)=0$ for $|x|=R$, $t_1<t<t_2$.

If the equal sign in (1.3) is replaced by $\geq$ ($\leq$) then we obtain the definition of a generalized subsolution (supersolution) of the equation (1.1) in S.

Definition. The function $u(x,t)$ is called a generalized solution of the Cauchy problem (1.1), (1.2) if it is a generalized solution of the equation (1.1) in S and condition (1.2) is satisfied.

In the present paper we investigate the conditions when the generalized solution of the Cauchy problem (1.1), (1.2) vanishes at every point $x\in\mathbb{R}^N$ in a finite time $T_0(x)$ depending on x. For $n>(m+p)/2$ one-dimensional equation (1.1) is considered.

Behavior for large values of the time of unbounded generalized solutions of the Cauchy problem (1.1), (1.2) with $a_{ij}=1$ for $i=j$, $a_{ij}=0$ for $i\neq j$ and $b_i=0$ ($i,j=1,\dots,N$) has been studied in [1] and [2] for $m=1$ and $m>1$ respectively. The case $n=(m+p)/2$ has been considered in [3] in terms of the control theory.

The distribution of the paper is as follows. In the next section we introduce notations and give existence and uniqueness results which we need in the following. The condition on the initial data for the vanishing of solutions of the Cauchy problem (1.1), (1.2) at every point of $\mathbb{R}^N$ in a finite time in the case $n<(m+p)/2$ we point out in Section 3. The same results for the one-dimensional equation (1.1) in the cases $(m+p)/2<n<m$, $n=m$ and $n>m$ are established in Sections 4–6 respectively.

2. Existence and uniqueness

We begin with an existence theorem which reduces the vanishing problem to the construction of a suitable upper bound for the generalized solution. This statement can be proved in a similar way as the corresponding theorems in [4] – [6].

Theorem 2.1. *Suppose that $\Phi(x,t)\geq 0$ is a generalized supersolution of the equation (1.1) in S and $u_0(x)\leq\Phi(x,0)$. Then there exists a generalized solution $u(x,t)$ of the Cauchy problem (1.1), (1.2) in S, which is minimal in the set of solutions of this Cauchy problem, such that $0\leq u(x,t)\leq\Phi(x,t)$ in S.*

To construct a generalized supersolution we shall use the following lemma which is easily proved by integration by parts.

Lemma 2.2. *Let $v(x,t)$ be a continuous nonnegative function in $\overline{S}$ that satisfies the inequality*

$$-v_t + \sum_{i,j=1}^{N} a_{ij}(v^m)_{x_i x_j} + \sum_{i=1}^{N} b_i(v^n)_{x_i} - cv^p \leq 0 \quad (\geq 0)$$

and belongs to the space $C^{2,1}_{x,t}$ in S outside a set G that consists for each fixed $t \in (0,+\infty)$ of finitely many bounded closed hypersurfaces each of which is formed by finitely many piecewise smooth surfaces. Furthermore, suppose that $\nabla(v^m)$ is continuous on G. Then $v(x,t)$ is a generalized supersolution (subsolution) of the equation (1.1).

Part of existence and uniqueness classes of the Cauchy problem (1.1), (1.2) have been established in [5, 7, 8] and others can be obtained in a similar way. Let us formulate that results in the part which is necessary for our aim.
(a) Consider the case $n < (m+1)/2$. It is well known that a positive definite quadratic form $\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j$ reduces to the shape $\sum_{i=1}^{N} \eta_i^2$ by means of linear transformation

$$\xi_i = \sum_{j=1}^{N} c_{ij}\eta_j, \quad i = 1,\dots,N, \tag{2.1}$$

where $c_{ij} = c_{ji}$ $(i,j = 1,\dots,N)$. Put for $x \in \mathbb{R}^N$

$$\operatorname{dist}(x) = \Big[\sum_{i=1}^{N}\Big(\sum_{j=1}^{N} c_{ij}x_j\Big)^2\Big]^{1/2}. \tag{2.2}$$

Obviously, $\operatorname{dist}(x) > 0$ for $x \neq 0$. Denote $r = \operatorname{dist}(x)$.

We define the class $\mathcal{K}_1$ of nonnegative functions $\varphi(x,t)$ and $\varphi(x)$ which satisfy in arbitrary layer $S_T = \mathbb{R}^N \times [0,T]$ and $\mathbb{R}^N$ respectively the following condition

$$\varphi \leq M_1(\gamma_1 + r)^k, \quad 0 \leq k < 2/(m-1). \tag{2.3}$$

Here and below by M_i and γ_i $(i = 1,2,\dots)$ we shall denote positive and nonnegative constants respectively. Constants k, M_1 and γ_1 in (2.3) can depend on T and function φ.

Theorem 2.3. *Let $u_0(x) \in \mathcal{K}_1$. Then the Cauchy problem* (1.1), (1.2) *has a minimal generalized solution $u(x,t) \in \mathcal{K}_1$ in S. The generalized solution is unique in the class $\mathcal{K}_1$.*

(b) Assume now that $n = (m+1)/2$. In contrast to the previous case now the second term in the right hand side of (1.1) is essential for existence and uniqueness. Let us consider the equation (1.1) for the dimension $N = 1$

$$\mathcal{L}_1(u) \equiv -u_t + a(u^m)_{xx} + b(u^n)_x - cu^p = 0 \tag{2.4}$$

with the initial data (1.2). For definiteness here and below in the paper we shall suppose $b > 0$. Else the case $b = 0$ has been studied in [2] and for $b < 0$ space variable substitution x to $(-x)$ leads to (2.4) with $b > 0$.

Define the class $\mathcal{K}_2$ of nonnegative functions $\varphi(x,t)$ and $\varphi(x)$ satisfying in arbitrary strip $S_T = \mathbb{R} \times [0,T]$ and $\mathbb{R}$ respectively the following inequalities

$$\varphi \le M_2(\gamma_2 + x)^k \quad \text{for } x \ge 0, \quad 0 \le k < 2/(m-1), \tag{2.5}$$

$$\varphi \le \Big(\frac{b(m-1)}{2am}(\gamma_3 + |x|)\Big)^{2/(m-1)} \quad \text{for } x < 0. \tag{2.6}$$

Constants k, M_2, γ_2 and γ_3 in (2.5) and (2.6) can depend on T and function φ.

Theorem 2.4. *Let $u_0(x) \in \mathcal{K}_2$. Then the Cauchy problem* (2.4), (1.2) *has a minimal generalized solution $u(x,t)$ in S. The generalized solution is unique in the class $\mathcal{K}_2$.*

(c) Consider the case $(m+1)/2 < n < m$. Equation (2.4) has a family of stationary solutions $u_s(x)$ satisfying the ordinary differential equation

$$a(u_s^m)'' + b(u_s^n)' - cu_s^p = 0. \tag{2.7}$$

By $o(h(s))$ we shall denote the functions with the following property

$$\lim_{s\to+\infty} \frac{o(h(s))}{h(s)} = 0.$$

It is known (see [9]) that for $u_s(x)$ satisfying the conditions

$$u_s(x_0) = M_3, \quad u_s'(x_0) = 0, \quad x_0 \in \mathbb{R}, \tag{2.8}$$

the following equalities are true

$$u_s(x) = c_+ x^{1/(n-p)} + o(x^{1/(n-p)}) \quad \text{for } x \ge 0, \tag{2.9}$$

$$u_s(x) = c_-|x|^{1/(m-n)} + o(|x|^{1/(m-n)}) \quad \text{for } x < 0, \tag{2.10}$$

where

$$c_+ = (\frac{c(n-p)}{bn})^{1/(n-p)}, \quad c_- = (\frac{b(m-n)}{am})^{1/(m-n)}. \tag{2.11}$$

Note that asymptotic formulas (2.9) and (2.10) are valid for any $(m+p)/2 < n < m$. Denote

$$c_-^* = (\frac{bn(m-n)}{am(2m-n)})^{1/(m-n)}. \tag{2.12}$$

Obviously, $c_-^* < c_-$ for $n < m$. Define the class $\mathcal{K}_3$ of nonnegative functions $\varphi(x,t)$ and $\varphi(x)$ satisfying in arbitrary strip $S_T = \mathbb{R} \times [0,T]$ and $\mathbb{R}$ respectively the following inequalities

$$\varphi \le M_4(\gamma_4 + x)^k \quad \text{for } x \ge 0, \quad 0 \le k < 1/(n-1), \tag{2.13}$$

$$\varphi \le M_5(\gamma_5 + |x|)^{1/(m-n)} \quad \text{for } x < 0, \quad M_5 < c_-^*. \tag{2.14}$$

Constants k, M_4, M_5, γ_4, γ_5 in (2.13) and (2.14) can depend on T and function φ.

Theorem 2.5. *Assume that $u_0(x) \le u_s(x)$ for some function $u_s(x)$ satisfying* (2.7)*. Then the Cauchy problem* (2.4), (1.2) *has a minimal generalized solution $u(x,t)$ in S. If additionally $u_0(x) \in \mathcal{K}_3$ then $u(x,t) \in \mathcal{K}_3$ and the generalized solution is unique in the class $\mathcal{K}_3$.*

It isn't known if the constant c_-^* is optimal with respect to the uniqueness. One can find in [5] the example of nonuniqueness of generalized solution of the Cauchy problem (2.4), (1.2) with $c = 0$ in the class of functions satisfying (2.13) and (2.14) with $M_5 = c_-$. But there isn't example of non-uniqueness for $c_-^* \le M_5 < c_-$.

In [6], [10] for the similar Cauchy problem for the equation (2.4) with $b = 0$ and $1 < p < m$ the optimality of the uniqueness result is not established too.
(d) Suppose that $n = m$. Then for the solutions of the problem (2.7), (2.8) asymptotic formulas (2.9) and

$$u_s(x) = M_6 \exp(-\frac{b}{am}x) + o(\exp(-\frac{b}{am}x)) \quad \text{for } x < 0 \tag{2.15}$$

are true (see [9]) with some positive constant M_6 depending on x_0, M_3 and parameters of the equation (2.7). Also one can verify that the functions

$$w_s(x) = [M_7 \exp(-\frac{b}{a}x) + \gamma_6]^{1/m}$$

with arbitrary M_7 and γ_6 are stationary classical supersolutions of the equation (2.4).

Define the class $\mathcal{K}_4$ of nonnegative functions $\varphi(x,t)$ and $\varphi(x)$ satisfying in arbitrary strip $S_T = \mathbb{R} \times [0,T]$ and $\mathbb{R}$ respectively the inequalities (2.13) and

$$\varphi \le \delta(x) \exp(-\frac{b}{am}x) \quad \text{for } x < 0, \tag{2.16}$$

where $\delta(x) \ge 0$ and $\lim_{x\to-\infty} \delta(x) = 0$. Constants M_4, γ_4, k in (2.13) and function $\delta(x)$ in (2.16) can depend on T and function φ.

Theorem 2.6. *Assume that $u_0(x)$ satisfies for $x < 0$ the inequality $u_0(x) \le w_s(x)$ and (2.13) holds with $\varphi = u_0(x)$. Then the Cauchy problem (2.4), (1.2) has a minimal generalized solution $u(x,t)$ in S. The generalized solution is unique in the class $\mathcal{K}_4$.*

One can find in [5] for the equation (2.4) with $c = 0$ the example which indicates the impossibility of replacement in (2.16) $\delta(x)$ to any positive constant without loss of uniqueness for the Cauchy problem.
(e) And finally we consider the case $n > m$. Define the class $\mathcal{K}_5$ of nonnegative functions $\varphi(x,t)$ and $\varphi(x)$ satisfying in arbitrary strip $S_T = \mathbb{R} \times [0,T]$ and $\mathbb{R}$ respectively the inequality (2.13) where the constants M_4, γ_4 and k can depend on T and function φ.

The proof of the following theorem is very similar to the arguments from [8, Theorems 1 and 3].

Theorem 2.7. *Let $u_0(x) \in \mathcal{K}_5$. Then the Cauchy problem (2.4), (1.2) has a minimal generalized solution $u(x,t) \in \mathcal{K}_5$ in S. The generalized solution is unique in the class $\mathcal{K}_5$.*

Note that no assumption has to be made on the behavior of $u_0(x)$ as $x \to -\infty$ in Theorem 2.7.

Remark 2.8. If $u_0(x)$ satisfies (2.3) for $n < (m+1)/2$ or (2.5) for $n = (m+1)/2$ with $k = 2/(m-1)$ or (2.13) for $n > (m+1)/2$ with $k = 1/(n-1)$ then a minimal generalized solution of the Cauchy problem (1.1), (1.2) for $n < (m+1)/2$ and (2.4), (1.2) for $n \ge (m+1)/2$ may blow up in a finite time (see, for example, [5] for the equation (2.4) with $c = 0$).

3. The case $n < (m+p)/2$

In this section we prove the vanishing of generalized solutions of the Cauchy problem (1.1), (1.2) with initial data having definite growth at infinity. In the end of the section we show certain optimality of obtained results. Put $c_N = \{\frac{c(m-p)^2}{2m(2p+N(m-p))}\}^{1/(m-p)}$.

Theorem 3.1. *Assume that $n < (m+p)/2$ and $u_0(x)$ satisfies the inequality*

$$u_0(x) \leq Ar^{2/(m-p)} + o(r^{2/(m-p)}), \tag{3.1}$$

where $r = \operatorname{dist}(x)$ is defined in (2.2) and $0 \leq A < c_N$. Then the generalized solution of the Cauchy problem (1.1), (1.2) *from the class $\mathcal{K}_1$ vanishes at every point $y \in \mathbb{R}^N$ in a finite time $T_0(y)$.*

Proof. Fix arbitrary $y \in \mathbb{R}^N$ and construct a generalized supersolution $W(x,t)$ of the equation (1.1) in S satisfying the conditions

$$u_0(x) \leq W(x,0) \tag{3.2}$$

and $W(y,t) = 0$ for $t \geq T_0$ where T_0 is finite and depends on y. We shall seek a function $W(x,t)$ in the form $W(x,t) = w(r,t)$. Using (2.2), Lemma 2.2 and the inequality $|\sum_{j=1}^N c_{ij}x_j| \leq r$ $(i = 1,\dots,N)$ we conclude that $W(x,t)$ is a generalized supersolution of the equation (1.1) if for $w(r,t)$ outside of finitely many curves of the form $r = \zeta(t)$ the following inequality holds

$$-w_t + (w^m)_{rr} + \frac{N-1}{r}(w^m)_r + d|(w^n)_r| - cw^p \leq 0, \quad r \geq 0, \tag{3.3}$$

with $d = \sum_{i,j=1}^N |c_{ij}b_i|$ and condition $(w^m)_r(0,t) = 0$ takes place. At the points where (3.3) is not valid we suppose that the derivative $(w^m)_r$ is continuous. Choose ε in the following way

$$0 < \varepsilon < 1 - (A/c_N)^m. \tag{3.4}$$

Set

$$w(r,t) = \{\varepsilon g^m(t) + (1-\varepsilon)z^m(r)\}^{1/m}, \tag{3.5}$$

where

$$g(t) = [K - \varepsilon^{(m-1)/m}c(1-p)t]_+^{1/(1-p)}, \tag{3.6}$$

and a positive constant K and nonnegative nondecreasing function $z(r)$ will be defined below. In (3.6) the notation $s_+ = \max\{s, 0\}$ is used. Using the convexity of the function $h(s) = s^{p/m}$ we obtain

$$w^p \geq \varepsilon g^p + (1-\varepsilon)z^p. \tag{3.7}$$

It is not difficult to show the validity of the following relations:

$$-w_t \leq c\varepsilon g^p, \tag{3.8}$$

$$|(w^n)_r| = (1-\varepsilon)(z^n)'D(r,t), \tag{3.9}$$

where

$$D(r,t) = \frac{z^{m-n}(r)}{\{\varepsilon g^m(t) + (1-\varepsilon)z^m(r)\}^{(m-n)/m}}. \tag{3.10}$$

Since $z(r)$ and $g(t)$ are nonnegative functions we have $0 \leq D(r,t) \leq (1-\varepsilon)^{-(m-n)/m}$ for $r \geq 0$ and $t \geq 0$. Therefore,

$$d|(w^n)_r| \leq B(1-\varepsilon)(z^n)', \tag{3.11}$$

where $B = d(1-\varepsilon)^{-(m-n)/m}$. Thus from (3.5)–(3.11) it follows that (3.3) holds when

$$\mathcal{L}_2(z) \equiv (z^m)'' + \frac{N-1}{r}(z^m)' + B(z^n)' - cz^p \le 0. \tag{3.12}$$

Moreover, we suppose that $z'(0) = 0$ and at the points where (3.12) is not true the derivative $(z^m)'$ is continuous. Let us verify that the following function

$$z(r) = c_N (r^l - M_8)_+^{2/[l(m-p)]} \tag{3.13}$$

with

$$0 < l \le \min\Big\{\frac{2(n-p)}{m-p}, \frac{m+p-2n}{m-p}\Big\} \tag{3.14}$$

and sufficiently large M_8 satisfies the above conditions. Remark that sum of the positive numbers in right hand side of (3.14) is equal to one. Hence we have $l \le 1/2$. Obviously, $\mathcal{L}_2(z) = 0$ for $r < M_8^{1/l}$. For $r > M_8^{1/l}$ elementary calculations give us

$$\begin{aligned} \mathcal{L}_2(z) = cc_N^p (r^l - M_8)^{2p/[l(m-p)]} \Big\{ & \frac{2m - l(m-p)}{2p + N(m-p)} (1 - M_8 r^{-l})^{-2+2/l} \\ & + \frac{(m-p)(l+N-2)}{2p+N(m-p)} (1 - M_8 r^{-l})^{-1+2/l} \\ & + \frac{2Bnc_N^{n-p}}{c(m-p)} (r^l - M_8)^{-1+2(n-p)/[l(m-p)]} r^{l-1} - 1 \Big\}. \end{aligned} \tag{3.15}$$

For $N \ge 2$ we apply the inequality

$$s^\alpha < s \quad \text{for } 0 < s < 1 \text{ and } \alpha > 1 \tag{3.16}$$

to the first and second terms in the braces of (3.15) and conclude that $\mathcal{L}_2(z) \le 0$ if

$$-\delta M_8 + \frac{2Bnc_N^{n-p}}{c(m-p)} (r^l - M_8)^{-1+2(n-p)/[l(m-p)]} r^{2l-1} \le 0, \tag{3.17}$$

where $\delta = 1$. For $N = 1$ transforming the second term in the braces of (3.15) and using (3.16) we get that $\mathcal{L}_2(z) \le 0$ if (3.17) holds with $\delta = [2p + l(m-p)]/(m+p) > 0$. Choosing sufficiently large M_8 and using (3.14) we obtain that (3.17) is correct for $r > M_8^{1/l}$. Moreover, $(z^m)'$ is continuous at the point $r = M_8^{1/l}$ and $z'(0) = 0$. Thus due to Lemma 2.2 constructed function $W(x,t)$ is the generalized supersolution of the equation (1.1).

Now let us choose K from (3.6) to satisfy the inequality (3.2). By virtue of (3.1), (3.4), (3.5) and (3.13) there exists the maximal root R of the equation

$$(1-\varepsilon)^{1/m} z(s) = \max_{\operatorname{dist}(x) \le s} u_0(x). \tag{3.18}$$

From (3.1), (3.5) and (3.18) we conclude that (3.2) is correct for $\operatorname{dist}(x) \ge R$. To satisfy (3.2) for $\operatorname{dist}(x) < R$ it is sufficient that $\varepsilon^{1/m} g(0) = \max_{\operatorname{dist}(x) \le R} u_0(x)$. Hence we can set

$$K = [\varepsilon^{-1/m} \max_{\operatorname{dist}(x) \le R} u_0(x)]^{1-p}.$$

Now Theorems 2.1 and 2.3 applied to the generalized solution $u(x,t)$ from the class $\mathcal{K}_1$ and to the generalized supersolution $W(x,t)$ give us the estimate

$$u(x,t) \le W(x,t) \quad \text{in } S. \tag{3.19}$$

Setting $M_8 \geq [\mathrm{dist}(y)]^l$ we have

$$z(\mathrm{dist}(y)) = 0. \tag{3.20}$$

As a consequence of (3.5), (3.6) and (3.20) we get $W(y,t) = 0$ for $t \geq T_0 = K/[(1-p)c\varepsilon^{(m-1)/m}]$. This completes the proof. $\square$

Remark 3.2. Let us show that Theorem 3.1 is optimal in a certain sense. Indeed, the equation (1.1) with $a_{ij} = 1$ for $i = j$, $a_{ij} = 0$ for $i \neq j$ and $b_i = 0$ $(i,j = 1,\dots,N)$ has explicit stationary solution from the class $\mathcal{K}_1$

$$u_s(r) = c_N r^{2/(m-p)}.$$

It satisfies (3.1) with $A = c_N$ and does not vanish at every point of $\mathbb{R}^N$.

4. The case $(m+p)/2 < n < m$

For the rest of the paper we shall consider one-dimensional equation (2.4). Let $w_i(\xi)$ $(i = 1,2)$ be nonnegative functions satisfying for $\xi \geq 0$ the differential inequalities

$$\mathcal{T}_i(w_i) \equiv w_i' + a(w_i^m)'' + d_i(w_i^n)' - cw_i^p \leq 0 \tag{4.1}$$

with constants $d_1 = b$ and $d_2 = -b$ and the conditions

$$w_i(0) = M_9, \quad w_i'(0) = 0. \tag{4.2}$$

For each of functions w_i the inequality (4.1) can be not fulfilled at finitely many points $\xi_{j_i}^{(i)}$ $(j_i = 1,\dots,k_i;\ i = 1,2)$ where the derivative $(w_i^m)'$ is continuous. Note that Cauchy problems for the equations $\mathcal{T}_i(w_i) = 0$ with initial conditions (4.2) have unique solutions with the following asymptotic behavior

$$w_1(\xi) = c_+\xi^{1/(n-p)} + o(\xi^{1/(n-p)}), \quad w_2(\xi) = c_-\xi^{1/(m-n)} + o(\xi^{1/(m-n)}), \tag{4.3}$$

where the constants c_+ and c_- were defined in (2.11) (see [9] for similar problems).

For $\overline{x} \in \mathbb{R}$ define the auxiliary function

$$w_{\overline{x}}(x) = \begin{cases} w_1(x-\overline{x}) & \text{for } x \geq \overline{x}, \\ w_2(\overline{x}-x) & \text{for } x < \overline{x}. \end{cases} \tag{4.4}$$

Now we can formulate the main result of this section.

Theorem 4.1. *Assume that $(m+p)/2 < n < m$ and $u_0(x)$ satisfies the inequality*

$$u_0(x) \leq w_{\overline{x}}(x) \tag{4.5}$$

for some $\overline{x} \in \mathbb{R}$ and some function $w_{\overline{x}}(x)$ constructed in (4.4). Suppose also that $u_0(x) \in \mathcal{K}_i$ $(i = 1,2,3)$ for $n < (m+1)/2$, $n = (m+1)/2$ and $n > (m+1)/2$ respectively. Then the generalized solutions of the Cauchy problem (2.4), (1.2) from the classes $\mathcal{K}_i$ $(i = 1,2,3)$ for $n < (m+1)/2$, $n = (m+1)/2$ and $n > (m+1)/2$ respectively vanish at every point $y \in \mathbb{R}$ in a finite time $T_0(y)$.

Proof. We prove the theorem in two steps. At first we show that the generalized solution of the Cauchy problem (2.4), (1.2) is bounded in $\Omega \times [0,+\infty)$ where Ω is any bounded domain in $\mathbb{R}$. Then we establish the vanishing of the generalized solution at every point of $\mathbb{R}$ in a finite time.

We start with construction in S a traveling-wave generalized supersolution of the equation (2.4). Put

$$W(x,t)=\begin{cases} w_1(x-\overline{x}-t) & \text{for } x\geq \overline{x}+t,\\ M_9 & \text{for } \overline{x}-t<x<\overline{x}+t,\\ w_2(\overline{x}-x-t) & \text{for } x\leq \overline{x}-t. \end{cases} \tag{4.6}$$

In view of (4.1) we have $\mathcal{L}_1(W)\leq 0$ everywhere except the lines $x\pm t=\overline{x}$, $x-t=\overline{x}+\xi^{(1)}_{j_1}$ and $x+t=\overline{x}-\xi^{(2)}_{j_2}$ ($j_i=1,\dots,k_i;\ i=1,2$) where the derivative $(W^m)_x$ is continuous. Moreover, from (4.5) we obtain that (3.2) holds since $W(x,0)=w_{\overline{x}}(x)$. Applying Theorem 2.1 and one of Theorems 2.3–2.5 for $n<(m+1)/2$, $n=(m+1)/2$ and $n>(m+1)/2$ respectively, we obtain the estimate (3.19). From (3.19) and (4.6) we conclude that for all $y\in\mathbb{R}$

$$u(y,t)\leq M_9 \quad \text{for } t\geq |y-\overline{x}|. \tag{4.7}$$

Let us consider the function

$$w(x,t)=\{\varepsilon g^m(t-t_0)+(1-\varepsilon)z^m(\sigma)\}^{1/m},\quad \sigma=|x-y|, \tag{4.8}$$

where $g(t)$ was defined in (3.6), $t_0>0$ and nonnegative nondecreasing function $z(\sigma)$ will be defined below, ε is arbitrary number from the interval $(0,1)$. It is easy to see that relations (3.7)–(3.11) with $d=b$ remain true after replacement r to σ. Thus to satisfy the inequality $\mathcal{L}_1(w)\leq 0$ it is sufficient to require that

$$a(z^m)''+B(z^n)'-cz^p\leq 0. \tag{4.9}$$

We define now $z(\sigma)$ as follows

$$z(\sigma)=M_{10}(\sigma^l-1)_+^{1/[l(n-p)]}, \tag{4.10}$$

where $l<(m-p)/[2(n-p)]$ and M_{10} is small enough. Then the function $z(\sigma)$ satisfies (4.9) and the condition $z'(0)=0$. Obviously, the equation $z(\sigma)=M_9(1-\varepsilon)^{-1/m}$ has a unique root

$$\sigma_0=[(M_9(1-\varepsilon)^{-1/m}/M_{10})^{l(n-p)}+1]^{1/l}. \tag{4.11}$$

Fix arbitrary $y\in\mathbb{R}$. Choose in (4.8) t_0 in the following way:

$$t_0=|y-\overline{x}|+\sigma_0. \tag{4.12}$$

The relations (4.8), (4.10) – (4.12) yield

$$w(y\pm\sigma_0,t)\geq M_9,\quad t\geq t_0. \tag{4.13}$$

Setting in (3.6) $K=(\varepsilon^{-1/m}M_9)^{1-p}$ we obtain

$$w(x,t_0)\geq M_9,\quad y-\sigma_0\leq x\leq y+\sigma_0. \tag{4.14}$$

From (4.7), (4.13), (4.14) we conclude that on the parabolic boundary of the domain $Q=[y-\sigma_0,y+\sigma_0]\times[t_0,+\infty)$ the inequality

$$u(x,t)\leq w(x,t) \tag{4.15}$$

holds. Moreover, $\mathcal{L}_1(w)\leq 0$ in Q. Applying the comparison theorem (see, for example, [11]) we obtain the estimate (4.15) in Q. But by virtue of (4.8), (4.10) and (3.6) $w(y,t)=0$ for $t\geq T_0=t_0+K/[(1-p)c\varepsilon^{(m-1)/m}]$. Theorem is proved. $\square$

Remark 4.2. Without assumptions $u(x,t) \in \mathcal{K}_i$ $(i = 1,2,3)$ in Theorem 4.1 we can conclude only the vanishing of the *minimal* generalized solution of the Cauchy problem (2.4), (1.2) since we know nothing about its uniqueness (see item (c) of Section 2).

Remark 4.3. Let us show a certain optimality of Theorem 4.1. Passing to the integral equation and using Schauder-Tychonoff theorem one can show that there exists a stationary solution $u_1(x)$ of the equation (2.4) such that $u_1(x) = 0$ for $x \le 0$ and $u_1(x) > 0$ for $x > 0$. Arguing as in [9] one can obtain asymptotic formula (2.9) for this solution. Therefore, $u_1(x)$ belongs to the class $\mathcal{K}_1$ for $n < (m+1)/2$, to the class $\mathcal{K}_2$ for $n = (m+1)/2$ and to the class $\mathcal{K}_3$ for $n > (m+1)/2$ but it doesn't vanish at every point of $\mathbb{R}$ in a finite time. Note that $u_1(x)$ has the same first term of asymptotic behavior as $x \to +\infty$ as $w_{\overline{x}}(x)$.

In a similar way we can construct a stationary solution $u_2(x)$ of the equation (2.4) such that $u_2(x) = 0$ for $x \ge 0$ and $u_2(x) > 0$ for $x < 0$. This solution has the same first term of asymptotic behavior as $x \to -\infty$ as $w_{\overline{x}}(x)$ and belongs to the class $\mathcal{K}_1$ for $n < (m+1)/2$ and to the class $\mathcal{K}_2$ for $n = (m+1)/2$. Note that for $n > (m+1)/2$ the function $w_{\overline{x}}(x)$ grows as $x \to -\infty$ faster than any function from the class $\mathcal{K}_3$.

5. The case $n = m$

The main result of this section is the following.

Theorem 5.1. *Let $n = m$ and initial data satisfy the inequalities* (2.16) *with $\varphi = u_0(x)$ and*

$$u_0(x) \le A_+ x^{1/(n-p)} + o(x^{1/(n-p)}) \quad \textit{for } x \ge 0, \tag{5.1}$$

where $0 \le A_+ < c_+$ and the constant c_+ was defined in (2.11). *Then the generalized solution of the Cauchy problem* (2.4), (1.2) *from the class $\mathcal{K}_4$ vanishes at every point $y \in \mathbb{R}$ in a finite time $T_0(y)$.*

Proof. Fix an arbitrary $y \in \mathbb{R}$. We construct a supersolution $w(x,t)$ of the equation (2.4) in the form (3.5), (3.6), where $r = x$,

$$0 < \varepsilon < 1 - (A_+/c_+)^m \tag{5.2}$$

and function $z(x)$ will be determined below. Since the inequalities (3.7) and (3.8) remain true and $(w^n)_x = (1-\varepsilon)(z^m)'$ we conclude that $\mathcal{L}_1(w) \le 0$ when

$$a(z^m)'' + b(z^m)' - cz^p \le 0. \tag{5.3}$$

At the points where (5.3) is not valid we suppose that the derivative $(z^m)'$ is continuous. For $x \ge y$ these conditions are fulfilled for the following function

$$z(x) = c_+[(x-y)^l - M_{11}]_+^{1/[l(m-p)]}, \tag{5.4}$$

where $l < 1/2$ and M_{11} is large enough. Clearly,

$$z(y) = 0. \tag{5.5}$$

For $x \in [y-\delta, y)$ $(\delta > 0)$ put $z(x) = (y-x)^\gamma$, $\gamma > 2/(m-p)$, and note that (5.3) holds here if δ is small enough. Further, for $x < y - \delta$ set

$$z(x) = \left(M_{12} \exp(-\frac{b}{a}x) + \beta\right)^{1/m}, \tag{5.6}$$

where the constants $M_{12} > 0$ and β are determined from the corresponding continuity conditions of $z(x)$ and $(z^m)'(x)$ at the point $x = y - \delta$. It is not difficult to check that the function (5.6) satisfies (5.3). Applying Lemma 2.2 we have that $w(x,t)$ is the generalized supersolution of the equation (2.4) in S.

Taking into account hypotheses of Theorem 5.1, (5.2), (5.4) and (5.6) we conclude that there exists the minimal R_- and the maximal R_+ roots of the equation

$$(1-\varepsilon)^{1/m} z(x) = u_0(x). \tag{5.7}$$

Thus from (3.5) and (5.7) we have

$$w(x,0) \geq u_0(x) \tag{5.8}$$

for $x \geq R_+$ and $x \leq R_-$. To satisfy (5.8) for $R_- < x < R_+$ we put in (3.6)

$$K = [\varepsilon^{-1/m} \max_{R_- \leq x \leq R_+} u_0(x)]^{1-p}.$$

Applying Theorems 2.1 and 2.6, (3.5), (3.6) and (5.5) we complete the proof. □

Remark 5.2. Slightly modifying the proof of Theorem 5.1 we can construct a generalized supersolution of the Cauchy problem (2.4), (1.2) vanishing at every point of $\mathbb{R}$ in a finite time as well for initial data satisfying (5.1) and the inequality

$$u_0(x) \leq M_{13} \exp(-\frac{b}{am}x) + o(\exp(-\frac{b}{am}x)) \quad \text{for } x < 0 \tag{5.9}$$

with arbitrary positive constant M_{13}. Now we can conclude only that the *minimal* generalized solution of the Cauchy problem (2.4), (1.2) vanishes at every point of $\mathbb{R}$ in a finite time since this solution may be non-unique (see item (d) of Section 2). Note that solutions $u_s(x)$ of the problem (2.7), (2.8) have asymptotic representation for $x < 0$ as right hand side of (5.9) where M_{13} depends on M_3.

Remark 5.3. From (2.9) it follows that for the stationary solution $u_1(x)$ of the equation (2.4) constructed as in Remark 4.3 the inequality (5.1) with $A_+ = c_+$ holds. This fact demonstrates a certain optimality of Theorem 5.1. Note that stationary solution $u_2(x)$ of the equation (2.4) constructed as in Remark 4.3 grows as $x \to -\infty$ faster than any function from the class $\mathcal{K}_4$.

6. The case $n > m$

We show here that some generalized solutions of the Cauchy problem (2.4), (1.2) with any growing as $x \to -\infty$ initial function vanish at every point of $\mathbb{R}$ in a finite time.

Theorem 6.1. *Assume that $n > m$ and $u_0(x)$ satisfies the inequality* (5.1)*. Then the generalized solution of the Cauchy problem* (2.4), (1.2) *from the class $\mathcal{K}_5$ vanishes at every point $y \in \mathbb{R}$ in a finite time $T_0(y)$.*

Proof. Let $W(x,t)$ be a travelling-wave generalized supersolution of the equation (2.4) of the form (4.6) where $\overline{x} = 0$ and functions $w_i(\xi)$ $(i = 1,2)$ satisfy (4.1) and (4.2). It isn't difficult to check that the function

$$w_1(\xi) = B_+(M_9 + \xi^2)^{1/[2(n-p)]}, \quad \xi \geq 0,$$

where $A_+ < B_+ < c_+$, satisfy the above requirements and (3.2) holds for $x \geq 0$ if M_9 is large enough.

Let us construct $w_2(\xi)$. Suppose that

$$M_9 \geq (\frac{2}{bn})^{1/(n-1)}. \tag{6.1}$$

At first we prove the following auxiliary result. □

Lemma 6.2. *The solution $g(\xi)$ of the equation*

$$\mathcal{T}_2(g) = 0 \tag{6.2}$$

satisfying the conditions (4.2) *and* (6.1) *exists only in a finite half-interval* $[0, \xi_0)$ *and* $\xi_0 \to 0$ *as* $M_9 \to +\infty$. *Moreover,*

$$\lim_{\xi \to \xi_0 - 0} g'(\xi)[g(\xi)]^{-(n-m+1)} = \frac{b}{am}. \tag{6.3}$$

Proof. It is easy to see that

$$g(\xi) > M_9 \quad \text{and} \quad g'(\xi) > 0 \quad \text{for } \xi > 0. \tag{6.4}$$

Using (6.1) and (6.4) we have

$$(g^m)'' \geq \frac{b}{2a}(g^n)' + \frac{c}{a}M_9^p. \tag{6.5}$$

Integrating (6.5) over $(0, \xi)$ and taking into account (4.2) we get

$$(g^m)' \geq \frac{b}{2a}g^n - \frac{b}{2a}M_9^n + \frac{c}{a}M_9^p\xi. \tag{6.6}$$

Putting $v = g^m - M_9^m$ and using the inequality

$$r^\alpha - s^\alpha > (r-s)^\alpha, \quad 0 < s < r, \quad \alpha > 1,$$

with $\alpha = n/m$ we obtain for $\xi > 0$

$$v' \geq \frac{b}{2a}v^{n/m}. \tag{6.7}$$

As a consequence of (6.4) and (6.6) we have

$$v \geq \frac{c}{2a}M_9^p\xi^2. \tag{6.8}$$

Fixing arbitrary $\varepsilon_1 > 0$, integrating (6.7) over (ε_1, ξ) and using (6.8) we deduce the inequality

$$v(\xi) > \Big[(\frac{c}{2a}M_9^p\varepsilon_1^2)^{-(n-m)/m} - \frac{b(n-m)}{2am}(\xi - \varepsilon_1)\Big]^{-m/(n-m)}. \tag{6.9}$$

The first part of Lemma 6.2 follows from (6.9) by virtue of arbitrariness of ε_1. Integrating (6.2) over $(0, \xi)$, $\xi < \xi_0$, it is easy to obtain (6.3). Lemma is proved. □

Pass in (4.1) to new unknown function $f(\xi) = [w_2(\xi)]^{-(n-m)}$. If we multiply obtained inequality for $f(\xi)$ by $(n-m)f^{(2n-m)/(n-m)}/m$ the relations (4.1), (4.2) can be written in the form

$$\begin{aligned} \mathcal{L}_3(f) \equiv -aff'' + \frac{an}{n-m}(f')^2 + \frac{bn}{m}f' - \frac{1}{m}f^{(n-1)/(n-m)}f' \\ - \frac{c(n-m)}{m}f^{(2n-m-p)/(n-m)} \leq 0, \end{aligned} \tag{6.10}$$

$$f(0) = M_9^{-(n-m)}, \quad f'(0) = 0. \tag{6.11}$$

Using Lemma 6.2 and (6.4) it is not difficult to verify that solutions of the Cauchy problem for the equation

$$\mathcal{L}_3(f) = 0 \tag{6.12}$$

with initial conditions (6.11) have the following properties:

$$\begin{gathered} f(\xi) > 0, \quad -b(n-m)/(am) < f'(\xi) < 0, \quad f''(\xi) < 0 \quad \text{for } 0 < \xi < \xi_0, \\ \lim_{\xi \to \xi_0 - 0} f(\xi) = 0, \quad \lim_{\xi \to \xi_0 - 0} f'(\xi) = -b(n-m)/(am), \\ \xi_0 \to 0 \text{ as } M_9^{-(n-m)} \to 0. \end{gathered} \tag{6.13}$$

Put $\overline{u}_0(\xi) = [u_0(-\xi) + 1]^{-(n-m)}$, $\xi \geq 0$. Obviously, for $x \leq 0$ (3.2) follows from the inequality

$$f(\xi) \leq \overline{u}_0(\xi). \tag{6.14}$$

Let a constant M_{14} be so large that $\xi_0 < 1/2$ when $M_9 \geq M_{14}$. Choose M_9 as follows

$$M_9 \geq \Big[\min\{M_{14}^{-(n-m)}, \frac{b(n-m)}{4am}, \overline{u}_0(2)\} \Big]^{-1/(n-m)}. \tag{6.15}$$

In the interval $[0, \xi_*]$ we put the function $f(\xi)$ equal to the solution of the Cauchy problem (6.12), (6.11) where the point $\xi_* < \xi_0$ is defined in such a way that the function $q(\xi)$, linear for $\xi \leq 1$, which passes through the points $(\xi_*, f(\xi_*))$ and $(1, \min\{f(0)/3, \overline{u}_0(3)\})$, satisfies the equality $q'(\xi_*) = f'(\xi_*)$. We define the sequence $\{q(k)\}$ by the recurrence relation

$$q(k) = \min\{q(k-1)/3, \overline{u}_0(k+2)\}, \quad k = 2, 3, \dots. \tag{6.16}$$

Let function $q(\xi)$ be piecewise-linear for $\xi \geq 1$ and have a graphical representation obtained by joining points $(k, q(k))$, $k = 1, 2, \dots$. Denote the mollification of $q(\xi)$ by $q_h(\xi)$. Assume that $h < 1/2$. Using the definition of $q(\xi)$ and the properties of mollifiers we have

$$\begin{gathered} q_h(\xi) \in C^\infty(\xi_*, \infty), \quad q_h(\xi) = q(\xi) \text{ for } \xi_* < \xi < 1-h, \\ k-1+h < \xi < k-h, \quad k = 2, 3, \dots, \\ q_h''(\xi) \geq 0, \quad q_h'(\xi) \leq 0, \quad q(\xi) \leq q_h(\xi) \leq \overline{u}_0(\xi) \quad \text{for } \xi > \xi_*. \end{gathered} \tag{6.17}$$

For $\xi \geq \xi_*$ we put $f(\xi) = q_h(\xi)$. Due to (6.1), (6.15) – (6.17) inequality (6.10) is valid for $\xi \neq \xi_*$. Nevertheless $f'(\xi)$ is continuous at the point $\xi = \xi_*$. The function $w_2(\xi)$ is constructed.

The rest of the proof completely repeats the same arguments as in the proof of Theorem 4.1 except for the inequality (4.9). Instead of it we require that

$$a(z^m)'' + \frac{bn}{m}\{\varepsilon K^{m/(1-p)} + (1-\varepsilon) z^m(r)\}^{(n-m)/m}(z^m)' - cz^p \leq 0, \tag{6.18}$$

but the same function $z(\sigma)$, which is defined by (4.10), satisfies (6.18).

Remark 6.3. Let us show a certain optimality of Theorem 6.1. Let $u_1(x)$ be the stationary solution of the equation (2.4) constructed as in Remark 4.3. This solution belongs to the class $\mathcal{K}_5$ and satisfies (5.1) with $A_+ = c_+$ but it doesn't vanish at every point of $\mathbb{R}$ in a finite time.

References

[1] M. A. Herrero and J. J. L. Velazquez, On the dynamics of a semilinear heat equation with strong absorption, *Comm. Part. Diff. Equat.,* **14** (1989), no. 12, 1653-1715.

[2] A. L. Gladkov, Behavior of solutions of certain quasilinear parabolic equations with power-type nonlinearities, *Matematicheskii sbornik,* **191** (2000), no. 3, 25-42 (in Russian). English translation in *Sbornik: Mathematics,* **191** (2000), no. 3, 341-358.

[3] O. V. Khramtsov, Relative stabilization of a nonlinear degenerating parabolic equation, *Differentsial'nye uravneniya,* **37** (2001), no. 12, 1650-1654 (in Russian). English translation in *Differential equations,* **37** (2001), no. 12, 1736-1741.

[4] A. S. Kalashnikov, The effect of absorption on heat propagation in a medium with thermal conductivity depending on temperature, *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki,* **16** (1976), no. 3, 689-696 (in Russian).

[5] A. L. Gladkov, On the Cauchy problem in growing functions classes for a diffusion equation with convection, *Matematicheskii sbornik,* **186** (1995), no. 6, 35-56 (in Russian). English translation in *Sbornik: Mathematics,* **186** (1995), no. 6, 803-825.

[6] S. Kamin, L. A. Peletier and J. L. Vazquez, A nonlinear diffusion-absorption equation with unbounded initial data, *Nonlinear Diffusion Equations and Their Equilibrium States,* **3** (1992), 243-263.

[7] S. A. Prokhozhy, On the unique solvability of the Cauchy problem for a diffusion equation with convection and absorption, *Bulletin of Vitebsk State University,* **32** (2004), 84-91 (in Russian).

[8] A. L. Gladkov, Unbounded solutions of the nonlinear heat-conduction equation with strong convection at infinity, *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki,* **36** (1996), no. 10, 73-86 (in Russian). English translation in *Comp. Maths Math. Phys.,* **36** (1996), no. 10, 1381-1391.

[9] S. A. Prokhozhy, On the behavior at infinity of solutions of some nonlinear ODE, *Bulletin of Vitebsk State University,* **14** (1999), 83-90 (in Russian).

[10] J. B. McLeod , L. A. Peletier and J. L. Vazquez, Solutions of a nonlinear ODE appearing in the theory of diffusion with absorption, *Differential Integral Equations,* **4** (1991), 1-14.

[11] J. I. Diaz and R. Kersner, On a nonlinear degenerate parabolic equation in infiltration or evaporation through a porous medium, *J. Diff. Equat.,* **69** (1987), 368-403.

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