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COMMENTARY ON LOCAL AND BOUNDARY REGULARITY OF WEAK SOLUTIONS TO NAVIER-STOKES EQUATIONS

ZDENĚK SKALÁK

Abstract. We present results on local and boundary regularity for weak solutions to the Navier-Stokes equations. Beginning with the regularity criterion proved recently in [14] for the Cauchy problem, we show that this criterion holds also locally. This is also the case for some other results: We present two examples concerning the regularity of weak solutions stemming from the regularity of two components of the vorticity ([2]) or from the regularity of the pressure ([3]). We conclude by presenting regularity criteria near the boundary based on the results in [10] and [16].

1. Introduction

Let $\Omega = \mathbb{R}^3$ or Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, let $T > 0$ and $Q_T = \Omega \times (0,T)$. Consider the Navier-Stokes initial-boundary value problem describing the evolution of the velocity $u(x,t)$ and the pressure $p(x,t)$ in Q_T:

$$\frac{\partial u}{\partial t} - \nu\Delta u + u \cdot \nabla u + \nabla p = 0 \quad \text{in } Q_T, \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in } Q_T, \tag{1.2}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T) \quad \text{if } \Omega \text{ is bounded}, \tag{1.3}$$

$$u\big|_{t=0} = u_0, \tag{1.4}$$

where $\nu > 0$ is the viscosity coefficient. The initial data u_0 satisfy the compatibility conditions $u_0|_{\partial\Omega} = 0$ and $\operatorname{div} u_0 = 0$. Let us stress that the condition (1.3) does not apply if $\Omega = \mathbb{R}^3$.

As is usual in the standard theory of the Navier-Stokes equations, define $D(\Omega) = \{\psi \in C_0^\infty(\Omega)^3; \nabla \cdot \psi = 0 \text{ in } \Omega\}$ and let $L^2_\sigma(\Omega)$ be the completion of $D(\Omega)$ in $L^2(\Omega)^3$. Define also $D_T = \{\varphi \in C_0^\infty(\Omega \times [0,T))^3; \nabla \cdot \varphi = 0 \text{ in } \Omega \times [0,T)\}$.

Definition 1.1. Let $u_0 \in L^2_\sigma(\Omega)$. A measurable function $u : Q_T \to \mathbb{R}^3$ is called a weak solution of the problem (1.1)–(1.4) if $u \in L^2(0,T,W^{1,2}(\Omega)) \cap L^\infty(0,T,L^2_\sigma(\Omega))$

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and

$$\int_0^T \int_\Omega \Big[u \cdot \frac{\partial \varphi}{\partial t} - \nu \nabla u \cdot \nabla \varphi - u \cdot \nabla u \cdot \varphi\Big]\, dxdt = -\int_\Omega u_0 \cdot \varphi(\cdot, 0)\, dx$$

for all $\varphi \in D_T$.

The existence of weak solutions is generally known but their uniqueness and regularity remains an open problem (see for example [18]).

Leray-Hopf weak solutions of (1.1)–(1.4) are those weak solutions from Definition 1.1 which satisfy the energy inequality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2 ds \le \|u_0\|_2^2$$

for every $t \in (0, T]$.

A condition stronger than the energy inequality is the so called strong energy inequality. We say that a weak solution of (1.1)–(1.4) satisfies the strong energy inequality, if

$$\|u(t_2)\|_2^2 + 2\int_{t_1}^{t_2} \|\nabla u(s)\|_2^2 ds \le \|u(t_1)\|_2^2 \tag{1.5}$$

for every $t_2 \in (0, T]$ and almost every $0 < t_1 \le t_2$.

The pair (u, p) is called a suitable weak solutions of (1.1)–(1.4) if u is a weak solution of (1.1)–(1.4) from Definition 1.1 and together with the pressure p satisfy the so called generalized energy inequality, that is

$$2\nu \int_0^T \int_\Omega |\nabla u|^2 \phi \le \int_0^T \int_\Omega \Big[|u|^2 \Big(\frac{\partial \phi}{\partial t} + \nu \Delta \phi\Big) + (|u|^2 + 2p) u \cdot \nabla \phi\Big] \tag{1.6}$$

for every non-negative real-valued function $\phi \in C_0^\infty(Q_T)$. There is also an equivalent form of (1.6):

$$\int_{\Omega \times \{t\}} |u|^2 \phi + 2\nu \int_0^t \int_\Omega |\nabla u|^2 \phi \le \int_0^t \int_\Omega |u|^2 \Big(\frac{\partial \phi}{\partial t} + \nu \Delta \phi\Big) + \int_0^t \int_\Omega (|u|^2 u + 2pu) \cdot \nabla \phi \tag{1.7}$$

which holds for every non-negative real-valued function $\phi \in C_0^\infty(Q_T)$ and every $t \in (0, T)$.

The suitable weak solutions were thoroughly studied in [1] and used in [9, 11, 13]. Their existence is known under the assumption of a sufficient regularity of the initial condition u_0 (see [1]). We use the suitable weak solutions in this paper in Theorems 2.2, 2.3 and 2.6. The main reason for their use is that the local boundary integrals which appear in the proofs of these theorems are easily controllable due to the boundedness of the velocity u and its derivatives and sufficient regularity of the pressure p near the boundary.

It was proved in [6] that if the initial condition u_0 is sufficiently smooth then there exists a suitable weak solution of (1.1)–(1.4) which satisfies the generalized energy inequality for every smooth test function. More precisely, the inequality

$$\begin{aligned} &\int_{\Omega \times \{t_2\}} |u|^2 \phi + 2\nu \int_{t_1}^{t_2} \int_\Omega |\nabla u|^2 \phi \\ &\le \int_{\Omega \times \{t_1\}} |u|^2 \phi + \int_{t_1}^{t_2} \int_\Omega |u|^2 \Big(\frac{\partial \phi}{\partial t} + \nu \Delta \phi\Big) + \int_{t_1}^{t_2} \int_\Omega (|u|^2 u + 2pu) \nabla \phi \end{aligned} \tag{1.8}$$

holds for every $\phi \in C^\infty(\overline{Q}_T), \phi \geq 0$ and every $0 \leq t_2 < T$ and almost every $0 \leq t_1 \leq t_2$. By the choice of $\phi \equiv 1$ on Q_T we see that such solutions satisfy the strong energy inequality which does not seem to be known for the suitable weak solutions satisfying the "ordinary" generalized energy inequality (1.7). We will use these solutions in Theorem 3.2, where the assumption of the strong energy inequality is needed.

Recall that a point $(x_0, t_0) \in \overline{\Omega} \times (0, T)$ is called a regular point of u if u is essentially bounded in a space-time neighbourhood U of (x_0, t_0), that is if $u \in L^\infty(U)$. A point $(x_0, t_0) \in \overline{\Omega} \times (0, T)$ is called singular if it is not regular.

In this paper we use the following regularity criterion proved in [9, Theorem 2.2]. Let us present here only a simplified version for $f \equiv 0$.

Theorem 1.2. *Let (u, p) be a suitable weak solution of* (1.1)–(1.4). *Then there exists a positive number ε_* with the following property. Assume that for a point $z_0 = (x_0, t_0) \in Q_T$ the inequality*

$$\limsup_{r\to 0} \frac{1}{r} \int_{Q(z_0,r)} |\nabla u|^2 < \varepsilon_* \tag{1.9}$$

holds, where $Q(z_0, r) = B(x_0, r) \times (t_0 - r^2, t_0)$. Then z_0 is a regular point of u.

In fact, Theorem 2.2 in [9] is still stronger than Theorem 1.2. It even says that the velocity u is Hölder continuous function in some space-time neighborhood of z_0. The famous criterion proved in [1, Proposition 2], is weaker then Theorem 1.2, since it uses $Q^*(z_0, r) = B(x_0, r) \times (t_0 - \frac{7}{8}r^2, t_0 + \frac{1}{8}r^2)$ instead of $Q(z_0, r)$.

Let us note that an analogous result to Theorem 1.2 was proved in [15] for the case of $x_0 \in \partial\Omega$ and $\partial\Omega \cap B_r(x_0)$ lying in a plane for some $r > 0$.

In what follows, we use the standard notation for the Lebesgue and Sobolev spaces (L^p and $W^{k,p}$, respectively) and for the corresponding norms ($\|\cdot\|_p$ and $\|\cdot\|_{k,p}$, respectively). To simplify the notation we often write $\int f$ instead of $\int_\Omega f(x)\,dx$ or $\int f(x)\,dx$ instead of $\int f(x,t)\,dx$. We do not distinguish between $(L^p)^m$ and L^p. The outer normal vector is denoted by n. In the paper c stands for a generic constant.

2. Local regularity

The main goal of this section is to show that some criterions on regularity of Leray-Hopf weak solutions of the Cauchy problem for the Navier-Stokes equations hold also locally. We will present three examples of such local regularity results. The results are formulated for the suitable weak solutions. The boundary integrals appearing during the computations are then easily controllable due to the boundedness of the velocity u and all its space derivatives near the boundary.

We will suppose in this section that $D \subset Q_T$ is an open set and $(x_0, t_0) \in D$ is an arbitrary point. If (u, p) is a suitable weak solution of (1.1)–(1.4) and if one wants to show that (x_0, t_0) is a regular point of u then it is possible to suppose that the following conditions are fulfilled (for a detailed discussion see [11, 7, 8, 12, 13]).

There exist positive numbers $\varepsilon_1 < \varepsilon_2$ and τ such that $\overline{B_2} \times [t_0 - \tau, t_0 + \tau] \subset D$ and $(\overline{B_2} \setminus B_1) \times [t_0 - \tau, t_0 + \tau] \cap S(u) = \emptyset$, where $B_i = B_{\varepsilon_i}(x_0)$, $i = 1, 2$ and $S(u)$ is a set of all singular points of u from Q_T. Further, there are no singular points of u in $\overline{B_2} \times [t_0 - \tau, t_0)$, u and all its space derivatives are continuous in $(\overline{B_2} \setminus B_1) \times [t_0 - \tau, t_0 + \tau]$ and $\frac{\partial u}{\partial t}$ and p and all their space derivatives are in

$L^\beta((\overline{B_2}\setminus B_1)\times[t_0-\tau,t_0+\tau])$ for every $\beta\in(1,2)$. Moreover, if $\Omega=\mathbb{R}^3$ then $\frac{\partial u}{\partial t}$ and p and all their space derivatives are in $L^\infty((\overline{B_2}\setminus B_1)\times[t_0-\tau,t_0+\tau])$.

For further developments, we denote $B_3=B_{\varepsilon_3}(x_0)$, where $\varepsilon_3=(\varepsilon_1+\varepsilon_2)/2$. As an inspiration for this section served the following regularity criterion proved recently by M. Pokorný in [14].

Theorem 2.1. *Let $u_0\in W^{1,2}(\mathbb{R}^3)$ with div $u_0=0$, $\Omega=\mathbb{R}^3$ and let u be a weak solution of the Navier-Stokes equations* (1.1)–(1.4) *satisfying the energy inequality. Assume moreover that $\nabla u_3\in L^\alpha(0,T;L^\gamma)$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le\frac{3}{2}$, $\frac{4}{3}\le\alpha\le\infty$, $2\le\gamma\le\infty$. Then u and the corresponding pressure p is the smooth solution of the Navier-Stokes equations, i.e. $u\in L^\infty(0,T;W^{1,2}(\mathbb{R}^3))\cap L^2(0,T;W^{2,2}(\mathbb{R}^3))$, $\nabla p\in L^2(0,T;W^{1,2}(\mathbb{R}^3))$. Moreover, $u\in C^\infty([\delta,T)\times\mathbb{R}^3)$ and $p\in C^\infty([\delta,T)\times\mathbb{R}^3)$ for any small positive δ.*

The following result is a local version of Theorem 2.1. Its proof uses the same ideas as the proof of Theorem 2.3 below.

Theorem 2.2. *Let (u,p) be a suitable weak solution of* (1.1)–(1.4)*) and $D\subset Q_T$ be an open set. Assume moreover that $\nabla u_3\in L^{\alpha,\gamma}(D)$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le\frac{3}{2}$, $\frac{4}{3}\le\alpha\le\infty$, $2\le\gamma\le\infty$. Then u has no singular points in D.*

Let $\omega=\operatorname{curl}u=(\omega_1,\omega_2,\omega_3)=(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2},\frac{\partial u_3}{\partial x_1}-\frac{\partial u_1}{\partial x_3},\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1})$ denote the vorticity field. The two-component vorticity field is denoted $\tilde\omega=\omega_1e_1+\omega_2e_2$, where $e_1=(1,0,0),e_2=(0,1,0)$. The following theorem is a local version of Theorem 1 proved in [2].

Theorem 2.3. *Let (u,p) be a suitable weak solution of* (1.1)*)*–(1.4)*. Let $D\subset Q_T$ be an open set and $\tilde\omega\in L^{\alpha,\gamma}(D)$ with $\frac{2}{\alpha}+\frac{3}{\gamma}\le 2$, $1<\alpha<\infty$, $\frac{3}{2}<\gamma<\infty$ or the norm of $\tilde\omega$ in the space $L^{\infty,\frac{3}{2}}(D)$ is sufficiently small. Then u has no singular points in D.*

Proof. We follow the lines of the proof in [2]. The boundary integrals can be handled because the suitable weak solution is considered. We begin with the equation

$$\frac{\partial\omega}{\partial t}-\nu\Delta\omega+u\cdot\nabla\omega-\omega\cdot\nabla u=0. \tag{2.1}$$

Multiplying (2.1) by ω and integrating it over B_3 we get that

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_2^2+\nu\|\nabla\omega(t)\|_2^2=\int_{B_3}\omega\cdot\nabla u\cdot\omega+\int_{\partial B_3}\frac{\partial\omega}{\partial n}\cdot\omega-\frac{1}{2}\int_{\partial B_3}u\cdot n|\omega|^2 \tag{2.2}$$

holds for almost every $t\in(t_0-\tau,t_0)$. The last two boundary integral can be estimated by a constant c independent of time.

We estimate the integral $\int_{B_3}\omega\cdot\nabla u\cdot\omega$. We can express u by means of ω:

$$\begin{aligned}u(x)=\frac{1}{4\pi}\int_{B_2}\frac{\operatorname{rot}\omega(\xi)}{|x-\xi|}\,d\xi+\frac{1}{4\pi}\int_{\partial B_2}\frac{\partial u}{\partial n_\xi}(\xi)\frac{1}{|x-\xi|}\,d_\xi S\\ -\frac{1}{4\pi}\int_{\partial B_2}u(\xi)\frac{\partial}{\partial\xi}\frac{1}{|x-\xi|}\,d\xi.\end{aligned} \tag{2.3}$$

Equation (2.3) holds for every $t\in(t_0-\tau,t_0)$ and every $x\in B_2$. Therefore, $u=\bar u+\bar{\bar u}$, where $\bar u$ is defined by the first integral on the right hand side of (2.3) and $\bar{\bar u}$ is defined by sum of the second and the third integral on the right hand side of (2.3). Since $D_x^\gamma\bar{\bar u}\in L^\infty(B_3\times(t_0-\tau,t_0))$ for every multiindex $\gamma=(\gamma_1,\gamma_2,\gamma_3)$, $\gamma_i\ge 0$, $i=1,2,3$, it is possible to notice that there are no problems with $\bar{\bar{u}}$ in the following procedures. Therefore, we work only with $\bar{u}$ and for the sake of simplicity denote it as u.

From the definition of u we have for every $x \in B_3$ that

$$\begin{aligned} \frac{\partial u_i}{\partial x_j}(x) &= \lim_{\varepsilon \to 0_+} \int_{\partial B_\varepsilon(x)} \frac{\partial}{\partial \xi_j} \frac{1}{|x-\xi|} \epsilon_{ilk} n_k(\xi) \omega_l(\xi) d_\xi S \\ &\quad - \int_{\partial B_2} \frac{\partial}{\partial \xi_j} \frac{1}{|x-\xi|} \epsilon_{ilk} n_k(\xi) \omega_l(\xi) d_\xi S \\ &\quad + \lim_{\varepsilon \to 0_+} \int_{B_{2,\varepsilon}(x)} \frac{\partial^2}{\partial \xi_j \partial \xi_k} \frac{1}{|x-\xi|} \epsilon_{ilk} \omega_l(\xi) d\xi \\ &= I_1(x) + I_2(x) + I_3(x), \end{aligned} \tag{2.4}$$

where $B_{2,\varepsilon}(x) = B_2 \setminus B_\varepsilon(x)$ and ϵ_{ijk} is the Levi-Civita's tensor. After short computation one can get

$$I_1(x) = -\frac{4\pi}{3} \epsilon_{ijl} \omega_l(x)$$

and if we put $I_1(x)$ into the integral $\int_{B_3} \omega \cdot \nabla u \cdot \omega$ instead of ∇u we get

$$\int_{B_3} \omega \cdot \nabla u \cdot \omega = -\frac{4\pi}{3} \epsilon_{ijl} \int_{B_3} \omega_i \omega_j \omega_l = 0. \tag{2.5}$$

Since $|x-\xi| > (\varepsilon_2 - \varepsilon_1)/2$ for every $x \in B_3$ and every $\xi \in \partial B_2$, I_2 is bounded in $B_3 \times (t_0 - \tau, t_0)$ and for the sake of simplicity we do not consider this term any further.

If we put $I_3(x)$ into the integral $\int_{B_3} \omega \cdot \nabla u \cdot \omega$ instead of ∇u and decompose $\omega = \tilde{\omega} + \tilde{\tilde{\omega}}$, $\tilde{\tilde{\omega}} = (0, 0, \omega_3)$, we get

$$\begin{aligned} &\int_{B_3} \omega_j(x) \cdot \Big(\lim_{\varepsilon \to 0_+} \int_{B_{2,\varepsilon}(x)} \frac{\partial^2}{\partial \xi_j \partial \xi_k} \frac{1}{|x-\xi|} \epsilon_{ilk} \omega_l(\xi) d\xi \Big) \cdot \omega_i(x)\, dx \\ &= \int_{B_3} \omega_j P_{ij}(\omega) \omega_i \\ &= \int_{B_3} \omega_j P_{ij}(\tilde{\omega}) \tilde{\omega}_i + \int_{B_3} \omega_j P_{ij}(\tilde{\omega}) \tilde{\tilde{\omega}}_i + \int_{B_3} \omega_j P_{ij}(\tilde{\tilde{\omega}}) \tilde{\omega}_i + \int_{B_3} \omega_j P_{ij}(\tilde{\tilde{\omega}}) \tilde{\tilde{\omega}}_i. \end{aligned} \tag{2.6}$$

where $P(\cdot) = (P_{ij}(\cdot))_{i,j=1}^3$ denotes the singular integral operator defined by the third integral in (2.4). The last integral is equal to zero. The remaining three integrals on the right hand side of (2.6) can be estimated by

$$\begin{aligned} &\int_{B_3} |\omega||P(\tilde{\omega})||\tilde{\tilde{\omega}}| + \int_{B_3} |\omega||P(\tilde{\omega})||\tilde{\omega}| + \int_{B_3} |\omega||P(\tilde{\tilde{\omega}})||\tilde{\omega}| \\ &\le c \int_{B_3} |\omega|^2 |P(\tilde{\omega})| + c \int_{B_3} |\omega||P(\tilde{\tilde{\omega}})||\tilde{\omega}| = J_1 + J_2. \end{aligned} \tag{2.7}$$

We have by the Hölder inequality, the Calderon-Zygmund inequality, the interpolation inequality, the Sobolev inequality and the Young inequality that

$$J_1 \le \|P(\tilde{\omega})\|_\gamma \|\omega\|^2_{\frac{2\gamma}{\gamma-1}} \le c\|\tilde{\omega}\|_\gamma \|\omega\|_2^{\frac{2\gamma-3}{\gamma}} \|\nabla\omega\|_2^{\frac{3}{\gamma}} \le \frac{1}{4}\nu\|\nabla\omega\|_2^2 + C\|\tilde{\omega}\|_\gamma^\alpha \|\omega\|_2^2. \tag{2.8}$$

The same estimate can be obtained for J_2:

$$J_2 \le \frac{1}{4}\nu\|\nabla\omega\|_2^2 + C\|\tilde{\omega}\|_\gamma^\alpha \|\omega\|_2^2. \tag{2.9}$$

If $\alpha = \infty$ and $\gamma = 3/2$ then both J_1 and J_2 can be estimated by

$$C\|\tilde{\omega}\|_{3/2}\|\nabla\omega\|_2^2. \tag{2.10}$$

We get from (2.2)–(2.10) that

$$\frac{d}{dt}\|\omega(t)\|_2^2 + \nu\|\nabla\omega(t)\|_2^2 \le C\|\tilde{\omega}\|_\gamma^\alpha\|\omega\|_2^2 + C \tag{2.11}$$

and by the Gronwall lemma we have

$$\omega \in L^\infty(t_0-\tau, t_0; L^2(B_3)) \cap L^2(t_0-\tau, t_0; W^{1,2}(B_3)). \tag{2.12}$$

We can write for every $x \in B_3$ that

$$\begin{aligned} u(x) = \frac{1}{4\pi}\int_{B_2}\nabla_\xi\frac{1}{|x-\xi|}\times\omega(\xi)\,d\xi + \frac{1}{4\pi}\int_{\partial B_2}\frac{1}{|x-\xi|}\Big(\omega(\xi)\times n(\xi) + \frac{\partial u}{\partial n}(\xi)\Big)\,d_\xi S \\ - \frac{1}{4\pi}\int_{\partial B_2}\frac{\partial}{\partial n_\xi}\frac{1}{|x-\xi|}u(\xi)\,d_\xi S. \end{aligned} \tag{2.13}$$

The boundary integrals on the right hand side of (2.13) cause no problems because they are from $L^\infty(t_0-\tau, t_0; L^\infty(B_3))$. Applying now the famous results (see e.g. [5]) on the first integral in (2.13) we get that

$$u \in L^\infty(t_0-\tau, t_0; L^6(B_3))$$

and from this we are going to derive that

$$u \in L^\infty(t_0-\tau, t_0; W^{1,2}(B_3)). \tag{2.14}$$

In fact we will show a stronger result which we will need further in the proof of Theorem 2.4: if $u \in L^\infty(t_0-\tau, t_0; L^s(B_3))$, $s \in (3,6]$, then $u \in L^\infty(t_0-\tau, t_0; W^{1,2}(B_3))$. Let us multiply the equation (1.1) by $-\Delta u$, integrate it over B_3 and use the integration by parts. We get

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + \nu\|\Delta u\|_2^2 = \int_{B_3} u\cdot\nabla u\cdot\Delta u + \int_{\partial B_3} p\, n\cdot\Delta u + \int_{\partial B_3} n\cdot\nabla u\cdot\frac{\partial u}{\partial t}. \tag{2.15}$$

The last two boundary integrals are from $L^\beta(t_0-\tau, t_0)$ for any $\beta \in (1,2)$. Let us estimate the integral $\int_{B_3} u\cdot\nabla u\cdot\Delta u$.

$$\begin{aligned} \Big|\int_{B_3} u\cdot\nabla u\cdot\Delta u\Big| &\le \|u\|_s\|\nabla u\|_{\frac{2s}{s-2}}\|\Delta u\|_2 \\ &\le \|u\|_s\|\nabla u\|_2^{\frac{s-3}{s}}\|\nabla u\|_6^{\frac{3}{s}}\|\Delta u\|_2 \\ &\le c\|u\|_s\|\nabla u\|_2^{\frac{s-3}{s}}\|\nabla^2 u\|_2^{\frac{s+3}{s}} \\ &\le \frac{\nu}{2}\|\nabla^2 u\|_2^2 + c\|u\|_s^{\frac{2s}{s-3}}\|\nabla u\|_2^2, \end{aligned} \tag{2.16}$$

where we used the Hölder inequality, the interpolation inequality, the Sobolev inequality and the Young inequality. (2.14) now follows from (2.15) and (2.16). It means that the assumption (1.9) is fulfilled and Theorem 1.2 gives that (x_0, t_0) is a regular point of u. Since (x_0, t_0) was an arbitrary point in D, Theorem 2.3 is proved. □

We will now turn our attention to the paper [3]. The main result of this paper is the following theorem:

Theorem 2.4. *Let $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ for some $q > 3$ with $\operatorname{div} u_0 = 0$ in the sense of distributions and $\Omega = \mathbb{R}^3$. Suppose that u is a Leray-Hopf weak solution of (1.1)–(1.4) in $[0, T)$. If $p \in L^{\alpha,\gamma}(\mathbb{R}^3)$ with $\frac{2}{\alpha} + \frac{3}{\gamma} < 2$ and $1 < \alpha \le \infty$, $\frac{3}{2} < \gamma < \infty$, or $p \in L^{1,\infty}$, or else $\|p\|_{L^{\infty,3/2}}$ is sufficiently small, then u is a regular solution in $[0, T)$.*

For some α, γ it is possible to prove a local version of Theorem 2.4 - see Theorem 2.6. In the proof of Theorem 2.6 we will use the following Gronwall lemma (see [17, p. 88]).

Lemma 2.5. *Let g, h and y be locally integrable nonnegative functions on $[0, \infty)$ that satisfy the differential inequality*

$$y'(t) \le g(t)y(t) + h(t) \quad \text{on } [0, \infty), \; y(0) = y_0.$$

Let the function $y'(t)$ be also locally integrable. Then

$$y(t) \le y(0) \exp\Big(\int_0^t g(\tau)\,d\tau\Big) + \int_0^t h(s) \exp\Big(-\int_t^s g(\tau)\,d\tau\Big)\,ds, \; t \ge 0.$$

Theorem 2.6. *Let (u, p) be a suitable weak solution of (1.1))–1.4. Let $D \subset Q_T$ is an open set and $p \in L^{\alpha,\gamma}(D)$ with $1 \le \alpha \le \infty$, $\frac{3}{2} \le \gamma \le \infty$. Then u has no singular points in D if one of the following conditions is fulfilled:*

(i) $\frac{2}{\alpha} + \frac{3}{\gamma} < 2$, $\gamma \in (3, \infty)$ *and* $\alpha \ge 2$
(ii) $\frac{2}{\alpha} + \frac{3}{\gamma} < 2$, $\gamma \in (3, \infty)$ *and* $\Omega = \mathbb{R}^3$
(iii) $\frac{2}{\alpha} + \frac{3}{\gamma} < 2$ *and* $\gamma \in (\frac{3}{2}, 3]$
(iv) $\gamma = \infty$, $\alpha = 1$ *and* $\Omega = \mathbb{R}^3$,
(v) $\gamma = \frac{3}{2}$, $\alpha = \infty$ *and the norm* $\|p\|_{\infty,\frac{3}{2}}$ *is sufficiently small.*

Proof. Let $s > 3$. The proof is based on the inequality

$$\begin{aligned} \frac{d}{dt}\int_{B_3} |u|^s + 2\int_{B_3} |\nabla |u|^{s/2}|^2 &\le 2(s-2)\int_{B_3} |p||u|^{\frac{s-2}{2}} |\nabla |u|^{s/2}| \\ &\quad + \Big|\int_{\partial B_3} spu \cdot n|u|^{s-2}\Big| + \text{boundary integrals}, \end{aligned} \tag{2.17}$$

which can be obtained multiplying the equation (1.1) by $su|u|^{s-2}$ and using the integration by parts. The boundary integrals and the boundary integral with p on the right hand side of (2.17) are from the space $L^1(t_0 - \tau, t_0)$. Using the Gronwall lemma they play the role of the function h. We can write

$$\begin{aligned} p(x) &= \lim_{\varepsilon \to 0_+} \frac{1}{4\pi} \int_{B_{2,\varepsilon}(x)} u_i(y)u_j(y) \frac{\partial^2}{\partial y_i \partial y_j} \frac{1}{|x-y|}\,dy - \frac{1}{3}|u(x)|^2 \\ &\quad + \frac{1}{4\pi} \int_{\partial B_2} \Big\{ \frac{\partial}{\partial y_j}(u_i(y)u_j(y))n_i(y) \frac{1}{|x-y|} - u_i(y)u_j(y)n_j(y) \frac{\partial}{\partial y_i} \frac{1}{|x-y|} \\ &\quad + \frac{\partial p}{\partial n}(y) \frac{1}{|x-y|} - p(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|} \Big\}\, d_y S \\ &= p_1(x) + p_2(x), \end{aligned} \tag{2.18}$$

for every $x \in B_3$, where p_2 is defined by the boundary integral in (2.18). Let us note that in the following estimates p_1 will be handled using the Calderon-Zygmund inequality and $p_2 \in L^\beta(t_0-\tau,t_0;L^\infty(B_3))$ for every $\beta \in (1,2)$ if Ω is a bounded domain and $p_2 \in L^\infty(t_0-\tau,t_0;L^\infty(B_3))$ if $\Omega = \mathbb{R}^3$.

Let us estimate the integral $I = \int_{B_3} |p||u|^{\frac{s-2}{2}}|\nabla|u|^{s/2}|$ on the right hand side of (2.17). We will discus the assumptions (i)–(v).
(i) $\frac{2}{\alpha}+\frac{3}{\gamma}<2$, $\gamma\in(3,\infty)$ and $\alpha \ge 2$. Then there exists $s\in(3,\gamma)$, such that

$$2-\frac{2}{\alpha}-\frac{3}{\gamma} \ge \frac{s-3}{\gamma}. \tag{2.19}$$

By the Hölder inequality and the Young inequality we have

$$\begin{aligned} I &\le \|p\|_s\|u\|_s^{\frac{s-2}{2}}\|\nabla|u|^{\frac{s}{2}}\|_2 \\ &\le \|\nabla|u|^{\frac{s}{2}}\|_2^2 + c\|p\|_s^2\|u\|_s^{s(1-\frac{2}{s})} \\ &\le \|\nabla|u|^{\frac{s}{2}}\|_2^2 + c\|p\|_\gamma^\alpha\|u\|_s^{s(1-\frac{2}{s})}. \end{aligned} \tag{2.20}$$

Without loss of generality we can suppose that $\|u\|_s > 1$ for almost every $t\in(t_0-\tau,t_0)$. The inequality (2.20) then gives that $I \le \|\nabla|u|^{s/2}\|_2^2 + c\|p\|_\gamma^\alpha\|u\|_s^s$ and by the use of (2.17) and Lemma 2.5 we get that

$$u\in L^\infty(t_0-\tau,t_0;L^s(B_3)). \tag{2.21}$$

We will show that (2.21) holds also in the case of conditions (ii)–(v).
(ii) $\frac{2}{\alpha}+\frac{3}{\gamma}<2$, $\gamma\in(3,\infty)$ and $\Omega=\mathbb{R}^3$. Again, as in (i), there exists $s\in(3,\gamma)$ such that (2.19) is satisfied. Then

$$I \le \|p\|_s\|u\|_s^{\frac{s-2}{2}}\|\nabla|u|^{\frac{s}{2}}\|_2 \le \|p\|_{\frac{s}{2}}^{\frac{\gamma-s}{2\gamma-s}}\|p\|_\gamma^{\frac{\gamma}{2\gamma-s}}\|u\|_s^{\frac{s-2}{2}}\|\nabla|u|^{\frac{s}{2}}\|_2 \le c(I_1+I_2), \tag{2.22}$$

where we used the interpolation inequality and

$$I_k = \|p_k\|_{\frac{s}{2}}^{\frac{\gamma-s}{2\gamma-s}}\|p\|_\gamma^{\frac{\gamma}{2\gamma-s}}\|u\|_s^{\frac{s-2}{2}}\|\nabla|u|^{\frac{s}{2}}\|_2, \quad k=1,2.$$

Due to the definition of p_1 in (2.18) and the Calderon-Zygmund inequality we have

$$\begin{aligned} I_1 &\le \|u\|_s^{\frac{2\gamma-2s}{2\gamma-s}+\frac{s-2}{2}}\|p\|_\gamma^{\frac{\gamma}{2\gamma-s}}\|\nabla|u|^{\frac{s}{2}}\|_2 \le \|\nabla|u|^{\frac{s}{2}}\|_2^2 + c\|u\|_s^{s(1-\frac{2}{2\gamma-s})}\|p\|_\gamma^{\frac{2\gamma}{2\gamma-s}} \\ &\le \|\nabla|u|^{\frac{s}{2}}\|_2^2 + c\|u\|_s^{s(1-\frac{2}{2\gamma-s})}\|p\|_\gamma^\alpha, \end{aligned} \tag{2.23}$$

since $\frac{2\gamma}{2\gamma-s}\le\alpha$ as a consequence of the inequality (2.19). Furthermore,

$$I_2 \le \|\nabla|u|^{\frac{s}{2}}\|_2^2 + c\|u\|_s^{s-2}\|p\|_\gamma^{\frac{2\gamma}{2\gamma-s}}\|p_2\|_{\frac{s}{2}}^{\frac{2\gamma-2s}{2\gamma-s}}. \tag{2.24}$$

Since $\Omega=\mathbb{R}^3$, we know that $p_2\in L^\infty(t_0-\tau,t_0;L^\infty(B_3))$ and

$$I_2 \le \|\nabla|u|^{\frac{s}{2}}\|_2^2 + c\|u\|_s^{s-2}\|p\|_\gamma^{\frac{2\gamma}{2\gamma-s}} \le \|\nabla|u|^{\frac{s}{2}}\|_2^2 + c\|u\|_s^{s(1-\frac{2}{s})}\|p\|_\gamma^\alpha. \tag{2.25}$$

In the same way as in (i) we can conclude from (2.17), (2.22), (2.23), (2.25) and Lemma 2.5 that (2.21) is satisfied.
(iii) $\frac{2}{\alpha}+\frac{3}{\gamma}<2$ and $\gamma\in(\frac{3}{2},3]$. Let us suppose firstly that

$$\gamma\in(9/4,3]. \tag{2.26}$$

Then there exists $s>3$ such that

$$\gamma > \frac{3s}{s+1} \quad \text{and} \quad 2-\frac{2}{\alpha}-\frac{3}{\gamma} \ge 1-\frac{3}{s}. \tag{2.27}$$

The Hölder inequality, the interpolation inequality, the Sobolev inequality and the Young inequality give

$$\begin{aligned} I &\le \|p\|_\gamma \|u\|_{\frac{\gamma(s-2)}{\gamma-2}}^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \le \|p\|_\gamma \|u\|_s^{\frac{\gamma s-3s+\gamma}{2\gamma}} \|u\|_{3s}^{\frac{3s-3\gamma}{2\gamma}} \|\nabla |u|^{\frac{s}{2}}\|_2 \\ &\le \|p\|_\gamma \|u\|_s^{\frac{\gamma s-3s+\gamma}{2\gamma}} \|\nabla |u|^{\frac{s}{2}}\|_2^{\frac{3s-3\gamma}{\gamma s}+1} \le \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|p\|_\gamma^{\frac{2\gamma s}{\gamma s-3s+3\gamma}} \|u\|_s^{s(1-\frac{2\gamma}{\gamma s-3s+3\gamma})} \\ &\le \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|p\|_\gamma^{\alpha} \|u\|_s^{s(1-\frac{2\gamma}{\gamma s-3s+3\gamma})}. \end{aligned} \tag{2.28}$$

The last inequality follows from $\frac{2\gamma s}{\gamma s-3s+3\gamma} \le \alpha$ (which is a consequence of the second inequality in (2.27)) and (2.21) is satisfied.

Secondly, let

$$\gamma \in (3/2, 9/4]. \tag{2.29}$$

Then there exists $s > 3$ such that

$$2 - \frac{2}{\alpha} - \frac{3}{\gamma} > \frac{s-3}{\alpha}. \tag{2.30}$$

The Hölder inequality and the interpolation inequality give

$$I \le \|p\|_{\frac{3s}{s+1}} \|u\|_{3s}^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \le \|p\|_\gamma^{\frac{\gamma(s-1)}{3s-2\gamma}} \|p\|_{\frac{3s}{2}}^{\frac{3s-\gamma-\gamma s}{3s-2\gamma}} \|u\|_{3s}^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \le c(I_1 + I_2), \tag{2.31}$$

where

$$I_k = \|p\|_\gamma^{\frac{\gamma(s-1)}{3s-2\gamma}} \|p_k\|_{\frac{3s}{2}}^{\frac{3s-\gamma-\gamma s}{3s-2\gamma}} \|u\|_{3s}^{\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2, \quad k = 1, 2.$$

By the Hölder inequality, the Sobolev inequality, the Young inequality and the fact that $\frac{\gamma s-\gamma}{2\gamma-3} \le \alpha$ (which follows from the inequality (2.30)) we have

$$\begin{aligned} I_1 &\le \|p\|_\gamma^{\frac{\gamma(s-1)}{3s-2\gamma}} \|u\|_{3s}^{\frac{6s-2\gamma-2\gamma s}{3s-2\gamma}+\frac{s-2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \le \|p\|_\gamma^{\frac{\gamma(s-1)}{3s-2\gamma}} \|\nabla |u|^{\frac{s}{2}}\|_2^{\frac{6s-8\gamma+6}{3s-2\gamma}} \\ &\le c\|p\|_\gamma^{\frac{\gamma(s-1)}{2\gamma-3}} + \|\nabla |u|^{\frac{s}{2}}\|_2^2 \le c\|p\|_\gamma^\alpha + \|\nabla |u|^{\frac{s}{2}}\|_2^2. \end{aligned} \tag{2.32}$$

Furthermore, by the Sobolev inequality and the Young inequality

$$I_2 \le \|p\|_\gamma^{\frac{\gamma(s-1)}{3s-2\gamma}} \|p_2\|_{\frac{3s}{2}}^{\frac{3s-\gamma-\gamma s}{3s-2\gamma}} \|\nabla |u|^{\frac{s}{2}}\|_2^{\frac{s-2}{s}+1} \le \|\nabla |u|^{\frac{s}{2}}\|_2^2 + c\|p\|_\gamma^{\frac{\gamma s(s-1)}{3s-2\gamma}} \|p_2\|_{\frac{3s}{2}}^{\frac{s(3s-\gamma-\gamma s)}{3s-2\gamma}}. \tag{2.33}$$

Now we show the integrability in time of the second part of the right hand side of the last inequality. The Hölder inequality gives for some $\beta \in (1, 2)$ sufficiently close to 2 that

$$\int_{t_0-\tau}^{t_0} \|p\|_\gamma^{\frac{\gamma s(s-1)}{3s-2\gamma}} \|p_2\|_{\frac{3s}{2}}^{\frac{s(3s-\gamma-\gamma s)}{3s-2\gamma}} \, dt \le \|p\|_{\frac{\beta s\gamma(s-1)}{\beta(3s-2\gamma)-s(3s-\gamma-s\gamma)},\gamma}^{\frac{\gamma s(s-1)}{3s-2\gamma}} \|p_2\|_{\beta,\frac{3s}{2}}^{\frac{s(3s-\gamma-\gamma s)}{3s-2\gamma}}. \tag{2.34}$$

Since $p_2 \in L^\beta(t_0-\tau, t_0; L^\infty(B_3))$, we have $\|p_2\|_{\beta,\frac{3s}{2}}^{\frac{s(3s-\gamma-\gamma s)}{3s-2\gamma}} < \infty$. To show the boundedness of the integral on the left hand side of the inequality (2.34), it is now sufficient to realize that

$$\frac{\beta s\gamma(s-1)}{\beta(3s-2\gamma) - s(3s-\gamma-s\gamma)} \le \alpha. \tag{2.35}$$

We know that

$$\frac{2\gamma}{2\gamma-3} < \alpha. \tag{2.36}$$

If we denote

$$g_\gamma(\beta,s)=\frac{\beta s\gamma(s-1)}{\beta(3s-2\gamma)-s(3s-\gamma-s\gamma)}, \tag{2.37}$$

then for every $\gamma\in(\frac{3}{2},\frac{9}{4}]$, g_γ is a continuous function in β,s defined in an open neighbourhood of the point $(2,3)$. It can be easily verified that

$$g_\gamma(2,3)\le\frac{2\gamma}{2\gamma-3}\quad\text{for every }\gamma\in\Big(\frac{3}{2},\frac{9}{4}\Big]. \tag{2.38}$$

For β sufficiently close to 2 and s sufficiently close to 3 the inequality (2.35) now follows from (2.36), (2.37), (2.38) and the continuity of g_γ at the point $(2,3)$. Therefore, the value of the integral on the left hand side of the inequality (2.34) is less than ∞ and (2.21) now follows from (2.17), (2.31), (2.32), (2.33) and Lemma 2.5.
(iv) $\gamma=\infty$, $\alpha=1$ and $\Omega=\mathbb{R}^3$. Then by the Hölder inequality

$$I\le\|p\|_\infty^{\frac12}\|p\|_{\frac{s}{2}}^{\frac12}\|u\|_s^{\frac{s-2}{2}}\|\nabla|u|^{\frac{s}{2}}\|_2\le c(I_1+I_2), \tag{2.39}$$

where

$$I_k=\|p\|_\infty^{\frac12}\|p_k\|_{\frac{s}{2}}^{\frac12}\|u\|_s^{\frac{s-2}{2}}\|\nabla|u|^{\frac{s}{2}}\|_2,\quad k=1,2.$$

In a standard way we can estimate

$$I_1\le\|p\|_\infty^{\frac12}\|u\|_s^{\frac{s}{2}}\|\nabla|u|^{\frac{s}{2}}\|_2\le c\|p\|_\infty\|u\|_s^s+\|\nabla|u|^{\frac{s}{2}}\|_2^2, \tag{2.40}$$

$$I_2\le\|\nabla|u|^{\frac{s}{2}}\|_2^2+c\|p\|_\infty\|p_2\|_{\frac{s}{2}}\|u\|_s^{s(1-\frac{2}{s})}. \tag{2.41}$$

The term $\|p\|_\infty\|p_2\|_{s/2}$ above is integrable in time since $p_2\in L^\infty(t_0-\tau,t_0;L^\infty(B_3))$ and (2.21) now follows from (2.39), (2.40), (2.41), (2.17) and Lemma 2.5.
(v) $\gamma=\frac32$, $\alpha=\infty$ and the norm $\|p\|_{\infty,\frac32}$ is sufficiently small. Then for $s>3$

$$I\le\|p\|_{\frac{3s}{s+1}}\|u\|_{3s}^{\frac{s-2}{2}}\|\nabla|u|^{\frac{s}{2}}\|_2\le\|p\|_{\frac32}^{\frac12}\|p\|_{\frac{3s}{2}}^{\frac12}\|u\|_{3s}^{\frac{s-2}{2}}\|\nabla|u|^{\frac{s}{2}}\|_2\le c(I_1+I_2), \tag{2.42}$$

where

$$I_k=\|p\|_{\frac32}^{\frac12}\|p_k\|_{\frac{3s}{2}}^{\frac12}\|u\|_{3s}^{\frac{s-2}{2}}\|\nabla|u|^{\frac{s}{2}}\|_2,\quad k=1,2.$$

The Calderon-Zygmund inequality gives

$$I_1\le\|p\|_{\frac32}^{\frac12}\|u\|_{3s}^{\frac{s}{2}}\|\nabla|u|^{\frac{s}{2}}\|_2\le\|p\|_{\frac32}^{\frac12}\|\nabla|u|^{\frac{s}{2}}\|_2^2 \tag{2.43}$$

and we see that to finish the proof in this case the sufficient smallness of the norm $\|p\|_{\infty,\frac32}$ is necessary. Furthermore,

$$\begin{aligned}I_2&=\|p\|_{\frac32}^{\frac12}\|p_2\|_{\frac{3s}{2}}^{\frac12}\|u\|_{3s}^{\frac{s-2}{2}}\|\nabla|u|^{\frac{s}{2}}\|_2\le\|p\|_{\frac32}^{\frac12}\|p_2\|_{\frac{3s}{2}}^{\frac12}\|\nabla|u|^{\frac{s}{2}}\|_2^{\frac{2(s-1)}{s}}\\&\le\|\nabla|u|^{\frac{s}{2}}\|_2^2+c\|p\|_{\frac32}^{\frac{s}{2}}\|p_2\|_{\frac{3s}{2}}^{\frac{s}{2}}.\end{aligned} \tag{2.44}$$

The term $\|p\|_{\frac32}^{s/2}\|p_2\|_{\frac{3s}{2}}^{\frac{s}{2}}$ is clearly integrable in time if $s\in(3,4)$. Let us notice that for the estimate of I_2 we did not need the assumption on the smallness of the norm $\|p\|_{\infty,\frac32}$. Again, (2.21) now follows from (2.17), (2.42), (2.43), (2.44) and Lemma 2.5.

Thus, (2.21) holds for every condition (i) - (v) for some $s>3$. As was shown in the proof of Theorem 2.3, we then have $u\in L^\infty(t_0-\tau,t_0;W^{1,2}(B_3))$ and the proof of Theorem 2.6 can be concluded by the use of Theorem 1.2. $\square$

3. Boundary regularity

In this section Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. For $r > 0$ we denote

$$U_r = U_r(\partial\Omega) = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) < r\}.$$

In [10] J. Neustupa proved the following result.

Theorem 3.1. *Let u be a weak solution of* (1.1)–(1.4) *that satisfies the strong energy inequality. Let $0 \le t_1 < t_2 \le T$ and one of the two following conditions be fulfilled:*

(i) *$u \in L^p(t_1, t_2; L^{q^*}(U_r)^3)$ for some $r > 0$, $\frac{2}{p} + \frac{3}{q^*} \le 1$, $p \in [2, \infty]$, $q^* \in (3, \infty]$,*

(ii) *$u \in L^\infty(t_1, t_2; L^3(U_r)^3)$ and $\|u\|_{L^\infty(t_1,t_2;L^3(U_r)^3)}$ is sufficiently small.*

Let $\zeta > 0$ be such that $t_1 + \zeta < t_2 - \zeta$. Then $u \in L^\infty(t_1 + \zeta, t_2 - \zeta; W^{2+\delta,2}(U_\rho)^3)$ and both $\frac{\partial u}{\partial t}$ and ∇p belong to $L^\infty(t_1 + \zeta, t_2 - \zeta; W^{\delta,2}(U_\rho)^3)$ for each $\delta \in [0, \frac{1}{2})$ and $\rho \in (0, r)$.

It follows from Theorem 3.1 that $u \in L^\infty(U_\rho \times (t_1 + \zeta, t_2 - \zeta))$ and there are no singular points near or on the boundary $\partial\Omega$.

We show that the same result as in Theorem 3.1 can be proved if the conditions (i), (ii) are replaced by conditions on the vorticity field ω. To this purpose we are going to work with the suitable weak solutions which satisfy the generalized energy inequality for every smooth test function and were discussed in Introduction. We will prove the following theorem.

Theorem 3.2. *Let (u, p) be a suitable weak solution of* (1.1)–(1.4) *that satisfies the generalized energy inequality* (1.8) *for every $\phi \in C^\infty(\overline{Q}_T), \phi \ge 0$ and every $0 < t_1 \le t_2 < T$. Let*

$$\omega \in L^p(t_1, t_2; L^q(U_R)) \tag{3.1}$$

for some $R > 0$, $\frac{2}{p} + \frac{3}{q} \le 2$, $p \in [2, \infty]$, $q \in [\frac{3}{2}, 3]$. If $p = \infty$ and $q = \frac{3}{2}$ we still suppose that the norm of ω in $L^\infty(t_1, t_2; L^{\frac{3}{2}}(U_R))$ is sufficiently small.

Then either the condition (i) or the condition (ii) from Theorem 3.1 is fulfilled for any $0 < r < R$ and thus all the conclusions of Theorem 3.1 hold. Especially, there exist no singular points of u in $\overline{U}_r \times (t_1 - \zeta, t_2 + \zeta)$ for any $r \in (0, R)$ and $\zeta > 0$.

Proof. Let $0 < r < \eta < R$. We start with the equality (2.13) with U_η instead of B_2 which holds for every $x \in U_r$. Moreover, η can be chosen in such a way that there are no singular points of u on $(\partial U_\eta \cap \Omega) \times (0, T)$. It follows from the fact that u is a suitable weak solution - for a detailed discussion see [11], [12], [13]. The boundary of U_η consists of two parts, $\partial\Omega$ and $\partial U_\eta \cap \Omega$. Let us investigate firstly the boundary integrals in (2.13) over $\partial\Omega$. The second integral is equal to zero due to the homogeneous boundary conditions (1.3). After short computation the first integral can be written as

$$\int_{\partial\Omega} \frac{1}{|x - \xi|} \nabla u(\xi) \cdot n(\xi) \, d_\xi S. \tag{3.2}$$

Let $\xi \in \partial\Omega$ is an arbitrary point. Then $\frac{\partial u_j}{\partial x_i}(\xi) n_k(\xi)$ is a tensor of the third order and therefore $\frac{\partial u_j}{\partial x_i}(\xi) n_j(\xi)$ is a vector. Let us choose a new coordinate system with the x_1 axis in the direction of the outer normal vector to $\partial\Omega$ in the point ξ. Then the vector $\frac{\partial u_j}{\partial x_i}(\xi) n_j(\xi)$ has the form $(\frac{\partial u_1}{\partial x_1}, 0, 0)$ due to the homogeneous boundary conditions (1.3) and the fact that in the new coordinate system $n_2(\xi) = n_3(\xi) = 0$. Using the continuity equation (1.2) and the homogeneous boundary conditions (1.3) we obtain $\frac{\partial u_1}{\partial x_1}(\xi) = -\frac{\partial u_2}{\partial x_2}(\xi) - \frac{\partial u_3}{\partial x_3}(\xi) = 0$. Thus, $\frac{\partial u_j}{\partial x_i}(\xi) n_j(\xi)$ is a zero vector in any coordinate system. We can conclude that the integral (3.2) is equal to zero and the equality (2.13) can be written as

$$\begin{aligned} u(x) &= \frac{1}{4\pi} \int_{U_\eta} \nabla_\xi \frac{1}{|x-\xi|} \times \omega(\xi)\, d\xi \\ &+ \frac{1}{4\pi} \int_{\partial U_\eta \cap \Omega} \frac{1}{|x-\xi|} \Big(\omega(\xi) \times n(\xi) + \frac{\partial u}{\partial n}(\xi) \Big)\, d_\xi S \\ &- \frac{1}{4\pi} \int_{\partial U_\eta \cap \Omega} \frac{\partial}{\partial n_\xi} \frac{1}{|x-\xi|} u(\xi)\, d_\xi S = u_1(x) + u_2(x). \end{aligned} \tag{3.3}$$

Since $u \in C^\infty(\overline{\Omega})$ for almost every $t \in (0,T)$ (see e.g. [4], Theoreme de Structure), the equality (3.3) holds for almost every $t \in (t_1, t_2)$.

Now, since $\frac{1}{|x-\xi|} \le \frac{1}{\eta - r}$ for every $x \in U_r$ and every $\xi \in \partial U_\eta \cap \Omega$ and u is bounded on $(\partial U_\eta \cap \Omega) \times (t_1, t_2)$, we have that

$$u_2 \in L^\infty(U_r \times (t_1, t_2)). \tag{3.4}$$

For the first integral in (3.3) we use the result from [5], Lemma 7.12, and get that $u_1 \in L^p(t_1, t_2; L^{q^*}(U_r))$, where $\frac{1}{q} - \frac{1}{q^*} = \frac{1}{3}$ and $\frac{2}{p} + \frac{3}{q^*} = 1$. Therefore, if $\frac{3}{2} < q \le 3$ and $3 < q^* \le \infty$, then u_1 and also u (see (3.4)) satisfy the condition (i) from Theorem 3.1. If $q = \frac{3}{2}$ and $q^* = 3$, then u_1 satisfies the condition (ii) from Theorem 3.1. The proof of Theorem 3.2 can now be concluded by the use of (3.4) and Theorem 2.3. □

In the final part of this paper, we discuss the paper [16], in which the following result on the boundary regularity of weak solutions was proved.

Theorem 3.3. *Let u be a weak solution of* (1.1)–(1.4), *$x_0 \in \partial\Omega$, $0 < t_1 \le t_2 < T$ and $\delta > 0$. We denote $\Omega_1 = B_\delta(x_0) \cap \Omega$ and suppose that $\partial\Omega_1 \cap \partial\Omega$ is a part of a plane. Let one of the two following conditions be fulfilled:*

(i) *$u \in L^p(t_1, t_2; L^{q^*}(\Omega_1))$, $\frac{2}{p} + \frac{3}{q^*} \le 1$, $p \in [2, \infty]$, $q^* \in (3, \infty]$*

(ii) *$u \in L^\infty(t_1, t_2; L^3(\Omega_1))$ and $\|u\|_{L^\infty(t_1,t_2;L^3(\Omega_1))}$ is sufficiently small.*

Let $\zeta > 0$ be such that $t_1 + \zeta < t_2 - \zeta$. Then u has no singular points in $(B_{\delta'}(x_0) \cap \overline{\Omega}) \times (t_1 + \zeta, t_2 - \zeta)$ for every $\delta' \in (0, \delta)$.

As for Theorem 3.2, we prove the following version of Theorem 3.3, involving the conditions on the vorticity ω.

Theorem 3.4. *Let u be a weak solution of* (1.1)–(1.4) *that satisfies the strong energy inequality* (1.5), *$x_0 \in \partial\Omega$, $0 < t_1 \le t_2 < T$ and $\delta > 0$. We denote $\Omega_1 = B_\delta(x_0) \cap \Omega$ and suppose that $\partial\Omega_1 \cap \partial\Omega$ is a part of a plane. Let*

$$\omega \in L^p(t_1, t_2; L^q(\Omega_1)) \quad \text{and} \quad \nabla u \in L^p(t_1, t_2; L^1(\Omega_1)), \tag{3.5}$$

with $\frac{2}{p} + \frac{3}{q} \le 2$, $p \in [2, \infty]$, $q \in [\frac{3}{2}, 3]$. If $p = \infty$ and $q = \frac{3}{2}$ we still suppose that the norm of ω in $L^\infty(t_1, t_2; L^{\frac{3}{2}}(\Omega_1))$ is sufficiently small.

Let $\zeta > 0$ be such that $t_1 + \zeta < t_2 - \zeta$. Then u has no singular points in $(B_{\delta'}(x_0) \cap \overline{\Omega}) \times (t_1 + \zeta, t_2 - \zeta)$ for every $\delta' \in (0, \delta)$.

Proof. We proceed as in the proof of Theorem 3.2. Given $0 < \delta' < \delta$ there exist η_1, $\eta = \eta(t)$ and δ'' such that $0 < \delta' < \delta'' < \eta_1 < \eta < \delta$. We start with the equality (2.13) with B_2 replaced by $B_\eta(x_0) \cap \Omega$ which holds for every $x \in B_{\delta''}(x_0) \cap \Omega$. In the same way as in the proof of Theorem 3.2 we get that the boundary integrals over $\partial\Omega$ are equal to zero and we have

$$\begin{aligned} u(x) &= \frac{1}{4\pi} \int_{B_\eta(x_0)\cap\Omega} \nabla_\xi \frac{1}{|x-\xi|} \times \omega(\xi)\, d\xi \\ &\quad + \frac{1}{4\pi} \int_{\partial B_\eta(x_0)\cap\Omega} \frac{1}{|x-\xi|} \Big(\omega(\xi) \times n(\xi) + \frac{\partial u}{\partial n}(\xi) \Big)\, d_\xi S \\ &\quad - \frac{1}{4\pi} \int_{\partial B_\eta(x_0)\cap\Omega} \frac{\partial}{\partial n_\xi} \frac{1}{|x-\xi|} u(\xi)\, d_\xi S = u_1(x) + u_2(x), \end{aligned} \tag{3.6}$$

which holds for every $x \in B_{\delta''}(x_0) \cap \Omega$ and almost every $t \in (t_1, t_2)$. Now, u_1 can be treated in the same way as in Theorem 3.2 and we get that u_1 fulfills the conditions (i) (if $q > \frac{3}{2}$) or (ii) (if $p = \infty$ and $q = \frac{3}{2}$) from Theorem 3.3, where $\frac{1}{q} - \frac{1}{q^*} = \frac{1}{3}$.

Let us discuss u_2. Let $t \in (t_1, t_2)$. There exists $\eta = \eta(t) \in (\eta_1, \delta)$ such that

$$\int_{\partial B_\eta(x_0)\cap\Omega} |\nabla u|\, d_x S \le \frac{1}{\delta - \eta_1} \int_{(B_\delta(x_0)\setminus B_{\eta_1}(x_0))\cap\Omega} |\nabla u|\, dx. \tag{3.7}$$

Thus, for every $x \in B_{\delta''}(x_0) \cap \Omega$ we have due to the definition u_2 and (3.7) that

$$|u_2(x)| \le c \int_{\partial B_\eta(x_0)\cap\Omega} |\nabla u|\, d_x S \le c \int_{\Omega_1} |\nabla u|\, dx \tag{3.8}$$

and

$$\|u_2\|_{L^\infty(B_{\delta''}(x_0)\cap\Omega)} \le c \|\nabla u\|_{L^1(\Omega_1)}. \tag{3.9}$$

It follows from (3.5) and (3.9) that

$$u_2 \in L^p(t_1, t_2; L^\infty(B_{\delta''}(x_0) \cap \Omega)) \tag{3.10}$$

and the conclusion of Theorem 3.4 follows from (3.10), the facts proved above on u_1 and Theorems 3.3, and 2.3. □

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Institute of Hydrodynamics, Czech Academy of Sciences, Pod Paťankou 30/5, 166 12 Prague 6, Czech Republic

E-mail address: skalak@mat.fsv.cvut.cz, skalak@ih.cas.cz