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DIFFUSIVE PREDATOR-PREY MODELS WITH FEAR EFFECT IN SPATIALLY HETEROGENEOUS ENVIRONMENT

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ABSTRACT. This article concerns diffusive predator-prey models incorporating the cost of fear and environmental heterogeneity. Under homogeneous Neumann boundary conditions, we establish the uniform boundedness of global solutions and global stability of the trivial and semi-trivial solutions for the parabolic system. For the corresponding steady-state problem, we obtain sufficient conditions for the existence of positive steady states, and then study the effects of functional responses and the cost of fear on the existence, stability and number of positive steady states. We also discuss the effects of spatial heterogeneity and spatial diffusion on the dynamic behavior and establish asymptotic profiles of positive steady states as the diffusion rate of prey or predator individuals approaches zero or infinity. Our theoretical results suggest that fear plays a very important role in determining the dynamic behavior of the models, and it is necessary to revisit existing predator-prey models by incorporating the cost of fear.

1. INTRODUCTION

In 2011, Zanette et al. [30] conducted a manipulation on female song sparrows (Melospiza melodia) during an entire breeding season. This is the first experimental evidence demonstrating that the perceived predation risk can affect the populations of terrestrial vertebrates although many biologists early realized that the cost of fear should be considered except for direct killing in the predator-prey interactions (for example, see [5, 16, 19] and references therein). Since then, fear effect in the predator-prey interactions has attracted considerable attention (for example, see [7, 18, 21, 24, 25, 26, 27] and references therein).

In 2016, Wang, Zanette and Zou [25] initially proposed and analyzed the following ODE model:

$$\begin{aligned} \frac{du}{dt} &= r_0 u f(k,v) - du - au^2 - g(u)v, \\ \frac{dv}{dt} &= -mv + cg(u)v, \end{aligned} \tag{1.1}$$

which models the fear effect in predator-prey interactions. Here $u(t)$ and $v(t)$ denote the population densities of respective species at time $t > 0$; $r_0 > 0$ is the birth rate of the prey; $d > 0$ is the natural death rate of the prey; $a > 0$ is the intra-specific

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pressure of the prey; $m > 0$ is the natural death rate of the predator; $c > 0$ is the conversion rate; $g(u)$ represents the predator functional response which is a continuously differentiable function of $u \in \mathbb{R}^+$; $f(k, v)$ represents the cost of fear, where $k \geq 0$ accounts for the level of fear, and $f(k, v)$ satisfies

$$\begin{aligned} f(0, v) &= 1, \quad \frac{\partial f(k, v)}{\partial k} < 0, \quad \lim_{k \to \infty} f(k, v) = 0, \\ f(k, 0) &= 1, \quad \frac{\partial f(k, v)}{\partial v} < 0, \quad \lim_{v \to \infty} f(k, v) = 0. \end{aligned}$$

The following 3 functions satisfy all the above hypotheses: $f(k, v) = 1/(1 + kv)$, $f(k, v) = e^{-kv}$, and $f(k, v) = 1/(1 + kv + k'v^2)$. These functions describe different decreasing rates. By theoretical and numerical analyses, Wang, Zanette and Zou [25] have demonstrated that high levels of fear can exclude the existence of periodic solutions, while the low levels of fear can induce multiple limit cycles. These results quantitatively reveal the effect of the cost of fear on the dynamics of (1.1).

When the spatial distribution of respective species and the intra-specific pressure of the predator are considered, Wang and Zou [27] proposed and analyzed the following reaction-diffusion-advection predator-prey model incorporating the cost of fear and avoidance behaviors of the prey:

$$\begin{gathered} \frac{\partial u}{\partial t} = d_u \Delta u + \alpha \nabla \cdot (\beta(u) u \nabla v) + f_0(k_0 \alpha, v) r_0 u \\ - du - au^2 - up(u, v)v, \quad x \in \Omega,\ t > 0, \\ \frac{\partial v}{\partial t} = d_v \Delta v + v(-m(v) + cup(u, v)), \quad x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \mu} = \frac{\partial v}{\partial \mu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, \quad x \in \Omega. \end{gathered} \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 1$) with smooth boundary $\partial\Omega$; $u(x, t)$ and $v(x, t)$ are the population densities of respective species at location $x \in \Omega$ and time $t > 0$; $d_u > 0$ and $d_v > 0$ are the random diffusion coefficients of respective species; $m(v) = m_1 + m_2 v$, where $m_1 > 0$ is the death rate of the predator and $m_2 \geq 0$ is the intra-specific pressure of the predator; $up(u, v)$ is the predator functional response; $\mu(x)$ is the outer unit normal vector at $x \in \partial\Omega$ and $\partial u/\partial \mu = \mu(x) \cdot \nabla u$ is the out-flux of u; $\alpha \nabla \cdot (\beta(u) u \nabla v)$ is the diffusion of the prey which represents a directed movement towards lower density of the predator (i.e., predator-taxis), where α measures the strength of predator-taxis, and $\beta(u) = 1 - u/M$ for $0 \leq u \leq M$, $= 0$ for $u > M$, where $M > 0$ measures the maximum number of the prey that a unit volume can accommodate; $f_0(k_0 \alpha, v) = 1/(1 + k_0 \alpha v)$ satisfies the same hypotheses as $f(k, v)$ with k_0 as a nonnegative constant. By theoretical and numerical analyses, Wang and Zou [27] have established the sufficient and necessary conditions of spatial pattern formation for different functional response, and showed that the cost of fear and functional responses play an important role in spatial pattern formation.

For most biological species, the natural environment where they live is usually spatially heterogeneous. Therefore, it is reasonable to expect the dynamic behavior to be influenced by the environmental heterogeneity, apart from the direct effect (through predation) and indirect effect (fear effect) between the species. Moreover, the impact of environmental change on the dynamics of the predator-prey interactions is increasingly recognized (for example, see [2, 6, 10, 9, 15] and references therein). Therefore, it seems imperative to include such the environmental heterogeneity while modelling the predator-prey interactions incorporating fear effect. This constitutes our first motivation of the present paper.

Note that in particular that when modeling the predator-prey interactions incorporating fear effect in the literatures mentioned above, the predator species is always assumed to be specialist predators. However, most predator-prey interactions are generally comprised of a wide variety of predators, both specialists and generalists. Moreover, ecologists have studied the impacts of generalist versus specialist predators separately along with the potential outcome of interactions between them in various environments, and have shown many different dynamic behaviors (for example, see [12, 13, 23] and references therein). Therefore, the predator species should not only be assumed to be specialists, but should include both specialists and generalists. This constitutes our second motivation of the present paper.

As far as we know, although lots of mathematical models (mostly ODE models) have been proposed and studied to quantitatively investigate the effect of fear cost in the predator-prey interactions, there are few works to study a diffusive predator-prey model incorporating fear cost in spatially heterogeneous environment. Therefore, based on these considerations, we consider the following reaction-diffusion predator-prey model incorporating fear effect in spatially heterogeneous environment:

$$
\begin{gathered}
\frac{\partial u}{\partial t} = d_u\Delta u + \frac{ru}{1+kv} - du - au^2 - b(x)p(u,v)v, \quad x\in\Omega,\ t>0,\\
\frac{\partial v}{\partial t} = d_v\Delta v + mv - v^2 + cb(x)p(u,v)v, \quad x\in\Omega,\ t>0,\\
\frac{\partial u}{\partial \mu} = \frac{\partial v}{\partial \mu} = 0, \quad x\in\partial\Omega,\ t>0,\\
u(x,0)=u_0(x)\ge 0,\ v(x,0)=v_0(x)\ge 0, \quad x\in\Omega.
\end{gathered}
\tag{1.3}
$$

Note that we extend model (1.2) by assuming the predator-prey interaction $b(x)$ to be spatially dependent function instead of constant, and assuming the growth rate of the predator to be either positive or negative. In particular, $m>0$ means that predator individuals are generalists and $m<0$ means that predator individuals are specialists. The function $b(x)$ is positive and Hölder continuous in $\overline{\Omega}$. The functional response $p(u,v)$ is a non-negative C^1-function of $(u,v)\in[0,\infty)\times[0,\infty)$ such that

$$
\begin{gathered}
p(0,v)=0, \quad 0\le p_u(u,v)<\infty, \quad p_{uu}(u,v)\le 0,\\
p_v(u,v)\le 0, \quad \frac{\partial(p(u,v)v)}{\partial v}\ge 0
\end{gathered}
$$

for $(u,v)\in[0,\infty)\times[0,\infty)$. The most widely used forms of $p(u,v)v$ in the literatures are $p(u,v)v=uv$ (Linear functional response), $p(u,v)v=uv/(1+qu)$ (Holling-type II functional response) and $p(u,v)v=uv/(1+qu+fv)$ (Beddington-DeAngelis functional response). Here q,f are positive constants. Note in particular that compared with (1.2), we compromise a little bit in the diffusion term for the prey by considering the random movement instead of the directed movement towards lower density of the predator. Indeed, in spatially heterogeneous environment, such directed movement for the prey is interesting, biologically important but yet mathematically challenging, and we have to leave it for future research project.

The main purpose of this paper is to reveal the impact of fear cost, spatial diffusion and environmental heterogeneity on the dynamics of (1.3). In order to better demonstrate their influence, we also need to consider the non-negative steady states of (1.3) which satisfy the following nonlinear elliptic equations:

$$\begin{gathered}
-d_u\Delta u = \frac{ru}{1+kv} - du - au^2 - b(x)p(u,v)v, \quad x\in\Omega,\\
-d_v\Delta v = mv - v^2 + cb(x)p(u,v)v, \quad x\in\Omega,\\
\frac{\partial u}{\partial\mu} = \frac{\partial v}{\partial\mu} = 0, \quad x\in\partial\Omega.
\end{gathered} \tag{1.4}$$

Here $u(x)$ and $v(x)$ denote the density of the prey and predator individuals at equilibrium, respectively. It is, naturally, the dynamics in the biologically meaningful region $u\ge 0$, $v\ge 0$ that are of interest. It is clear that (1.4) admits a trivial solution $(u,v)=(0,0)$, two semi-trivial solutions $(u,v)=((r-d)/a,0)$ with $r>d$ and $(u,v)=(0,m)$ with $m>0$, and positive solutions (u,v) with no component identically zero. From now on, (u,v) will be called a positive solution of (1.4) or a positive steady state of (1.3) if (u,v) is a classical solution satisfying $u>0$ and $v>0$ in Ω.

The rest of this paper is organized as follows. In Sect. 2, we establish the long-time behavior of solutions to (1.3) and the sufficient conditions for the existence of positive solutions to (1.4). In Sect. 3, we obtain some insights on how fear cost affects the population dynamics of (1.3) by choosing different functional responses and different sets of parameters. Section 4 is dedicated to the effect of spatial diffusion and environmental heterogeneity on the dynamics of (1.3). Finally, the significance of current studies is outlined in Sect. 5.

2. Solution and equilibria of system (1.3)

The purpose of this section is to investigate the long-time behavior of solutions to (1.3) and the existence of positive solutions to (1.4).

2.1. Long-time behavior of solutions to (1.3).

For any given function $\phi\in C(\overline{\Omega})$, we denote

$$\phi^* =: \max_{\overline{\Omega}}\phi, \quad \phi_* =: \min_{\overline{\Omega}}\phi, \quad \overline{\phi} = \frac{1}{|\Omega|}\int_\Omega \phi dx,$$

where $|\Omega|$ represents the volume of the region Ω. Let $X=\{\phi\in W^{2,p}_\mu(\Omega):\partial\phi/\partial\mu = 0\}$ and $Y=L^p(\Omega)$ with $p>n$. Define a closed linear operator $\mathbb{D}$ in $Y\times Y$ by $\mathbb{D}(u,v)=(-d_u\Delta u,-d_v\Delta v)$, where $(u,v)\in D(\mathbb{D}) =: X\times X$. Here $D(\mathbb{D})$ represents the domain of the operator $\mathbb{D}$. By [29], the closed linear operator $-\mathbb{D}$ generates an analytic semigroup $\{e^{(-t\mathbb{D})}\}_{t\ge 0}$ in $Y\times Y$. Then we can use the similar argument to that of Proposition 3.1 in [29] to prove the following global solvability theorem.

Theorem 2.1. *Assume that the initial values $u_0(x)$ and $v_0(x)$ are non-negative functions of class $C(\overline{\Omega})$. There exists a unique solution $(u(x,t),v(x,t))$ of (1.3) in $C([0,\infty);Y\times Y)\cap C^1((0,\infty);D(\mathbb{D}))$, and it satisfies*

$$\begin{gathered}
0\le u(x,t)\le \max\{(r-d)/a, u_0^*\} =: M_1,\\
0\le v(x,t)\le \max\{m+cb^*p(M_1,0), v_0^*\}
\end{gathered}$$

in $\Omega \times [0, \infty)$.

For later discussion, we now collect some results on linear eigenvalue problem (see e.g., [1, 3]). For any $q(x) \in C^{\nu}(\overline{\Omega})$ with $\nu \in (0,1)$, the eigenvalue problem

$$-d\Delta\phi + q(x)\phi = \lambda\phi,\ x \in \Omega, \quad \frac{\partial \phi}{\partial \mu} = 0,\ x \in \partial\Omega \tag{2.1}$$

has an infinite number of eigenvalues. Let them be $\{\lambda_i(d, q(x))\}_{i=1}^{\infty}$ satisfying $\lambda_i(d, q(x)) \geq \lambda_j(d, q(x))$ for $i > j \geq 1$, where $\lambda_1(d, q(x))$ is the least eigenvalue and is called the principal eigenvalue. In particular, $\lambda_1(d, 0) = 0$. Moreover, $\lambda_1(d, q(x))$ is a simple eigenvalue and the corresponding eigenfunction does not change sign in Ω. It follows from the variational characterization that $\lambda_1(d, q(x))$ is given by

$$\lambda_1(d, q(x)) = \inf_{\phi \in H^1(\Omega), \phi \neq 0} \int_{\Omega} \left(d|\nabla\phi|^2 + q(x)\phi^2 \right) dx \Big/ \int_{\Omega} \phi^2 dx.$$

Next we call some properties of $\lambda_1(d, q(x))$.

Proposition 2.2. *The following assertions hold.*

1. $\lambda_1(d, q(x))$ *is continuous and monotone increasing with respect to* $q(x)$ *in the sense that* $q_1 \leq q_2$ *and* $q_1 \not\equiv q_2$ *implies* $\lambda_1(d, q_1(x)) < \lambda_1(d, q_2(x))$.
2. $\lambda_1(d, q(x))$ *is strictly monotone increasing with respect to* $d > 0$ *such that* $\lambda_1(d, q(x)) \to q_*$ *as* $d \to 0^+$ *and* $\lambda_1(d, q(x)) \to \overline{q}$ *as* $d \to \infty$.

Following [29], we say that any non-negative solution of (1.4) is locally asymptotically stable provided that the spectrum of the corresponding linearized operator lies in the right-hand side of the imaginary axis and it is unstable provided that there are some points in the spectrum with negative real parts. Then the (local) stability of trivial and semi-trivial solutions reads as follows.

Theorem 2.3. *The following assertions hold.*

1. $(0, 0)$ *is locally asymptotically stable if* $r < d$ *and* $m < 0$*; unstable if* $r > d$ *or* $m > 0$.
2. $(\frac{r-d}{a}, 0)$ *is locally asymptotically stable if* $\lambda_1\left(d_v, -m - cb(x)p\left(\frac{r-d}{a}, 0\right)\right) > 0$*; unstable if* $\lambda_1\left(d_v, -m - cb(x)p\left(\frac{r-d}{a}, 0\right)\right) < 0$.
3. $(0, m)$ *is locally asymptotically stable if* $\lambda_1\left(d_u, -\frac{r}{1+km} + d + b(x)p_u(0, m)m\right) > 0$*; unstable if* $\lambda_1\left(d_u, -\frac{r}{1+km} + d + b(x)p_u(0, m)m\right) < 0$.

Proof. Since the proofs of (1)-(3) are similar, we only prove (3) here. The linearized operator of (1.4) at $(u, v) = (0, m)$ is

$$\mathcal{L}_{(0,m)} = \begin{pmatrix} -d_u\Delta - \frac{r}{1+km} + d + b(x)p_u(0, m)m & 0 \\ -cb(x)p_u(0, m) & -d_v\Delta + m \end{pmatrix}.$$

It follows from the Riesz-Schauder theory that the spectrum $\sigma(\mathcal{L}_{(0,m)})$ of the operator $\mathcal{L}_{(0,m)}$ is composed of real eigenvalues, moreover

$$\sigma(\mathcal{L}_{(0,m)}) = \sigma\Big(- d_u\Delta - \frac{r}{1 + km} + d + b(x)p_u(0, m)m\Big) \cup \sigma\left(-d_v\Delta + m\right).$$

Since $\lambda_i(d_v, m) \geq \lambda_1(d_v, m) = m > 0$, there is no point with a negative real part in the spectrum $\sigma(-d_v\Delta + m)$. In addition, the spectrum

$$\sigma\Big(- d_u\Delta - \frac{r}{1 + km} + d + b(x)p_u(0, m)m\Big)$$

lies on the real axis and the least eigenvalue is given by

$$\lambda_1\Big(d_u, -\frac{r}{1+km} + d + b(x)p_u(0,m)m\Big).$$

Hence, the stability semi-trivial solution $(0,m)$ is determined by the sign of this eigenvalue. This completes the proof. $\square$

By applying the comparison principle, the global attractivity of trivial and semi-trivial solutions can be established.

Theorem 2.4. *The following assertions hold.*

(1) *Assume that $m < 0$ and $r \le d$. Then any non-negative solution of* (1.3) *converges to* $(0,0)$ *uniformly in* $\overline{\Omega}$ *as* $t \to \infty$.
(2) *Assume that $m < 0$, $r > d$ and $\lambda_1\left(d_v, -m - cb(x)p\left((r-d)/a, 0\right)\right) > 0$. Then any non-negative solution of* (1.3) *with* $u_0(x) \ge (\not\equiv) 0$ *converges to* $((r-d)/a, 0)$ *uniformly in* $\overline{\Omega}$ *as* $t \to \infty$.
(3) *Assume that $m > 0$. Then any non-negative solution of* (1.3) *with* $v_0(x) \ge (\not\equiv) 0$ *converges to* $(0,m)$ *uniformly in* $\overline{\Omega}$ *as* $t \to \infty$ *if one of the following hypotheses holds:*
   (i) $\frac{r}{1+km} \le d$;
   (ii) $\frac{r}{1+km} > d$ *and* $\lambda_1\Big(d_u, -\frac{r}{1+km} + d + b(x)p_u\Big(\big(\frac{r}{1+km} - d\big)/a, m\Big)m\Big) > 0$.

Proof. Since the proofs of (1)–(3) are similar and the proof of (3) is a little more complicated, we only give prove (3). It follows from equation (1.3) for $v(x,t)$ that

$$\frac{\partial v}{\partial t} = d_v\Delta v + mv - v^2 + cb(x)p(u,v)v \ge d_v\Delta v + mv - v^2$$

in $\Omega \times (0,\infty)$. Let $V(x,t)$ be the solution of

$$\begin{gathered} V_t = d_v\Delta V + mV - V^2, \quad x \in \Omega,\ t > 0, \\ \frac{\partial V}{\partial \mu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ V(x,0) = v_0(x) \ge (\not\equiv) 0, \quad x \in \Omega. \end{gathered}$$

It is well known that $V(x,t) \to 0$ uniformly in $\overline{\Omega}$ as $t \to \infty$ if $m \le 0$, and $V(x,t) \to m$ uniformly in $\overline{\Omega}$ as $t \to \infty$ if $m > 0$. Thus, we apply the comparison principle to derive that $v(x,t) \ge V(x,t)$ in $\overline{\Omega} \times (0,\infty)$. This ensures that for any $\varepsilon_1 > 0$, there exists $T_{\varepsilon_1} > 0$ such that $v(x,t) \ge m - \varepsilon_1$ in $\overline{\Omega} \times [T_{\varepsilon_1}, \infty)$ since $m > 0$. In addition, it follows from the equation for $u(x,t)$ of (1.3) that

$$\frac{\partial u}{\partial t} = d_u\Delta u + \frac{ru}{1+kv} - du - au^2 - b(x)p(u,v)v \le d_u\Delta u + \big(\frac{r}{1+k(m-\varepsilon_1)} - d\big)u - au^2$$

in $\overline{\Omega} \times [T_{\varepsilon_1}, \infty)$. As above, we apply the comparison principle to derive that for any $\varepsilon_2 > 0$, there exists $T_{\varepsilon_2} > T_{\varepsilon_1}$ such that

$$u(x,t) \le \max\big\{\Big(\frac{r}{1+k(m-\varepsilon_1)} - d\Big)/a, 0\big\} + \varepsilon_2$$

in $\Omega\times[T_{\varepsilon_2},\infty)$. This and (1.3) show that

$$\begin{aligned}\frac{\partial u}{\partial t}&=d_u\Delta u+\frac{ru}{1+kv}-du-au^2-b(x)p(u,v)v\\&\le d_u\Delta u+\frac{ru}{1+k(m-\varepsilon_1)}-du-au^2-b(x)p(u,m-\varepsilon_1)(m-\varepsilon_1)\\&=d_u\Delta u+\frac{ru}{1+k(m-\varepsilon_1)}-du-au^2-b(x)p_u(u',m-\varepsilon_1)(m-\varepsilon_1)u\\&\le d_u\Delta u+\Lambda u-au^2\end{aligned}\tag{2.2}$$

in $\Omega\times[T_{\varepsilon_2},\infty)$, where $u'\in[0,u]$ and

$$\begin{aligned}\Lambda:=&\frac{r}{1+k(m-\varepsilon_1)}-d\\&-b(x)p_u\Big(\max\big\{\big(\frac{r}{1+k(m-\varepsilon_1)}-d\big)/a,0\big\}+\varepsilon_2,m-\varepsilon_1\Big)(m-\varepsilon_1).\end{aligned}$$

Hence, the comparison principle shows that $u(x,t)\le U(x,t)$ in $\overline{\Omega}\times[T_{\varepsilon_2},\infty)$, where $U(x,t)$ is the solution of

$$\begin{gathered}U_t=d_u\Delta U+\Lambda U-aU^2,\quad x\in\Omega,\ t>T_{\varepsilon_2},\\ \frac{\partial U}{\partial\mu}=0,\quad x\in\partial\Omega,\ t>T_{\varepsilon_2},\\ U(x,T_{\varepsilon_2})=u(x,T_{\varepsilon_2}),\quad x\in\Omega.\end{gathered}$$

We consider two possibilities:

(i) If $r/(1+km)\le d$, then it is clear that

$$\lambda_1\Big(d_u,-\frac{r}{1+km}+d+b(x)p_u(0,m)m\Big)>0.$$

By choosing enough small $\varepsilon_1>0$ and $\varepsilon_2>0$, we find that $\lambda_1(d_u,-\Lambda)\ge0$, and hence $U(x,t)\to0$ uniformly in $\overline{\Omega}$ as $t\to\infty$. Thus, for any $\varepsilon_3>0$, there exists $T_{\varepsilon_3}>T_{\varepsilon_2}$ such that $u(x,t)\le\varepsilon_3$ in $\overline{\Omega}\times[T_{\varepsilon_3},\infty)$. Letting $\varepsilon_3\to0$, we obtain that $\lim_{t\to\infty}u(x,t)=0$ uniformly in $\overline{\Omega}$. This, together with the equation for $v(x,t)$ of (1.3), shows that

$$\frac{\partial v}{\partial t}=d_v\Delta v+mv-v^2+cb(x)p(u,v)v\le d_v\Delta v+(m+cb^*p(\varepsilon_3,m-\varepsilon_1))\,v-v^2$$

in $\Omega\times[T_{\varepsilon_3},\infty)$. As above, the comparison principle shows that for any $\varepsilon_4>0$, there exists $T_{\varepsilon_4}>T_{\varepsilon_3}$ such that $v(x,t)\le m+cb^*p(\varepsilon_3,m-\varepsilon_1)+\varepsilon_4$ in $\overline{\Omega}\times[T_{\varepsilon_4},\infty)$. Letting $\varepsilon_i\to0$ with $i=1,2,3,4$, we have that $\lim_{t\to\infty}v(x,t)=m$ uniformly in $\overline{\Omega}$.

(ii) If $r/(1+km)>d$, then for any $\varepsilon_1>0$ small, we have

$$\Lambda=\frac{r}{1+k(m-\varepsilon_1)}-d-b(x)p_u\Big(\big(\frac{r}{1+k(m-\varepsilon_1)}-d\big)/a+\varepsilon_2,m-\varepsilon_1\Big)(m-\varepsilon_1).$$

Moreover, by choosing enough small $\varepsilon_1>0$ and $\varepsilon_2>0$ if necessary, we obtain $\lambda_1(d_u,-\Lambda)\ge0$ by the hypothesis

$$\lambda_1\Big(d_u,-\frac{r}{(1+km)}+d+b(x)p_u\Big(\big(\frac{r}{(1+km)}-d\big)/a,m\Big)m\Big)>0.$$

The rest of the argument is the same as that of the case (i) and hence is omitted. $\square$

Remark 2.5. (i) Choosing $p(u,v)=u/(1+qu)$. By a straightforward modification to the expression of (2.2), that is,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= d_u\Delta u+\frac{ru}{1+kv}-du-au^2-\frac{b(x)uv}{1+qu}\\
&\le d_u\Delta u+\frac{ru}{1+k(m-\varepsilon_1)}-du-au^2-\frac{b(x)(m-\varepsilon_1)u}{1+qu}\\
&\le d_u\Delta u+\frac{ru}{1+k(m-\varepsilon_1)}-du-au^2\\
&\quad -\frac{b(x)(m-\varepsilon_1)u}{1+q\big(\max\{(\frac{r}{1+k(m-\varepsilon_1)}-d)/a,0\}+\varepsilon_2\big)}\\
&\le d_u\Delta u+\Big(\frac{r}{1+k(m-\varepsilon_1)}-d\\
&\quad -\frac{b(x)(m-\varepsilon_1)}{1+q\big(\max\{(\frac{r}{1+k(m-\varepsilon_1)}-d)/a,0\}+\varepsilon_2\big)}\Big)u-au^2,
\end{aligned}$$

the hypothesis

$$\lambda_1\Big(d_u,-\frac{r}{1+km}+d+b(x)p_u\big((\frac{r}{1+km}-d)/a,m\big)m\Big)>0$$

becomes

$$\lambda_1\Big(d_u,-\frac{r}{1+km}+d+\frac{amb(x)}{a+q(r/(1+km)-d)}\Big)>0.$$

(ii) Choosing $p(u,v)=u/(1+qu+fv)$. By a similar modification to the expression of (2.2) to that of (i), the hypothesis

$$\lambda_1\Big(d_u,-\frac{r}{1+km}+d+b(x)p_u\Big((\frac{r}{1+km}-d)/a,m\Big)m\Big)>0$$

can be rewritten as

$$\lambda_1\Big(d_u,-\frac{r}{1+km}+d+\frac{amb(x)}{a+q(r/(1+km)-d)+afm}\Big)>0.$$

2.2. **Existence of positive solutions to** (1.4)**.** In this subsection, we establish the sufficient conditions for the existence of positive solutions to (1.4) by applying degree theory in cones. The following lemma gives the $L^\infty(\Omega)$-estimate for any positive solution which is independent of the diffusion coefficients d_u and d_v.

Lemma 2.6. *Assume that $r>d$ and (u,v) is any positive solution of* (1.4)*. Then $0\le u\le (r-d)/a$ and $\max\{m,0\}\le v\le m+cb^*p\left((r-d)/a,0\right)$ in $\overline{\Omega}$. Moreover, if $m>0$ and $r/(1+km)>d$, then $0\le u\le (r/(1+km)-d)/a$ in $\overline{\Omega}$.*

The proof of the above lemma is standard by a simple comparison argument, hence is omitted. When the diffusion coefficients d_u and d_v are away from 0, we provice the $W^{2,p}(\Omega)$-estimate with $p\in(1,\infty)$ for any positive solution of (1.4).

Lemma 2.7. *Assume that $r>d$ and (u,v) is any positive solution of* (1.4) *with $d_u\ge\varepsilon$ and $d_v\ge\varepsilon$, where $\varepsilon>0$ is any small number. Then there is a positive number $C=C(\varepsilon,r,d,a,m,c,b(x),|\Omega|)$ such that $\|u\|_{W^{2,p}(\Omega)}\le C$ and $\|v\|_{W^{2,p}(\Omega)}\le C$ for any $p\in(1,\infty)$.*

Proof. Suppose that $(u,v)=(u(x),v(x))$ is any positive solution of (1.4) with $d_u\geq\varepsilon$ and $d_v\geq\varepsilon$. Let $C_i=:C_i(\varepsilon,n,r,d,a,m,c,b(x),|\Omega|)$ for simplicity. Multiplying (1.4) by u and integrate the resulting expression we obtain

$$\begin{aligned}\int_\Omega|\nabla u|^2dx&=\frac{1}{d_u}\int_\Omega\Big(\frac{ru^2}{1+kv}-du^2-au^3-b(x)p(u,v)uv\Big)dx\\&\leq\frac{1}{\varepsilon}\int_\Omega(r-d-au)u^2dx.\end{aligned}$$

Note that $(r-d-au)u^2\leq 4(r-d)^3/27a^2$ for $0\leq u\leq(r-d)/a$. It follows from Lemma 2.6 that $\int_\Omega|\nabla u|^2dx\leq\frac{4(r-d)^3}{27\varepsilon a^2}|\Omega|$. Moreover, Lemma 2.6 shows that $\frac{ru}{1+kv}-du-au^2-b(x)p(u,v)v$ is bounded for any $p\in(1,\infty)$. By the standard L^p-estimate for elliptic equation, we have

$$\|u\|_{H^2(\Omega)}\leq C_2\Big(\|u\|_{H^1(\Omega)}+\frac{1}{\varepsilon}\Big\|\frac{ru}{1+kv}-du-au^2-b(x)p(u,v)v\Big\|_{L^2(\Omega)}\Big)\leq C_3.$$

Therefore, we apply the Sobolev embedding theorem to have

$$u\in H^2(\Omega)\subset\begin{cases}C^1(\overline{\Omega}) & \text{if } n=1;\\ W^{1,p_1}(\Omega),\ \forall p_1\in[1,\infty) & \text{if } n=2;\\ W^{1,p_1}(\Omega),\ \forall p_1\in[1,\frac{2n}{n-2}] & \text{if } n>2.\end{cases}$$

Here n is the dimension of space. For the case that $n>2$, the standard L^p-estimate for elliptic equation yields

$$\|u\|_{W^{2,p_1}(\Omega)}\leq C_4\Big(\|u\|_{W^{1,p_1}(\Omega)}+\frac{1}{\varepsilon}\Big\|\frac{ru}{1+kv}-du-au^2-b(x)p(u,v)v\Big\|_{L^{p_1}(\Omega)}\Big)\leq C_5.$$

Thus, applying the Sobolev embedding theorem we have

$$u\in\|u\|_{W^{2,p_1}(\Omega)}\subset\begin{cases}C^1(\overline{\Omega}) & \text{if } n<p_1;\\ W^{1,p_2}(\Omega),\ \forall p_2\in[1,\infty) & \text{if } n=p_1;\\ W^{1,p_2}(\Omega),\ \forall p_2\in[1,\frac{p_1n}{n-p_1}] & \text{if } n>p_1.\end{cases}$$

Consequently, for any $p\in(1,\infty)$, we can repeat the above argument if necessary to find a positive constant C_6 such that $\|u\|_{W^{2,p}(\Omega)}\leq C_6$.

Similarly, we can obtain the desired estimate for $\|v\|_{W^{2,p}(\Omega)}$. □

Assume that E is a real Banach space. For a closed convex set W in E, it is said to be a wedge if $\alpha W\subset W$ for all $\alpha\geq 0$. Moreover, when $W\cap\{-W\}=0$, W is called a cone. Define $W_y=:\mathrm{cl}\{x\in E: y+\omega x\in W\}$, where $y\in W$, $\omega>0$ is some constant and "cl" represents the closure of the set. Clearly, W_y is a wedge, and its maximal linear subspace is denoted by S_y. Let $\mathbb{A}$ be a Fréchet differentiable compact operator in E such that $y\in W$ is a fixed point of $\mathbb{A}$ and $\mathbb{A}(W)\subseteq W$. Then W_y and S_y are both invariant under $L(y)$, where $L(y)$ is the Fréchet derivative of $\mathbb{A}$ at y. Let U be an open subset of W. Define $\mathrm{index}_W(\mathbb{A},U)=\mathrm{index}(\mathbb{A},U,W)=\deg_W(I-\mathbb{A},U,0)$, where I is the identity map. Assume that y is an isolated fixed point of $\mathbb{A}$. Then the index of $\mathbb{A}$ at y in W is given by $\mathrm{index}_W(\mathbb{A},y)=\mathrm{index}(\mathbb{A},U(y),W)$, where $U(y)$ is a small open neighborhood of y in W.

Assume that E has the decomposition $E=E_y\oplus S_y$, where E_y is a closed linear subspace of E, and W_y is generating. Then the index of $\mathbb{A}$ at y can be calculated by the following proposition (see [20, Theorems 2.2 and 2.3]).

Proposition 2.8. *Suppose that $Q : E \to E_y$ is a projection operator of E_y along S_y. If $L(y)$ has no non-zero fixed point on W_y, then* $\operatorname{index}(\mathbb{A}, y)$ *exists. Moreover, the following assertions hold.*

1. $\operatorname{index}_W(\mathbb{A}, y) = 0$ *if $Q \circ L(y)$ has an eigenvalue greater than 1 on W_y;*
2. $\operatorname{index}_W(\mathbb{A}, y) = \operatorname{index}_{S_y}(L(y), 0) = (-1)^\sigma$ *if $Q \circ L(y)$ has no eigenvalue greater than 1 on W_y, where σ is the sum of algebraic multiplicities of the eigenvalues of $L(y)$ restricted in S_y which are greater than 1.*

Let $E = C(\overline{\Omega}) \oplus C(\overline{\Omega})$ and $W = P \oplus P$, where $P = \{\phi \in C(\overline{\Omega}) : \phi(x) \geq 0 \text{ in } \overline{\Omega}\}$. Set $D = \big\{(u, v) \in W : u \leq (r - d)/a + 1, v \leq m + 1 + cb^* p\left((r - d)/a, 0\right)\big\}$. For each $t \in [0, 1]$, we define a Fréchet differentiable compact operator $\mathbb{A}_t : D \to E$ by

$$\mathbb{A}_t \begin{pmatrix} u \\ v \end{pmatrix} = (-\Delta + M)^{-1} \begin{pmatrix} \frac{1}{d_u}\big(\frac{ru}{1+kv} - du - au^2 - tb(x)p(u, v)v\big) + Mu \\ \frac{1}{d_v}\big(mv - v^2 + tcb(x)p(u, v)v\big) + Mv \end{pmatrix}.$$

Here M is a suitably large number such that

$$(d_u)^{-1}\Big(\frac{ru}{1 + kv} - du - au^2 - tb(x)p(u, v)v\Big) + Mu > 0$$

and $(d_v)^{-1}\big(mv - v^2 + tcb(x)p(u, v)v\big) + Mv > 0$ for any $(u, v) \in D$.

Assume that $r > d$ and $m > 0$. Then $(0, 0)$, $((r - d)/a, 0)$ and $(0, m)$ are the only non-negative fixed points of $\mathbb{A}_1$ which are not positive. We can apply Proposition 2.8 to calculate their indices.

Lemma 2.9. *Assume that $r > d$ and $m > 0$. Then*

1. $\operatorname{index}_W(\mathbb{A}_1, (0, 0)) = 0$*;*
2. $\operatorname{index}_W(\mathbb{A}_1, ((r - d)/a, 0)) = 0$*;*
3. $\operatorname{index}_W(\mathbb{A}_1, (0, m)) = \begin{cases} 0, & \text{if } \lambda_1\big(d_u, -\frac{r}{1+km} + d + b(x)p_u(0, m)m\big) < 0, \\ 1, & \text{if } \lambda_1\big(d_u, -\frac{r}{1+km} + d + b(x)p_u(0, m)m\big) > 0; \end{cases}$
4. $\operatorname{index}_W(\mathbb{A}_1, D) = 1$ *if $r/(1 + km) \neq d$.*

Proof. Since the proofs of (1)–(4) are similar and the proof of (3) is a little more complicated, we only prove (3). By the definitions of $W_{(0,m)}$ and $S_{(0,m)}$, it is easy to check that $W_{(0,m)} = P \oplus C(\overline{\Omega})$ and $S_{(0,m)} = \{0\} \oplus C(\overline{\Omega})$. Choosing $E_{(0,m)} = C(\overline{\Omega}) \oplus \{0\}$. Then $E = S_{(0,m)} \oplus E_{(0,m)}$. Define

$$L_1(0, m) = (-\Delta + M)^{-1} \begin{pmatrix} \frac{1}{d_u}\big(\frac{r}{1+km} - d - b(x)p_u(0, m)m\big) + M & 0 \\ \frac{1}{d_v}\left(cb(x)p_u(0, m)m\right) & -\frac{1}{d_v}m + M \end{pmatrix}.$$

Then $L_1(0, m)$ is the Fréchet derivative of $\mathbb{A}_1$ with respect to (u, v) at $(0, m)$. Assume that $(\phi, \psi) \in W_{(0,m)}$ is an eigenfunction of $L_1(0, m)$ and κ is the corresponding eigenvalue. Then

$$\begin{gathered} (-\Delta + M)^{-1}\Big\{\frac{1}{d_u}\Big(\frac{r}{1 + km} - d - b(x)p_u(0, m)m\Big) + M\Big\}\phi = \kappa\phi, \quad x \in \Omega, \\ (-\Delta + M)^{-1}\Big\{\frac{1}{d_v}\big(cb(x)p_u(0, m)m\big)\Big\}\phi \\ + (-\Delta + M)^{-1}\Big\{-\frac{1}{d_v}m + M\Big\}\psi = \kappa\psi, \quad x \in \Omega, \\ \frac{\partial\phi}{\partial\mu} = \frac{\partial\psi}{\partial\mu} = 0, \quad x \in \partial\Omega. \end{gathered}$$

When $\lambda_1\left(d_u, -r/(1+km)+d+b(x)p_u(0,m)m\right) \neq 0$ and $m>0$, it is easy to check that $I - L_1(0,m)$ is invertible on $W_{(0,m)}$ by contradiction. Hence, $L_1(0,m)$ has no non-zero fixed point on $W_{(0,m)}$. Consequently, it follows from Proposition 2.8 that $\operatorname{index}_W(\mathbb{A}_1,(0,m))$ exists.

To obtain the value of $\operatorname{index}_W(\mathbb{A}_1,(0,m))$, we now analyze the eigenvalues of $Q\circ L_1(0,m)$, where $Q: E = C(\overline{\Omega})\oplus C(\overline{\Omega}) \to E_{(0,m)} = C(\overline{\Omega})\oplus\{0\}$ is a projection operator of $E_{(0,m)}$ along $S_{(0,m)}$. By the definition of Q, we see that every eigenfunction of $Q\circ L_1(0,m)$ has the form $(\phi,0)$, where ϕ is a non-zero solution of

$$\begin{gathered}(-\Delta+M)^{-1}\Big\{\frac{1}{d_u}\Big(\frac{r}{1+km}-d-b(x)p_u(0,m)m\Big)+M\Big\}\phi=\kappa\phi,\quad x\in\Omega,\\ \frac{\partial\phi}{\partial\mu}=0,\quad x\in\partial\Omega.\end{gathered}$$

By [14, Lemma 2.4], it is well known that

$$r\Big[(-\Delta+M)^{-1}\Big\{\frac{1}{d_u}\Big(\frac{r}{1+km}-d-b(x)p_u(0,m)m\Big)+M\Big\}\Big]>1 \text{ (resp. } <1)$$

if $\lambda_1\big(d_u,-\frac{r}{1+km}+d+b(x)p_u(0,m)m\big)<0$ (resp.>0). Therefore, $Q\circ L_1(0,m)$ has an eigenvalue larger than 1 when $\lambda_1\big(d_u,-\frac{r}{1+km}+d+b(x)p_u(0,m)m\big)<0$, and thus $\operatorname{index}_W(\mathbb{A}_1,(0,m))=0$ by Proposition 2.8(1). On the other hand, if $\lambda_1\big(d_u,-\frac{r}{1+km}+d+b(x)p_u(0,m)m\big)>0$, then $Q\circ L_1(0,m)$ has no eigenvalue larger than or equal to 1. Hence, by Proposition 2.8(2), we see that $\operatorname{index}_W(\mathbb{A}_1,(0,m)) = (-1)^\sigma$, where σ is the sum of algebraic multiplicities of the eigenvalues of $L_1(0,m)$ restricted in $S_{(0,m)}$ which are greater than 1. Indeed, we can prove $\sigma=0$. Assume that $(\phi,\psi)\in S_{(0,m)}$ is an eigenfunction of $L_1(0,m)$ and κ is the corresponding eigenvalue. Then $\phi=0$ and ψ is a non-zero solution of

$$(-\Delta+M)^{-1}\big\{-\frac{m}{d_v}+M\big\}\psi=\kappa\psi,\ \in\Omega,\quad \frac{\partial\psi}{\partial\mu}=0,\ x\in\partial\Omega.$$

Since $\lambda(d_v,m)=m>0$, we derive from [14, Lemma 2.4] that $r[(-\Delta+M)^{-1}(-\frac{m}{d_v}+M)]<1$. This implies that $L_1(0,m)$ has no eigenvalue larger than or equal to 1 in $S_{(0,m)}$, and thus $\sigma=0$. Consequently, $\operatorname{index}_W(\mathbb{A}_1,(0,m))=1$. □

Assume that $r>d$ and $m<0$. Then $(0,0)$ and $((r-d)/a,0)$ are the only non-negative fixed points of $\mathbb{A}_1$ which are not positive. There is a result corresponding to Lemma 2.9 for the case $m<0$.

Lemma 2.10. *Assume that $r>d$ and $m<0$. Then*

1. $\operatorname{index}_W(\mathbb{A}_1,(0,0))=0$;
2. $\operatorname{index}_W(\mathbb{A}_1,((r-d)/a,0)) = \begin{cases}0, & \text{if } \lambda_1(d_v,-m-cb(x)p(\frac{r-d}{a},0))<0,\\ 1, & \text{if } \lambda_1(d_v,-m-cb(x)p(\frac{r-d}{a},0))>0;\end{cases}$
3. $\operatorname{index}_W(\mathbb{A}_1,D)=1$.

With the help of Lemmas 2.9 and 2.10, we use the excision property of fixed point index to obtain the existence of positive solutions to (1.4) as follows.

Theorem 2.11. *The following assertions hold.*

1. *Assume that $r/(1+km)>d$ and $m>0$. Then system (1.4) admits at least one positive solution if $\lambda_1\left(d_u,-r/(1+km)+d+b(x)p_u(0,m)m\right)<0$.*
2. *Assume that $r>d$ and $m<0$. Then system (1.4) admits at least one positive solution if and only if $\lambda_1\left(d_v,-m-cb(x)p\left((r-d)/a,0\right)\right)<0$.*

Proof. We argue indirectly. For the case that $r/(1+km) > d$ and $m > 0$. Assume that (1.4) has no positive solution when

$$\lambda_1\Big(d_u, -\frac{r}{1+km} + d + b(x)p_u(0,m)m\Big) < 0.$$

Then from Lemma 2.9, the additivity property of fixed point indices yields

$$\begin{aligned} 1 &= \mathrm{index}_W(\mathbb{A}_1, D) \\ &= \mathrm{index}_W(\mathbb{A}_1, (0,0)) + \mathrm{index}_W(\mathbb{A}_1, ((r-d)/a, 0)) + \mathrm{index}_W(\mathbb{A}_1, (0,m)) = 0. \end{aligned}$$

This contradiction shows that when

$$\lambda_1\Big(d_u, -\frac{r}{1+km} + d + b(x)p_u(0,m)m\Big) < 0,$$

(1.4) admits at least one positive solution. Similarly, we can show that (1.4) admits at least one positive solution when $\lambda_1\left(d_v, -m - cb(x)p\left((r-d)/a, 0\right)\right) < 0$.

It remains to prove that (1.4) has no positive solution if

$$\lambda_1\Big(d_v, -m - cb(x)p\big(\frac{r-d}{a}, 0\big)\Big) \geq 0.$$

On the contrary, assume that (1.4) has a positive solution (u, v). Then from the equation for u and Lemma 2.6, we deduce that

$$\begin{aligned} 0 &= \lambda_1\left(d_v, -m + v - cb(x)p(u,v)\right) \\ &> \lambda_1(d_v, -m - cb(x)p(u,0)) \\ &\geq \lambda_1\left(d_v, -m - cb(x)p((r-d)/a, 0)\right) \geq 0. \end{aligned}$$

This is impossible which proves the desired result. $\square$

3. Effect of fear

The purpose of this section is to obtain some insights on how fear affects the dynamics of (1.3). To make the analysis more explicit, we consider three particular forms for the functional response $p(u, v)$.

3.1. **Linear functional response.** In this subsection, we choose the linear functional response (i.e., $p(u, v) = u$) to demonstrate the effect of fear on the dynamics of (1.3).

The zero level curve of $\lambda_1\left(d_u, -r/(1+km) + d + mb(x)\right) = 0$ is given by $r = r(k;m) =: (1+km)\lambda_1\left(d_u, d + mb(x)\right)$. By Proposition 2.2(1), we have the following lemma.

Lemma 3.1. *For any $k \geq 0$, the function $r(k;m)$ is continuously differentiable and monotone increasing with respect to $m > 0$, moreover it satisfies $\lim_{m\to 0^+} r(k;m) = d$ and $\lim_{m\to\infty} r(k;m) = \infty$. Furthermore, $\lambda_1\left(d_u, -r/(1+km) + d + mb(x)\right) < 0$ if $r > r(k;m)$; > 0 if $r < r(k;m)$.*

Denote $m = m(r) =: \lambda_1\left(d_v, -c(r-d)b(x)/a\right)$. Then we have the following properties of $m(r)$.

Lemma 3.2. *For $r > d$, the function $m(r)$ is continuously differentiable and monotone decreasing with respect to r, moreover it satisfies $\lim_{r\to d^+} m(r) = 0$ and $\lim_{r\to\infty} m(r) = -\infty$. Furthermore, $\lambda_1\left(d_v, -m - c(r-d)b(x)/a\right) < 0$ if $m > m(r)$; > 0 if $m < m(r)$.*

Suppose that (1.4) with $p(u,v) = u$ has a positive solution (u,v). Then by Lemma 2.6, we have

$$0 = \lambda_1\Big(d_u, -\frac{r}{1+kv} + d + au + b(x)v\Big) > \lambda_1\Big(d_u, -\frac{r}{1+km} + d + mb(x)\Big)$$

for any $m > 0$. Therefore, combined with Theorems 2.4 and 2.11, we summarize the above discussion to obtain the following result which can be described in the rm-plane (see Figure 1).

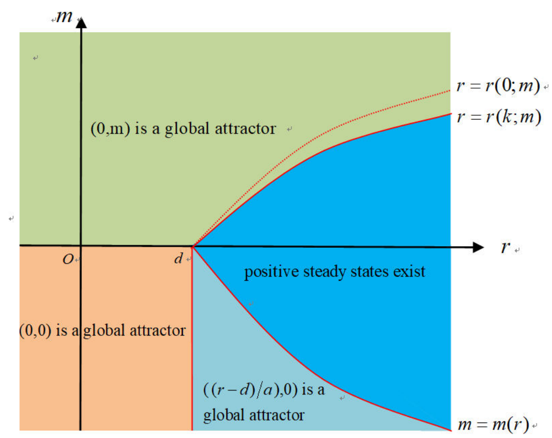


FIGURE 1. Dynamics of (1.3) with linear functional response.

Theorem 3.3. *Let $p(u,v) = u$. Then the following assertions hold.*

(1) Assume that $m < 0$. Then any non-negative solution of (1.3) converges to $(0,0)$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for $r \le d$, any non-negative solution of (1.3) with $u_0(x) \ge (\not\equiv) 0$ converges to $((r-d)/a, 0)$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for $d < r < m^{-1}(r)$, where $m^{-1}(r)$ is the inverse function of $m(r) =: \lambda_1(d_v, -cb(x)(r-d)/a)$, and (1.3) admits at least one positive steady state for $r > m^{-1}(r)$.

(2) Assume that $m > 0$. Then any non-negative solution of (1.3) with $v_0(x) \ge (\not\equiv) 0$ converges to $(0,m)$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for $r < r(k;m) =: (1 + km)\lambda_1(d_u, d + b(x)m)$, and (1.3) admits at least one positive steady state for $r > r(k;m)$.

3.2. **Holling-type II functional response.** In this subsection, we choose the Holling-type II functional response (i.e., $p(u,v) = u/(1+qu)$) to demonstrate the effect of fear on the dynamics of (1.3).

It is clear that the zero level curve of $\lambda_1(d_v, -m - c(r-d)b(x)/(a + q(r-d))) = 0$ is given by $m = \overline{m}(r) =: \lambda_1(d_v, -c(r-d)b(x)/(a+q(r-d)))$. By Proposition 2.2(1), one can easily obtain the following properties of $\overline{m}(r)$.

Lemma 3.4. *For $r > d$, the function $\overline{m}(r)$ is continuously differentiable and monotone decreasing with respect to r, moreover it satisfies $\lim_{r\to d^+} \overline{m}(r) = 0$ and $\lim_{r\to\infty} \overline{m}(r) = \lambda_1(d_v, -cb(x)/q)$. Furthermore, $\lambda_1\Big(d_v, -m - \frac{(r-d)cb(x)}{a+q(r-d)}\Big) < 0$ if $m > \overline{m}(r)$; > 0 if $m < \overline{m}(r)$.*

From the equation for u and Lemma 2.6, a necessary condition for the existence of positive solutions to (1.4) with $p(u,v) = u/(1+qu)$ is

$$\Psi(r,m) =: \lambda_1\Big(d_u, -\frac{r}{1+km} + d + \frac{amb(x)}{a+q(r/(1+km)-d)}\Big) < 0.$$

The following lemma gives the zero level curve of $\Psi(r,m)$.

Lemma 3.5. *For any $k \geq 0$ and $r/(1+km) > d$, there exists a monotone increasing function $\overline{r}(k;m)$ with respect to m such that $\Psi(\overline{r}(k;m),m) = 0$ for any $m > 0$, moreover it satisfies $d(1+km) < \overline{r}(k;m) < r(k;m)$ for any $m > 0$, $\lim_{m\to 0^+}\overline{r}(k;m) = d$ and $\lim_{m\to\infty}\overline{r}(k;m) = \infty$. Furthermore, $\Psi(r,m) < 0$ if $r > \overline{r}(k;m)$; > 0 if $r < \overline{r}(k;m)$.*

Proof. Define

$$\psi(r,m) =: -\frac{r}{1+km} + d + \frac{amb(x)}{a+q(r/(1+km)-d)}.$$

Then $\Psi(r,m) = \lambda_1\left(d_u, \psi(r,m)\right)$. Since $\psi(r,m)$ is monotone decreasing with respect to r and is monotone increasing with respect to m, it follows from Proposition 2.2(1) that $\Psi(r,m)$ is monotone decreasing with respect to r and is monotone increasing with respect to m. Since $\psi(r,m) \to -\infty$ as $r \to \infty$ and $\psi(r,m) \to mb(x) > 0$ as $r \to (d(1+km))^+$, we deduce from the variational characterization that

$$\lim_{r\to\infty}\Psi(r,m) = -\infty$$

and

$$\lim_{r\to(d(1+km))^+}\Psi(r,m) = \int_\Omega \left(d_u|\nabla\phi|^2 + mb(x)\phi^2\right)dx \Big/ \int_\Omega \phi^2 dx > 0$$

for some $\phi \in H^1(\Omega)$, where ϕ does not change sign in Ω. By the intermediate value theorem, there exists a unique number $\overline{r}(k;m)$ such that $\Psi(\overline{r}(k;m),m) = 0$ for any $m > 0$. Moreover, it follows from implicit function theorem that $\overline{r}(k;m) \in C^1(0,\infty)$ is monotone increasing with respect to m.

Since

$$\lambda_1\Big(d_u, -\frac{r}{1+km} + d + \frac{amb(x)}{a+q(r/(1+km)-d)}\Big) < \lambda_1\Big(d_u, -\frac{r}{1+km} + d + mb(x)\Big)$$

we have $\lambda_1\left(d_u, -\overline{r}(k;m)/(1+km) + d + mb(x)\right) > 0$. Thus, it follows from Lemma 3.1 that $\overline{r}(k;m) < r(k;m)$ for any $m > 0$. On the other hand,

$$\begin{aligned}\lambda_1\Big(d_u, -\frac{r}{1+km} + d + \frac{amb(x)}{a+q(r/(1+km)-d)}\Big) &> \lambda_1\Big(d_u, -\frac{r}{1+km} + d\Big) \\ &= -\frac{r}{1+km} + d,\end{aligned}$$

we deduce from this inequality to obtain $\overline{r}(k;m) > d(1+km)$ for any $m > 0$. Hence, we proved that $d(1+km) < \overline{r}(k;m) < r(k;m)$ for any $m > 0$. This, together with Lemma 3.1, implies that $\lim_{m\to 0^+}\overline{r}(k;m) = d$ and $\lim_{m\to\infty}\overline{r}(k;m) = \infty$. This completes the proof. $\square$

Therefore, combining Theorems 2.4 with 2.11, we obtain the following result which can be described in the rm-plane (see Figure 2).

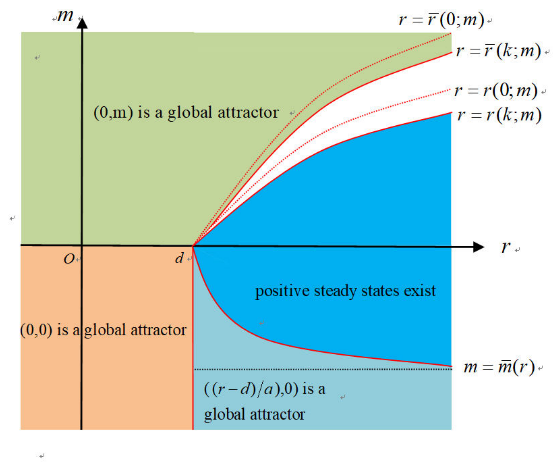


FIGURE 2. Dynamics of (1.3) with Holling-type II functional response.

Theorem 3.6. *Let $p(u,v) = u/(1+qu)$. The following assertions hold.*

(1) Assume that $m \le \lambda_1(d_v, -cb(x)/q)$. Then any non-negative solution of (1.3) converges to $(0,0)$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for $r \le d$, and any non-negative solution of (1.3) with $u_0(x) \ge (\not\equiv) 0$ converges to $((r-d)/a, 0)$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for $r > d$.

(2) Assume that $\lambda_1(d_v, -cb(x)/q) < m < 0$. Then any non-negative solution of (1.3) converges to $(0,0)$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for $r \le d$, any non-negative solution of (1.3) with $u_0(x) \ge (\not\equiv) 0$ converges to $((r-d)/a, 0)$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for $d < r < \overline{m}^{-1}(r)$, where $\overline{m}^{-1}(r)$ is the inverse function of $\overline{m}(r) =: \lambda_1(d_v, -c(r-d)b(x)/(a+q(r-d)))$, and (1.3) admits at least one positive steady state for $r > \overline{m}^{-1}(r)$.

(3) Assume that $m > 0$. Then any non-negative solution of (1.3) with $v_0(x) \ge (\not\equiv) 0$ converges to $(0,m)$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for $r < \overline{r}(k;m)$ given in Lemma 3.5, and (1.3) admits at least one positive steady state for $r > r(k;m)$ given in Lemma 3.1.

For any fixed $m > 0$, Theorem 3.6 provides no information on the existence and non-existence of positive steady states to (1.3) with $p(u,v) = u/(1+qu)$ for $r \in (\overline{r}(k;m), r(k;m))$. A further study is therefore necessary in order to better understand the dynamics of (1.3) with $p(u,v) = u/(1+qu)$.

We now apply the bifurcation result of Crandall-Rabinowitz in [4] to obtain a branch of positive solutions to (1.4) with $p(u,v) = u/(1+qu)$ emanating from $(r,u,v) = (r(k;m), 0, m)$ with $m > 0$. Recall that $X = \{u \in W^{2,p}_\mu(\Omega) : \partial u/\partial\mu = 0\}$ and $Y = L^p(\Omega)$. Let $\mathbb{B} : \mathbb{R} \times X \times X \to Y \times Y$ be given by

$$\mathbb{B}(r,u,v) = \begin{pmatrix} d_u\Delta u + \frac{ru}{1+kv} - du - au^2 - \frac{b(x)uv}{1+qu} \\ d_v\Delta v + mv - v^2 + \frac{cb(x)uv}{1+qu} \end{pmatrix}.$$

It follows from the Krein-Rutman theorem that $\mathbb{B}_{(u,v)}(r,0,m)(\phi,\psi) = 0$ has a solution with $\phi > 0$ in $\overline{\Omega}$ if and only if $r = r(k;m)$. Moreover, the kernel

$\mathcal{N}(\mathbb{B}_{(u,v)}(r(k;m),0,m)) = \text{span}\{(\phi_1,\psi_1)\}$, where ϕ_1 is a positive solution of

$$-d_u\Delta\phi + \Big(-\frac{r(k;m)}{1+km} + d + b(x)m\Big)\phi = 0,\ x\in\Omega,\quad \frac{\partial\phi}{\partial\mu} = 0,\ x\in\partial\Omega,$$

and $\psi_1 = (-d_v\Delta + mI)^{-1}(cmb(x)\phi_1)$. Moreover, it follows from Fredholm alternative theorem that the range $\mathcal{R}(\mathbb{B}_{(u,v)}(r(k;m),0,m)) = \{(\phi,\psi)\in Y^2 : \int_\Omega \phi\phi_1 dx = 0\}$. In addition, a simple calculation yields

$$\mathbb{B}_{r(u,v)}(r(k;m),0,m)(\phi_1,\psi_1) = (\phi_1/(1+km),0) \notin \mathcal{R}(\mathbb{B}_{(u,v)}(r(k;m),0,m)).$$

Therefore, by applying the local bifurcation theorem in [4], we conclude that positive solutions of (1.4) with $p(u,v) = u/(1+qu)$ near $(r(k;m),0,m)$ lie in a smooth curve $\Sigma_\epsilon = \{(r(s),u(s),v(s)) : s\in(0,\epsilon)\}$, where $(r(s),u(s),v(s))$ is continuously differentiable with respect to s and $(r(0),u(0),v(0)) = (r(k;m), s\phi_1 + o(s), m + s\psi_1 + o(s))$. Furthermore, by the formula (4.5) in [22], we see that

$$\begin{aligned} r'(0) = \Big(\int_\Omega \frac{\phi_1^2}{1+km}dx\Big)^{-1}\Big[&\int_\Omega (a - qmb(x))\,\phi_1^3 dx \\ &+ \int_\Omega \Big(\frac{k\lambda_1(d_u, d+b(x)m)}{1+km} + b(x)\Big)\phi_1^2\psi_1 dx\Big]. \end{aligned}$$

Consequently, the bifurcation of Σ_ϵ at $(r(k;m),0,m)$ is subcritical ($r'(0)<0$) if $q > q_0(k)$ and it is supercritical ($r'(0)>0$) if $0\le q < q_0(k)$, where

$$q_0(k) = \Big(\int_\Omega mb(x)\phi_1^3 dx\Big)^{-1}\int_\Omega \Big(a\phi_1 + \Big(b(x) + \frac{k\lambda_1(d_u,d+b(x)m)}{1+km}\Big)\psi_1\Big)\phi_1^2 dx.$$

Remark 3.7. When $q\in(q_0(0),q_0(k))$, the level of fear k alters the direction of bifurcation from subcritical ($r'(0)<0$) to supercritical ($r'(0)>0$). This, together with Theorem 3.6(3), indicates that the cost of fear will not only affect the existence of positive solutions to (1.4), but also change the direction of steady-state bifurcation.

A global bifurcation consideration, together with the maximum principle, proves that the local curve Σ_ϵ is contained in a global branch of positive solutions to (1.4) with $p(u,v) = u/(1+qu)$ which is denoted by $\Sigma = \{(r,u,v)\}$. Furthermore, by the similar argument to that of Theorem 2.4 in [15], we can show that the global branch Σ is unbounded in $\mathbb{R}\times X\times X$. It follows from Lemma 2.6 that any positive solution (u,v) of (1.4) with $p(u,v) = u/(1+qu)$ satisfies $0\le u\le \frac{1}{a}(r/(1+km)-d)$ and $m\le v\le m + cb^*(r-d)/(a+q(r-d))$ for any $m>0$. Thus, the only possible is $(r(k;m),\infty)\subset\{r : (r,u,v)\in\Sigma\}$. Summarizing the above discussion, we obtain the following result.

Proposition 3.8. *Assume that $r/(1+km) > d$ and $m>0$. Then there exists a global unbounded continuum Σ of positive solutions to* (1.4) *with $p(u,v) = u/(1+qu)$ which contains Σ_ϵ such that Σ emanates from $(r,u,v) = (r(k;m),0,m)$ and tends to infinity as r goes to infinity. Moreover, the bifurcation of Σ_ϵ at $(r(k;m),0,m)$ is subcritical* ($r'(0)<0$) *if $q>q_0(k)$ and it is supercritical* ($r'(0)>0$) *if $0\le q<q_0(k)$.*

Based on Lemma 2.9 and Proposition 3.8, we establish the multiplicity of positive solutions to (1.4) with $p(u,v) = u/(1+qu)$ by the similar argument to that of Theorem 2.3 in [15].

Theorem 3.9. *Assume that $r/(1+km) > d$ and $m > 0$. If $q > q_0(k)$, then there exists a constant $\mathbf{r} \in (\overline{r}(k;m), r(k;m))$ such that (1.4) with $p(u,v) = u/(1+qu)$ has at least two positive solutions for $r \in (\mathbf{r}, r(k;m))$ and has at least one positive solution for $r \in [r(k;m), \infty)$.*

3.3. **Beddington-DeAngelis functional response.** In this subsection, we choose the Beddington-DeAngelis functional response (i.e., $p(u,v) = u/(1+qu+fv)$) to demonstrate the effect of fear on the dynamics of (1.3).

The zero level curve of $\lambda_1\left(d_u, -r/(1+km) + d + mb(x)/(1+fm)\right) = 0$ is given by $r = \widetilde{r}(k;m) =: (1+km)\lambda_1\left(d_u, d + mb(x)/(1+fm)\right)$. We have the following lemma.

Lemma 3.10. *For any $k \geq 0$, the function $\widetilde{r}(k;m)$ is continuously differentiable and monotone increasing with respect to $m > 0$, moreover it satisfies $d(1+km) < \widetilde{r}(k;m) < r(k;m)$ for any $m > 0$, and*

$$\lim_{m\to 0^+} \widetilde{r}(0;m) = d, \quad \lim_{m\to\infty} \widetilde{r}(0;m) = \lambda_1\left(d_u, d + b(x)/f\right),$$
$$\lim_{m\to 0^+} \widetilde{r}(k;m) = d, \quad \lim_{m\to\infty} \widetilde{r}(k;m) = \infty \quad \text{with } k > 0.$$

Furthermore, $\lambda_1\left(d_u, -r/(1+km) + d + mb(x)/(1+fm)\right) < 0$ if $r > \widetilde{r}(k;m)$; > 0 if $r < \widetilde{r}(k;m)$.

From the equation for u and Lemma 2.6, a necessary condition for the existence of positive solutions to (1.4) with $p(u,v) = u/(1+qu+fv)$ is $\Phi(r,m) =: \lambda_1\big(d_u, -\frac{r}{1+km} + d + \frac{amb(x)}{a+q(r/(1+km)-d)+afm}\big) < 0$. By the similar argument to that of Lemma 3.5, we have the following lemma.

Lemma 3.11. *For any $k \geq 0$ and $r/(1+km) > d$, there exists a monotone increasing function $\widehat{r}(k;m)$ with respect to m such that $\Phi(\widehat{r}(k;m), m) = 0$ for any $m > 0$, moreover it satisfies $d(1+km) < \widehat{r}(k;m) < \widetilde{r}(k;m)$ for any $m > 0$, and*

$$\lim_{m\to 0^+} \widehat{r}(0;m) = d, \quad \lim_{m\to\infty} \widehat{r}(0;m) = \lambda_1\left(d_u, d + b(x)/f\right),$$
$$\lim_{m\to 0^+} \widehat{r}(k;m) = d, \quad \lim_{m\to\infty} \widehat{r}(k;m) = \infty \quad \text{with } k > 0.$$

Furthermore, $\Phi(r,m) < 0$ if $r > \widehat{r}(k;m)$; and $\Phi(r,m) > 0$ if $r < \widehat{r}(k;m)$.

Combined this with Theorems 2.4 and 2.11, we obtain the following result which can be described in the rm-plane (see Figure 3).

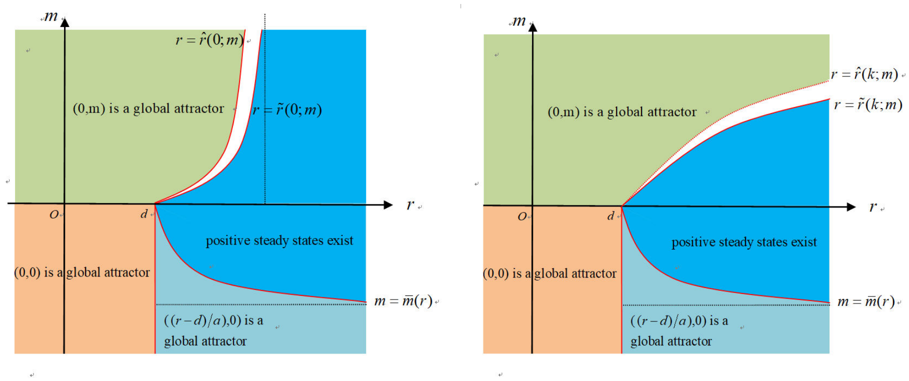
Theorem 3.12. *Let $p(u,v) = u/(1+qu+fv)$.*

(1) Assume that $m \leq \lambda_1\left(d_v, -cb(x)/q\right)$. Then any non-negative solution of (1.3) converges to $(0,0)$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for $r \leq d$, and any non-negative solution of (1.3) with $u_0(x) \geq (\not\equiv) 0$ converges to $((r-d)/a, 0)$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for $r > d$.

(2) Assume that $\lambda_1\left(d_v, -cb(x)/q\right) < m < 0$. Then any non-negative solution of (1.3) converges to $(0,0)$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for $r \leq d$, any non-negative solution of (1.3) with $u_0(x) \geq (\not\equiv) 0$ converges to $((r-d)/a, 0)$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for $d < r < \overline{m}^{-1}(r)$, where $\overline{m}^{-1}(r)$ is the inverse function of $\overline{m}(r) =: \lambda_1\left(d_v, -c(r-d)b(x)/(a+q(r-d))\right)$, and (1.3) admits at least one positive steady state for $r > \overline{m}^{-1}(r)$.

(3) Assume that $m > 0$. Then any non-negative solution of (1.3) with $v_0(x) \geq (\not\equiv) 0$ converges to $(0,m)$ uniformly in $\overline{\Omega}$ as $t \to \infty$ for $r < \widehat{r}(k;m)$ given in Lemma

3.11, and (1.3) *admits at least one positive steady state for* $r > \widetilde{r}(k; m)$ *given in Lemma 3.10.*



(a) $k = 0$ (b) $k > 0$

FIGURE 3. Dynamics of (1.3) with Beddington-DeAngelis functional response.

Our next goal is to investigate the uniqueness and stability of positive solutions to (1.4) with $p(u, v) = u/(1 + qu + fv)$ when the competition of the predator is strong (i.e., f is large). Since the analyses are similar to those of subsection 2.3.2 in [15], we leave the details of the proof to the interested reader and only state the main results. For the case $m < 0$, the following result holds.

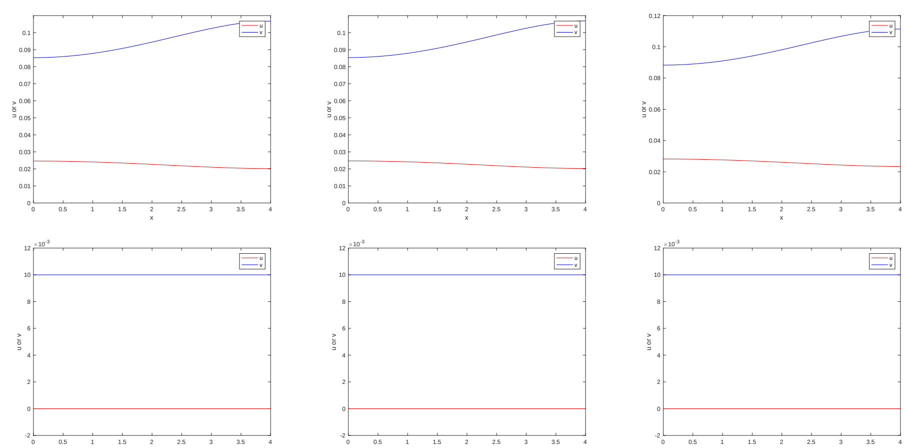
Theorem 3.13. *Assume that* $r > d$ *and* $m \in \big(\lambda_1\big(d_v, -\frac{c(r-d)b(x)}{a+q(r-d)}\big), 0\big)$. *There exists* F *large such that if* $f > F$, *then* (1.4) *with* $p(u, v) = u/(1 + qu + fv)$ *has a unique and asymptotically stable positive solution* (u_f, v_f), *moreover* (u_f, v_f) *is close to* $((r - d)/a, 0)$ *and* fv_f *is close to* V, *where* V *is the unique positive solution of*

$$-d_v\Delta V = \Big(m + \frac{c(r-d)b(x)}{a + q(r-d) + aV}\Big)V,\ x \in \Omega, \quad \frac{\partial V}{\partial \mu} = 0,\ x \in \partial\Omega. \tag{3.1}$$

We want to point out that the cost of fear has no qualitative impact on the behavior of positive solutions (1.4) for the case $m < 0$. However, for the case $m > 0$, the cost of fear decreases the densities of the prey population. More precisely, we have the following result.

Theorem 3.14. *Assume that* $m > 0$ *and* $r/(1+km) > d$. *There exists* F *large such that if* $f > F$, *then* (1.4) *with* $p(u, v) = u/(1 + qu + fv)$ *has no positive solution for* $r < \widetilde{r}(k; m)$ *and has a unique and asymptotically stable positive solution* (u_f, v_f) *for* $r > \widetilde{r}(k; m)$ *which is close to* $((r/(1 + km) - d)/a, m)$.

To conclude this section, we state the effects of fear on the population dynamics of (1.3). In view of Theorem 3.3, 3.6 and 3.12, one finds that a common feature is that the cost of fear has no qualitative impact on the dynamics of (1.3) for $m < 0$, while the cost of fear has obvious impact on the dynamics of (1.3) for $m > 0$. More precisely, when the predator individuals are generalists, the cost of fear makes the predator-only steady state $(0, m)$ more stable by excluding the existence of positive steady states under certain conditions. A special numerical simulation example is presented in Figure 4 to verify this result. In particular, the cost of fear seems to have a more profound effect on the dynamics of (1.3) by choosing $p(u,v) = u/(1+qu+fv)$. To be more specific, when the cost of fear is ignored, the predator-only steady state $(0,m)$ is never a stable one for any $r > \lambda_1(d_u, d+b(x)/f)$ and all $m>0$, and a positive steady state always exists, this implies that two species can coexist; while when the cost of fear is considered, the predator-only steady state $(0,m)$ is globally asymptotically stable for any fixed $r \in \mathbb{R}$ and large m, and hence no positive steady state exists, this shows that the prey will become extinct unconditionally and the predator will persist unconditionally for any initial population distribution. Additionally, Theorem 3.14 shows that when the competition of the predator is strong (i.e., f is large), the cost of fear makes the total population of the prey drop form $(r-d)/a \cdot |\Omega|$ to $((r/(1+km)-d)/a, m) \cdot |\Omega|$, where $|\Omega|$ represents the area of the region Ω. This implies that the cost of fear not only makes it more difficult for the prey to survive, but also reduce the total population of the prey even though the prey population survives.



(a) $q=0$ and $f=0$ (b) $q=0.1$ and $f=0$ (c) $q=0.1$ and $f=1$

Figure 4. Fix $\Omega=(0,4)$, $b(x)=1+x^2$, $u_0(x)=0.1+0.1\sin(5x)$ and $v_0(x)=0.01+0.01\sin(5x)$. By choosing $d_u=11$, $d_v=1$, $r=6$, $d=5.17$, $a=10$, $m=0.01$, $c=0.6$, the effect of the cost of fear is revealed for $k=0$ in the first line and for $k=16$ in the second line.

4. Effect of spatial diffusion and environmental heterogeneity

The purpose of this section is to investigate the impact of spatial diffusion and environmental heterogeneity on the dynamics of (1.3). To make the analysis more explicit, we choose the linear functional response (i.e., $p(u,v)=u$) in the equations (1.3) and (1.4).

By Proposition 2.2, we can easily derive the following properties of

$$\lambda_1\Big(d_u, -\frac{r}{1+km}+d+b(x)m\Big)$$

with respect to the diffusion rate of the prey individuals.

Lemma 4.1. *Assume that $r/(1+km) > d$ and $m > 0$. Then*

(1) $\lambda_1(d_u, -r/(1+km)+d+b(x)m)$ *is strictly monotone increasing with respect to $d_u > 0$.*
(2) *If $d < r/(1+km) \le d + b_* m$, then $\lambda_1(d_u, -r/(1+km) + d + b(x)m) > 0$ for all $d_u > 0$.*
(3) *If $d + b_* m < r/(1+km) < d + \overline{b(x)}m$, then there is a unique value $D_u = D_u(r, m, d, k, b(x)) \in (0, \infty)$ such that $\lambda_1(d_u, -r/(1+km)+d+b(x)m) < 0$ for each $d_u \in (0, D_u)$ and $\lambda_1(d_u, -r/(1+km) + d + b(x)m) > 0$ for each $d_u \in (D_u, \infty)$.*
(4) *If $r/(1+km) \ge d + \overline{b(x)}m$, then $\lambda_1(d_u, -r/(1+km) + d + b(x)m) < 0$ for all $d_u > 0$.*

Based on Theorem 2.4(3), Theorem 2.11(1) and Lemma 4.1, the dynamic behavior of (1.3) with $p(u, v) = u$ with respect to the diffusion rate d_u is as follows.

Theorem 4.2. *Let $p(u, v) = u$, and assume that $r/(1+km) > d$ and $m > 0$. Then*

(1) *If $d < r/(1+km) \le d + b_* m$, then $(0, m)$ is globally asymptotically stable for all $d_u > 0$.*
(2) *If $d+b_* m < r/(1+km) < d+\overline{b(x)}m$, then $(0, m)$ is globally asymptotically stable for each $d_u \in (D_u, \infty)$; $(0, m)$ is unstable and (1.3) admits at least one positive steady state for each $d_u \in (0, D_u)$.*
(3) *If $r/(1+km) \ge d+\overline{b(x)}m$, then $(0, m)$ is unstable and (1.3) admits at least one positive steady state for all $d_u > 0$.*

Next we investigate the asymptotic profiles of positive steady states to (1.3) with $p(u, v) = u$ as as the diffusion rate of the prey or predator individuals approaches zero or infinity. The following theorem gives the asymptotic behavior of any positive solution to (1.4) with $p(u, v) = u$ as d_u goes to ∞, and a special numerical simulation example is presented in Figure 5.

Theorem 4.3. *Assume that $m > 0$ and $r/(1+km) > d + m\overline{b(x)}$. Then for fixed $d_v > 0$, any positive solution (u, v) of (1.4) with $p(u, v) = u$ satisfies*

$$(u, v) \to (u_\infty, v_\infty) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \text{ as } d_u \to \infty,$$

where

$$u_\infty = \frac{1}{a|\Omega|} \int_\Omega \Big(\frac{r}{1+kv_\infty} - d - b(x)v_\infty\Big)dx$$

and v_∞ is a positive solution of

$$\begin{gathered} -d_v \Delta v_\infty = \Big(m + \frac{cb(x)}{a|\Omega|} \int_\Omega \Big(\frac{r}{1+kv_\infty} - d - b(x)v_\infty\Big)dx\Big)v_\infty - v_\infty^2, \quad x \in \Omega, \\ \frac{\partial v_\infty}{\partial \mu} = 0, \quad x \in \partial\Omega. \end{gathered} \tag{4.1}$$

Proof. From Theorem 4.2(3), our assumption conditions ensure that (1.4) with $p(u, v) = u$ admits at least one positive solution for all $d_u > 0$. Hence, we may assume that $(u_{d_{u,n}}, v_{d_{u,n}})$ is a positive solution of (1.4) with $p(u_{d_{u,n}}, v_{d_{u,n}}) = u_{d_{u,n}}$ and $d_u = d_{u,n}$, where $d_{u,n} \to \infty$ as $n \to \infty$. By Lemma 2.7 and the embedding theorem, we can assume that

$$(u_{d_{u,n}}, v_{d_{u,n}}) \to (u_\infty, v_\infty) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \text{ as } n \to \infty.$$

As $n \to \infty$, it follows from the equation for $u_{d_{u,n}}$ and Lemma 2.6 that u_∞ is a non-negative constant. If we assume that $u_\infty = 0$ in $\overline{\Omega}$, then from the equation for $v_{d_{u,n}}$, we find that v_∞ satisfies

$$-d_v \Delta v_\infty = m v_\infty - v_\infty^2,\ x \in \Omega, \quad \frac{\partial v_\infty}{\partial \mu} = 0,\ x \in \partial\Omega.$$

By Lemma 2.6, $v_{d_{u,n}} \geq m > 0$ for all $n \in \mathbb{N}^+$, and thus $v_\infty \geq m > 0$ in $\overline{\Omega}$. Hence, we have $v_\infty = m$ in $\overline{\Omega}$. Let $\widetilde{u}_{d_{u,n}} = u_{d_{u,n}}/\|u_{d_{u,n}}\|_{L^\infty(\Omega)}$. Then from the equation for $u_{d_{u,n}}$ it follows that

$$\begin{gathered}
-d_{u,n} \Delta \widetilde{u}_{d_{u,n}} = \Big(\frac{r}{1+kv_{d_{u,n}}} - d - a u_{d_{u,n}} - b(x) v_{d_{u,n}}\Big) \widetilde{u}_{d_{u,n}}, \quad x \in \Omega, \\
\frac{\partial \widetilde{u}_{d_{u,n}}}{\partial \mu} = 0, \quad x \in \partial\Omega.
\end{gathered}$$

By the standard elliptic regularity theory, we can assume that $\widetilde{u}_{d_{u,n}} \to 1$ in $C^1(\overline{\Omega})$ since $d_{u,n} \to \infty$ as $n \to \infty$. Moreover, for all $n \in \mathbb{N}^+$, we integrate the equation for $\widetilde{u}_{d_{u,n}}$ over Ω to obtain

$$\int_\Omega \Big(\frac{r}{1+kv_{d_{u,n}}} - d - a u_{d_{u,n}} - b(x) v_{d_{u,n}}\Big) \widetilde{u}_{d_{u,n}} dx = 0.$$

Letting $n \to \infty$, we obtain $r/(1+km) = d + m\overline{b(x)}$. This contradiction implies that u_∞ is a positive constant. Hence, we deduce from the equation for $u_{d_{u,n}}$ that

$$u_\infty = \frac{1}{a|\Omega|} \int_\Omega \Big(\frac{r}{1+kv_\infty} - d - b(x) v_\infty\Big) dx.$$

This implies that v_∞ is a non-negative solution of (4.1).

It remains to prove that v_∞ is a positive solution of (4.1). Otherwise, we apply Lemma 2.6 and the Harnack inequality to (4.1) to derive that $v_\infty \equiv 0$ in $\overline{\Omega}$. This shows that $u_\infty = (r-d)/a$. Let $\widetilde{v}_{d_{u,n}} = v_{d_{u,n}}/\|v_{d_{u,n}}\|_{L^\infty(\Omega)}$. Then it follows from the equation for $v_{d_{u,n}}$ that

$$-d_v \Delta \widetilde{v}_{d_{u,n}} = \big(m - v_{d_{u,n}} + c b(x) u_{d_{u,n}}\big) \widetilde{v}_{d_{u,n}},\ x \in \Omega, \quad \frac{\partial \widetilde{v}_{d_{u,n}}}{\partial \mu} = 0,\ x \in \partial\Omega.$$

By the standard elliptic regularity theory, we can assume that $\widetilde{v}_{d_{u,n}} \to \widetilde{v}_\infty$ in $C^1(\overline{\Omega})$ as $n \to \infty$. Moreover, $\widetilde{v}_\infty$ satisfies

$$-d_v \Delta \widetilde{v}_\infty = (m + c b(x)(r-d)/a)\, \widetilde{v}_\infty,\ x \in \Omega, \quad \frac{\partial \widetilde{v}_\infty}{\partial \mu} = 0,\ x \in \partial\Omega.$$

Since $\|\widetilde{v}_\infty\|_{L^\infty(\Omega)} = 1$, we apply the Harnack inequality to obtain $\widetilde{v}_\infty > 0$ in $\overline{\Omega}$. Thus, it follows from the Krein-Rutman theorem that $\lambda_1(d_v, -m - cb(x)(r-d)/a) = 0$. This is a contradiction, and thus we derive the desired result. This completes the proof. □

We next investigate the asymptotic behavior of any positive solution to (1.4) with $p(u,v) = u$ as d_u goes to 0, and a special numerical simulation example is presented in Figure 6.

Theorem 4.4. *Assume that $r/(1+km) > d$, $m > 0$ and*

$$\frac{r}{1+k(m+cb^*(r-d)/a)} > d + b^* \Big(m + \frac{cb^*(r-d)}{a}\Big). \tag{4.2}$$

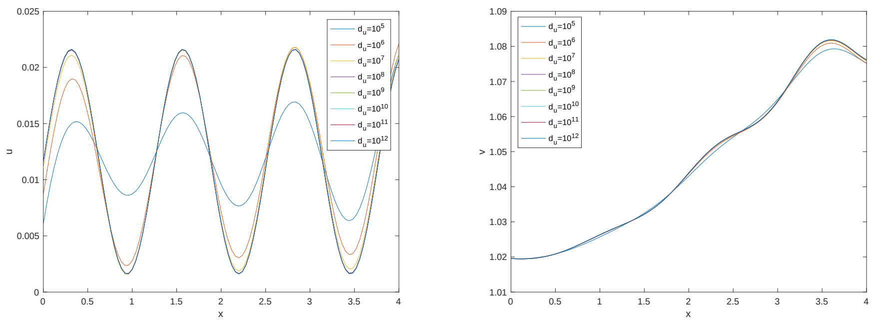


FIGURE 5. Fix $\Omega = (0,4)$, $b(x) = 1+x^2$, $u_0(x) = 0.01+0.01\sin(5x)$ and $v_0(x) = 1+\sin(5x)$. By choosing $d_v = 1$, $r = 24$, $k = 1$, $d = 5$, $a = 1$, $m = 1$, $c = 0.6$, each curve represents the solution (u,v) of (1.4) with $p(u,v) = u$ for large d_u.

Then for fixed $d_v > 0$, any positive solution (u,v) of (1.4) with $p(u,v) = u$ satisfies

$$(u,v) \to \Big(\frac{1}{a}\Big(\frac{r}{1+kv_\infty} - d - b(x)v_\infty\Big), v_\infty\Big) \quad \text{in } L^\infty(\Omega)\times C^1(\overline{\Omega}) \text{ as } d_u \to 0,$$

where v_∞ is the unique positive solution of

$$\begin{gathered}
-d_v\Delta v_\infty = \Big(m + \frac{c(r-d)b(x)}{a} - \Big(\frac{crb(x)k}{a(1+kv_\infty)} + \frac{cb^2(x)}{a} + 1\Big)v_\infty\Big)v_\infty, \quad x \in \Omega, \\
\frac{\partial v_\infty}{\partial \mu} = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{4.3}$$

Proof. By (4.2), it is clear that $r/(1+km) > d + mb_*$. Hence, from Theorem 4.2(2) and (3), we see that (1.4) with $p(u,v) = u$ admits at least one positive solution for small $d_u > 0$. Let $(u_{d_{u,n}}, v_{d_{u,n}})$ be any positive solution of (1.4) with $p(u_{d_{u,n}}, v_{d_{u,n}}) = u_{d_{u,n}}$ and $d_u = d_{u,n}$, where $d_{u,n} \to 0$ as $n \to \infty$. Then for fixed $d_v > 0$, Lemma 2.7 shows that $\|v_{d_{u,n}}\|_{W^{2,p}(\Omega)}$ is bounded for any $p > 1$. Hence, it follows from the Sobolev embedding theorem that the sequence $\{v_{d_{u,n}}\}_{n=1}^\infty$ is compact in $C^1(\overline{\Omega})$. We may assume that $v_{d_{u,n}} \to v_\infty \geq m > 0$ in $C^1(\overline{\Omega})$ since $v_{d_{u,n}} \geq m > 0$ for all $n \in \mathbb{N}^+$. This result ensures that there exists some small $\varepsilon > 0$ such that

$$0 < v_\infty - \varepsilon \leq v_{d_{u,n}} \leq v_\infty + \varepsilon \quad \text{in } \overline{\Omega} \tag{4.4}$$

for all large n.

By (4.4) and the equation for $u_{d_{u,n}}$, we find that

$$-d_{u,n}\Delta u_{d_{u,n}} \leq \Big(\frac{r}{1+k(v_\infty - \varepsilon)} - d - au_{d_{u,n}} - b(x)(v_\infty - \varepsilon)\Big)u_{d_{u,n}}, \quad x \in \Omega.$$

A standard comparison argument yields $u_{d_{u,n}} \leq \overline{U}_{d_{u,n}}$ in $\overline{\Omega}$ for all large n, where $\overline{U}_{d_{u,n}}$ is the unique positive solution of

$$-d_{u,n}\Delta\overline{U}_{d_{u,n}} = \Big(\frac{r}{1+k(v_\infty - \varepsilon)} - d - b(x)(v_\infty - \varepsilon) - a\overline{U}_{d_{u,n}}\Big)\overline{U}_{d_{u,n}}, \quad x \in \Omega,$$

$$\frac{\partial \overline{U}_{d_{u,n}}}{\partial \mu} = 0, \quad x \in \partial\Omega.$$

Indeed, when (4.2) holds, we use Proposition 2.2(1) and Lemma 2.6 to obtain

$$\lambda_1\Big(d_{u,n}, -\frac{r}{1+k(v_\infty-\varepsilon)} + d + b(x)(v_\infty-\varepsilon)\Big) < 0.$$

Thus, the existence and uniqueness of $\overline{U}_{d_{u,n}}$ is clear by [11, Theorems 3.5 and 3.7]. Moreover, from the above analysis, we find that the hypothesis (4.2) ensures that $r/(1+k(v_\infty-\varepsilon)) - d - b(x)(v_\infty-\varepsilon) > 0$ in $\overline{\Omega}$. Hence, by [17, Theorem 1.1(c)], we see that

$$\overline{U}_{d_{u,n}} \to \frac{1}{a}\Big(\frac{r}{1+k(v_\infty-\varepsilon)} - d - b(x)(v_\infty-\varepsilon)\Big) \quad \text{in } L^\infty(\Omega) \text{ as } n \to \infty.$$

This shows that

$$\limsup_{n\to\infty} u_{d_{u,n}} \le \frac{1}{a}\Big(\frac{r}{1+k(v_\infty-\varepsilon)} - d - b(x)(v_\infty-\varepsilon)\Big). \tag{4.5}$$

On the other hand, by (4.4) and the equation for $u_{d_{u,n}}$, we find that

$$-d_{u,n}\Delta u_{d_{u,n}} \ge \Big(\frac{r}{1+k(v_\infty+\varepsilon)} - d - a u_{d_{u,n}} - b(x)(v_\infty+\varepsilon)\Big) u_{d_{u,n}}, \quad x \in \Omega.$$

It follows from the standard comparison argument that $u_{d_{u,n}} \ge \underline{U}_{d_{u,n}}$ in $\overline{\Omega}$ for all large n, where $\underline{U}_{d_{u,n}}$ is the unique positive solution of

$$-d_{u,n}\Delta \underline{U}_{d_{u,n}} = \Big(\frac{r}{1+k(v_\infty+\varepsilon)} - d - b(x)(v_\infty+\varepsilon) - a\underline{U}_{d_{u,n}}\Big)\underline{U}_{d_{u,n}}, \quad x \in \Omega,$$
$$\frac{\partial \underline{U}_{d_{u,n}}}{\partial \mu} = 0, \quad x \in \partial\Omega.$$

Note that the existence and uniqueness of $\underline{U}_{d_{u,n}}$ can be verified as above. From [17, Theorem 1.1(c)] it follows that

$$\underline{U}_{d_{u,n}} \to \frac{1}{a}\Big(\frac{r}{1+k(v_\infty+\varepsilon)} - d - b(x)(v_\infty+\varepsilon)\Big) \quad \text{in } L^\infty(\Omega) \text{ as } n \to \infty.$$

This shows that

$$\liminf_{n\to\infty} u_{d_{u,n}} \ge \frac{1}{a}\Big(\frac{r}{1+k(v_\infty+\varepsilon)} - d - b(x)(v_\infty+\varepsilon)\Big). \tag{4.6}$$

Letting $\varepsilon \to 0$ in (4.5) and (4.6), we derive

$$u_{d_{u,n}} \to \frac{1}{a}\Big(\frac{r}{1+kv_\infty} - d - b(x)v_\infty\Big) \quad \text{in } L^\infty(\Omega) \text{ as } n \to \infty.$$

This means that v_∞ satisfies (4.3).

Since $\lambda_1(d_v, -m - c(r-d)b(x)/a) < 0$, [11, Theorems 3.5 and 3.7] imply that (4.3) has a unique positive solution. Moreover, by (4.4), v_∞ is a positive solution of (4.3). Therefore, v_∞ is the unique positive solution of (4.3). The proof is complete. $\square$

Now we study the effect of the slow or fast movement of the predator individuals on the profiles of positive solutions of (1.4) with $p(u,v) = u$. The following theorem gives the asymptotic profiles of positive solutions as the predator diffusion coefficient d_v goes to 0, and a special numerical simulation example is presented in Figure 7.

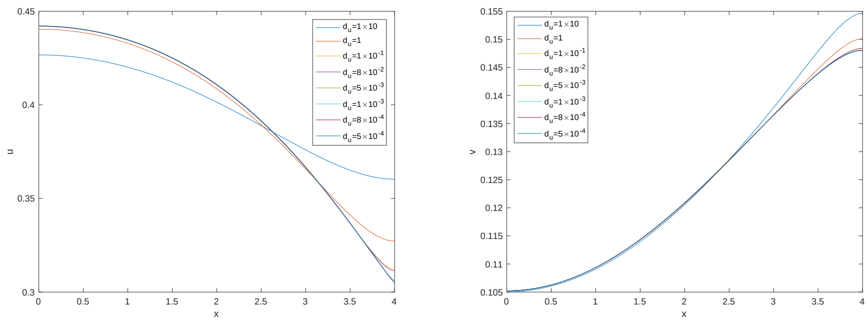


FIGURE 6. Fix $\Omega = (0,4)$, $b(x) = 1 + x^2$, $u_0(x) = 0.01 + 0.01\sin(5x)$ and $v_0(x) = 1 + \sin(5x)$. By choosing $d_v = 0.01$, $r = 10$, $k = 1$, $d = 0.1$, $a = 20$, $m = 0.1$, $c = 0.01$, each curve represents the solution (u,v) of (1.4) with $p(u,v) = u$ for small d_u.

Theorem 4.5. *Assume that $m > 0$ and either $d + b_* m < r/(1+km) < d + \overline{b(x)}m$ and $d_u \in (0, D_u)$ or $r/(1+km) \geq d + \overline{b(x)}m$ and $d_u > 0$. Then for fixed $d_u > 0$, any positive solution (u,v) of (1.4) with $p(u,v) = u$ satisfies*

$$(u,v) \to (u_\infty, m + cb(x)u_\infty) \quad \text{in } C^1(\overline{\Omega}) \times L^\infty(\Omega) \text{ as } d_v \to 0,$$

where u_∞ is the unique positive solution of

$$\begin{gathered} -d_u \Delta u_\infty = \frac{r u_\infty}{1 + k\left(m + cb(x)u_\infty\right)} - (d + mb(x))u_\infty - (a + cb^2(x))u_\infty^2, \quad x \in \Omega, \\ \frac{\partial u_\infty}{\partial \mu} = 0, \quad x \in \partial\Omega. \end{gathered} \tag{4.7}$$

Proof. From Theorem 4.2(2) and (3), our assumption conditions ensure that (1.4) with $p(u,v) = u$ admits at least one positive solution for all $d_v > 0$. Hence, we may assume that $(u_{d_{v,n}}, v_{d_{v,n}})$ is a positive solution of (1.4) with $p(u_{d_{v,n}}, v_{d_{v,n}}) = u_{d_{v,n}}$ and $d_v = d_{v,n}$, where $d_{v,n} \to 0$ as $n \to \infty$. Then by Lemma 2.7, we find that $\|u_{d_{v,n}}\|_{W^{2,p}(\Omega)}$ is bounded for any $p > 1$. By the Sobolev embedding theorem, $\{u_{d_{v,n}}\}_{n=1}^\infty$ is compact in $C^1(\overline{\Omega})$. Thus, we may assume that $u_{d_{v,n}} \to u_\infty \geq 0$ in $C^1(\overline{\Omega})$. This ensures that there exists some small $\varepsilon > 0$ such that

$$u_\infty - \varepsilon \leq u_{d_{v,n}} \leq u_\infty + \varepsilon \text{ in } \overline{\Omega} \tag{4.8}$$

for all large n.

From the equation for $v_{d_{v,n}}$ and (4.8), it follows from

$$\begin{gathered} -d_{v,n}\Delta v_{d_{v,n}} \leq \left(m + cb(x)(u_\infty + \varepsilon) - v_{d_{v,n}}\right) v_{d_{v,n}}, \ x \in \Omega, \quad \frac{\partial v_{d_{v,n}}}{\partial \mu} = 0, \ x \in \partial\Omega, \\ -d_{v,n}\Delta v_{d_{v,n}} \geq \left(m + cb(x)(u_\infty - \varepsilon) - v_{d_{v,n}}\right) v_{d_{v,n}}, \ x \in \Omega, \quad \frac{\partial v_{d_{v,n}}}{\partial \mu} = 0, \ x \in \partial\Omega. \end{gathered}$$

It follows from a comparison argument that $\underline{V}_{d_{v,n}} \le v_{d_{v,n}} \le \overline{V}_{d_{v,n}}$ in $\overline{\Omega}$ for all large n, where $\overline{V}_{d_{v,n}}$ is the unique positive solution of

$$-d_{v,n}\Delta\overline{V}_{d_{v,n}} = \big(m + cb(x)(u_\infty + \varepsilon) - \overline{V}_{d_{v,n}}\big)\overline{V}_{d_{v,n}},\ x \in \Omega, \quad \frac{\partial \overline{V}_{d_{v,n}}}{\partial \mu} = 0,\ x \in \partial\Omega,$$

and $\underline{V}_{d_{v,n}}$ is the unique positive solution of

$$-d_{v,n}\Delta\underline{V}_{d_{v,n}} = \big(m + cb(x)(u_\infty - \varepsilon) - \underline{V}_{d_{v,n}}\big)\underline{V}_{d_{v,n}},\ x \in \Omega, \quad \frac{\partial \underline{V}_{d_{v,n}}}{\partial \mu} = 0,\ x \in \partial\Omega.$$

From [11, Theorems 3.5 and 3.7], the uniqueness of $\overline{V}_{d_{v,n}}$ or $\underline{V}_{d_{v,n}}$ is clear since $\lambda_1(d_{v,n}, -m - cb(x)u_\infty \pm \varepsilon cb(x)) < 0$ for $m > 0$ and small $\varepsilon > 0$. We can apply [17, Theorem 1.1(c)] to conclude that

$$\begin{gathered}\overline{V}_{d_{v,n}} \to m + cb(x)(u_\infty + \varepsilon) \quad \text{in } L^\infty(\Omega) \text{ as } \ n \to \infty,\\ \underline{V}_{d_{v,n}} \to m + cb(x)(u_\infty - \varepsilon) \quad \text{in } L^\infty(\Omega) \text{ as } n \to \infty.\end{gathered}$$

Therefore, we let $\varepsilon \to 0$ to derive

$$v_{d_{v,n}} \to m + cb(x)u_\infty \quad \text{in } L^\infty(\Omega) \text{ as } n \to \infty.$$

This shows that u_∞ is a non-negative solution of (4.7).

We next claim that u_∞ is a positive solution of (4.7). Argue by a contradiction. We may assume that there exists some point $x_0 \in \overline{\Omega}$ such that $u_\infty(x_0) = 0$. Thus, we apply the Harnack inequality to obtain that $u_\infty \equiv 0$ in $\overline{\Omega}$. Hence, we find that $v_{d_{v,n}} \to m$ uniformly in $\overline{\Omega}$ as $n \to \infty$. Denote $\widetilde{u}_{d_{v,n}} = u_{d_{v,n}}/\|u_{d_{v,n}}\|_{L^\infty(\Omega)}$. From the equation for $u_{d_{v,n}}$, it follows that

$$\begin{gathered}-d_u\Delta\widetilde{u}_{d_{v,n}} = \Big(\frac{r}{1 + kv_{d_{v,n}}} - d - au_{d_{v,n}} - b(x)v_{d_{v,n}}\Big)\widetilde{u}_{d_{v,n}}, \quad x \in \Omega,\\ \frac{\partial \widetilde{u}_{d_{v,n}}}{\partial \mu} = 0, \quad x \in \partial\Omega.\end{gathered}$$

Then we apply the standard elliptic regularity theory to derive $\widetilde{u}_{d_{v,n}} \to \widetilde{u}_\infty$ in $C^1(\overline{\Omega})$ as $n \to \infty$, where $\widetilde{u}_\infty$ is a non-negative function and satisfies

$$-d_u\Delta\widetilde{u}_\infty = \Big(\frac{r}{1 + km} - d - b(x)m\Big)\widetilde{u}_\infty,\ x \in \Omega, \quad \frac{\partial \widetilde{u}_\infty}{\partial \mu} = 0,\ x \in \partial\Omega.$$

Since $\|\widetilde{u}_\infty\|_{L^\infty(\Omega)} = 1$, the Harnack inequality implies that $\widetilde{u}_\infty > 0$ in $\overline{\Omega}$. Consequently, we apply the Krein-Rutman theorem to obtain that

$$\lambda_1\Big(d_u, -\frac{r}{1 + km} + d + b(x)m\Big) = 0,$$

a contradiction. Therefore, $u_\infty > 0$ in $\overline{\Omega}$.

Since our assumption ensure that $\lambda_1\left(d_u, -r/(1 + km) + d + b(x)m\right) < 0$, and (4.7) is equivalent to

$$\begin{aligned}-d_u\Delta u_\infty &= \Big(\frac{r}{1 + km} - d - mb(x)\Big)u_\infty\\ &\quad - \Big(\frac{crkb(x)}{(1 + km)(1 + km + ckb(x)u_\infty)} + a + cb^2(x)\Big)u_\infty^2, \quad x \in \Omega,\\ &\qquad \frac{\partial u_\infty}{\partial \mu} = 0, \quad x \in \partial\Omega,\end{aligned}$$

we derive from [11, Theorems 3.5 and 3.7] that the above equation has a unique positive solution. Thus (4.7) has a unique positive solution u_∞. The proof is complete. $\square$

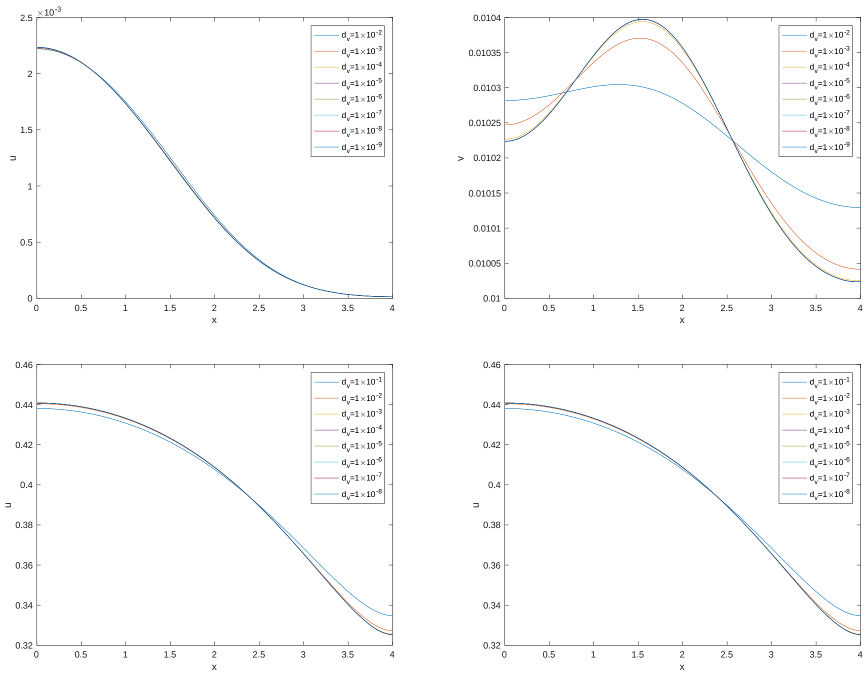


FIGURE 7. Fix $\Omega = (0, 4)$, $b(x) = 1+x^2$, $u_0(x) = 0.01+0.01\sin(5x)$ and $v_0(x) = 1 + \sin(5x)$, each curve represents the solution (u, v) of (1.4) with $p(u, v) = u$ for small d_v by choosing $d_u = 0.01$, $r = 5$, $k = 2$, $d = 4.84$, $a = 20$, $m = 0.01$, $c = 0.1$ in the first line, and by choosing $d_u = 1$, $r = 10$, $k = 1$, $d = 0.1$, $a = 20$, $m = 0.1$, $c = 0.01$ in the second line.

As d_v goes to ∞, the asymptotic behavior of positive solutions of (1.4) with $p(u, v) = u$ reads as follows, and a special numerical simulation example is presented in Figure 8.

Theorem 4.6. *Assume that $m > 0$ and either $d + b_* m < r/(1 + km) < d + \overline{b(x)}m$ and $d_u \in (0, D_u)$ or $r/(1 + km) \geq d + \overline{b(x)}m$ and $d_u > 0$. Then for fixed $d_u > 0$, any positive solution (u, v) of (1.4) with $p(u, v) = u$ satisfies*

$$(u, v) \to (u_\infty, v_\infty) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \text{ as } d_v \to \infty,$$

where

$$v_\infty = \frac{1}{|\Omega|} \int_\Omega (m + cb(x)u_\infty) dx$$

and u_∞ is a positive solution of

$$
\begin{aligned}
-d_u \Delta u_\infty = \Big(&\frac{r}{1+\frac{k}{|\Omega|}\int_\Omega (m+cb(x)u_\infty)dx} - d \\
&- \frac{b(x)}{|\Omega|}\int_\Omega (m+cb(x)u_\infty)dx\Big)u_\infty - au_\infty^2, \quad x \in \Omega, \\
&\frac{\partial u_\infty}{\partial \mu} = 0, \quad x \in \partial\Omega.
\end{aligned}
$$

Since the proof of the above theorem is similar to that of Theorem 4.3, we omit it.

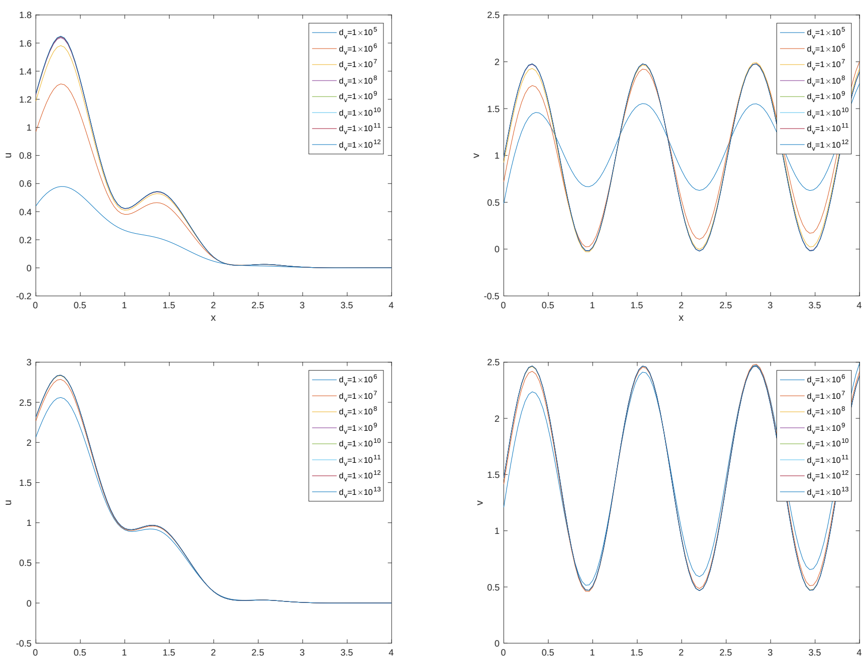


FIGURE 8. Fix $\Omega = (0,4)$, $b(x) = 1 + x^2$, $u_0(x) = 0.01 + 0.01\sin(5x)$ and $v_0(x) = 1 + \sin(5x)$, each curve represents the solution (u, v) of (1.4) with $p(u, v) = u$ for large d_v by choosing $d_u = 1$, $r = 24$, $k = 1$, $d = 5$, $a = 1$, $m = 1$, $c = 0.6$ in the first line, and by choosing $d_u = 1$, $r = 24$, $k = 1$, $d = 9$, $a = 1$, $m = 1$, $c = 0.6$ in the second line.

Finally, we consider the case $m < 0$ to investigate the effects of spatial diffusion and environmental heterogeneity on the dynamics of (1.3). By Proposition 2.2, the following properties of $\lambda_1 (d_v, -m - c(r-d)b(x)/a)$ hold with respect to the diffusion rate of the predator individuals.

Lemma 4.7. *Assume that $r > d$ and $m < 0$. Then*

(1) $\lambda_1 (d_v, -m - c(r-d)b(x)/a)$ *is strictly monotone increasing with respect to* $d_v > 0$.

(2) *If* $m \le -cb^*(r-d)/a$, *then* $\lambda_1(d_v, -m - c(r-d)b(x)/a) > 0$ *for all* $d_v > 0$.
(3) *If* $-cb^*(r-d)/a < m < -c\overline{b(x)}(r-d)/a$, *then there is a unique value* $D_v = D_v(r, m, d, a, c, b(x)) \in (0, \infty)$ *such that* $\lambda_1(d_v, -m - c(r-d)b(x)/a) < 0$ *for* $d_v \in (0, D_v)$ *and* $\lambda_1(d_v, -m - c(r-d)b(x)/a) > 0$ *for* $d_v \in (D_v, \infty)$.
(4) *If* $-c\overline{b(x)}(r-d)/a \le m < 0$, *then* $\lambda_1(d_v, -m - c(r-d)b(x)/a) < 0$ *for all* $d_v > 0$.

In view of Theorems 2.4(2) and 2.11(2) and Lemma 4.7, the dynamic behavior of (1.3) with $p(u, v) = u$ with respect to the diffusion rate d_v is as follows.

Theorem 4.8. *Let* $p(u, v) = u$. *Assume that* $r > d$ *and* $m < 0$. *Then*
(1) *If* $m \le -cb^*(r-d)/a$, *then* $((r-d)/a, 0)$ *is globally asymptotically stable for all* $d_v > 0$.
(2) *If* $-cb^*(r-d)/a < m < -c\overline{b(x)}(r-d)/a$, *then* $((r-d)/a, 0)$ *is globally asymptotically stable for each* $d_v \in (D_v, \infty)$; $((r-d)/a, 0)$ *is unstable and* (1.3) *admits at least one positive steady state for each* $d_v \in (0, D_v)$.
(3) *If* $-c\overline{b(x)}(r-d)/a \le m < 0$, *then* $((r-d)/a, 0)$ *is unstable and* (1.3) *admits at least one positive steady state for all* $d_v > 0$.

5. Summary and discussion

Motivated by some recent experimental field study and mathematical model analysis on the fear effect of prey, we proposed a reaction-diffusion system to demonstrate the impact of fear cost on the population dynamics of prey. The novelty lies in the incorporation of fear cost, spatial diffusion and environmental heterogeneity. We have theoretically analyzed the dynamics of the model and obtained some insights on how fear cost, spatial diffusion and environmental heterogeneity affects the population dynamics.

Our theoretical results established in section 3 indicate that when the predator individuals are generalists, the cost of fear makes the prey more likely to become extinct. Moreover, our results also indicate that the impact of fear cost on the dynamics of the model is closely related to the functional response. For the case with the linear functional response, it seems that only the effect described above is observed. For the case with the Holling-type II functional response, we find that the cost of fear will not only affect the existence of positive steady states but also change the direction of steady-state bifurcation. For the case with the Beddington-DeAngelis functional response, a more profound effect is observed: when the cost of fear is ignored, a positive steady state always exists for any $r > \lambda_1(d_u, d + b(x)/f)$ even though the predator is very strong; while when the cost of fear is considered, $(0, m)$ will become globally asymptotically stable, and no positive steady state exists. Another effect of fear cost is that when the competition of the predator is strong, the cost of fear makes the total population of the prey drop form $(r-d)/a\cdot|\Omega|$ to $((r/(1+km) - d)/a, m) \cdot |\Omega|$. This implies that the cost of fear not only makes the prey more difficult to survive, but also reduce the total population of the prey even though the prey population survives.

Our theoretical results established in section 4 indicate that spatial diffusion and environmental heterogeneity significantly influence the dynamics of (1.3). When the predator individuals are generalists, the predator can always invade (i.e., $(0, m)$ is globally asymptotically stable) for any diffusion rate of the prey if the birth rate of the prey is small; the predator can invade for large diffusion rate of the prey and will coexist with the prey for small diffusion rate of the prey if the birth rate of the prey is intermediate; the predator and the prey can always coexist for any diffusion rate of the prey if the birth rate of the prey is large. When the predator individuals are specialists, the predator can rarely invade (i.e., $((r-d)/a,0)$ is globally asymptotically stable) for any diffusion rate of the predator if the death rate of the predator is large; the predator can rarely invade for large diffusion rate of the predator and will coexist with the prey for small diffusion rate of the predator if the death rate of the predator is intermediate; the predator and the prey can always coexist for any diffusion rate of the predator if the death rate of the predator is small. These results shows that for the generalist predator, predator invasion or coexistence with the prey mainly depends on the diffusion rate and the birth rate of the prey; while for the specialist predator, predator invasion or coexistence with the prey mainly depends on the diffusion rate and the death rate of the predator.

We next compare the findings here with those in [27] where system (1.2) were studied where the coefficients are positive constants. We find that the analytical methods and main results are quiet different from those in [27]. In particular, spatially dependent coefficients bring the theoretical analysis lots of difficult and challenging. In terms of results, we summarize and highlight according to various functional response. When the linear functional response is adopted, consideration of spatial heterogeneity can lead to spatial pattern formation. When the Holling-type II or Beddington-DeAngelis functional response is adopted, we explicitly establish the sufficient conditions of spatial pattern formation, rather than just by numerical simulation as in [27]. More importantly, the multiplicity and uniqueness of non-constant positive steady states to (1.3) are established by choosing different sets of parameters. In addition, by comparing our results with those results obtained in [27], we find that the cost of fear has obvious impact on the dynamics of (1.2) for $m<0$, but has no qualitative impact on the dynamics of (1.3). This indicates that whether the cost of fear affects the dynamics (especially for $m<0$) may be dependent on spatial heterogeneity.

Our theoretical results suggest that factors such as the level of fear, functional responses, environmental heterogeneity and movement of individuals play vital but subtle roles in the dynamics of (1.3). Therefore, a good understanding of these factors could be helpful in understanding the dynamic behavior of the model and be important in designing effective species conservation measures.

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